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TECHNICAL MEMORANDUM

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 1002

CONTRIBUTION TO THE IDEAL EFFICIENCY
OF SCREW PROPELLERS

By Wilhelm Hoff

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The stipulation of best thrust distribution is applied to the annular elements of the screw propeller with infinitely many blades in frictionless, incompressible flow and an ideal jet propulsion system derived possessing hyperbolic angular velocity distribution along the blade radius and combining the advantage of uniform thrust distribution over the section with minimum slipstream and rotation losses. This system is then compared with a propeller possessing the same angular velocity at all blade elements and the best possible thrust distribution secured by means of an induced efficiency varying uniformly over the radius. Lastly, the case of the lightly loaded propeller also is discussed.

SUMMARY

The assumptions of the simple momentum theory providing uniform thrust distribution over the swept-disk area and minimum slipstream losses cannot be fulfilled with a system the power of which is derived from a rotating shaft with blades set at right angles to the direction of air flow. Rotation losses are involved.

Assuming a hyperbolic distribution over the radius of angular velocities of the blade elements for the jet propulsion system - which, admittedly, is difficult to realize technically - the uniformly distributed axial losses in the slipstream remain unchanged. Uniformly distributed rotation losses also occur; a low pressure is produced in the slipstream. The system is thus termed "ideal."

*"Bermerkungen zum idealen Wirkungsgrad von Schraubenpropellern." Luftfahrtforschung, vol. 18, no. 4, April 22, 1941, pp. 114-121.

But propellers have the same angular velocity in all annular elements. The condition of best thrust distribution with minimum losses is fulfilled by an arrangement in which the induced efficiency of the annular elements changes between an inner and an outer limit. The thrust and power of this propeller are compared with the ideal case; a slightly inferior efficiency, necessarily associated with the high coefficient of advance for elements near the propeller hub, is noted.

Lightly loaded propellers have been extensively explored in the theory. Their effect is included in the comparison with the other two arrangements.

The derivations are explained by graphs, particularly of the thrust factor k_s and the power factor k_l in relation to the induced efficiency η_i and the outside coefficient of advance Λ.

NOTATION

r	radius of annular element of the propeller
R	radius of outer boundary of the propeller
S	propeller thrust
dA	element of lift
dT	element of tangential force
dM	element of torque
N	propeller power
N_v	power loss
v	translatory speed
w	speed parallel to thrust axis
w^*	radial component of speed
ω	angular velocity of annular element of slipstream

Ω angular velocity of annular element of propeller

Ω_a angular velocity of outside boundary of propeller

$\lambda = \frac{v}{\Omega r}$ coefficient of advance of annular element of the propeller

$\Lambda = \frac{v}{\Omega_a R}$ coefficient of advance of the outside boundary of the propeller

$k_s = \frac{2S}{\pi \rho \Omega_a^2 R^4}$ thrust factor

$k_l = \frac{2N}{\pi \rho \Omega_a^3 R^5}$ power factor

ρ air density

$q = (\rho/2) v^2$ dynamic pressure

p pressure

η_a axial efficiency

η_u rotation efficiency

η_i induced efficiency

α ratio of speed increase of slipstream element at the point of an annular element of propeller at distance r from the axis to the speed v: $w = (1 + \alpha)v$, and $\alpha = \frac{w}{v} - 1$, respectively,

β_i induced angle of advance

γ ratio of half the angular velocity increase of slipstream element at the point of the propeller to the angular velocity of the blade element: $\omega = 2\gamma\Omega$

δ	ratio of half the speed increase in annular element at the point of the developed slipstream behind the propeller to the speed: $w_1 = (1 + 2\delta)v$
$\epsilon = (1 - \eta_{ir})$	substitute mathematical quantity
$x = \frac{r}{R}$	ratio of radii
C	constant of best thrust distribution
k	constant in equations (5.2), (5.3), and (5.4).
J_1, J_{1m}, J_2	mathematical quantities in (5.8), (6.6), and (7.8).

Subscripts

0	for upstream from the propeller
1	for downstream from the propeller
B	relates to Bernoulli's equation
Z	relates to centrifugal pressure
r	relates to annular element at distance r from thrust axis
R	relates to annular outside propeller boundary
ri	relates to annular axis of slipstream
ra	relates to a very wide outside radius
i	relates to induced flow
m	averaged values

	For upstream in slipstream	In the arrangement		For down-stream in slipstream
		center	aft	
radius	R_0, r_0	R, r	R, r	R_1, r_1
speed	$w_0 = v$	w	w	w_1
angular velocity of propeller		Ω, Ω_a		
angular velocity of slipstream		$\frac{1}{2}\omega$	ω	ω_1
pressure	p_0	p'	p''	p_1

I. ELEMENTARY MOMENTUM THEORY

The technical problem of the system identified as "propeller" by reason of its shape and operating method is to so create a slipstream with minimum losses that its reaction provides the best possible thrust. If this task is visualized as achieved by a "jet propulsion system," which need not be closely identified and described, nor necessarily be a "propeller" in the ordinary sense, the best possible thrust and the least slipstream losses can be secured by means of a simple momentum consideration. They are achieved when thrust and losses are uniformly distributed over the slipstream section. Rotation and viscosity losses are discounted in this classical, simple momentum theory.

After Bendemann, Prandtl, and Madelung, in their work "Practical Propeller Calculation," 1917, (reference 1) had pointed out the significance of the (subsequently, (reference 2) so called) axial efficiency, many latter reports on propellers utilized the (Bendemann) limits given in this efficiency for comparison with practical results and for estimating its quality.

With the chosen system of notation the simple momentum theory defines the thrust factor at:

$$k_s = \frac{2S}{\pi\rho\Omega_a{}^2 R^4} = 4\,\Lambda^2 (1 + \alpha)\,\alpha = 4\,\Lambda^2 \frac{(1 - \eta_a)}{\eta_a{}^2} \qquad (1.1)$$

the power factor at

$$k_l = \frac{2N}{\pi\rho\Omega_a{}^3 R^5} = 4\,\Lambda^3 (1 + \alpha)^2\,\alpha = 4\,\Lambda^3 \frac{(1 - \eta_a)}{\eta_a{}^3} \qquad (1.2)$$

the axial efficiency at

$$\eta_a = \frac{vS}{N} = \frac{k_s}{k_l} = \frac{1}{1 + \alpha} \qquad (1.3)$$

the speeds at:

$$\alpha = \frac{1 - \eta_a}{\eta_a} = \text{const over-all radii of the propeller} \qquad (1.4)$$

$$\frac{\delta}{\alpha} = 1 = \text{const over-all radii of the slipstream} \qquad (1.5)$$

II. SPEED AND PRESSURE IN THE ENLARGED MOMENTUM THEORY (ANNULAR ELEMENT)

On the jet-propulsion system developed to its highest efficiency, the "propeller," the power is introduced by a revolving shaft whose axis points in the direction of the thrust-producing stream. This power transfer is accompanied by supplementary rotation losses defined by the distribution of the effective torque and the angular velocity of the annular elements of the propeller.

