NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1402

AEROELASTIC PROBLEMS OF AIRPLANE DESIGN

By H. G. Küssner

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1. INTRODUCTION

For thirty years aeroelastic problems in airplane design were practically unknown, owing to the low speed and rigid design method of aircraft at that time. The first contributions to the unsteady lifting surface theory appeared then, but they created only academic interest. In the meantime, the speed of the airplanes continued to increase, while the aerodynamic refinement dictated consistently thinner and slenderer wing configurations. As a result, the stiffness and the oscillation phenomena connected with it have proved to be a limit of the technique just like the strength.

The aeroelastic processes are complicated because they encompass the whole airplane with its many parts and parameters. Aeroelasticity is in a stage of rapid development of its theoretical and experimental methods. Its scope expands continuously. The development started with studies on wing and aileron flutter. They were followed later by studies on tail and tab flutter. At very high speed, two new phenomena appeared, fluttering of the skin and vibrations on approaching sonic velocity, which are based on the instability of compression shocks at the curved surfaces. An extreme case of flutter is the static instability at zero frequency. Relevant also is the distortion of the wing by the aerodynamic forces and the buckling and distortion of the skin. The reversal of the aileron effect due to the wing distortion, while concerning only the control, can be treated with the same formulas.

As regards the gust stress and the stability of flight motion, the stiffness of the structural components also becomes so much more significant as the wing shapes become thinner and slenderer. This is why these zones are now included in aeroelasticity and are treated in part by the methods developed for flutter. However, their discussion would go beyond the confines of the present report. It also applies to the aeroelastic problems of the rotating wings, the supersonic propellers, and the fan and turbine blades. The literature on aeroelasticity since 1945 is too voluminous to be enumerated here. Some more recent comprehensive descriptions on airplane flutter are presented in the reports by Küssner (ref. 1), Broadbent (ref. 2), Garrick (ref. 3), and

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*"Aeroelastische Probleme des Flugzeugbaus," Zeitschrift für Flugwissenschaften, 3. Jahrgang, Heft 1, Jan. 1955, pp. 1-18.

van de Vooren (ref. 4), and in the introductory article by Scanlan and Rosenbaum (ref. 5), which also contains further references.

2. INDICATIONS OF AIRPLANE VIBRATIONS (FLUTTER)

Flutter rests on the fact that the airplane as a vibratory system absorbs energy from the airstream. This is so much more easily possible as the speed of the airplane is greater. Both the damping energy and the consumption energy from the airstream increase with the flutter amplitude. Depending upon which of these energies grows faster, it may result in mild flutter with constant amplitude or vicious flutter with amplitude increasing steadily to failure.

Simplest of all are the natural vibrations of elastomechanical systems in vacuum. If the system exists in an incompressible fluid without moving in it, the Kelvin impulses are additive; they are the mass forces of the covibrating fluid masses. In the atmosphere, the Kelvin impulses generally amount to only a few percent of the mass forces of the airplane, hence do not change its vibratory behavior appreciably. For these so-called static vibrations, all points of the system pass almost simultaneously through zero position. The nodal lines, that is, the loci of disappearing vibration amplitude, are then so much more sharply pronounced as the damping energy is less. It is customary, although not exactly correct, to identify all static oscillation modes as "oscillations with one degree of freedom."

The static oscillation modes are not capable, as a rule, of taking energy from the airstream; they rather give off energy on the airstream, hence are damped additionally. Absorption of energy is generally possible only when a static oscillation mode is modified by the air forces which are additionally created on the oscillating wing during forward motion. In that event, there is no longer a sharp nodal line; the new oscillation has several degrees of freedom. However, it is possible that certain static oscillation modes with one degree of freedom take up energy from the airstream in certain speed ranges, hence flutter, especially at higher Mach numbers, and in the supersonic range. (Cf. Runyan (ref. 6), Cunningham (ref. 7), Watkins (ref. 8), Weber and Ruppel (ref. 8a).)

To maintain a static oscillation mode with constant amplitude calls for considerable energy input, even for an oscillating system in vacuum. The cause for it is the solid friction of the many individual components on each other, of which the airplane is riveted together, welded, or bonded, as well as the friction of the control bearings and control cables. Solid test rods have about 100 times lower damping than airplane wings of the conventional type, but even in these the damping is probably produced by solid friction on microscopic cracks, grain boundaries, and such. Thus surface effects in the widest sense are involved. It is not likely that these damping losses can be treated differentially; they can be taken into account only integrally and in time-average for the whole airplane. Since it is largely a solid friction that is involved, the damping energy per cycle at forced vibrations of a certain mode and amplitude in a wide range is nearly independent of the oscillation frequency. According to the Navier-Stokes theorem, fluid friction would yield damping proportional to frequency, hence must be excluded. Structural damping losses can be reliably determined only by measurements on the finished airplane. It is customary to allow for the damping energy in the theoretical flutter formula by introducing a constant damping phase angle g_i in all elastic stresses of a certain oscillation mode i; for conventional airplanes, g_i ranges from 0.01 to 0.10 (radians).

In still air, a linearly moving wing can exceed its critical speed without fluttering; it is then in unstable equilibrium. Any shock against such a wing initiates the entire natural frequency spectrum, that is, with a certain distribution of amplitudes and phase angles. But it does not signify that this distribution is suitable for absorbing energy from the airstream. When many shocks act on different parts of the airplane, however, it is increasingly possible that accidentally the right distribution of the natural oscillation modes for absorbing energy from the airstream and initial flutter is excited. This has been observed in wind-tunnel tests with dynamically similar models. Many cases of flutter were observed only in flight through gusty air, whereby a great variety of different shocks are exerted on the airplane. From these observations, it follows that a certain minimum size of shock is necessary to initiate flutter. The extent to which this rests on boundary-layer effects or overcoming solid friction is an open question.

When the flutter of an airplane or of a model wing is recorded, the record often reveals considerable departures from the harmonic or "sinusoidal" vibration, which may be due to aerodynamic causes and to the aforementioned solid friction. For example, there are complicated control surface forms with internal balance, for which the pressures are not linearly dependent on the control angle. Such controls may even flutter in two different partially foreign frequencies simultaneously. If the angle of attack of a wing at landing or pullout becomes so great that the flow starts to separate, there is no further clear coordination between angle of attack and pressure distribution in steady flow. A rotary motion of the wing with one degree of freedom is then able to take up energy from the airstream. A periodic state of oscillation is sustained, which, as a result of the complicated processes in the boundary layer, is not linear at breakway. This type of flutter has been known for a long time and has lately been investigated by Halfman, Johnson, and Haley (ref. 9).

The airplane engineer does not wish the fully developed flutter; he would rather prevent it, though it may please some experimenters. He wants to prove that all physically possible flutter phenomena, and even those with very low amplitude, lie outside the permissible speed of his airplane design. For this purpose, the harmonic, linearized oscillation theorem with simplified assumptions regarding air forces and structural damping has been utilized successfully up to now; it is particularly suitable for stability studies with arbitrary small amplitude. The first attempt involved the problem of making this simplest case of aeroelasticity amenable to mathematical treatment and theoretical solution. This problem is already very difficult and only approximately solvable, as will be shown in the following.

3. THEORY OF STATIC OSCILLATIONS

The airplane is replaced by an elastomechanical system of n point masses and their connecting elastic members, which satisfy the classical theory of elasticity. How this substitution is done will be disregarded here. The deformations of such a replacement system under given small forces and particularly the natural vibrations under the mass forces can be computed by assuming infinitely small displacements, hence linearization. A suitable aid for solving this linearized problem is in matrix form. Its application to real (self-adjoined) eigen-value problems in physics and engineering is generally known. Such a problem is involved here.

A one-row matrix corresponds exactly to a vector P_i, a square matrix to a tensor a_{ik}, whereby the first subscript (i) is always coordinated to the rows, the second subscript (k) to the columns. In the customary matrix calculus the subscripts are omitted. All these subscriptless "direct" calculi, to which the vector and tensor analysis themselves belong, are too little amenable to expansion; another drawback is that the significance of many operation signs must be marked and the sequence of the factors must not be changed. For the representation of the deformations of the lifting surface by polynomials or surface harmonics as well as their numerical integration by means of these functions, the subscripts are indispensable. Therefore we shall use subscripts, apart from a temporary application of the vector analysis in the flow theory. With respect to factors with identical subscripts or exponents i, k, and l, m is added up from 1 to n, so far as the subscripts are not bracketed. This is the only rule that needs to be followed.

Consider a plate-like wing of little thickness divided into n tables or "torsion boxes" (cf. Williams and Mech (ref. 10)); P_i are the forces and ζ_k the normal displacements of the n centers of gravity $i = 1$ to n of these tables. The matrix equations read then

$$P_i = a_{ik}\zeta_k \qquad a_{ik} = a_{ki} \tag{1}$$

$$\zeta_k = b_{kl}P_l \qquad b_{kl} = b_{lk} \tag{2}$$

where P_i, ζ_k one-row matrices with n terms, a_{ik} and b_{kl} square matrices with n^2 terms; a_{ik} is the stiffness matrix and b_{kl} the deformation matrix; both are symmetric. By equations (1) and (2)

$$\begin{aligned} a_{ik}b_{kl} &= \delta_{il} \\ &= \begin{cases} 0 \text{ when } i \gtrless l \\ 1 \text{ when } i = l \end{cases} \end{aligned} \tag{3}$$

The deformation matrix b_{kl} is obtained by solving the linear equation system (1) with respect to ζ_k, when a_{ik} is known. For abbreviation

$$b_{ki} = a_{ik}^{-1} \qquad a_{ik} = b_{ki}^{-1} \tag{4}$$

For harmonic oscillation, the displacement of point k is

$$\zeta_k = A_k \exp j\nu t \qquad j = \sqrt{-1} \tag{5}$$

where A_k denotes the complex oscillation amplitude and ν the cyclic frequency. The mass force is therefore

$$\begin{aligned} P_k &= M_{(k)} \frac{\partial^2 \zeta_k}{\partial t^2} \\ &= -M_{(k)}A_k\nu^2 \exp j\nu t \end{aligned} \tag{6}$$

with M_k denoting the masses concentrated in point k. The manner by which a given continuous mass distribution may be best replaced by n separate masses M_k in n given points, will be shown later. If equilibrium is to exist, the sum of the forces in every point must disappear. Multiplying equation (6) by b_{ik}/ν^2 gives, in the absence of air forces, according to equations (2), (5), (6), the principal equation

$$\lambda A_i - (bM)_{ik} A_k = 0 \qquad \lambda = 1/\nu^2 \tag{7}$$

where

$$(bM)_{ik} = b_{ik} M_{(k)} \tag{8}$$

is termed dynamic matrix and $\lambda = 1/\nu^2$ real eigen value. Equation (7) contains an nth order algebraic equation for the n eigen values λ_i of the elastomechanic replacement system, the so-called secular equation. It is obtained by equating to zero the denominator determinant of the linear equation system (7). The secular equation is solved either by using the Graeff process or, if n is great, by iteration. First estimate an amplitude distribution A_i' of the fundamental frequency, then resolve the matrix equation

$$\lambda' A_i' = (bM)_{ik} A_k' \tag{9}$$

with respect to λ'. With this approximate value, compute a new amplitude distribution A_i'' and repeat the process until the values of λ and A_i/A_n do no longer vary. This gives the exact solution of the fundamental frequency mode $A_i^{(1)}$ and of the eigen value λ_1. Now the orthogonality condition

$$M_i A_i A_i^{(1)} = 0 \tag{10}$$

is valid for all harmonic oscillations.

