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THE CHARACTERISTICS METHOD APPLIED TO STATIONARY TWO-DIMENSIONAL AND ROTATIONALLY SYMMETRICAL GAS FLOWS

By F. Pfeiffer and W. Meyer-König

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THE CHARACTERISTICS METHOD APPLIED TO STATIONARY TWO-DIMENSIONAL AND ROTATIONALLY SYMMETRICAL GAS FLOWS*

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SUMMARY

By means of characteristics theory, formulas for the numerical treatment of stationary compressible supersonic flows for the two-dimensional and rotationally symmetrical cases have been obtained from their differential equations.

INTRODUCTION

The auxiliary means for the theoretical treatment of stationary gas flows at supersonic velocity are the characteristics of the partial differential equations governing the motion. The Busemann graphic methods[1] for the treatment of such potential flows is based on the network of the characteristics λ = const., μ = const. in the velocity field (u, v-plane) and the fact, that at corresponding points of the velocity and the flow field (x, y-plane) corresponding characteristics are perpendicular to one another. Guderley[2] has extended the Busemann method to two-dimensional and rotationally

*"Die Charakteristikenmethode bei stationären ebenen und rotationssymmetrischen Gasströmungen." Zentrale für wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (ZWB) Berlin-Adlershof. Forschungsbericht Nr. 1581, March 20, 1942.

[1]Busemann, A.: "Gasdynamik." Handbuch d. Experimentalphysik Bd. 4, 1. Teil, 1931, pp. 341-460, particularly p. 421 and the following pages.

[2]Guderley, G.: Die charakteristikenmethode für ebene und achsensymmetrische Überschallströmungen. Jahrbuch der Deutschen Luftfahrtforschung 1940, pp. I 522 - I 535.

symmetrical, turbulent flows. On the basis of physical considerations, he defined the changes $d\lambda$, $d\mu$ which the quantities λ and μ experience in an advance from one point of the flow field to an adjacent point along a characteristic.

For cases more general than Guderley postulated, it is possible to get the differential equations for the characteristics by pure mathematics directly from the differential equations for the gas motion. They have a very simple form, so that they are also suitable for the graphic, numerical treatment of special problems.

SYMBOLS

x, y	rectangular coordinates in the flow plane
$\underline{v}$	velocity vector
u, v	rectangular coordinates in the velocity plane
w, ϑ	polar coordinates in the velocity plane
ρ	density
p	pressure
s	entropy
T	absolute temperature
c_p, c_v	specific heat at constant pressure, or at constant volume
a	sonic velocity
a^x	critical sonic velocity
i	heat content
i_o	heat content at $w = 0$
α	Mach angle
$w', u', v', a', \rho', p'$	dimensionless quantities instead of w, u, v, a, ρ, p

ABBREVIATIONS

$$\sigma = \ln \rho', \quad \tau = \ln (a'^2), \quad S = \frac{s}{c_v}, \quad S^x = \frac{S}{k}, \quad \Phi = S^x - \tau,$$

$$A = \frac{k-1}{\sin \alpha \cos \alpha}, \quad k = \frac{c_p}{c_v}, \quad R = c_p - c_v$$

I: THE TWO-DIMENSIONAL FLOW

From the fundamental hydrodynamic equations for compressible, stationary flow[3]

$$\left.\begin{aligned} (\underline{v}\nabla)\underline{v} &= -\frac{1}{\rho}\,\mathrm{grad}\; p \\ \mathrm{div}\,(\rho\underline{v}) &= 0 \end{aligned}\right\} \tag{1}$$

for the velocity vector $\underline{v}$, the pressure p, and the density ρ, the following equations are obtained for two-dimensional flow with velocity coordinates u, v

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0, \quad u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = 0 \tag{2}$$

$$\rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0 \tag{3}$$

In addition for an ideal gas there occurs the equation

$$p = Ce^S\rho^k \tag{4}$$

where $S = \frac{s}{c_v}$, s the entropy, $k = \frac{c_p}{c_v}$ the ratio of the specific heats, C represents a constant, which is derivable from the equation of state

$$\frac{p}{\rho} = R\,T \tag{4a}$$

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[3]Webster, A. G., and Szegö, G.: Partielle Differentialgleichungen der mathematischen Physik, 1930, p. 34.