Now the processes are no longer as clear (reference 4) as in the afore-mentioned simple momentum theory and are expediently treated at the annular element (fig. 1), since the integration over all annular elements of the propeller and in the slipstream is beset with many difficulties as explained elsewhere by Betz and Helmbold. (See reference 5.)

Glauert (reference 6) has given a lucid derivation of the processes on the annular elements patterned after an old and recently reprinted article by N. E. Joukowsky. (See reference 7.)

From far upstream to the point of the arrangement (propeller) the annular element follows the Bernoulli equation

$$p_o + \frac{1}{2}\rho v^2 = p' + \frac{1}{2}\rho\ (w^2 + w^{*2})$$

and from the rear of the system far into the wake:

$$p'' + \frac{1}{2}\ \rho\ (w^2 + w^{*2} + \omega^2 r^2) = p_{1B} + \frac{1}{2}\ \rho\ (w_1^{\ 2} + \omega_1^{\ 2} r_1^{\ 2})$$

whence the pressure difference between the points for upstream and downstream follows at

$$(p_o - p_{1B}) = \frac{1}{2}\rho(w_1^{\ 2} - v^2) + \frac{1}{2}\rho(\omega_1^{\ 2} r_1^{\ 2} - \omega^2 r^2) - (p'' - p') \qquad (2.1)$$

The blade elements rotate with Ω, setting the annular elements in rotation. The relative angular velocity of the blade element to the rotating slipstream is $(\Omega - \omega)$; consequently the pressure jump produced at the propeller is

$$(p'' - p') = \frac{1}{2}\rho[\Omega^2 - (\Omega - \omega)^2]\ r^2 = \frac{1}{2}\rho\ [2\Omega - \omega]\ \omega\ r^2 \qquad (2.2)$$

Combining (2,1) and (2,2) affords

$$(p_o - p_{1B}) = \frac{1}{2}\ \rho\ (w_1^{\ 2} - v^2) - \frac{1}{2}\ \rho\ [2\Omega\omega r^2 - \omega_1^{\ 2} r_1^{\ 2}]$$

On passing down the slipstream the vortex strength

$$\omega r^2 = \omega_1 r_1^{\ 2} \qquad (2.3)$$

is maintained.

Next, (2.3) is entered and rearranged:

$$\frac{p_o - p_{1B}}{q} = \left[\left(\frac{w_1}{v}\right)^2 - 1\right] - \frac{2\Omega\omega_1 r_1^{\ 2}}{v^2} + \frac{\omega_1^{\ 2} r_1^{\ 2}}{v^2} \qquad (2.4)$$

whence the pressure rise over the slipstream radius for downstream follows at

$$-\frac{1}{q}\frac{dp_{1B}}{dr_1} = \frac{d\left[\left(\frac{w_1}{v}\right)^2 - 1\right]}{dr_1} - 2\,\frac{d[\Omega\omega_1 r_1{}^2]}{v^2 dr_1} + \frac{d[\omega_1{}^2 r_1{}^2]}{v^2 dr_1} \qquad (2.5)$$

The wake rotation causes centrifugal pressures, the rise of which over the radius is

$$\frac{1}{q}\frac{dp_{1Z}}{dr_1} = \frac{\rho 2\pi r_1 dr_1 \; 1 \; \omega_1{}^2 r_1{}^2}{2\pi r_1 \; 1 \; r_1 \; dr_1 \; \rho/2 \; v^2} = \frac{2\omega_1{}^2 r_1}{v^2} \qquad (2.6)$$

The difference between the rise due to the centrifugal pressure and that due to the dynamic pressure brings the total pressure rise to

$$\frac{1}{q}\frac{dp_1}{dr_1} = \frac{1}{q}\left[\frac{dp_{1Z}}{dr_1} - \frac{dp_{1B}}{dr_1}\right] \qquad (2.7)$$

If, as will be explained later on, the angular velocities Ω and ω_1, are assumed variable over the radius, the insertion of (2.5) and (2.6) followed by differentiation, gives:

$$\frac{1}{q}\frac{dp_1}{dr_1} = + \frac{d\left[\left(\frac{w_1}{v}\right)^2 - 1\right]}{dr_1} - \frac{2[\Omega - \omega_1]}{v^2}\,\frac{d(\omega_1 r_1{}^2)}{dr_1} - \frac{2\omega_1 r_1{}^2}{v^2}\,\frac{d\Omega}{dr_1} \qquad (2.8)$$

and with it the differential equation for the pressures in the slipstream (reference 6, page 192, equation (1.9) extended)

$$\frac{dp_1}{q} = d\left[\left(\frac{w_1}{v}\right)^2 - 1\right] - \frac{2(\Omega - \omega_1)}{v^2}\,d(\omega_1 r_1{}^2) - \frac{2\omega_1 r_1{}^2}{v^2}\,d\Omega \qquad (2.9)$$

Applying the arguments of the simple momentum theory to the annular element, the thrust element becomes, on the one hand,

$$dS = \rho 2\pi r_1 dr_1 (w_1 - v)\, w_1 - 2\pi r_1 dr_1 (p_0 - p_{1B}) \qquad (2.10)$$

and with (2.2):

$$dS = 2\pi r dr \ (p'' - p') = \frac{1}{2}\, \rho [2\Omega - \omega]\, \omega r^2 2\pi r dr \qquad (2.11)$$

on the other.

For each annular element the equation of continuity

$$v r_0 dr_0 = w\ r\ dr = w_1 r_1 dr_1 \qquad (2.12)$$

holds true.

Combining (2.10) and (2.11) and entering (2.4) and (2.12) gives

$$(2\Omega - \omega)\omega r^2 = 2w(w_1 - v) - \frac{w}{w_1}\left[(w_1{}^2 - v^2) - (2\Omega - \omega_1)\omega_1 r_1{}^2\right]$$

and rearranged:

$$\frac{1}{2}\ \frac{(w_1 - v)^2}{v^2} = \left[\frac{(\Omega - 1/2\omega)}{w} - \frac{(\Omega - 1/2\omega_1)}{w_1}\right] \frac{w_1 \omega_1 r_1{}^2}{v^2} \qquad (2.13)$$

This formula (2.13) must be satisfied for the speeds for each annular element of the slipstream. (See reference 6, p. 193.)

A further fact to be borne in mind is that half of the ultimate angular velocity of the still uncontracted slipstream is not reached at the place of the propeller.