From equations (7) and (10) follows then, by the same iteration process, the first harmonic oscillations of the system and so forth, until all n eigen values λ_k and oscillation modes $A_i^{(k)}$ of the system are obtained.

The matrix method is particularly suitable with automatic electronic computers; they can compute all the aforementioned matrix operations and iterations very quickly. Even for $n = \binom{11}{2} = 55$ points the operations take only a few hours. This assures a wide range of adaptation of the replacement system to the actual airplane. The assumption of elastic beams is no longer necessary; even plate-like systems can be treated. As a rule, it is more convenient to fix the stiffness matrix a_{ik} according to the design data, as indicated by Williams (ref. 10). The elasticity theory of thin plates has been described by Reissner and Stein (ref. 11), Fung (ref. 12), and Theodorides (ref. 13).

4. THEORY OF AIR FORCES OF THE OSCILLATING LIFTING SURFACE

To resolve aeroelastic problems, a theory of unsteady surface is necessary. Some new advances in this direction are described in the following based on references 1, 14, 15, and 16. Assuming perfect, compressible fluid, the Euler equations for the velocity potential Φ and pressure p read

$$\rho_0 \frac{d\Phi}{dt} + p = p_0 \tag{11}$$

$$\rho_0 c_0{}^2 \Delta^2 \Phi + \frac{dp}{dt} = 0 \tag{12}$$

ρ_0 is air density, c_0 velocity of sound in still air. Equation (12) is applicable to very small pressure changes only. The velocity of the fluid is $\upsilon = \Delta\Phi$. The substantial derivation with respect to time is

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \upsilon\Delta \tag{13}$$

Let

$$\upsilon = v + \upsilon'$$

where v is the velocity of undisturbed flow along the positive x axis and υ' the interference flow. Because the flight speed v changes rather slowly, as a rule, $v = \text{constant}$ can be assumed. Then

$$\frac{d\upsilon}{dt} = \frac{d\upsilon'}{dt} \qquad \Delta\upsilon = \Delta\upsilon' \tag{14}$$

In aeroelastic problems it is sufficient to know the corresponding very small air forces, since the aim is limited to exploration of the start of the flutter mechanism in order to avoid flutter. Hence, the linearization by assuming $|\upsilon'| \ll v$; the substantial derivation is then

$$\frac{d}{dt} \sim \frac{\partial}{\partial t} + v\frac{\partial}{\partial x} \tag{15}$$

Equations (14) and (15) must be inserted in equations (11) and (12). These equations and their solutions can be simplified by the following substitutions

$$\beta = \frac{v}{c_0} \qquad \omega = \frac{j\nu L}{v(1 - \beta^2)} \qquad \kappa = \frac{\nu L}{c_0(1 - \beta^2)} \tag{16}$$

$$X = Lx \qquad Y = \frac{L}{\sqrt{1 - \beta^2}}\, y \qquad Z = \frac{L}{\sqrt{1 - \beta^2}}\, z \tag{17}$$

X, Y, Z are ordinary Cartesian coordinates, ω is the reduced frequency, κ the number of waves; L is a characteristic length, such as half maximum wing chord, for example.

For harmonic oscillations

$$\phi = Lv \exp\left(j\nu t + \omega\beta^2 x\right)\phi^*(x,y,z) + Lvx \tag{18}$$

$$p = \rho_0 v^2 \exp\left(j\nu t + \omega\beta^2 x\right)p^*(x,y,z) + p_0 \tag{19}$$

$$\zeta = L \exp\left(j\nu t + \omega\beta^2 x\right)\zeta^*(x,y) + L\zeta_0(x,y) \tag{20}$$

With these new variables, the Euler equations for harmonic oscillations become

$$\left(\omega + \frac{\partial}{\partial x}\right)\phi^*(x,y,z) + p^*(x,y,z) = 0 \tag{21}$$

$$\left(\Delta^2 + \kappa^2\right)\Phi^*(x,y,z) = 0 \tag{22}$$

Assume, for the present, an infinitely thin lifting surface, lying approximately in the plane $z = 0$, hence as having only infinitely small deflections ζ. As boundary condition for the solution of equations (21) and (22), there is the given downwash on the lifting surface

$$\begin{aligned} w &= \frac{d\zeta}{dt} \\ &= \left.\frac{\partial\Phi}{\partial z}\right|_{z=0} \end{aligned} \tag{23}$$

By equations (17), (18), (20), and (23) the reduced downwash on the lifting surface is

$$\begin{aligned} w^*(x,y) &= \Phi_z^*(x,y,0) \\ &= \frac{1}{\sqrt{1 - \beta^2}}\left(\omega + \frac{\partial}{\partial x}\right)\zeta^*(x,y) \end{aligned} \tag{24}$$

The solutions of equations (21) and (22) are usually represented in integral form by means of Green's functions or difference kernels K

$$\Phi^*(x,y,z) = \iint K_1(x - x',y - y',z)p^*(x',y',+0)dx'\ dy' \tag{25}$$

$$p^*(x,y,z) = \iint K_2(x - x',y - y',z)w^*(x',y',0)dx'\ dy' \tag{26}$$

The integrals are extended over the lifting surfaces; $w^*(x',y',0)$ is the given reduced downwash on the lifting surface; $p^*(x',y',+0)$ is the reduced pressure on the upper side of the lifting surface. The pressure $p^*(x',y',-0)$ on the bottom side is inversely equal. The Kernels K_1 and K_2 must satisfy the wave equation (22), as is readily apparent. Let $F(x,y,z)$ be a function that satisfies the wave equation

$$\left(\Delta^2 + \kappa^2\right)F(x,y,z) = 0 \tag{27}$$

and disappears at infinity. By equation (17) the reduced coordinates are

$$\bar{y} = -jy \qquad \bar{z} = -jz \quad \text{real, if} \quad \beta > 1 \tag{28}$$

Then the Kernel K_1 is generally given by

$$K_1(x,y,z) = \int_{-\infty}^{x} \exp \omega(a - x)da \frac{\partial}{\partial z} F(a,y,z) \tag{29}$$

$$F(x,y,z) = \frac{\exp\left(-j\kappa \sqrt{x^2 + y^2 + z^2}\right)}{2\pi \sqrt{x^2 + y^2 + z^2}}, \text{ if } \beta \lessdot 1 \tag{30}$$

$$F(x,\bar{y},\bar{z}) = \frac{\cos\left(\kappa \sqrt{x^2 - \bar{y}^2 - \bar{z}^2}\right)}{\pi \sqrt{x^2 - \bar{y}^2 - \bar{z}^2}}, \text{ if } \beta > 1 \tag{31}$$

Integration of equations (30) and (31), with respect to y and $\bar{y}$ from $-\infty$ to $+\infty$ gives the corresponding functions for plane flow

$$F(x,z) = -\frac{j}{2} H_0^{(2)}\left(\kappa \sqrt{x^2 + z^2}\right), \text{ if } \beta < 1 \tag{32}$$

$$F(x,\bar{z}) = -jJ_0\left(\kappa\sqrt{x^2 - \bar{z}^2}\right), \text{ if } \beta > 1 \tag{33}$$

where J_0, H_0 are cylindrical functions of zero order.

The surface equation

$$x^2 - \bar{y}^2 - \bar{z}^2 = 0 \tag{34}$$

gives a cone envelope, the so-called Mach cone. As the roots in equations (31) and (33) must always remain real, the range of integration of equation (25) in the supersonic range $\beta > 1$ must be limited to the Mach cone. The Kernel K_2 in the supersonic range is

$$K_2(x,\overline{y},\overline{z}) = \left(\omega + \frac{\partial}{\partial x}\right)\frac{\cos\left(\kappa\sqrt{x^2 - \overline{y}^2 - \overline{z}^2}\right)}{\pi\sqrt{x^2 - \overline{y}^2 - \overline{z}^2}} \quad \text{when} \quad \beta > 1 \tag{35}$$

By equation (35), integration with respect to $\overline{y}$ in plane flow

$$K_2(x,\overline{z}) = -j\left(\omega + \frac{\partial}{\partial x}\right)J_0\left(\kappa\sqrt{x^2 - \overline{z}^2}\right) \quad \text{when} \quad \beta > 1 \tag{36}$$

In the subsonic range $\beta < 1$, no simple formula in Cartesian coordinates can be given for Kernel K_2, since it is impossible to satisfy the Kutta wake condition in simple manner by this procedure. Curvilinear orthogonal coordinates (u,v,w) must be introduced, making the lifting surface itself a member of the family of orthogonal surfaces.

Let

$$x = x(u,v,w) \qquad y = y(u,v,w) \qquad z = z(u,v,w) \tag{37}$$

The auxiliary functions

$$\left.\begin{aligned}
U^{-2} &= \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2 \\
V^{-2} &= \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2 \\
W^{-2} &= \left(\frac{\partial x}{\partial w}\right)^2 + \left(\frac{\partial y}{\partial w}\right)^2 + \left(\frac{\partial z}{\partial w}\right)^2
\end{aligned}\right\} \tag{38}$$

are defined.