because of the first and second laws of thermodynamics ($R = c_p - c_v$, T = absolute temperature). Lastly there is the equation

$$u\frac{\partial S}{\partial x} + v\frac{\partial S}{\partial y} = 0 \tag{5}$$

which expresses the hypothesis that the entropy along each individual streamline is constant. From (4) for the (variable) sonic velocity a

$$a^2 = \frac{\partial p}{\partial \rho} = kCe^S\rho^{k-1} = k\frac{p}{\rho} = kRT \tag{6}$$

further

$$\left.\begin{aligned} \frac{\partial p}{\partial x} &= \frac{a^2}{k}\,\rho\frac{\partial S}{\partial x} + a^2\,\frac{\partial \rho}{\partial x} \\ \frac{\partial p}{\partial y} &= \frac{a^2}{k}\,\rho\frac{\partial S}{\partial y} + a^2\,\frac{\partial \rho}{\partial y} \end{aligned}\right\} \tag{6a}$$

If equations (2) are written thusly:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial y} + \frac{a^2}{\rho}\,\frac{\partial \rho}{\partial x} + \frac{a^2}{k}\,\frac{\partial S}{\partial x} = 0$$

$$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} - u\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial x} + \frac{a^2}{\rho}\,\frac{\partial \rho}{\partial y} + \frac{a^2}{k}\,\frac{\partial S}{\partial y} = 0$$

multiplication by dx or dy and addition (with $w^2 = u^2 + v^2$)

$$w\,dw + \frac{a^2}{\rho}\,d\rho + \frac{a^2}{k}\,dS = (u\,dy - v\,dx)\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$$

that is, for laminar flow the following prevails everywhere

$$w\,dw + \frac{a^2}{\rho}\,d\rho + \frac{a^2}{k}\,dS = 0 \tag{7}$$

for turbulent flow, on account of (5), on each streamline

$$w\,dw + \frac{a^2}{\rho}\,d\rho = 0 \tag{7a}$$

On each streamline $p = C_1\rho^k$ (C_1 constant) and $a^2 = \frac{kp}{\rho}$, therefore $\frac{a^2}{\rho}\,d\rho = \frac{k}{k-1}\,d\left(\frac{p}{\rho}\right)$. Accordingly, it follows from (7a) by integration for each streamline

$$\frac{w^2}{2} + \frac{a^2}{k-1} = i_o \tag{8}$$

or

$$\frac{w^2}{2} + i = i_o \tag{8a}$$

if $i = c_pT = \frac{k}{k-1}RT = \frac{a^2}{k-1}$ is the heat content of the ideal gas. The constant i_o is, in general, different from streamline to streamline. The condition (8a) also holds in the final form if the streamline passes through a compression shock[4]; first, only the region that does not include a compression shock will be considered.

Equations (2), (3), and (5), where $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$ in (2) are replaced in accordance with (6a), together with

$$\left.\begin{aligned} du &= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy, \quad dv = \frac{\partial v}{\partial x}\,dx + \frac{\partial v}{\partial y}\,dy \\ d\rho &= \frac{\partial \rho}{\partial x}\,dx + \frac{\partial \rho}{\partial y}\,dy, \quad dS = \frac{\partial S}{\partial x}\,dx + \frac{\partial S}{\partial y}\,dy \end{aligned}\right\} \tag{9}$$

from eight linear equations for $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial \rho}{\partial x}, \frac{\partial \rho}{\partial y}, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}$ with the matrix

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[4] Busemann, A., (See footnote 1) p. 433.

$$\begin{array}{cccccccc|c}
\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial \rho}{\partial x} & \frac{\partial \rho}{\partial y} & \frac{\partial S}{\partial x} & \frac{\partial S}{\partial y} & = \\
\hline
u & v & 0 & 0 & \frac{a^2}{\rho} & 0 & \frac{a^2}{k} & 0 & 0 \\
0 & 0 & u & v & 0 & \frac{a^2}{\rho} & 0 & \frac{a^2}{k} & 0 \\
\rho & 0 & 0 & \rho & u & v & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & u & v & 0 \\
dx & dy & 0 & 0 & 0 & 0 & 0 & 0 & du \\
0 & 0 & dx & dy & 0 & 0 & 0 & 0 & dv \\
0 & 0 & 0 & 0 & dx & dy & 0 & 0 & d\rho \\
0 & 0 & 0 & 0 & 0 & 0 & dx & dy & dS
\end{array}$$