For the subsequent considerations the proportionality factors α, δ, γ are introduced in place of the speeds and the coefficient of advance λ for the annular element at the place of the propeller.

Then the pressure equation (2.9) takes the form:

$$\frac{dp_1}{q} = 4\left\{(1+2\delta)d\delta - \Omega\left[1 - 2\gamma\frac{(1+2\delta)}{(1+\alpha)}\right]d\left(\frac{\gamma}{\Omega\lambda^2}\right) - \frac{\gamma}{\lambda^2}\frac{d\Omega}{\Omega}\right\} \quad (2.14)$$

while the speed equation (2.13) reads:

$$\lambda^2 = \frac{\gamma(2\delta - \alpha)}{\delta^2(1+\alpha)} \quad (2.15)$$

This equation (2.15) can also be solved, if necessary, according to the ratios α, δ, and γ. It should be remembered that the ratio δ is markedly affected by the disposition of adjacent annular elements in the slipstream.

III. GEOMETRICAL RELATIONSHIP AND EFFICIENCY IN THE ENLARGED MOMENTUM THEORY (ANNULAR ELEMENTS)

With the omission of the radial velocity component the speed diagram for the annular element at the propeller is plotted as in figure 2. The flow loss $\sqrt{\alpha^2 v^2 + \gamma\,\Omega^2 r^2}$ is at right angles to the principal inflow. As is seen $\alpha = 0$ for $\gamma = 0$. The ratios α and γ increase according to geometrically defined condition:

$$\gamma = 1/2\left(1 - \sqrt{1 - 4\lambda^2\alpha(1+\alpha)}\right) \quad (3.1)$$

reaching a maximum at $\gamma = \frac{1}{2}$, after $\lambda^2 = \frac{1}{4\alpha(1+\alpha)}$.

This, however, indicates that $\omega = 2\gamma\Omega = \Omega$, itself has reached the maximum value (slipstream rotation equal to propeller rotation). Added to this the absence of viscosity losses by reason of the assumedly frictionless flow, it follows that the lift element dA must be at right angles to the principal inflow. Bearing in mind the components of the flow loss, the three similar triangles of figure 2 afford the relationship at the place of the propeller for

forces: $$\tan\beta_i = \frac{dT}{dS} \tag{3.2}$$

speed losses: $$\tan\beta_i = \frac{\gamma\Omega r}{\alpha v} = \frac{\gamma}{\alpha\lambda} \tag{3.3}$$

flow velocities: $$\tan\beta_i = \frac{(1+\alpha)v}{(1-\gamma)\Omega r} = \frac{(1+\alpha)}{(1-\gamma)}\lambda \tag{3.4}$$

Herewith the induced efficiency of the annular element (reference 6, p. 198) becomes

$$\eta_{ir} = \frac{vdS}{\Omega dM} = \frac{vdS}{\Omega rdT} = \lambda\frac{dS}{dT} = \frac{(1-\gamma)}{(1+\alpha)} = \eta_{ar}\,\eta_{ur} \tag{3.5}$$

where, as in equation (1.3), the axial efficiency is

$$\eta_{ar} = \frac{1}{(1+\alpha)} \text{ and } \alpha = \frac{1-\eta_{ar}}{\eta_{ar}} \tag{3.6}$$

and the rotation efficiency

$$\eta_{ur} = (1-\gamma) \text{ and } \gamma = (1-\eta_{ur}) \tag{3.7}$$

After equating (3.3) and (3.4) for $\tan\beta_i$ in conjunction with equation (3.5) the proportionality factors can be expressed (reference 6, p. 198) by

$$\alpha = \frac{(1-\eta_{ir})\,\eta_{ir}}{\lambda^2+\eta_{ir}^2} \tag{3.8}$$

$$\gamma = \frac{(1-\eta_{ir})\,\lambda^2}{\lambda^2+\eta_{ir}^2} \tag{3.9}$$

and, in connection with (2.15), by

$$\delta = \frac{(1 - \eta_{ir})}{(\lambda^2 + \eta_{ir})}\left[1 - \sqrt{\frac{\lambda^2(1 - \eta_{ir})}{\lambda^2 + \eta_{ir}^2}}\right] \tag{3.10}$$

Combining (3.6) and (3.8) affords

$$\eta_{ar} = \frac{\eta_{ir}^2 + \lambda^2}{\eta_{ir} + \lambda^2} \tag{3.11}$$

and, by combination of (3.7) and (3.9):

$$\eta_{ur} = \frac{\eta_{ir} + \lambda^2}{\eta_{ir}^2 + \lambda^2}\,\eta_{ir} \tag{3.12}$$

For very high coefficients of advance (inside radii!) λ becomes $\eta_{ar} = 1$ and $\eta_{ur} = \eta_{ir}$; for very small coefficients of advance (outside radii!), $\eta_{ar} = \eta_{ir}$ and $\eta_{ur} = 1$. The intermediate vallues for η_{ar} and η_{ur} in relation to $\frac{1}{\lambda}$ and η_{ir} are shown in figure 3.

According to the simple momentum theory (equation (1,1)) the thrust component at the annular element is

$$dS = \pi\rho\Omega^2 r^2 k_s r\ dr = 4\pi\rho v^2\ (1 + \alpha)\ \alpha r dr \tag{3.13}$$

Following the application of the momentum theorem to the rotational motion the element of the initiated power is

$$dN = \Omega dM = 4\pi\rho v\Omega^2\ (1 + \alpha)\ \gamma r^3 dr \tag{3.14}$$

the thrust coefficient for the annular element is

$$k_s = \frac{2dS}{2\pi\rho\Omega^2 r^3 dr} = 4\lambda^2(1 + \alpha)\ \alpha = 4\lambda^2\frac{(1 - \eta_{ir})(\lambda^2 + \eta_{ir})}{(\lambda^2 + \eta_{ir}^2)^2}\,\eta_{ir}$$

$$\ldots \quad (3.15)$$

and the power factor

$$k_l = \frac{2dN}{2\pi\rho\Omega^3 r^4 dr} = 4\lambda^3(1 + \alpha)\gamma = 4\lambda^3\frac{(1 - \eta_{ir})(\lambda^2 + \eta_{ir})}{(\lambda^2 + \eta_{ir}^2)^2} \qquad (3.16)$$

IV. THE BEST THRUST DISTRIBUTION BY THE ENLARGED MOMENTUM THEORY (ANNULAR ELEMENT)

The solution of (3.5) for the induced efficiency gives

$$\Omega dM - vdS = \alpha vdS + \gamma\Omega dM = dN_v \qquad (4.1)$$

The left-hand side gives the power required to cover the power loss in the slipstream which itself consists of the flow losses in axial direction and from the slipstream rotation. After insertion of

$$dS = \frac{\alpha v}{\gamma\Omega r}\, dT = \frac{\alpha v}{\gamma\Omega r^2}\, dM \qquad (4.2)$$

in equation (3.14) for dM the power loss (reference 6, p. 196) becomes

$$dN_v = 4\pi\rho v^3 \ (1 + \alpha)\left(\frac{\gamma}{\lambda^2} - \alpha\right) rdr \qquad (4.3)$$

The best thrust distribution (reference 4) over the propeller cross section is governed by the condition that a thrust increment $\Delta(dS)$ at any point of the propeller section is accompanied by an equal power decrement $\Delta(dN_v)$.