For the three-axial ellipsoid with the half axes a, b, c we get, for example

$$\left.\begin{aligned}
x^2 &= \frac{\left[\underline{p}(u) - e_1\right]\left[\underline{p}(v) - e_1\right]\left[\underline{p}(w) - e_1\right]}{(e_1 - e_2)(e_1 - e_3)} \\
y^2 &= \frac{\left[\underline{p}(u) - e_2\right]\left[\underline{p}(v) - e_2\right]\left[\underline{p}(w) - e_2\right]}{(e_2 - e_1)(e_2 - e_3)} \\
z^2 &= \frac{\left[\underline{p}(u) - e_3\right]\left[\underline{p}(v) - e_3\right]\left[\underline{p}(w) - e_3\right]}{(e_3 - e_1)(e_3 - e_2)}
\end{aligned}\right\} \tag{39}$$

$$\left.\begin{aligned} 3e_1 &= b^2 + c^2 - 2a^2 \\ 3e_2 &= c^2 + a^2 - 2b^2 \\ 3e_3 &= a^2 + b^2 - 2c^2 \end{aligned}\right\} \tag{40}$$

$$\left.\begin{aligned} U^{-2} &= \left[\underline{p}(u) - \underline{p}(v)\right]\left[\underline{p}(u) - \underline{p}(w)\right] \\ V^{-2} &= \left[\underline{p}(v) - \underline{p}(w)\right]\left[\underline{p}(v) - \underline{p}(u)\right] \\ W^{-2} &= \left[\underline{p}(w) - \underline{p}(u)\right]\left[\underline{p}(w) - \underline{p}(v)\right] \end{aligned}\right\} \tag{41}$$

$\underline{p}$ is the elliptic function of Weierstrass. For the elliptic lifting surface $c = 0$. In orthogonal coordinates, wave equation (27) becomes

$$\left\{UVW\left[\frac{\partial}{\partial u}\left(\frac{U}{VW}\frac{\partial}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{V}{UW}\frac{\partial}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{W}{UV}\frac{\partial}{\partial w}\right)\right] + \kappa^2\right\}F = 0 \tag{42}$$

In the cases involved here, this equation can be resolved by separation of the variables

$$F = R_n(u)S_n(v,w) \tag{43}$$

The coordinate u = constant is to denote a convex surface, $u = 0$ the lifting surface, upper and lower side, $R_n(u)$ a retarded solution of the wave equation (42), which disappears for $u = \infty$. The functions $S_n(v,w)$ are then so-called surface harmonics of the surface u = constant; they satisfy the orthogonality relation

$$\iint S_n(v,w)S_m(v,w)\frac{U}{VW}\,dv\,dw = \begin{cases} 0, & \text{if } n \gtrless m \\ k_n, & \text{if } n = m \end{cases} \tag{44}$$

The plane element is

$$d\sigma = dx\,dy = \frac{dv}{V}\frac{dw}{W}$$

The integral can be extended over an arbitrary surface u = constant, because U/VW is independent of u, if the separation of the variables is possible. (Cf. eq. (41).) The surface coordinates v,w shall be unequivocal at least in certain ranges, as for example, the geographic coordinates of the sphere in the ranges

$$0 < v < 2\pi, \quad -\frac{\pi}{2} < w < +\frac{\pi}{2}$$

The coordinate $w = 0$ denotes the plane of symmetry, normal to the lifting surface, parallel to the flight direction, the equatorial plane, the coordinate $u = v = 0$ the leading edge and $u = 0$, $v = \pi$ the trailing edge of the lifting surface. The elliptic coordinates given in equation (39) still are ambiguous, but can be made unequivocal by introduction of the Weierstrass elliptic σ functions.

Introducing, for abbreviation, the differential operators

$$\left.\begin{aligned} D_x &= \frac{\partial}{\partial x} + \frac{\partial}{\partial x'} \\ D_y &= \frac{\partial}{\partial y} + \frac{\partial}{\partial y'} \end{aligned}\right\} \tag{45}$$

Then there follows for the difference kernels K_1 and K_2 of the integrals (25), (26) the differential equations

$$D_x K = D_y K = 0 \tag{46}$$

To satisfy these conditions in curvilinear orthogonal coordinates too, the characteristic function

$$G(u,v,w;v',w') = \sum_{n=1}^{\infty} \frac{R_n(u)}{R_n'(0)} S_n(v,w) S_n(v',w') \tag{47}$$

is introduced.

By definition this function satisfies wave equation (42); it further shall satisfy the compatibility condition

$$C_1 D_x G = C_2 D_y G = \begin{vmatrix} G_v(0,\pi,0;v',w'), & G_{v'}(u,v,w;0,0) \\ G_v(0,0,0;v',w'), & G_{v'}(u,v,w;\pi,0) \end{vmatrix} \tag{48}$$

By appropriate normalization of G the constant C_1 can be made equal to unity. Then the Kernel K_2 in the subsonic range $\beta < 1$ can be represented as follows. (Cf. ref. 16.)

$$\left.\begin{aligned} K_2 &= -\Lambda_1\left\{\left(\omega - \frac{\partial}{\partial x'}\right)G(u,v,w;v',w') + \right. \\ &\quad \left. G_{v'}(u,v,w;0,0)\left[G_v(0,\pi,0;v',w')T_1(\kappa,\omega) - G_v(0,0,0;v',w')\right]\right\} \\ K_2 &= -\Lambda_2\left\{\left(\omega - \frac{\partial}{\partial x'}\right)G(u,v,w;v',w') + \right. \\ &\quad \left. G_{v'}(u,v,w;\pi,0)\left[G_v(0,\pi,0;v',w') - G_v(0,0,0;v',w')T_2(\kappa,\omega)\right]\right\} \end{aligned}\right\} \tag{49}$$

$$T_1(\kappa,\omega) = \frac{\displaystyle\int_{-\infty+i\pi/2}^{\infty-i\pi/2} \exp\,\omega x(u,0,0)G_{vv'}(u,0,0;\pi,0)\frac{V(u,0,0)}{U(u,0,0)}\,du}{\displaystyle\int_{-\infty+i\pi/2}^{\infty-i\pi/2} \exp\,\omega x(u,0,0)G_{vv'}(u,0,0;0,0)\frac{V(u,0,0)}{U(u,0,0)}\,du} \tag{50}$$

$$T_2(\kappa,\omega) = \frac{\displaystyle\int_{-\infty+i\pi/2}^{\infty-i\pi/2} \exp\,\omega x(u,0,0)G_{vv'}(u,\pi,0;0,0)\frac{V(u,0,0)}{U(u,0,0)}\,du}{\displaystyle\int_{-\infty+i\pi/2}^{\infty-i\pi/2} \exp\,\omega x(u,0,0)G_{vv'}(u,\pi,0;\pi,0)\frac{V(u,0,0)}{U(u,0,0)}\,du} \tag{51}$$

These equations hold for complex values of ν, κ, ω corresponding to equation (16), i.e., for damped and amplified oscillations also.

For lifting surfaces having a symmetry axis $x = z = 0$, for the elliptic lifting surface, for example, $T_1 = T_2$. For the constants Λ_1 and Λ_2 in equation (49), the following condition is applicable

$$\left.\begin{aligned}\frac{1}{\Lambda_1+\Lambda_2} &= \iint dx\ dy \lim_{z=0}\frac{\partial}{\partial z}G(u,v,w;v',w') \\ &= \iint G_u(0,v,w;v',w')\ \frac{U}{VW}\ dv\ dw\end{aligned}\right\} \tag{52}$$

The integrand must be a Dirac δ function

$$G_u(0,v,w;v',w') = \left.\begin{matrix} 0, \text{ if } & v \gtrless v' & \text{or} & w \gtrless w' \\ \infty, \text{ if } & v = v' & \text{and} & w = w' \end{matrix}\right\} \tag{53}$$

The conditions (52), (53) are such that the Kernel closely related with K_2 of the integral representation of the velocity potential as function of the downwash always reproduces the given downwash on the lifting surface. (Cf. ref. 15.) If both the leading and trailing edge of the lifting surface are in uniform flow, we get

$$\Lambda_1 = \Lambda_2 \tag{54}$$

hence zero lift in steady flow. Unsteady motion of lifting surface creates additional members, among them the aforementioned Kelvin impulses, which are acceleration forces of the comoving fluid. The solution (54) is termed Kelvin solution; in the 19th century it was regarded as a paradox of hydrodynamics. Experiments prove that the Kelvin flow occurs at the start of the motion, but is soon changed by the boundary layer of the lifting surface. Kutta therefore made the phenomenological assumption of smooth flowoff at the trailing edge, which leads to the constant

$$\Lambda_2 = 0 \tag{55}$$

Kutta's solution (55) gives a 20- to 30-percent higher circulation of the lifting surface; every steady and unsteady lifting surface theory up to now rests on this solution. However, actually the trailing edge is in a weak flow, so that

$$\Lambda_1 > \Lambda_2 > 0 \tag{56}$$

A second conditional equation for the constants Λ_1 and Λ_2 of the general solution (49) might be gained either from an unsteady boundary-layer theory or from measurements of the unsteady pressure distribution. But the viscosity of the fluid discounted for the present makes itself felt considerably in the final result.

5. TWO-DIMENSIONAL SOLUTIONS

The wave functions R_n and S_n are not well known except for the case of the strip and the circular plate.

For the strip, there are the coordinates of the elliptic cylinder

$$\left.\begin{aligned} x &= -\cosh u \cos v \quad y = w \quad z = \sinh u \sin v \\ U^{-2} &= V^{-2} = \sinh^2 u + \sin^2 v, \quad W = 1 \end{aligned}\right\} \tag{57}$$

With these coordinates, the characteristic function for two-dimensional flow reads

$$G(u,v,v') = \sum_{n=1}^{\infty} \frac{Ne_n^{(2)}(\kappa,u)}{Ne_n^{(2)\prime}(\kappa,0)} se_n(\kappa,v)se_n(\kappa,v'), \text{ if } \kappa > 0 \tag{58}$$

$$G(u,v,v') = \frac{1}{4} \ln \frac{\cosh u - \cos(v - v')}{\cosh u - \cos(v + v')}, \text{ if } \kappa = 0 \tag{59}$$

Ne_n and Se_n are Mathieu functions in the Goldstein norm. McLachlan employs the parameter $q = \kappa^2/4$. The tables of the NBS (ref. 17) contain these functions in a different norm with the parameter $s = \kappa^2$.