The characteristics are here defined as those curves on the integral surfaces sought, along which the derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial \rho}{\partial x}$, $\frac{\partial \rho}{\partial y}$, $\frac{\partial S}{\partial x}$, $\frac{\partial S}{\partial y}$ do not result from these eight equations as definite, along which, therefore, discontinuities in these derivatives are possible. In order that the quantities $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial \rho}{\partial x}$, $\frac{\partial \rho}{\partial y}$, $\frac{\partial S}{\partial x}$, $\frac{\partial S}{\partial y}$ can be ambiguous, the determinants of eight rows of this matrix must equal zero. Setting these determinants equal to zero furnishes the four systems of characteristics (projection of the first counted twice)

$$\left.\begin{aligned} u\,dy - v\,dx &= 0 \\ dS &= 0 \qquad \text{(streamlines)} \\ w\,dw + a^2 \frac{d\rho}{\rho} &= 0 \end{aligned}\right\} \tag{10}$$

$$\left.\begin{aligned} &(v^2 - a^2)\,dx^2 - 2uv\,dx\,dy + (u^2 - a^2)\,dy^2 = 0 \\ &(u\,dy - v\,dx)(u\,dv - v\,du) + \left(\frac{d\rho}{\rho} + \frac{dS}{k}\right) a^2 (u\,dx + v\,dy) = 0 \end{aligned}\right\} \tag{11}$$

(Mach waves)

As a result of the introduction of polar coordinates with $u = w \cos \vartheta$, $v = w \sin \vartheta$ in the u, v-plane and of the Mach angle α through $\sin \alpha = \frac{a}{w}$ there is obtained

$$u\,dv - v\,du = w^2\,d\vartheta, \quad u\,du + v\,dv = w\,dw$$

and from the first equation (11)

$$\frac{dy}{dx} = \frac{-uv \pm a\sqrt{u^2 + v^2 - a^2}}{a^2 - u^2} = \tan\,(\vartheta \mp \alpha)$$

For $w > a$, that is, for the case of supersonic flow, which is the present problem, the characteristics become real. On account of $\frac{u\,dy - v\,dx}{u\,dx + v\,dy} = \mp \tan \alpha$ equations (10) and (11) become

$$\left.\begin{aligned} \frac{dy}{dx} &= \tan \vartheta \\ dS &= 0 \\ dw + a \sin \alpha \frac{d\rho}{\rho} &= 0 \end{aligned}\right\} \qquad (10a)$$

$$\left.\begin{aligned} \frac{dy}{dx} &= \tan\,(\vartheta \mp \alpha) \\ \mp \frac{d\vartheta}{\sin \alpha \cos \alpha} + \frac{d\rho}{\rho} + \frac{dS}{k} &= 0 \end{aligned}\right\} \qquad (11a)$$

(These equations are still valid, if the equation of state is in the more general form $p = f(\rho, S)$ provided in the second equation (11a) $\frac{1}{k}$ is replaced by $\frac{1}{a^2\rho}\frac{\partial f}{\partial S}$.)

The equations are made dimensionless, as a result of introducing $w = w^0 w'$, $a = w^0 a'$, $\rho = \rho^0 \rho'$, where w^0 is any reference velocity, ρ^0 any reference density. After setting $\ln \rho' = \sigma$ and $S^x = \frac{S}{k}$, the characteristic equations become

$$\left.\begin{aligned} \frac{dy}{dx} &= \tan\vartheta \\ dS &= 0 \\ dw' + a' \sin\alpha\, d\sigma &= 0 \end{aligned}\right\} \qquad (10b)$$

$$\left.\begin{aligned} \frac{dy}{dx} &= \tan(\vartheta \mp \alpha) \\ \mp \frac{d\vartheta}{\sin\alpha\cos\alpha} + d\sigma + dS^x &= 0 \end{aligned}\right\} \qquad (11b)$$

with $\sin\alpha = \frac{a'}{w'}$.