With equation (3.13) we get

$$\Delta(dS) = 4\pi\rho v^2 \ (1 + 2\alpha)\ \Delta\alpha rdr \qquad (4.4)$$

and with (4.3):

$$\Delta(dN_v) = 4\pi\rho v^3\left[\frac{1}{\lambda^2}(\gamma\Delta\alpha + (1+\alpha)\Delta\gamma) - (1+2\alpha)\Delta\alpha\right] rdr \tag{4.5}$$

With (4.4) and (4.5) the condition for the best thrust distribution becomes

$$\frac{\Delta(dN_v)}{v\Delta(dS)} = \frac{\gamma\Delta\alpha + (1+\alpha)\Delta\gamma}{\lambda^2 (1+2\alpha)\Delta\alpha} - 1 = C \tag{4.6}$$

From the combined equations (3.3) and (3.4) the differentiation gives

$$\lambda^2 (1+2\alpha)\Delta\alpha = (1-2\gamma)\Delta\gamma \tag{4.7}$$

which is entered in (4.6). The elimination of α and γ according to (3.8) and (3.9) leaves the condition for the best thrust distribution (reference 6, p. 197) at:

$$(1+C) = \frac{\gamma}{\lambda^2 (1+2\alpha)} + \frac{(1+\alpha)}{(1-2\gamma)}$$

$$= \frac{1-\eta_{ir}}{\lambda^2 + \eta_{ir}(2-\eta_{ir})} + \frac{\lambda^2 + \eta_{ir}}{\eta_{ir}^2 + \lambda^2 (2\eta_{ir} - 1)} \tag{4.8}$$

For many calculations it is convenient to introduce (reference 6, p. 198):

$$\eta_{ir} = 1 - \epsilon$$

which results in

$$(1+C) = \frac{\epsilon}{\lambda^2 + (1-\epsilon^2)} + \frac{\lambda^2 + (1-\epsilon)}{(1-\epsilon)^2 + \lambda^2 (1-2\epsilon)} \tag{4.9}$$

For annular elements with very small λ-values (outside radii!) we get

$$(1 + C) = \frac{(1 + 2\epsilon_a)}{(1 - \epsilon_a^2)} \text{ and}$$

$$\epsilon_a = \frac{1}{(1 + C)}\left[\sqrt{1 - (1 + C) + (1 + C)^2} - 1\right] \qquad (4.10)$$

for those with very great λ-values (inside radii!)

$$(1 + C) = \frac{1}{1 - 2\epsilon_i} \text{ and } \epsilon_i = \frac{1}{2}\left[1 - \frac{1}{(1 + C)}\right] \qquad (4.11)$$

Constant C can rise from 0 to ∞, thus affording the following limiting values:

C	ϵ_a	η_{ira}	ϵ_i	η_{iri}
0	0	1.0	0	1.0
∞	1	0	.5	.5

(4.12)

The relation of C, η_{ira}, and η_{iri} is illustrated in figure 4, from which the pertinent values of C, η_{ira}, and η_{iri} can be ascertained. It should be noted in particular that inside the value of η_{iri} is never less than 0.5.

Resolved with respect to $(1 + \lambda^2)$, equation (4.9) gives the relation

$$(1 + \lambda^2) = \frac{\epsilon^2}{2}\,\frac{[3 + 2\epsilon\,(1 + C)]}{[1 - (1 - 2\epsilon)\,(1 + C)]}$$

$$\left\{1 + \sqrt{1 - \frac{4[2 + \epsilon(1 + C)]\,[1 - (1 - 2\epsilon)\,(1 + C)]}{\epsilon[3 + 2\epsilon\,(1 + C)]^2}}\right\} \qquad (4.13)$$

The condition of best thrust distribution can be fulfilled in different fashion as shown hereinafter.

V. PROPELLER WITH EQUAL THRUST DISTRIBUTION OVER THE DISK AREA

The "jet propulsion system" can be visualized for the moment (although hardly realizable technically) as being designed with angular velocity alternating over the propeller radius. Keeping the product Ωr at each annular element constant, the angular velocities decrease hyperbolically from the innermost value indefinitely toward the outside. The speed triangles (fig. 2) are maintained for all annular elements, whence the coefficient of advance λ itself and the induced efficiency η_{ir} are constant for all annular elements.

With these assumptions the condition of best thrust distribution (4.8) for all annular elements is satisfied also.

According to equations (3.8), (3.9) and (3.10) the ratios α, γ, and δ themselves become constants for all annular elements.

With $d\delta = 0$, $d\gamma = 0$, and $d\lambda = 0$ the pressure distribution in the slipstream (cf. equation (2.14)) is:

$$\frac{dp_1}{q} = -4\frac{\gamma}{\lambda^2}\left[\Omega(1 - 2\gamma)\frac{(1 + 2\delta)}{(1 + \alpha)}\, d\left(\frac{1}{\Omega}\right) - \frac{d\Omega}{\Omega}\right] \tag{5.1}$$

If $\lambda = \text{const} = \Lambda$ and $\Omega = \frac{\Omega_a R}{r} = \frac{\Omega_a}{x}$ it simplifies

$$\frac{dp_1}{q} = +k\frac{dx}{x} \tag{5.2}$$

constant k denoting

$$k = 8\frac{\gamma^2}{\Lambda^2}\frac{(1 + 2\delta)}{(1 + \alpha)} \tag{5.3}$$

After posting the original values for α, δ, γ, and Λ while bearing in mind (2.3) and (2.12) we get

$$k = 8\ \frac{\omega_a^2\,\Omega_a^2\,R^2\,w_1 v}{4\Omega_a^2 v^2 v\ w} = 2\ \frac{\omega_{1a}^2 R_1^4\,R^2 R^2}{v^2 R^4 R_1^2} = 2\left[\frac{\omega_{1a} R_1}{v}\right]^2 \tag{5.4}$$

The value $\dfrac{v}{\omega_{1a} R_1}$ is to be treated as a kind of coefficient of advance of the rotating slipstream boundary.