By equations (26), (50), and (58), the Kutta condition (55) gives the reduced pressure. (Cf. ref. 14.)

$$p^*(u,v) = -\frac{2}{\pi} \oint_0^{\pi} w^*(0,v')\sin v' \, dv' \left\{ \left(\omega - \frac{1}{\sin v'} \frac{\partial}{\partial v'}\right) G(u,v,v') + G_{v'}(u,v,0) \left[G_v(0,\pi,v')T(\kappa,\omega) - G_v(0,0,v')\right] \right\} \tag{60}$$

$$T(\kappa,\omega) = \frac{\int_{-\infty+i\pi/2}^{\infty-i\pi/2} \exp\ (-\omega \cosh u) G_{vv'}(u,0,\pi) du}{\int_{-\infty+i\pi/2}^{\infty-i\pi/2} \exp\ (-\omega \cosh u) G_{vv'}(u,0,0) du} \tag{61}$$

For $\kappa = 0$, i.e., for incompressible fluid, the characteristic function (59) is inserted in equation (60), so that the solution becomes, term by term, the Küssner-Schwarz solution. (Cf. refs. 1 and 14.) For $\kappa = 0$, there are large five-place tables of the aerodynamic coefficients of the wing with aileron and tab on the assumption of the bent flat plate as substitute system. (Cf. ref. 11.) For $\kappa > 0$, Blanch and Fettis (ref. 18), as well as Timman, van de Vooren, and Greidanus (ref. 19), have published five-place tables of coefficients for the flat plate and NLL Amsterdam (ref. 19a) also for the bent flat plate.

For small aspect-ratio wings, such as tapered delta wings, the speed and pressure changes in x direction can be approximately disregarded and so eliminate compliance with the flowoff condition. By utilizing the orthogonal coordinates

$$y = \cosh u \cos v \qquad z = \sinh u \sin v \tag{62}$$

the Kernel K_2 becomes then approximately equal to the regular solution of the wave equation

$$K_2(u,v,v') = -\frac{2\omega}{\pi} G(u,v,v') \tag{63}$$

where G is the function given in equation (58). Equation (63) holds for all Mach numbers β. The lifting surface with its given downwash distribution is divided in separate strips parallel to the y axis; each strip being treated according to equations (26) and (63). Corresponding solutions and tables have been computed by Merbt and Landahl (ref. 20). How far the approximation (63), which equation (46) does no longer satisfy, is practicable must be left to exact solutions.

6. THREE-DIMENSIONAL SOLUTIONS

For $\kappa = 0$, the wave equation reduces to Laplace's potential equation. For the circular plate, the solutions are spherical functions and for the elliptic plate the Lamé functions, so that corresponding characteristic functions can be set up. For the circular plate, Schade (ref. 21) computed solutions with spherical functions by a complicated procedure.

When $\kappa > 0$, solutions for the circular plate can be computed with spheroid functions developed by Meixner. For the elliptic plate, which is of greater aeronautical interest, the corresponding wave functions are still lacking.

Küssner (ref. 21a), Reissner (refs. 22 and 23), and Dengler-Goland (ref. 23a) have published approximate solutions patterned after Prandtl's vortex filament theory for high aspect-ratio wings. The closed solution of the two-dimensional problem of airfoil theory is assumed known and a correction of this solution is computed by resolving one or more linear integral equations. When the Wronsky determinant of the cylindrical functions is taken into consideration, the theories of Küssner (ref. 21a) and Reissner (ref. 22) are identical in every respect. For incompressible fluid $\kappa = 0$, the method is tested. But for $\kappa > 0$ it is very complicated and has not been carried through numerically up to now. Compressibility effects, in aeroelastic problems, do not make themselves felt appreciably until $\beta > 0.7$ (ref. 3). This is the reason for the practical results obtained thus far with the wing-flutter theory at the pressures in the range $0 < \beta < 0.7$ computed for $\kappa = 0$.

In the supersonic range $\beta > 1$, the Kernel K_2 is known in the simple form (eqs. (35) and (36)); hence the many tables of the aerodynamic coefficients for plane flow available now (cf. Weber, refs. 23b and 5), as well as three-dimensional solutions for rectangular, triangular, and swept-wing configurations. (Cf. Watkins (ref. 8); Lomax, Heaslet, and Fuller (ref. 24); Nelson (ref. 25); Watkins and Berman (ref. 26); Miles (ref. 27); Walsh, Zartarian, and Voss (ref. 27a).) If the flutter study involves thin plates - a problem particularly posed in the supersonic range - the integrations of equations (26) and (35) must be made from case to case, corresponding to the wing contour and the number of points of the elasto-mechanical deformation matrix. This is apparent from the flutter theory developed further on. Solutions for $\beta = 1$ are most easily obtained from the supersonic solution by the boundary transition $\beta \longrightarrow 1$. (Cf. refs. 28, 29, and 30.)

7. IMPROVEMENT AND EXPERIMENTAL CHECK OF THEORY

In incompressible fluid ($\kappa = 0$) and two-dimensional flow, profiles of finite thickness can be treated as unsteady by the conventional methods of mapping (refs. 31 and 32). Numerical solutions for small oscillations have been computed by Couchet (ref. 33). The hope of obtaining aerodynamic forces that are in better agreement with experiments has not been fulfilled at all, or only in a small part. (Cf. ref. 34.) In fact, if a frictionless airfoil theory is to be maintained at all, it seems advisable to relinquish the Kutta condition. A possible improvement of theory by the general solution (49) with the phenomenological constant Λ_2/Λ_1 has been pointed out previously.

Drescher (ref. 35) measured the pressure distribution of an oscillating wing with control surface in plane flow in the water tunnel and compared it with lifting surface theory at $\kappa = 0$. The amount and the phase angle of the differences vary very little in the reduced frequencies range $\omega = 0$ to 2.5; this means that the differences correspond to those in steady flow. Similar results for total lift and moment of a rigid oscillating airfoil have been obtained by Greidanus, van de Vooren, and Bergh (ref. 36) in wind-tunnel tests. The Kelvin impulses were measured at $v = 0$; they are in fairly exact agreement with theory, up to a small additional damping portion. Dörr (refs. 34 and 37) studied the flutter of a wing with and without aileron in plane flow in the wind tunnel and found agreement in several cases between computed and measured critical speeds after multiplying the theoretical pressures by the reduction factor $\eta = 0.7$. However, this agreement is contingent upon the choice of mean camber line used as theoretical basis for the aileron with internal balance. The experimental findings may be attributable to profile shape effects or to boundary-layer effects. A new theory which accounts for both is therefore desirable.

8. MATRIX THEORY OF WING FLUTTER

The earlier flutter theory has been described in several comprehensive reports (refs. 1, 4 and 5); they chiefly rest on the replacement of the elasto-mechanical system of the wing by an elastic beam and on the assumption of plane flow past the individual wing sections. It results in a system of linear differential equations that can be solved with any desired degree of accuracy by iteration. These simplifying assumptions suggested themselves as a result of the wing design of the time on the one hand and the absence of large analog computers on the other. The demands of flutter theory are one of the driving forces in the evolution of the first automatic computers in Germany and the U.S.A., of which a considerable number is now available. This circumstance has led to a marked improvement of the elasto-mechanical substitution system and of the theory of natural oscillations of the airplane. And here is where the matrix calculation is particularly suited. The first step that must be taken is to improve the aerodynamic principles of the flutter theory. The aerodynamic forces which must be inserted in the principal equation (7) in order to be able to resolve aeroelastic problems, must also be represented in matrix form. This is theoretically possible, as soon as the Kernel K_2 in equation (26) is known.

To this end, the deflection of the lifting surface must be represented in appropriate manner. Assume a given class of functions

$$F(x,y;\ r,s) \tag{64}$$

which are regular in the range of the lifting surface, i.e., finite, continuous and differentiable; x and y are the reduced coordinates introduced in equation (17); r and s are integral parameters, starting with $r = s = 0$. The reduced displacement ζ^* of the lifting surface is developed with respect to a series of functions F by means of the formula

$$\zeta^*(x,y) = F(x,y;r_i,s_i)\,g_{ik}\zeta_k^*;\quad 0 \leq r_i + s_i < m \tag{65}$$

ζ_k^* are the given displacements in $n = \binom{m+1}{2}$ chosen points (x_k,y_k) of the lifting surface. The functions F and the point coordinates are so chosen that their determinant

$$\det\left|F(x_k,y_k;r_i,s_i)\right| \gtrless 0 \tag{65a}$$

By equation (65), the insertion of the coordinates $x = x_l$, $y = y_l$ gives the relation

$$F(x_l,y_l;r_i,s_i)\,g_{ik} = \delta_{lk}$$

and with the abbreviations set up in equations (3) and (4) the matrix

$$g_{ik} = \left[F(x_k,y_k;r_i,s_i)\right]^{-1} \tag{66}$$

The representations (65) and (66) are exact, when the function $\zeta^*(x,y)$ belongs to class (64) and of lower than mth order; otherwise, it is approximately valid. When the lifting surface displacement ζ^* contains a discontinuity due to a bend in the control surface, two separate relations must be established and then later combined along the control surface axis.

Integration of equation (65) with respect to the lifting surface gives the well known numerical integration formula

$$\iint \zeta^*(x,y)dx\,dy = a_k\zeta^*(x_k,y_k) = a_k\zeta_k^* \tag{67}$$

with the weight factors

$$a_k = g_{ik}\iint F(x,y;r_i,s_i)dx\,dy \tag{68}$$

When $f(x,y) = \zeta^*(x,y)$ is an arbitrary function of the class (64) and of lower than mth order, equation (67) is exactly valid, otherwise approximately, the error can be computed by Mises' method (ref. 38).

In numerical integrations polynomials, i.e., linear forms of the function

$$F(x,y;r,s) = x^r y^s \qquad 0 \leqq r + s < m \tag{69}$$

are generally employed. When the lifting surface $u = 0$ is given in curvilinear orthogonal coordinates, it is, however, more advantageous to utilize linear forms of the function class

$$F(x,y;r_k,s_k) = U(0,v,w)S_k(v,w) \tag{70}$$

in which S_k are the previously introduced surface harmonics. These functions too are regular in the entire lifting surface range.

Assume an arbitrary integrable mass distribution $m(x,y)$ over the lifting surface. This is to be replaced by n single masses M_k in the points (x_k,y_k) in such a way that the total masses, static moments, moments of inertia, etc. of both distributions correspond. This condition is met when

$$M_kF(x_k,y_k;r_i,s_i) = \iint F(x,y;r_i,s_i)m(x,y)dx\,dy \tag{71}$$

$$M_k = g_{ik} \iint F\left(x,y;r_i,s_i\right) m(x,y)dx\ dy \tag{72}$$

Equation (72) is similar to equation (68). In contrast to the aforementioned function $f(x,y)$ which must be regular, function $m(x,y)$ in equations (71) and (72) may also contain integrable singularities (single masses). These integrals are called Stieltjes integrals. When the reduced coordinates equation (17) are used, the right-hand side of equation (72) must be multiplied by $L^2/\sqrt{1 - \beta^2}$, in order to obtain physical masses on the left.