In the point-by-point numerical determination of the flow phase with the aid of the characteristics the Massau method[5] is made use of.

Let all the phase quantities at points 1 and 2 of the flow be known and provided with the subscript 1 or 2. A point 3 of the flow with its phase quantities is obtained thusly: According to the first equation (11b) the position x_3, y_3 of 3 is obtained from $y_3 - y_1 = \tan(\vartheta_1 - \alpha_1)(x_3 - x_1)$ and $y_3 - y_2 = \tan(\vartheta_2 + \alpha_2)(x_3 - x_2)$ graphically or numerically. From the second equation (11b) are obtained these equations as differences

$$-\frac{1}{\sin\alpha_1\cos\alpha_1}(\vartheta_3 - \vartheta_1) + (\sigma_3 + S_3^x) = \sigma_1 + S_1^x$$

and

$$\frac{1}{\sin\alpha_2\cos\alpha_2}(\vartheta_3 - \vartheta_2) + (\sigma_3 + S_3^x) = \sigma_2 + S_2^x$$

From this (numerically) ϑ_3 and $\sigma_3 + S_3^x$. By means of the first equation (10b)

$$y_3 - y_m = \tan\vartheta_3 (x_3 - x_m)$$

[5] Encyklopädie der Math. Wissenschaften II, 3, 1, p. 162.

either graphically or numerically point m is reached from point 3 by going in the direction of the streamline related to 3, m being in the connecting line 1–2, whose phase quantities are computed by linear interpolation from 1 and 2. Then $S_3^x = S_m^x$ and, therefore σ_3 also, is obtained. The third equation (10b) gives

$$w_3' - w_m' = -\alpha'_m \sin \alpha_m (\sigma_3 - \sigma_m)$$

from which w_3' follows.

With this x_3, y_3, ϑ_3, σ_3, S_3^x, w_3' are to be had. For further calculation a'_3 and α_3 from $\alpha_3^2 = kCe^{S_3}\rho_3^{k-1}$ and $\sin \alpha_3 = \frac{a'_3}{w'_3}$ are used. From the first approximations obtained in this way, better values can be obtained by iteration after the manner of Massau; often estimated, possible extrapolated values can be used for the first approximations.

In all cases graphic representations can be given for simplification of the calculation, for example, because

$$A = \frac{1}{\sin \alpha \cos \alpha} = \frac{1}{\frac{\alpha'}{w'}\sqrt{1 - \left(\frac{\alpha'}{w'}\right)^2}}$$

the ray system A = constant and α = constant from the origin of the a', w' plane, or the system of straight lines S = const. in a plane with log a and log ρ in the coordinate directions.

NOTE

In an isentropic laminar motion, $dS = 0$ and $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$, therefore, everywhere and

$$w \, dw + \frac{a^2}{\rho} d\rho = 0 \tag{7b}$$

The equations for the characteristics systems (11a) become

$$\left.\begin{aligned} \frac{dy}{dx} &= \tan\ (\vartheta \mp \alpha) \\ \mp\ d\vartheta - \cot\ \alpha \frac{dw}{w} &= 0 \end{aligned}\right\} \tag{12}$$

If P and P' are two corresponding points in the x, y- and the u, v-planes and two pairs of corresponding characteristics with their tangent directions are drawn through them, then

$$\tan\ \gamma = \frac{w}{dw/dv} = \mp \cot\ \alpha$$

also

$$\gamma = 90^\circ \pm \alpha$$

$$\beta = \vartheta + \gamma = 90^\circ + \vartheta \pm \alpha$$

By the first equation (12)

$$S = \vartheta \mp \alpha$$

In addition, there is the fact that is the basis for the Busemann graphic method for isentropic, laminar motion, that in two corresponding points of the u, v- and the x, y-planes the characteristics are mutually perpendicular crosswise.

II. THE ROTATIONALLY SYMMETRICAL FLOW

Equations (2) result from equations (1) again by application of cylindrical coordinates[6] and considering the rotational symmetry while the term $\rho\frac{u}{x}$ enters in equation (3) on the left side. In addition to that, equations (4) and (5) are again valid.