At the slipstream boundary, that is, for $x = 1$, $p_1 = p_{1a}$. Within the slipstream the pressure is

$$\frac{1}{q}\int_{p_{1a}}^{p_1} dp_1 = k\int_1^x \frac{dx}{x}$$

or, after integration,

$$\frac{p_1}{q} = \frac{p_{1a}}{q} + k \ln x \tag{5.5}$$

Since $0 < x < 1$, $k \ln x$ is always smaller than 0.

From a pressure of

$$\frac{p_{1i}}{q} = \frac{p_{1a}}{q} - k\,\infty \quad \text{within}$$

it rises to $\dfrac{p_{1a}}{q}$ on the outside.

The arrangement of a jet propulsion system assumed here is designated as "ideal." It should be aimed at if, as is ordinarily not the case and would be important only by ventilation of gears, the flow in hub proximity should be ascribed the same significance as that far outside.

The thrust for the chosen constants $\eta_{ir} = \eta_{iR} = \eta_i$ and $\lambda = \Lambda$ is according to (3.15):

$$k_s = 4 \Lambda^2 \frac{(1 - \eta_i)}{\eta_i^2} J_1 \tag{5.6}$$

and the power, according to (3.16):

$$k_l = 4 \Lambda^3 \frac{(1 - \eta_i)}{\eta_i^3} J_1 \tag{5.7}$$

where the quantity J_1

$$J_1 = \eta_i^3 \frac{(\eta_i + \Lambda^2)}{(\eta_i^2 + \Lambda^2)^2} \tag{5.8}$$

denotes a factor that drops with ascending Λ from 1 to 0. For $\Lambda = 0$, equations (5.6) and (5.7) change to (1.1) and (1.2).

Figure 5 illustrates the correlation of η_i, Λ with k_s and k_l. It applies thus to a jet-propulsion system with uniform thrust distribution over the disk area and simultaneously occurring minimum slipstream and rotation losses. The effect of this propulsion system comes closest to the conditions outlined in the simple momentum theory, the slipstream losses are supplemented by the rotation losses as it must be by the deflection of the force.

VI. PROPELLER WITH THRUST DISTRIBUTED NONUNIFORMLY OVER THE DISK AREA

The ordinary propeller operates with angular velocity Ω equal for all annular elements, consequently the coefficient of advance λ is dependent on the radius. On the inside $\lambda \longrightarrow \infty$ and on the outside $\lambda \longrightarrow 0$. If the condition for best thrust distribution conformably to equation (4.8) is to be maintained for certain constants

(1 + C), the variable coefficients of advance λ define variable induced efficiencies η_{ir} and values ϵ according to equation (4.13).

In figure 6 the efficiencies η_{ir}, η_{ar}, and η_{ur} are plotted against the reciprocal value of $\frac{1}{\lambda}$ for different constants C between C = 0 and C = ∞. As already seen from figure 3 the outside annular elements (small λ, great $\frac{1}{\lambda}$) are important for the production of thrust; in the inner annular elements the rotation losses are primarily induced. For higher thrust the increased angular velocity is lacking. This points toward the fact that the hub region of such a propeller can be discounted as a thrust producer. Nevertheless, this propeller satisfies the condition of best thrust distribution C = const.

As η_{ir} changes over the radius so η_{ar} and η_{ur} themselves assume different values. An average value can be formed only for propellers with given outside coefficient of advance Λ and outside induced efficiency η_{ir}, and even then not in general form.

With $\frac{1}{\lambda} = \frac{r}{R} = x$ we get

$$\eta_{im} = \frac{\sum_{0}^{R} 2\pi\ \eta_{ir}\ r\ \Delta r}{\pi R^2} = 2 \sum_{0}^{1} \eta_{ir} x \Delta x \qquad (6.1)$$

and accordingly

$$\eta_{am} = 2 \sum_{0}^{1} \eta_{ar}\ x\Delta x \qquad (6.2)$$

$$\eta_{um} = 2 \sum_{0}^{1} \eta_{ur}\ x\Delta x \qquad (6.3)$$

For small strips $\Delta x = \Lambda \Delta\left(\frac{1}{\lambda}\right)$ the summation can be made direct on figure 6. With the example

$$\Lambda = 0.333 \text{ and } \frac{1}{\Lambda} = 3 \text{ and}$$

$$\eta_{iR} = 0.658; \quad \eta_{aR} = 0.705; \quad \eta_{uR} = 0.930$$

the average values are

$$\eta_{im} = 0.679 \quad \eta_{am} = 0.775 \quad \eta_{um} = 0.901$$

Despite the choice of high constant $C = 1.0$, η_{iR} and η_{im} differ very little from each other (3.5 percent), while η_{aR} and η_{am} as well as η_{uR} and η_{um} do so much more because of their opposite distribution.

So, for small constants C, it is permissible without introducing abnormal errors to use without summation the values η_{ir}, η_{ar}, and η_{ur}, read off, say for $x = 0.7$, as η_{im}, η_{am}, and η_{um}.

With variable η_{ar} and η_{ur}, α and γ themselves change according to (3.8) and (3.9).

To determine the pressure differences in the propeller annular elements, equation (2.14) the secured values for α, δ, γ, λ, and $d\delta$, $d\gamma$, and $d\lambda$ would have to be inserted. It was omitted in this instance, since it is very tedious and theoretically does not differ from the study in section VII (fig. 10). The essential difficulty (reference 5) involving the arrangement of the annular elements in the slipstream and hence the correct size of δ and $d\delta$ has already been pointed out at the conclusion of section II.

With the average value η_{im} the thrust

$$k_s = 4\,\Lambda^2 \frac{(1 - \eta_{im})}{\eta_{im}^2} J_{1m} \tag{6.4}$$

and the power factor

$$k_l = 4\Lambda^3 \frac{(1 - \eta_{im})}{\eta_{im}^3} J_{1m} \tag{6.5}$$

are formed, where

$$J_{1m} = \eta_{im}^3 \frac{(\eta_{im} + \Lambda^2)}{(\eta_{im}^2 + \Lambda^2)^2} \tag{6.6}$$

according to equation (5.3).

As $\eta_{im} > \eta_{iR}$, smaller η_{iR} must be chosen to achieve the same thrust and power factors of section V. But this is synonymous with lighter propeller loading in section V for a given coefficient of advance Λ. Bearing in mind this difference, the representation (fig. 5) can equally be employed for dealing with correlated values η_{im}, Λ, k_s and k_l.