The same procedure is used for the replacement of the pressure distribution by n single forces. Assuming an aerodynamic replacement force of

$$P_k = \exp\left(j\nu t + \omega\beta^2 x_{(k)}\right)P_k^* \tag{73}$$

equations (19), (72), and (73) give the reduced replacement force

$$P_k^* = \frac{2\rho_0 v^2 L^2}{\sqrt{1 - \beta^2}} g_{ik} \iint F\left(x,y;r_i,s_i\right) p^*(x,y, +0)dx\ dy \tag{74}$$

Insertion of equations (24), (26), and (65) in equation (74) gives

$$P_k^* = \frac{\rho_0 v^2 L^2}{1 - \beta^2} g_{ik} h_{il} g_{lm} \zeta_m^* = \frac{\rho_0 v^2 L^2}{1 - \beta^2} H_{kl} \zeta_l^* \tag{75}$$

$$h_{il} = 2 \iint F\left(x,y;r_i,s_i\right)dx\ dy \times \iint \left(\omega + \frac{\partial}{\partial x'}\right)F\left(x',y';r_l,s_l\right)dx'\ dy' \times$$
$$K_2(x - x',y - y',0) \tag{76}$$

The factor 2 in equations (74) and (76) is based on the fact that the integration covers only the upper side ($z = +0$) of the lifting surface, while the pressures on both sides of the lifting surface are inversely equal. In the supersonic range $\beta > 1$, the Kernel $\overline{K_2}$ is always known and given by equations (35) and (36) in cartesian coordinates x and y. Therefore, it is best to use polynomials in x and y as development functions. (Cf. eq. (69).) In the subsonic range $\beta < 1$, the problem is more difficult. Curvilinear orthogonal coordinates (37) function class (70) must be used as development functions. Partial integration of equation (76) gives the aerodynamic matrix

$$h_{il} = \iint S_i(v,w)\frac{U}{VW}\,dv\,dw \times \iint S_l(v',w')\frac{U'}{V'W'}\,dv'\,dw' \times \left(\omega - \frac{\partial}{\partial x'}\right)K_2(0,v,w;v',w') \tag{77}$$

Due to the insertion of the Kernel K_2 according to equation (49) in equation (77), all integrations can be carried out in closed form by means of the orthogonality relations (44). The integrations are extended over the entire orthogonal surface $u = 0$, i.e., both sides of the lifting surface. Thus, matrix h_{il} can be computed from a limited number of k_n values, according to equation (44), and of the wave function values appearing in equation (49). In the case of a hinged aileron, hence unsteady deflection $\zeta*$, the integrations over both areas of the lifting surface must be made separately. Equation (44) is then no longer applicable and infinite series are obtained. Only in plane flow can those integrations be carried out in closed form.

If the Kernel K_2 is not known, the integrodifferential equation

$$\begin{aligned} w*(x,y) &= \frac{1}{\sqrt{1-\beta^2}}\left(\omega + \frac{\partial}{\partial x}\right)\zeta*(x,y) \\ &= \lim_{z=0}\frac{\partial}{\partial z}\iint K_1(x - x',y - y',z)p*(x,y, + 0)dx'\,dy' \end{aligned} \tag{78}$$

following from equations (24) and (25), must be resolved, where K_1 is given by equations (29) and (30)

Integrating while using equation (27) gives

$$\zeta*(x,y) = \iint K_0(x - x',y - y')p*(x',y', + 0)dx'\,dy' \tag{79}$$

$$K_0(x,y) = \frac{\sqrt{1 - \beta^2}}{2\pi} \int_{-\infty}^{x} da' \times$$

$$\int_{-\infty}^{a'} da \exp \omega(a - x)\left(\kappa^2 - \frac{\partial^2}{\partial a^2} - \frac{\partial^2}{\partial y^2}\right)\frac{\exp\left(-j\kappa\sqrt{a^2 + y^2}\right)}{\sqrt{a^2 + y^2}} \quad (80)$$

If the Kernel K_0 were regular everywhere, it could be used as development function F and equations (74), (75), and (79) would then immediately yield the aerodynamic matrix

$$H_{kl} = 2\left[K_0\left(x_k - x_l, y_k - y_l\right)\right]^{-1}$$

But Kernel $K_0(x,y)$ has a singularity at $x = y = 0$. The reverse problem must therefore be solved some other way, which cannot be discussed here.

The sum of the reduced mass and aerodynamic forces is, according to equations (6), (23), (73), and (75)

$$P_k^* = -LM_{(k)}\nu^2\zeta_k^* + \frac{\rho_0 v^2 L^2}{1 - \beta^2} H_{kl}\zeta_l^* \quad (81)$$

On the other hand, elasto-mechanically, we get by equations (2), (23), and (73)

$$L\zeta_i^* = b'_{ik}P_k^* \quad (82)$$

$$b'_{ik} = b_{(ik)}\exp \omega\beta^2\left(x_k - x_i\right) \quad (83)$$

Multiplying equation (81) by $b'_{ik}/\nu^2 L$ and inserting it in equation (82) gives the principal equation of the flutter theory

$$\lambda\zeta_i^* + \left[-b'_{il}M_{(l)} + \mu b'_{ik}H_{kl}\right]\zeta_l^* = 0 \quad (84)$$

$$\mu = \frac{\rho_0 v^2 L}{\nu^2(1 - \beta^2)} = - \frac{\rho_0 L^3}{\omega^2} \left(1 - \beta^2\right) \tag{85}$$

The parameter μ has the dimension of a mass. When $\mu = 0$, equation (84) becomes equation (7). Equation (84) is the desired matrix equation for defining the n eigen values λ_i, of the oscillating wing.

9. ANALYTICAL SOLUTION METHOD

The bracketed matrices in equation (84) can be computed for given values of κ and ω and are complex. Hence, it seems logical to consider equation (84) as a complex eigen-value problem, whereby the aerodynamic matrix is not self-adjoined. Prior to 1943, little was known about complex eigen-value problems. Since then, Wielandt (ref. 39) has developed appropriate iteration methods for calculating complex eigen values and proved their convergence. Recently, Gossard (ref. 40) applied this procedure and proved it again.

Equation (84) thus yields a set of complex eigen values whose azimuth angle is the assumedly known damping phase angle g_i

$$\lambda_i = \left|\lambda_i\right| \exp jg_i \qquad \left|\lambda_i\right| = \frac{1}{\nu_i^2} \tag{86}$$

These physical solutions are obtained from the group of mathematical solutions by graphical interpolation, $\lambda_i(\kappa,\omega)$ being plotted in the complex numerical plane. The critical speed v_i is computed from the obtained values ν_i, κ, and ω, according to equation (16). The check on whether

$$\beta = \frac{v_i}{c_0} = \frac{j\kappa}{\omega} \tag{87}$$

conforms with the assumption is made by graphical method. And, since the airplane is to fly at different altitudes, the air density ρ_0 must be varied too.

The conditions (86) and (87) confine the physically possible solutions materially, so that often no physical solution is found in the explored speed range. This is exactly what the aeronautical engineer wants. To be on the safe side, however, he would still have to vary the most important parameters of the airplane with their limits of error and manufacturing tolerances. This can be done by one of the customary disturbance calculations, without having to repeat the whole calculation (ref. 4). It might be noted that Wielandt (ref. 41) has developed mathematical methods for estimating the upper and lower limits of the complex eigen values λ_i, when the principal equation (84) is given. When these limits meet the technical requirements, the solution of the characteristic value problem can be omitted.

Bearing in mind the structural damping, the reduced elastic force follows from equations (1), (20), and (73) at

$$P_k^* = L\zeta_i^* a_{(ik)} \exp\left[\omega\beta^2\left(x_i - x_k\right) + jg\right] = L\zeta_i^* a_{ik}' \tag{88}$$

The sum of the elastic, mass and aerodynamic forces is then, according to equations (81) and (88)

$$La_{ik}'\zeta_i^* - LM_{(k)}\nu^2\zeta_k^* + \frac{\rho_0 v^2 L^2}{1 - \beta^2} H_{kl}\zeta_l^* = Q_k^* \tag{89}$$

Flutter is generally initiated by small external forces. Hence, the thought lies close to introduce a given periodic outside force Q_k^* as interference function in equation (89), such as $Q_k^* = Q_1^*$, for example, or a force distribution representing a given "sinusoidal" gust. The amplitudes $\zeta_k^*(\nu,v)$ of the lifting surfaces can then be computed as functions of the frequency and speed from the inhomogeneous equation system (89).

Near the critical flight attitude, the amplitudes increase considerably and become infinite in the flutter range by reason of the disappearance of the denominator determinant of equation (89). This method also affords a complete survey on the flutter behavior of an airplane. This method is especially favored in Russia. It may also be used for investigating the effect of atmospheric turbulence, by analyzing it harmonically and posting it on the right-hand side of equation (89). The harmonic components of the elastic stresses calculated from the wing deflections are statistically superposed.

The six degrees of freedom of the aircraft motion as a rigid body as well as the free control surface notations must also be borne in mind in every flutter study. Up to now, the theory of path motions of airplanes rested on these degrees of freedom. However, no sharp boundaries between flight stability, flutter, and gust stresses can be drawn; there are many transition modes. Collar (ref. 42) said: "In short, it is evident that we are no longer dealing with a series of subjects each in its own water-tight compartment: there is a definite coalescence of the subjects into an integrated whole, which may be defined as the dynamics of a deformable airplane. And we are faced with the question: what are to be our methods of treatment of this unified problem?"

Rea (ref. 43) demonstrated how by Laplacian transform all three domains of aeroelasticity can be jointly treated (Transfer Function-Fourier Method). The transfer function of a dynamic system transforms the input into output, for example, control hinge moment into tail surface force. For the numerical solution of these and, in fact, of all linearized problems, the use of electrical analogies is recommended. They rest on the formal similarity of the differential equations of an ordinary linear mechanical system with a finite number of degrees of freedom (Lagrange's equations) and the differential equations of a linear electric network (Kirchhoff's law). The displacements represent the electric voltages, the generalized forces, and the electric currents. For conservative systems, the analogous electrical network contains only passive elements, resistance, capacitance, inductance, and transformer. When air forces are involved too, amplifiers must be included.