In setting up the characteristics, instead of 0 in the ninth column, third place from the top $\rho\frac{u}{x}$ appears now and, as a result, the characteristic equations become

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[6] Frank, Ph., and V. Mises, R.: Die Differential- und Integralgleichungen der Mechanik und Physik I, second ed., 1930, p. 86.

$$\left.\begin{array}{ll} u\,dy - v\,dx = 0 & (v^2 - a^2)\,dx^2 - 2\,u\,v\,dx\,dy + (u^2 - a^2)\,dy^2 = 0 \\ dS = 0 & (u\,dy - v\,dx)^2 \frac{u}{x} + (u\,dy - v\,dx)\,(u\,dv - v\,du) \\ w\,dw + a^2 \frac{d\rho}{\rho} = 0 & \quad + \left(\frac{d\rho}{\rho} + \frac{dS}{k}\right) a^2\,(u\,dx + v\,dy) = 0 \end{array}\right\} \quad (13)$$

With the polar coordinates w, ϑ in the u, v-plane and the Mach angle α there is obtained because of

$$\frac{1}{a^2} \frac{(u\,dy - v\,dx)^2}{u\,dx + v\,dy} \frac{u}{x} = \frac{1}{a^2} (\mp \tan\alpha) \frac{w\cos\vartheta}{x} dx \left(w\cos\vartheta \tan(\vartheta \mp \alpha) - w\sin\vartheta\right)$$

$$= \frac{dx}{x} \frac{\cos\vartheta}{\cos\alpha\cos(\vartheta \mp \alpha)}$$

$$\left.\begin{array}{ll} \frac{dy}{dx} = \tan\vartheta & \frac{dy}{dx} = \tan(\vartheta \mp \alpha) \\ dS = 0 & \mp \frac{d\vartheta}{\sin\alpha\cos\alpha} + \frac{d\rho}{\rho} + \frac{dS}{k} + \frac{dx}{x} \frac{\cos\vartheta}{\cos\alpha\cos(\vartheta \mp \alpha)} = 0 \\ dw + a\sin\alpha\,d\sigma = 0 & \end{array}\right\} \quad (13a)$$

providing $x \neq 0$.

III. SPECIAL CASE: IDENTICAL i_o FOR ALL STREAMLINES

Guderley[7] restricts himself to the practically important special case that i_o in equation (8) is the same constant for all streamlines, that is, there is a curve (not a streamline) in the region of flow, which is intersected by all streamlines and along which i_o has a fixed value.

It follows from equation (8) then

$$w\,dw + \frac{2}{k-1} a\,da = 0$$

and from this, because of

$$a^2 = kCe^S \rho^{k-1}$$

$$w\,dw + a^2 \frac{d\rho}{\rho} + \frac{a^2}{k-1} dS = 0$$

[7] Guderley, (See footnote 2) p. I 524.

or

$$w\,dw + a^2 \frac{d\rho}{\rho} + \frac{a^2}{k}\,dS = -\frac{a^2}{k(k-1)}\,dS$$

in the entire region of flow.[a]

With the resultant equation

$$\frac{d\rho}{\rho} + \frac{dS}{k} = -\frac{1}{k(k-1)}\,dS + \frac{2}{k-1}\frac{da}{a}$$

the second equation (11a) becomes, if, in addition, $ln\,(a'^2) = \tau$

$$\mp \frac{k-1}{\sin\alpha\cos\alpha}\,d\vartheta - dS^x + d\tau = 0 \qquad (11c)$$

As a result, the numerical work is simpler than in the general case.