Equations (3.4) and (3.5) can also be combined to

$$\tan \beta_i = \frac{(1 + \alpha) v}{(1 - \gamma) \Omega r} = \frac{v}{\eta_{ir}\Omega r} \tag{6.7}$$

or, after introduction of $\frac{r}{R} = x$ and Λ:

$$\tan \beta_i = \frac{\frac{v}{\Omega R \eta_{ir}}}{\frac{r}{R}} = \frac{\frac{\Lambda}{\eta_{ir}}}{\frac{r}{R}} = \frac{\frac{\Lambda}{\eta_{ir}}}{x} \tag{6.8}$$

In figure 7 the values $\frac{r}{R} = x$ are plotted as abscissas and the values $\frac{\Lambda}{\eta_{ir}}$ as ordinates at the same scale for $C = 1.0$ and $\Lambda = 0.333$. The induced pitch angles β_i for the different radii are obtained by combining correlated values on the two coordinates.

VII. LIGHTLY LOADED PROPELLER

For small induced efficiencies, corresponding to lightly loaded propellers, higher powers than the first power of ϵ can be disregarded in equation (4.9) for the condition of best thrust distribution. Following the elimination of λ^2 the relation (reference 6, p. 198):

$$(1 + C) = \frac{1}{(1 - 2\epsilon)} = \sim (1 + 2\epsilon) = \sim (3 - 2\eta_{ir})$$

$$C = \sim 2\epsilon = 2(1 - \eta_{ir}) = 2(1 - \eta_i) \qquad (7.1)$$

is produced.

Thus the constants C for lightly loaded propellers are in simple manner tied to the induced efficiencies.

The lightly loaded propellers of this kind have been treated repeatedly and become general knowledge on the basis of a report by Betz (reference 4).

To complete the arguments of the preceding sections the proportional factors α, δ, δ/α, and γ for the constants $\Lambda = 0.2$ and $\eta_i = 0.9$ are shown plotted against $\frac{r}{R}$ in figure 8, and the ratios of the slipstream velocities $\frac{w}{v}$, $\frac{w_1}{v}$ and of $\frac{\omega}{\Omega_a}$, $\frac{\omega_a}{\Omega_a}$ and $\frac{\lambda}{\Lambda}$ for the same constants against $\frac{r}{R}$ in figure 9.

It will be noted that the values $\frac{w_1}{v}$ and $\frac{\omega_1}{\Omega_a}$ for the contracted slipstream were plotted against $\frac{r}{R}$ rather than against the ratio of the slipstream.

Introducing $\Lambda = \frac{R}{r}\Lambda = \frac{\Lambda}{x}$ while disregarding the interference of the annular elements in the slipstream, the pressure equation (2.14) gives

$$\frac{1}{q}\frac{dp}{dx} = 4\left\{(1+2\delta)\frac{d\delta}{dx} - \left[1-2\gamma\,\frac{(1+2\delta)}{(1+\alpha)}\right]\left(\frac{x}{\Lambda}\right)^2\left[2\frac{\gamma}{x} + \frac{d\gamma}{dx}\right]\right\} \qquad (7.2)$$

In explanation of this pressure equation the variation of the values $\frac{1}{q}\frac{dp_1}{dx}$ and their integration $\frac{p_1}{q}$ for the assumption $\frac{p_{1a}}{q} = 1.0$ have been plotted against $\frac{r}{R} = x$ for $\Lambda = 0.2$, $\Lambda = 0.4$, and $\eta_i = 0.9$ in figure 10.

Because a pressure balance, that is, $\frac{p_1}{q} = 1$ must exist at the boundary (x = 1), a low pressure results within the slipstream which increases with the coefficient of advance Λ (reference 5, p. 17, fig. 12).

For the propeller in question the thrust distribution over the ratio of radii follows at

$$\frac{2\,\frac{dS}{dx}}{\rho\, v^2\, \pi\, R^2} = 8\,\Lambda\, G \tag{7.3}$$

and the power distribution at

$$\frac{2\,\frac{dN}{dx}}{\rho\, v^2\, \pi\, R^2} = 8\,\eta_i\,\Lambda\, G \tag{7.4}$$

where, with $\lambda_i = \frac{\lambda}{\eta_i}$, the value G (reference 6, p. 200, fig. 14) reads:

$$G = \frac{(1 - \eta_i)\left[\eta_i + \left(\frac{1}{\lambda_i}\right)^2\right]\left(\frac{1}{\lambda_i}\right)^3}{\eta_i^{\,4}\left[1 + \left(\frac{1}{\lambda_i}\right)^2\right]^2} \tag{7.5}$$

Figure 11 shows G in relation to the reciprocal value of the induced coefficient of advance $1/\lambda_i$ for various η_i; it is substantially a straight line (reference 6, p. 200, fig. 14). So, for many calculations, it is permissible to assume a linear thrust, torque, and power by assumedly uniformly distributed high induced efficiency.

The integration of (7.5) affords (reference 6, p. 200; also, F. Lösch). the thrust

$$k_s = 4\,\Lambda^2\,\frac{(1-\eta_i)}{\eta_i^{\,2}}\,J_2 \tag{7.6}$$

and the power factor

$$k_l = 4\,\Lambda^3\,\frac{(1-\eta_i)}{\eta_i^{\,3}}\,J_2 \tag{7.7}$$

$\frac{\Lambda}{\eta_i} = \Lambda_i$ denoting the integral:

$$J_2 = 1 - \Lambda_i^2\,(2-\eta_i)\ln\left[1+\frac{1}{\Lambda_i^{\,2}}\right] + \frac{(1-\eta_i)}{\left(1+\frac{1}{\Lambda_i^{\,2}}\right)} \tag{7.8}$$

As counterpart to figure 5 the correlation of k_s, k_l, η_i, and Λ is shown for the lightly loaded propeller in figure 12. Comparison shows, as is to be expected, that the curves of figure 15 give somewhat better values than those of figure 12. The discrepancies would be greater if the last examined propeller could be superimposed inside by a higher thrust loading.

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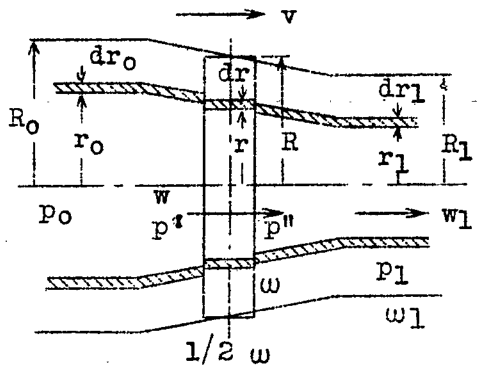
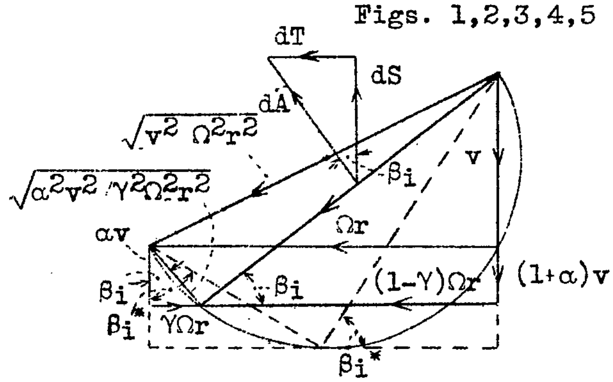