The solution, in characteristic value problems of flutter theory, is obtained by connecting periodic interference currents Q_k^* with the network and measuring the currents and voltages. One particular advantage of such analogy devices is that structural changes in the network can be easily copied in the design stage of the aircraft. The error of measurement increases nearly with the root of the degrees of freedom, therefore, is still conservative even for 100 degrees of freedom. McNeal, McCann, and Wilts (ref. 44) described the analogy setup developed at the California Institute of Technology which carried out flutter calculations with 70 degrees of freedom.

The stability of the solutions of linear differential equations is defined by Routh's criterion; but it becomes inconvenient with a large number of degrees of freedom. Nyquist, therefore, subjected the differential equations to a Laplacian transform and developed its characteristic equation in powers of Laplacian operators. The stability of the solutions of the particular system can then be proved by a simple theoretical function operation. Nyquist's method has been used for some time in electrotechnics, and has been applied also to flutter calculations by Dugundji (ref. 45).

10. FLUTTER OF SKIN PANELS (COVERING)

The covering of an airplane usually consists of thin sheet panels attached to frames; they can perform aeroelastic oscillations even on a rigid frame. To the extent that the problem can be linearized and the wing covering is approximately flat, this type of flutter could also be treated by the matrix method developed in section 8. However, this would require the assumption of a rather large number of degrees of freedom (points), and so the solution of very great matrices.

For this reason we examine a simpler, ideal case, namely, the infinitely flat thin elastic plate, pin supported on an infinite rectangular lattice framework with the spacings l_1 and l_2. The mean normal stresses T_1/h and T_2/h and the plate shearing stress S/h may be regarded as constant for very small deflections ζ, even if the lattice framework is immovable. The plate is of h thickness, and M is the mass per unit area. The linear differential equation (ref. 46) for ζ reads then

$$B\nabla^4\zeta + T_1 \frac{\partial^2\zeta}{\partial x^2} + 2S \frac{\partial^2\zeta}{\partial x \partial y} + T_2 \frac{\partial^2\zeta}{\partial y^2} + M \frac{\partial^2\zeta}{\partial t^2} = -\Pi(x,y,t) \tag{90}$$

x, y, and z are ordinary cartesian coordinates; z and ζ are counted positive upwards. Compressive stresses are computed positive. The bending stiffness

$$B = \frac{mG}{m - 1} \frac{h^3}{6} \exp ig \tag{91}$$

is assumed complex for periodic processes, to allow for structural damping. The pressure jump on the plate is

$$\Pi(x,y,t) = p(x,y, +0,t) - p(x,y, -0,t) + \sum_{n=-\infty}^{\infty} \delta\left(x - nl_1\right) Q_n(y,t) + \sum_{m=-\infty}^{\infty} \delta\left(y - ml_2\right) Q_m(x,t) \tag{92}$$

p = aerodynamic pressure, $\delta(x)$ Dirac's δ function, $Q_m(x,t)$ the normal force per unit length exerted on the mth bulkhead (longitudinal member) of the lattice framework. In the subsequently involved periodic solutions, the functions Q_n and Q_m repeat after each two steps. Equation (90) is solved by the substitution

$$\left.\begin{aligned} &\zeta(x,y,t) = \operatorname{Re}\operatorname{Re}' \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} A_{rs} \exp\left(i\nu t + irk_1 x + jsk_2 y\right) \\ &i^2 = ij = j^2 = -1 \end{aligned}\right\} \qquad (93)$$

A_{rs} are hypercomplex constants; in addition

$$k_1 = \pi/l_1 \qquad k_2 = \pi/l_2$$

Continual support of the plate on the lattice frame is contingent upon

$$\left.\begin{aligned} &\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} A_{2r,2s} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} A_{2r+1,2s} = 0 \\ &\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} A_{2r,2s+1} = \sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} A_{2r+1,2s+1} = 0 \end{aligned}\right\} \qquad (94)$$

The individual members of (93) represent traveling waves moving in x direction at speed $-\nu/rk_1$. The air strikes the upper side of the plate at constant speed v along the positive x axis, while still air prevails at the lower side. A new system of coordinates

$$\bar{x} = x + \frac{\nu t}{rk_1} \qquad \bar{y} = y \qquad \bar{z} = z \qquad (95)$$

that moves along with the traveling wave r is introduced. In the over-scored coordinate system, a steady flow problem exists, the upper side of the static wave r is in streamwise flow at speed

$$\overline{v}_0 = v + \frac{\nu}{rk_1}$$

the lower side at speed

$$\overline{v}_u = \frac{\nu}{rk_1}$$

For $\kappa = 0$, equation (22) gives the differential equation of the velocity potential

$$\left[\left(1 - \frac{\overline{v}^2}{c_0^2}\right)\frac{\partial^2}{\partial\overline{x}^2} + \frac{\partial^2}{\partial\overline{y}^2} + \frac{\partial^2}{\partial\overline{z}^2}\right]\Phi(\overline{x},\overline{y},\overline{z}) = 0 \tag{96}$$

from equation (23) follows the boundary condition for $\overline{z} = 0$

$$w(\overline{x},\overline{y}) = \Phi_{\overline{z}}(\overline{x},\overline{y},0) = \overline{v}\zeta_{\overline{x}}(\overline{x},\overline{y}) \tag{97}$$

By equations (11) and (15), the pressure on the upper side of the plate is

$$p(\overline{x},\overline{y}, +0) = -\rho_0\overline{v}\Phi_{\overline{x}}(\overline{x},\overline{y}, +0) + p_0 \tag{98}$$

The particular deflection of the plate is assumed as

$$\zeta(\overline{x},\overline{y}) = \exp\left(irk_1\overline{x} + jsk_2\overline{y}\right) \tag{99}$$

and the corresponding velocity potential for $\overline{z} \geqq 0$ as

$$\Phi(\overline{x},\overline{y},\overline{z}) = \frac{iB_{rs}}{\rho_0\overline{v}rk_1}\exp\left[irk_1\overline{x} + jsk_2\overline{y} - \overline{z}\sqrt{\left(1 - \frac{\overline{v}^2}{c_0^2}\right)r^2k_1^2 + s^2k_2^2}\right] \tag{100}$$

As a result of equations (96) to (100), the particular pressure is then

$$p(\bar{x},\bar{y}, +0) = B_{rs}(\bar{v})\exp\left(irk_1\bar{x} + jsk_2\bar{y}\right) \tag{101}$$

with

$$B_{rs}(\bar{v}) = -\frac{\rho_0\bar{v}^2r^2k_1^2}{\sqrt{\left(1 - \frac{\bar{v}^2}{c_0^2}\right)r^2k_1^2 + s^2k_2^2}} \tag{102}$$

when the radicand is >0, and

$$B_{rs}(\bar{v}) = -\frac{\rho_0\bar{v}r\left|\bar{v}r\right|k_1^2}{\sqrt{\left(1 - \frac{\bar{v}^2}{c_0^2}\right)r^2k_1^2 + s^2k_2^2}} \tag{103}$$

when the radicand is <0.

Putting equations (92), (93), (95), (99), and (101) in equation (90), multiplying by

$$\exp\left(-irk_1x - jsk_2y\right)$$

and integrating with respect to x,y from $-\infty$ to $+\infty$, yields the coefficients

$$A_{rs} = \left[C_0 + (-1)^rC_1 + (-1)^sC_2\right]\left(N_{rs}\right)^{-1} \tag{104}$$

$$N_{rs} = \left(r^2k_1^2 + s^2k_2^2\right)^2B - r^2k_1^2T_1 - 2rsk_1k_2S -$$
$$s_2k_2^2T_2 + B_{rs}\left(v + \frac{\nu}{rk_1}\right) + B_{rs}\left(\frac{\nu}{rk_1}\right) \tag{105}$$

The hypercomplex constants determine the type of the solutions. Corresponding to the support conditions (94), there are four different types of form changes, hence four types of flutter. Only the straight oscillation modes are considered here. By equations (94) and (104)

$$\sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty}\left(N_{2r,2s}\right)^{-1} = 0 \tag{106}$$

Energy absorption from the airstream and flutter is possible only when at least one of the coefficients B_{rs} becomes imaginary. (Cf eq. (103).) Equation (106) then becomes complex, from which follow two real equations for the determination of the eigen values v and ν. For $\nu = g = 0$, only static instability is possible in the subsonic range, but no flutter. By equation (102) all coefficients B_{rs} are then real, hence the sole result is a real equation (106) for computing the critical speed v. When at least two functions N_{rs} of the same parity disappear simultaneously, equation (106) itself is satisfied, since $\infty - \infty$ can be any number.

Hedgepeth, Budiansky, and Leonard (ref. 47) developed a representative theory for plane flow $\left(k_2 = 0\right)$ and computed the stability range of the fundamental oscillations $r = -2,0,2$ for a large number of parametric values on the basis of the Nyquist diagram (ref. 45). They found that structural damping g could lower the stability range under certain conditions. The limitation to fundamental oscillations for reasons of simplicity is a first approximation. The support reactions of the lattice frame cause discontinuities of the higher derivatives of the deformation area $\zeta(x,y,t)$, to whose representation Fourier series with infinitely many terms corresponding to (93) are required theoretically. Since N_{rs} is a 4th order polynomial in r and s, the series (106) converge fairly rapidly.

When external air forces $T > 0$ act on the plate or it is heated by skin friction, the plate can buckle statically even without air forces. By equation (90) the differential equation of the static plate (ref. 46) reads

$$B\nabla^4\zeta + T_1\frac{\partial^2\zeta}{\partial x^2} + 2S\frac{\partial^2\zeta}{\partial x\partial y} + T_2\frac{\partial^2\zeta}{\partial y^2} = 0 \tag{107}$$

Inserting equation (93) in (107) gives

$$A_{rs}\left[\left(r^2k_1^2 + s^2k_2^2\right)^2B - r^2k_1^2T_1 - 2rsk_1k_2S - s^2k_2^2T_2\right] = 0 \qquad (108)$$

Finite amplitudes A_{rs} are possible only when the bracketed term in equation (108) disappears, i.e., when T_1 and T_2 have a critical magnitude. If these increase gradually, buckling will follow at the smallest possible values of r and s, and a corresponding static wave field will be the result. Visualizing this wave field in a flow along x at speed v, its steady pressure distribution can be computed by equation (101), where $\overline{x} = x$, $\overline{v} = v$, $\nu = 0$. The aerodynamic forces continue to deform the given wavy plate until a new state of equilibrium is reached or, failing that, until the waves change into opposite position ($\zeta \longrightarrow -\zeta$). This loss of static stability is a supersonic phenomenon which comes into being through the change in sign of the radicand in equation (103). In the subsonic range the aerodynamic forces generally have a stabilizing effect on a static wave field, with exception of the aforementioned case of static instability, where a monotonic increase in wave height occurs until the linearized formula (90) becomes inapplicable.