Again, all phase quantities at points 1 and 2 (fig. 1) are known. The equations

$$y_3 - y_1 = \tan(\vartheta_1 - \alpha_1)(x_3 - x_1)$$

and

$$y_3 - y_2 = \tan(\vartheta_2 + \alpha_2)(x_3 - x_2)$$

again give (graphically or numerically) the position x_3, y_3 of point 3. From the equations (11c) expressed in differences

$$-A_1(\vartheta_3 - \vartheta_1) - (\Phi_3 - \Phi_1) = 0$$

and

$$A_2(\vartheta_3 - \vartheta_2) - (\Phi_3 - \Phi_2) = 0$$

[a] Crocco, L.: Eine neue Stromfunktion für die Erfoschung der Bewegung der Gase mit Rotation, ZAMM 17, 1937, pp. 1-7, particularly p. 2, equation (1")b.

with $A = \frac{k-1}{\sin\alpha\cos\alpha}$ and $S^x - \tau = \Phi$ becomes

$$\vartheta_3 = \frac{A_1\,\vartheta_1 + A_2\vartheta_2 + \Phi_1 - \Phi_2}{A_1 + A_2},$$

$$\Phi_3 = \Phi_1 - A_1\,(\vartheta_3 - \vartheta_1) = \Phi_2 + A_2\,(\vartheta_3 - \vartheta_2)$$

By means of the streamline through 3, m is obtained, as in the general case, and with $S_3^x = S_m^x$, τ_3' and a'_3. Because of equation (8) the third equation (10b) is not needed this time. Figure 4 furnishes w_3 and the quantities α_3 and A_3 needed for further calculation.

For comparison with Guderley the additional term which is to be put on the left side of the second equation (12) for the potential flow, is determined in order to obtain the left hand side of the second equation (11c)

$$\mp d\vartheta - \frac{\sin\alpha\cos\alpha}{k-1}\,dS^x + \frac{2\sin\alpha\cos\alpha}{k-1}\,\frac{da}{a}$$

for the turbulent flow. This additional term is

$$-\frac{\sin\alpha\cos\alpha}{k-1}\,dS^x + \frac{2\sin\alpha\cos\alpha}{k-1}\,\frac{da}{a} + \cot\alpha\,\frac{dw}{w} = -\frac{\sin\alpha\cos\alpha}{k-1}\,\frac{dS}{k}$$

$$-\cot\alpha\,\frac{dw}{w} + \cot\alpha\,\frac{dw}{w} = -\cot\alpha\,\frac{a^2}{w^2}\,\frac{ds}{(k-1)kc_v} = -\cot\alpha\,\frac{ds}{w^2}\,\frac{a^2}{kR}$$

$$= -\cot\alpha\,\frac{T}{w^2}\,ds$$ [9]

since $c_v\,(k-1) = R$ and $\frac{a^2}{kR} = T$

In the rotationally symmetrical case there is a further additional term; according to equation (13a) it is

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[9]Guderley, (See footnote 2), I 525, equation (16a) and (16b).

$$\frac{dx}{x}\,\frac{\sin\alpha\cos\vartheta}{\cos(\vartheta\mp\alpha)}$$

Introducing

$$dl = \sqrt{dx^2 + dy^2} = \frac{dx}{\cos(\vartheta\mp\alpha)}$$

it becomes

$$\frac{dl}{x}\sin\alpha\cos\vartheta$$

as in Guderley [10], if it is borne in mind that there, as opposed to our designations, y is used instead of x and ϑ there is the angle with the axis of rotation, $90^\circ - \vartheta$, therefore, in the case of this report.

IV. COMPRESSION SHOCKS

The treatment of a compression shock appearing in the two-dimensional flow with the variables used in this report is given for the example treated by Guderley[11].

For any point of the shock line let $\overline{w}$ be the velocity (parallel to the x-axis), $\overline{p}$ the pressure, $\overline{\rho}$ the density to the left of the shock line, w (coordinates u, v) velocity, p pressure, ρ density, a sonic velocity to the right of the shock line, let p_o, ρ_o be the pressure and density of the flowing gas at rest, a^x the (constant) critical sonic velocity and let

$$\frac{\overline{w}}{a^x} = \overline{w}', \quad \frac{\overline{p}}{p_o} = \overline{p}', \quad \frac{\overline{\rho}}{\rho_o} = \overline{\rho}', \quad \frac{w}{a^x} = w', \quad \frac{u}{a^x} = u', \quad \frac{v}{a^x} = v', \quad \frac{p}{p_o} = p',$$

$$\frac{\rho}{\rho_o} = \rho', \quad \frac{a}{a^x} = a',$$

[10] Guderley (See footnote 2), p. I 526.