Figure 1.-



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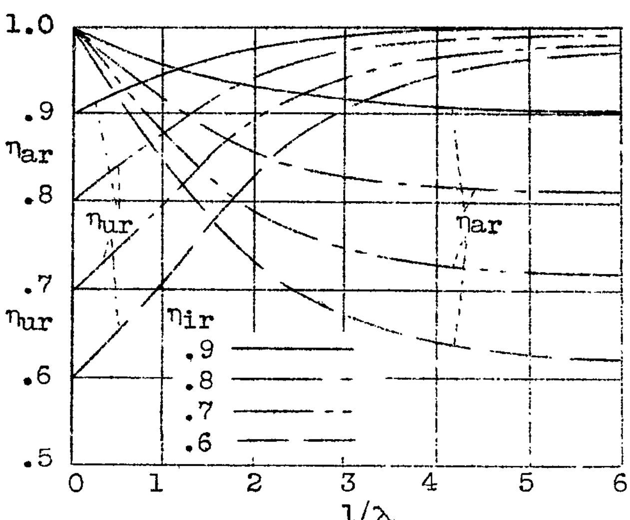

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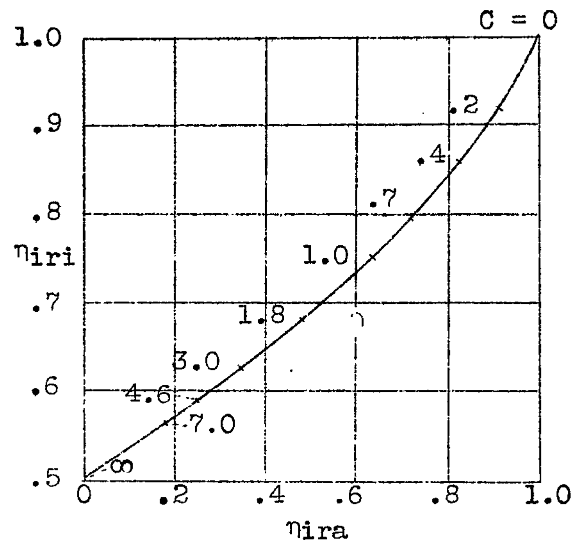

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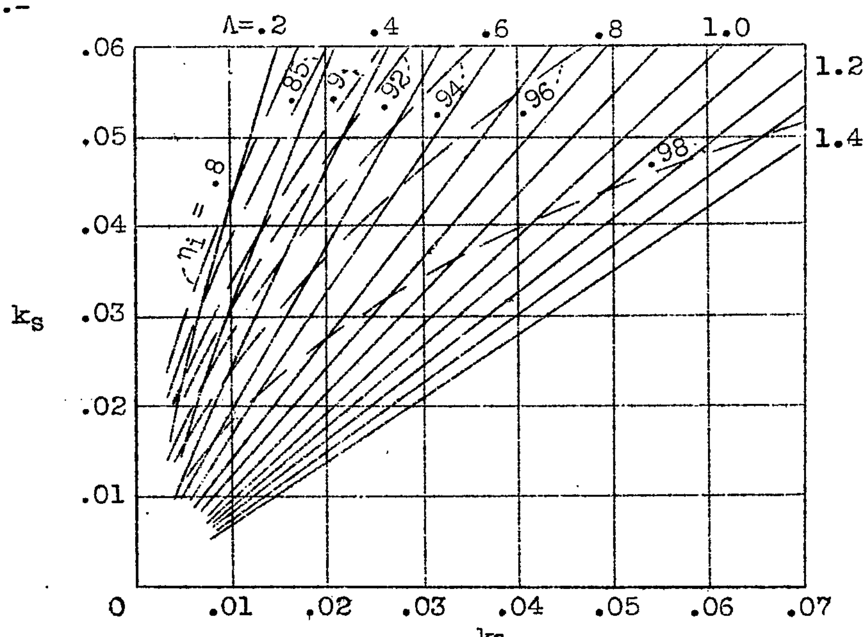

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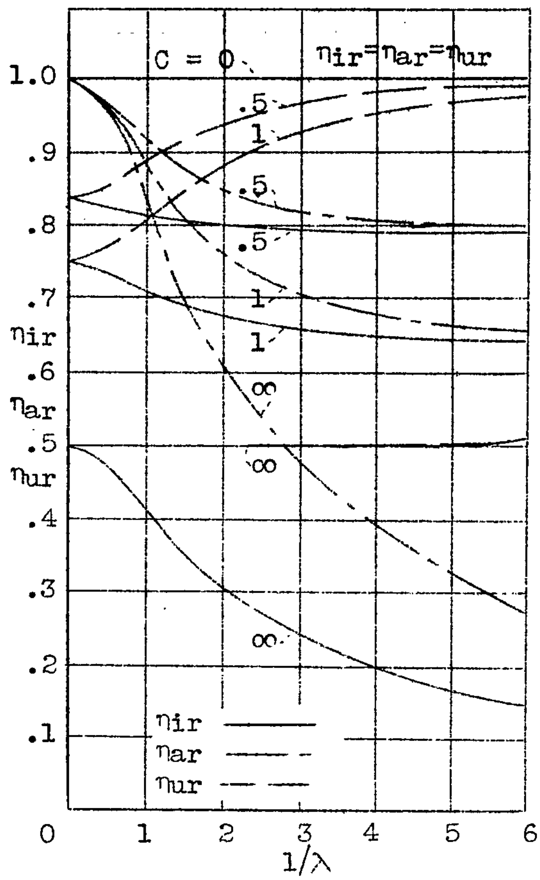
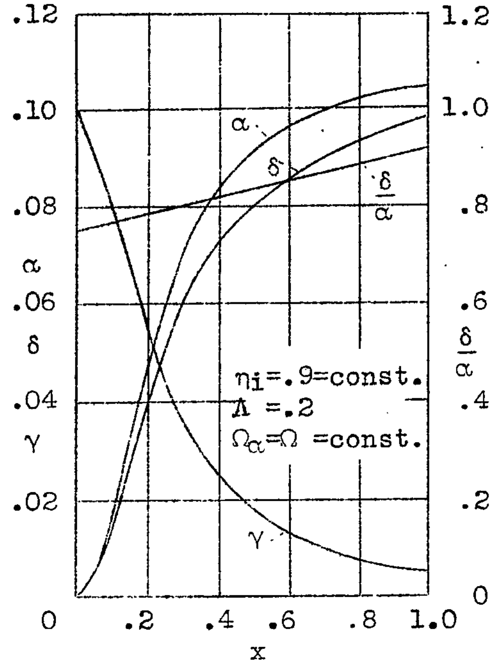