Fung (ref. 48) investigated the static stability of a sinusoidal half wave in two-dimensional supersonic flow $\left(k_2 = 0,\ \ v > c_0\right)$; however, his theory is identical with that of an infinite wave field. By assuming immovable supports and finite small amplitude, the problem becomes nonlinear with respect to the deflections, but still linear enough approximately as far as aerodynamic pressure is concerned. The critical speed of static instability is so much lower as the initial amplitude of the wave field is lower. Because the opposite position ($\zeta \longrightarrow -\zeta$) of the changed panel field has just as little stability, a complicated periodic process develops which is not harmonic and could not be calculated up to now. Such a process would lead to a rapid destruction of the shell (or skin) as observed on German V-2 rockets. Skin buckling on supersonic airplanes in all flight attitudes should therefore be prevented by appropriate design measures.

To the extent that the ideal case of the infinite plate with lattice frame investigated here is applicable to real aircraft, skin flutter can be considered as a typical supersonic phenomenon. Flutter processes can occur only below the critical speed of static instability.

11. STATIC OSCILLATION TEST

The static oscillation test on airplanes is the oldest and most extensively employed experimental method of aeroelasticity. Since flutter occurs mostly near static oscillation frequencies, the static oscillation test originally served to detect oscillation modes suspected of flutter, later on largely as experimental check of the design regulations I and II as a safeguard against flutter. Owing to the complicated structure of aircraft, there was no other possible way to meet these requirements.

The measured static oscillation modes $F_i(x,y)$ were utilized occasionally as starting function for the flutter theory, i.e., the wing deflections ζ were represented as linear modes of $F_i(x,y)$. The mass distribution $m(x,y)$ of the wing is comparatively easy to define from the design drawing, from which the substitute masses M_k in n selected points can be computed by equation (72). After inserting $F_i(x_k,y_k)$, M_k, and the measured natural frequencies ν_i in equations (1), (5), and (6), the stiffness matrix a_{ik} can be computed without having to make stiffness measurements or calculations with their attendant sources of errors and approximations. To be on the safe side, every measured static oscillation mode from the first to the nth should be introduced as degrees of freedom, which appears possible by electronic computers. In order to be able to carry out this procedure, the measured static oscillation modes must, first, be corrected and orthogonalized (ref. 48a).

For the static oscillation tests, the readied airplane is suspended in a hanger from very soft springs or else mounted on a suitable elastic base and excited in one or more points by oscillators with slowly increasing frequency. At the natural frequencies ν_i, the amplitudes $F_i(P)$ are maximum. These can be measured or recorded successively and thus give a complete survey of the static oscillation modes $F_i(P)$ of the airplane, P denoting any one point of the airplane.

The original oscillators were rotating unbalanced masses. But they were later replaced by electrodynamic oscillators, which are more accurate and easier to use. More recent oscillators are those developed by Herschel-Schweizerhof (ref. 49) and compared in an article by de Vries (ref. 50). Originally, the amplitudes were read from so-called oscillometers mounted at several points of the airplane, which were replaced later on by commercial electrical pickup and recording instruments (ref. 5). These pickups are either accelerometers operating with piezoelectric crystals, strain gages, or acceleration-responsive electron tubes; or else they are speedometers operating with induction coils. Accelerations and speeds are changed to deflections by electrical integration switches. Direct recording strain gages are suitable for elongation measurements.

These electrical instruments speed up the natural oscillation mode measurements, increase their accuracy, and make phase angle measurements possible (ref. 51). Only the natural oscillation excited in resonance has the phase position $\sim\pi/2 + k\pi$ relative to excitation, while other accidentally excited oscillations have the phase position $\sim 0 + k\pi$ and can be interpreted as disturbances and eliminated. The exact measurement of the excitation energy also enables the damping phase angle g_i to be determined when the generalized mass of the oscillating system

$$\mu_i = \sum_P F_i^2(P)m(P)$$

is known. This can be obtained by application of known additional masses on the oscillating system and by frequency measurements (refs. 52 and 52a).

12. MODEL TESTS

When an experimental clarification of the most important aeroelastic problems is demanded before a new type of aircraft is finished, a dynamically similar model of the planned airplane can be built. Its purpose may be the determination of the static oscillations, and can be met by the mathematical problem described in section 3. The outside of the model does not have to be geometrically similar to the full-scale design; the covering may be dispensed with under certain conditions.

But in wind-tunnel model tests for the determination of the critical speed or the proof of flutter freedom, an externally geometrically and internally dynamically similar model is necessary. This problem calls for a model scale that is not too small and generally also a far-reaching geometrical similarity of the internal construction. When compressibility effects are involved, the ratio of the sonic velocities of the structural material and of the flowing medium must agree. In wind tunnels with air media this practically indicates aluminum as model material.

It is difficult to meet all these requirements. The first dynamically similar models built in Germany were therefore still rather far from the dynamical similarity with full-scale design. Substantial improvements were attained by the tendency toward external and internal geometric similarity with the use of adhesive plastics such as Vinidur and adhesive thin aluminum sheets. They are available in the U.S.A. in finely gaged thickness for model designs. Even small forming rollers are utilized for such strips. Such models can actually be regarded as miniature copies of full-scale versions. Kinnanan (ref. 53) gives an insight into the model testing technique of the Boeing Co., which prefers a cheaper wood construction, as was customary in Germany. Melzer (ref. 53a) gives an outline of the laws of similarity and describes the technique of the Junkers Airplane Co.

However, it does not always call for a complete model. Experiments with a half-span wing restrained at the root can be very informative, when, say, the effect of individual masses on the wing (engines, fuel tanks) on the critical speed is involved. Such simplified model tests have also been used as experimental check on corresponding three-dimensional flutter calculations, since the approximate assumptions, which must be introduced in order to simplify the theoretical calculation, can already be proved on simplified systems (Runyan-Sewall (ref. 54), Woolston-Runyan (ref. 55), Gayman (ref. 56), Tuovila, Baker and Regier (ref. 56a) and Nelson and Rainey (ref. 56b)).

Since dynamically similar models of a complete airplane must not be too small, sufficiently great tunnels are required for their study, at airspeeds ranging within the permissible flying speeds of the full-scale design. This would be necessary if compressibility effects are to be accounted for. In favorable climate the expensive big tunnel can be replaced by a towing section in the open air, whereby a vehicle on rails, propelled by rockets, carries the model. Entire tail surfaces have been towed up to sonic velocity on such sleds, at the Edwards Air Force Base (U.S.A) and tested for flutter. A slight breeze, ground effect, and the inevitable vibrations are less disturbing in such tests than in steady profile measurements.

Flutter studies have also been made on dynamically similar models in free flight (ref. 3). The models can be studied in diving flight or can be equipped with their own rocket drive and launched seaward in flight without return. The salient data are generally radioed to the ground where they are recorded. Such free-flying models have been used repeatedly in the U.S. for aeroelasticity problems as well as other important flight characteristics. Free-drop tests of models on which flutter begins directly above sonic velocity have been described by Dat, Destruynder, Loiseau, and Trubert (ref. 57) and compared with theoretical calculations.

13. FLIGHT TESTS AT FULL SCALE

The flight safety of new airplane types must be proved by flight at maximum speed. Then it may happen that the critical speed is exceeded without initiating flutter, due to an accidentally absent outside impulse. On the other hand, sudden incipient flutter may lead to the destruction of the airplane before the pilot has time to observe the particulars of the event. Attempts have therefore been made to find some way by which a dangerous flutter possibility, that escaped theoretical or experimental detection, could still be spotted early enough in a flight test.

In Germany they mounted, for this purpose, oscillating masses on levers into the wing, which were excited with slowly increasing frequency, as in the static oscillation test. It was hoped that, at increasing oscillation amplitudes with increasing flying speed, an approach to the critical flutter oscillation could be identified at the right time. This is correct in principle. But in the execution of the flight tests it is very difficult and time-consuming to grade frequency and speed fine enough. The rise in amplitude can therefore be so fast that it is too late for a warning and the airplane crashes. However, favorable flutter modes whose amplitudes in a given speed range do not exceed a small, still nondangerous amount have been detected by this method. These were carefully recorded by flight tests, because they are suitable for checking theoretical methods of calculations.

On airplanes with servocontrol, it is easier to impress a periodic booster force on the control surface which is then transformed into a many times greater periodic wing load distribution by aerodynamic transformation. It is not difficult to install an additional corresponding servocontrol for acceptance testing. This method has been used in the U.S.A. Pepping (ref. 58) made an analytical study of wing flutter by artificially excited control surface oscillations. The flutter stability can be determined from the initiated torque of the oscillating control surface per unit of angle of rotation and observed periodically; when it is zero, flutter condition is reached. This dangerous condition can be avoided by installing a hydraulic damper, or even more effectively, a feedback coupling with the wing rotation in the servocontrol. This raises the critical speed, and makes it possible to draw conclusions about the critical speed which the aircraft would have without stabilizing servo mechanism from the measurements of the amplitudes and phase angles of the excitation and the wing motion. When the rate of reaction of the servo mechanism is high enough, it can be utilized to compensate any potential flutter by appropriate control motions with appropriate feedback. Even gust stresses can be compensated this way. Admittedly, it may seem risky to rely on the functioning of a sensitive servo mechanism for suppressing flutter. But it can be used to good advantage, at least in high-speed tests of a new airplane type.

14. PREVENTION OF FLUTTER BY DESIGN SPECIFICATIONS

In spite of the multitude of measurements of the stress frequency distributions of airplanes in flight already available, it has seldom been possible to use the physical process of the flight stress as basis of the strength calculations of airplanes. The strength of modern aircraft in flight rests rather on very primitive dimension-analytical formulas and empirical values on conventional aircraft compiled in the design specifications for airplanes.