[11] Guderley (See footnote 2), figure 14.

then the flow to the left of the shock line is given by $\bar{w}'$, and $\bar{p}' \times \bar{\rho}'$ follows from $\bar{p}' = \bar{\rho}'^k$. The following equations [12] hold for passing through the shock line.

$$v'^2 = (\bar{w}' - u')^2 \frac{u' - \frac{1}{\bar{w}'}}{\left(\frac{1}{\bar{w}'} + \frac{2}{k+1}\bar{w}'\right) - u'} \tag{14}$$

(shock polar; it is the same curve here for all points of the shock line).

$$p' = \bar{p}' + \bar{\rho}' (\bar{w}' - u') \bar{w}' \frac{2k}{k+1} \tag{15}$$

$$\frac{w'^2}{2} + \frac{1}{k-1} a'^2 = \frac{1}{2} \frac{k+1}{k-1} \tag{16}$$

$$p' = \frac{2}{k+1} a'^2 \rho' \tag{17}$$

$$p' = e^{S} \rho'^k \tag{18}$$

(Equation of state, if the entropy is set equal to zero for p_0, ρ_0.)

Point 1: The direction of the velocity after the shock is tangent to the profile at 1; as a result, from the shock polars are obtained (fig. 6) w_1', u_1', v_1', ϑ_1 and the direction of the shock line at 1, perpendicular to $\overline{AB}$.

Then with that, p_1' from (15), a_1' from (16), ρ_1' from (17), ϑ_1 from (18), and α_1 from $\sin \alpha_1 = \frac{a_1'}{w_1'}$.

Point 2: Point 2 is taken close to 1 on the shock line and all quantities (except x_2 and y_2) are taken as at point 1 (fig. 7).

Point 3: The coordinates x_3, y_3 of the point 3 which is the intersection of the characteristics 2–3 leaving from 2 with the contour of the profile, are obtained (graphically) by means of

[12] Busemann (See footnote 1) p. 436.

$$y_3 - y_2 = \tan (\vartheta_2 - \alpha_2) (x_3 - x_2)$$

Since 1–3 is a streamline, $S_3^x = S_1^x \vartheta_3$ is the direction of the tangent to the contour at 3. τ_3 is computed from

$$- A_2 (\vartheta_3 - \vartheta_2) - (S_3^x - S_2^x) + (\tau_3 - \tau_2) = 0$$

with τ_3, a'_3 is obtained from $ln (a'^2) = \tau$, w'_3 from

$$\frac{w_3'^2}{2} + \frac{a_3'^2}{k - 1} = \frac{1}{2} \frac{k + 1}{k - 1}$$

and α_3 with it.

Point 4: With $\bar{w}'$ and w'_2 the direction of the shock line at 2 is established (it still agrees with the direction at 1). The direction of the characteristics out from 3 gives, by means of

$$y_4 - y_3 = \tan (\vartheta_3 + \alpha_3) (x_4 - x_3)$$

the point 4 (x_4, y_4) (graphically) as the intersection of this characteristic with the direction of the shock line; ϑ_4 is estimated and w'_4, u'_4, v'_4 are obtained graphically from the shock polar; then, with that, p'_4 by (15), a'_4 by (16), ρ'_4 by (17), S_4 from (18), α_4 from $\sin \alpha_4 = \frac{a'_4}{w'_4}$, τ_4 from $\tau_4 = ln (a'_4)^2$.

Supplementing this, there is the second equation of the characteristics from 3:

$$A_3 (\vartheta_4 - \vartheta_3) - (S_4^x - S_3^x) + (\tau_4 - \tau_3) = 0$$

from which a new ϑ_4 is computed, with which an iteration is carried out.

Proceeding from point 4 along a characteristic of one family to a farther point on the contour, the quantities at this point are computed analogous to the manner at 3, from here proceeding along a characteristic of the other family to a farther point of the shock line and there computing the quantities analogous to the manner at 4.

As soon as the steps become too large, it becomes necessary to interpose points; new points in the region between shock line and contour are computed by the method in III. The related quantities for the points of departure a, b are determined by interpolation between 3 and 4 (fig. 8).