Figure 6.-



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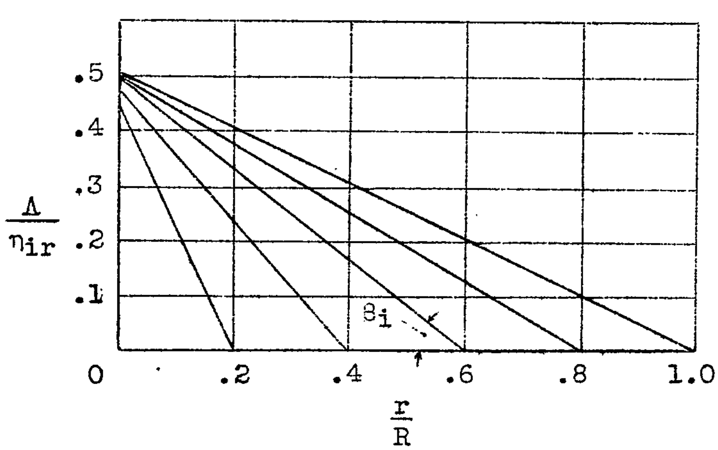

Figure 7.-

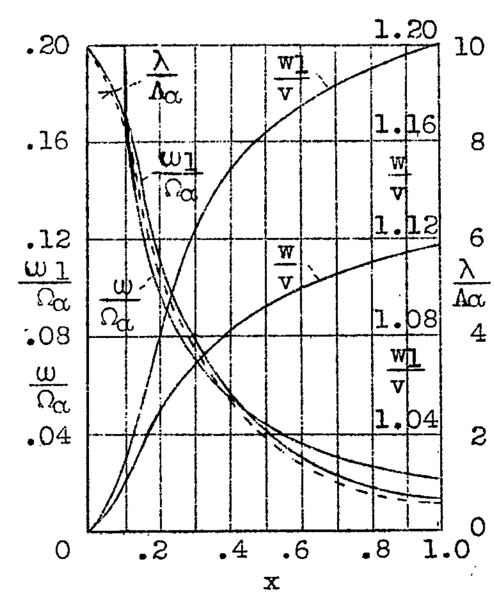

Figure 9.-

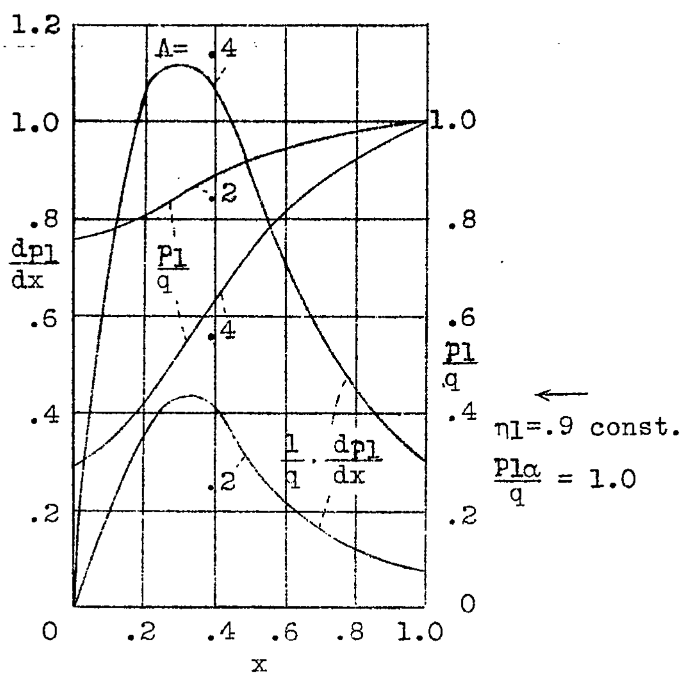

Figure 10.-

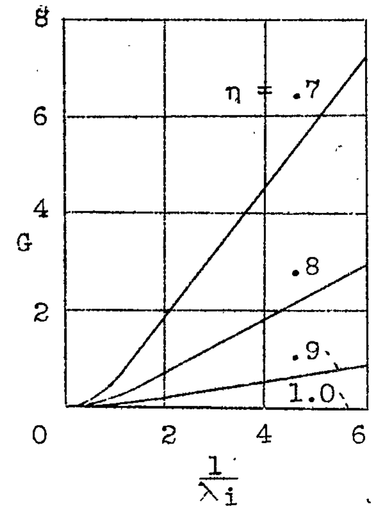

Figure 11.-

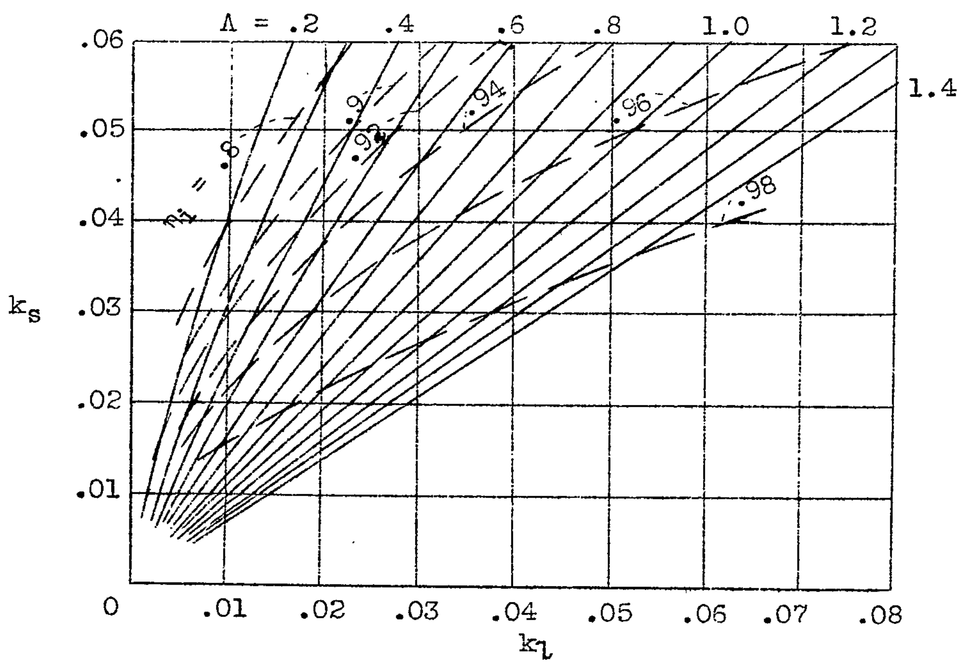

Figure 12.-