Therefore, it seemed justifiable to follow a similar path in the practical treatment of flutter in order to be able to build airplanes with less flutter risk at tolerable expense. The available theoretical and experimental aids are inadequate for a highly probable correct prediction of flutter freedom.

The convenient method developed in the early days of aviation, of piloting the airplane by free-swinging control surface hinged at the wing trailing edge, actually promoted flutter. The subsequent addition of auxiliary control surfaces multiplied the difficulty. This fundamental defect of all airplanes up to now can be largely removed by placing the centroidal axis of the control surface and auxiliary control surface in or directly before their axis of rotation. This way, the degrees of freedom "bending or translation of the wing" and "rotation of control surface" are uncoupled at least as regards the mass forces and Kelvin's impulses. The coupling of the degrees of freedom "wing rotation," "control surface rotation," and "auxiliary control surface rotation" is reduced also, but not eliminated. Complete uncoupling of the degrees of freedom is impossible for aerodynamic reasons.

The beneficial effect of this mass balance of control surfaces is voided when the control surface or auxiliary control surface with its control cables has static natural oscillations which are in resonance with those of the wing or with one another or approach it. The phase angle between wing oscillation and control surface oscillation is then very slight, i.e., variable by very slight outside forces, so that it can assume the value most favorable for energy absorption from the airstream and so lead to flutter. The second design rule therefore specifies the avoidance of such neighboring frequencies, the permissible clearance being defined by experiment and simplified flutter calculations.

The stepwise application of these construction rules for the design of new aircraft and their check on the finished airplane by static oscillation tests lowered the flutter probability considerably in Germany during the 1933 to 1944 period, where the effect of the individual measures could be proved statistically. Table I gives the cases of flutter investigated in Germany according to the statistics of Küssner (ref. 59) and Schwarzmann (ref. 60), for the periods a = 1925-1933, b = 1934-1940, and c = 1941-1944. Every spontaneously occurring case of flutter is counted once, arbitrarily inducible flutter altogether counted only once.

In these accidents more than 60 pilots and test engineers lost their lives, 41 of them due to auxiliary control surface flutter. The surprisingly high proportion of the auxiliary control surfaces is due to the fact that these small but vital structural components and their control cables and supports were not always given the necessary care in the design and manufacture. By themselves positively moving, sufficiently stiff auxiliary control surfaces require no mass balance. But if they or their connections are too weak or break, a new degree of freedom coupled by mass forces is produced. Hence, the requirement for mass balance of auxiliary control surfaces in the German specifications.

Looking back, it is apparent that a careful application of the rules:

1. Displacement of axis of gravity in or directly before its axis of rotation

2. Avoidance of frequencies approaching static oscillations on wings, control surfaces, and auxiliary control surfaces

would have prevented most of these flutter incidents. Experience indicates further that the permissible departures from these rules, i.e., the manufacturing tolerances of production aircraft, must be very narrow. Changes that may seem slight, even a new coat of paint etc., have caused flutter on proved airplane types, more than once.

For 39 of the observed cases of flutter calculations of the reduced frequency

$$\omega = \frac{\nu l_m}{v} \tag{109}$$

are available, where $2l_m$ is the mean wing chord of the wing of tail-surface portion under maximum flutter. Of particular interest are those values which are only rarely exceeded. Table II represents the highest, second highest, and third highest values. The highest values are of lesser interest, since the individual values may contain relatively great observation errors. The values $\omega_1 = 2.4$ was observed fairly accurately on a flying boat and represents a special case, because the aileron fluttered with two degrees of freedom.

The physical basis for the empirical fact that certain maximum values of the reduced frequency are rarely exceeded must be looked for in the structural damping of the aircraft. The damping phase angle on conventional aircraft is $g_i \geqq 0.01$. To be able to absorb the corresponding energy from the airstream requires certain minimum values of the wave length of the oscillatory motion even for optimum phase angle of degrees of freedom. On assuming that flutter frequency is always higher than the fundamental frequency ν_1 of the static oscillation, flutter should be avoidable when

$$\nu_1 \geqq \frac{v_0}{l_m} \omega_I \tag{110}$$

where ω_I indicates one of the values of table II and v_0 the maximum speed.

Originally, it was attempted in Germany to design airplanes according to equation (110). Since the degrees of freedom "wing bending" and "aileron rotation" have been decoupled since 1933 by the mass balance recommended in the German design specifications, the fundamental wing rotation served as lowest flutter responsive static oscillation mode and a corresponding torsional wing stiffness was demanded. But this requirement also proved itself as unnecessarily far-reaching for high-speed aircraft on which the two aforementioned design rules had been fulfilled, as numerous flutter calculations and model tests indicated. Auxiliary control surface flutter, which chiefly occurs with the degrees of freedom "control surface rotation" and "auxiliary control surface rotation" could not be prevented by wing stiffness no matter how great. For these reasons, the overall frequency (110) was stricken from the German Specifications 1936 (ref. 61). In its place a theoretical or experimental proof of flutter freedom in the permissible flight range was demanded. This was still afflicted with many uncertainties at that time, but it exerted, at least, an educational effect on the airplane builder and clarified the influence of the most important flutter parameters.

Through this flutter proof, which extended to the static limiting cases of wing deflection and reversal of control surface effect, the wing stiffness and control surface stiffness were given a lower limit.

It was found, repeatedly, that the stiffness prevailing for reasons of strength itself was above this lower limit, hence, that the particular airplane was safe against flutter and static instability when it satisfied the strength requirements and in addition satisfied the two aforementioned design regulations. Wittmeyer (ref. 62) advocated a corresponding airplane development while retaining certain basic design modes, which have proved themselves as flutter-safe in an extended series of developments. By prescribing certain dimensionless parameters and additional rules for the auxiliary control surfaces in the design of new types of aircraft, it should then be possible to attain flutter safety by way of compliance with the strength requirements. However, such design rules can be applicable only in a narrowly limited empirical range. When advancing into higher speed ranges and with the use of very thin airfoils, the matter becomes different. In section 9, there is given a third design rule which prevents skin buckling in any operating condition; de Vries (ref. 63) gives a critical summary of the design rules for the prevention of flutter.

In the British Specifications and in those of the ICAO (International Civil Aviation Organization (ref. 64)), overall stiffness requirements still take up a lot of space. It deals with recommendations for conventional aircraft, the compliance of which dispenses with a further flutter proof. These requirements call for minimum values of the mean torsional stiffness of wing, fuselage, tail unit, and control surfaces, which depend upon the permissible maximum speed and on certain dimensions of these structural components as follows

$$\frac{M}{\Theta} \geqq k^2 \rho_0 v_D^2 B S^2 \left(1 - \frac{v_D}{c_0}\right)^{-1/2} \qquad \frac{v_D}{c_0} \leqq 0.8 \tag{111}$$

M is a moment, Θ the angle of rotation during this moment at a certain point, v_D the maximum gliding speed at 30° path slope, B and S are lengths; on the wing, for example

$$B = 0.9b \qquad S = t_m$$

b = wing span from root to tip, t_m the mean geometric chord. The dimensionless constant k for the separate components is specified. This formula (111) is applied also to the flexural stiffness of the fuselage and the elevator overhang arm, with M and Θ defined accordingly.

Equation (111) follows from the condition of the static torsional stability of the wing by forming the mean value of the integral. For this case, the Prandtl factor is correct. The constant k^2 is then, however, proportional to the rearward position of the elastic axis[1] behind the neutral axis at quarter wing chord, but this rearward position is materially dependent upon the respective structural design of the wing.

So far as the overall stiffnesses according to equation (111) are to serve for the prevention of flutter, they are supportable only by equation (110) and table II, hence are disguised frequency requirements. (Cf. Collar, Broadbent, and Puttick (refs. 2 and 65).) But the static oscillation frequencies ν_i of an airplane are dependent in a rather complicated manner upon the size and distribution of masses and stiffnesses. To justify overall stiffness formulas like equation (111), the constructive tolerance, conceded to the "conventional" airplanes, would have to be very narrow, i.e., they all would have to be practically geometrically similar and partially dynamically similar, according to development series investigated by Wittmeyer (ref. 62). Or else the safety factor of the overall stiffness would have to be very great, since the actual differences of the airplanes of different manufacturers cause a correspondingly great variation of the decisive oscillation frequencies ν_i.

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[1]The calculating methods hitherto were based on the replacement of the wing by an elastic beam with elastic axis. A plate-like wing has no elastic axis.

Shifting the gravity axis and the elastic axis of the wing in the neutral axis, the degrees of freedom "wing bending" and "wing rotation" in the subsonic range can be decoupled, thus avoiding flutter. A mass-balanced control surface does not change this action very much as a rule. This mass balance of the wing has been used very rarely in the past. But it might be that future aircraft with very thin wings will be subject to a requirement regarding the position of the gravity axis of the wing, in order to lower the required torsional stiffness of the wing to a structurally tolerable level.

To assure safety of aircraft against flutter in the future, the young aeronautical engineers must be made familiar with aeroelastic problems now, so as to enable them to take the correct safeguards while the aircraft is being designed. Formerly, airplanes were generally designed or even built without regard to flutter hazards, and trying to find a remedy proved then a too long and expensive undertaking. The flight performances often deteriorated so much that the particular type had to be withdrawn from the competition. A subsequent elevator mass balance, for example, cost 10 percent of the pay load and more on balance weight, aside from the greater air resistance of the externally mounted balance weights.

As we enter the higher speed ranges, aeroelastic problems must receive particular attention. Empirical data and design rules, which are satisfactory or still acceptable for conventional aircraft, can no longer be relied upon, the physical process itself must be studied and mastered.

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TABLE I.- FLUTTER STATISTICS

Probable causes of flutter	Wing unit			Horizontal tail surfaces			Vertical tail surfaces			
	a	b	c	a	b	c	a	b	c	$\sum$
Rear balance of control surface	13	28	8	1	12	1	-	17	1	81
Neighboring frequencies	--	7	-	-	12	-	-	1	-	20
Auxiliary control surfaces	--	4	-	-	43	7	-	18	3	75
Landing flaps	--	2	-	-	--	-	-	--	-	2
Unexplained	--	1	1	-	1	-	-	--	-	3

TABLE II.- MAXIMUM VALUES OF OBSERVED REDUCED FREQUENCY

	ω_1	ω_2	ω_3
Wing unit	(2.4)	1.29	1.14
Horizontal tail surface	1.1	0.55	0.51
Vertical tail surface	0.4	0.40	0.38

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