Translated by Dave Feingold
National Advisory Committee
for Aeronautics

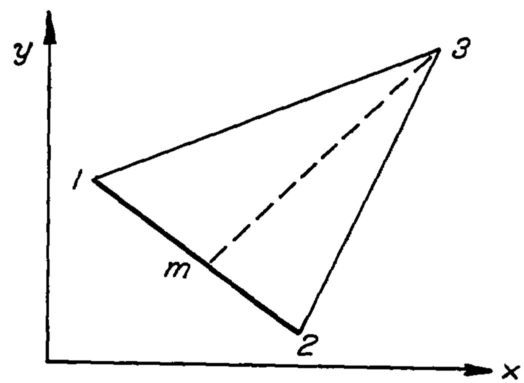


Figure 1.

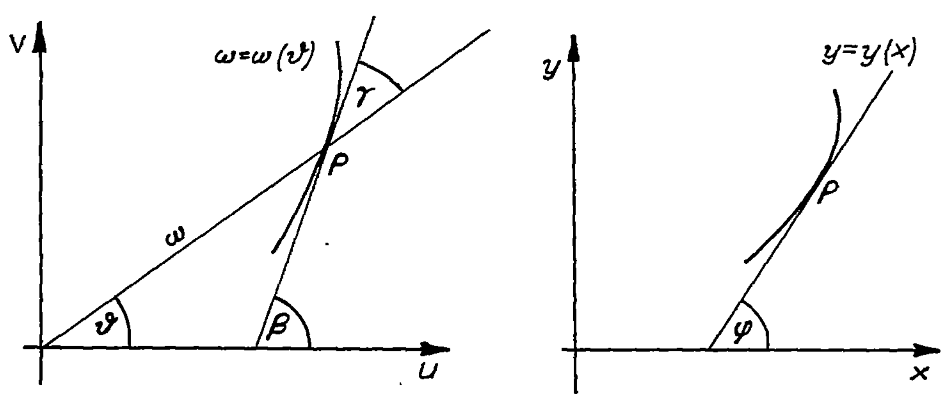


Figure 2.

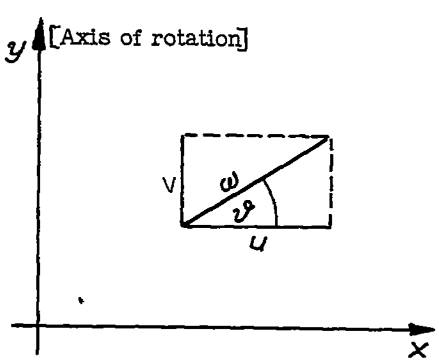
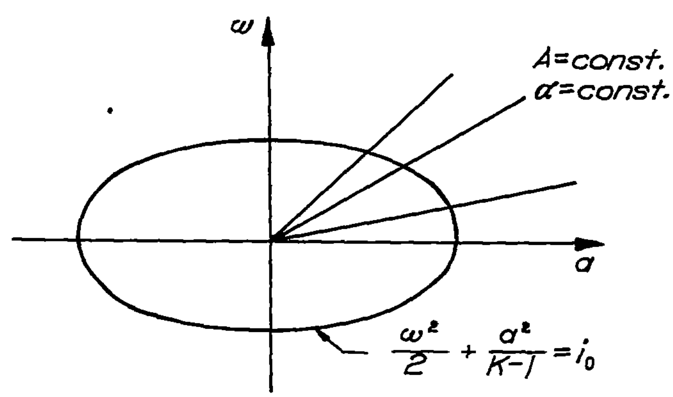


Figure 3.



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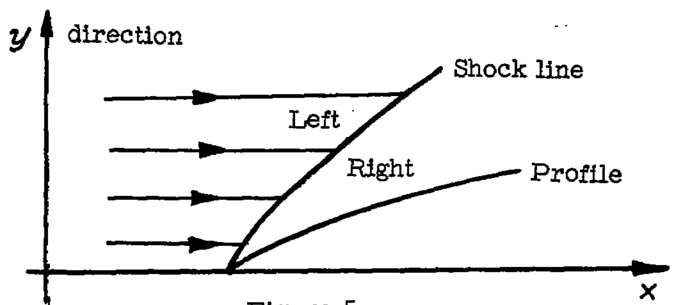

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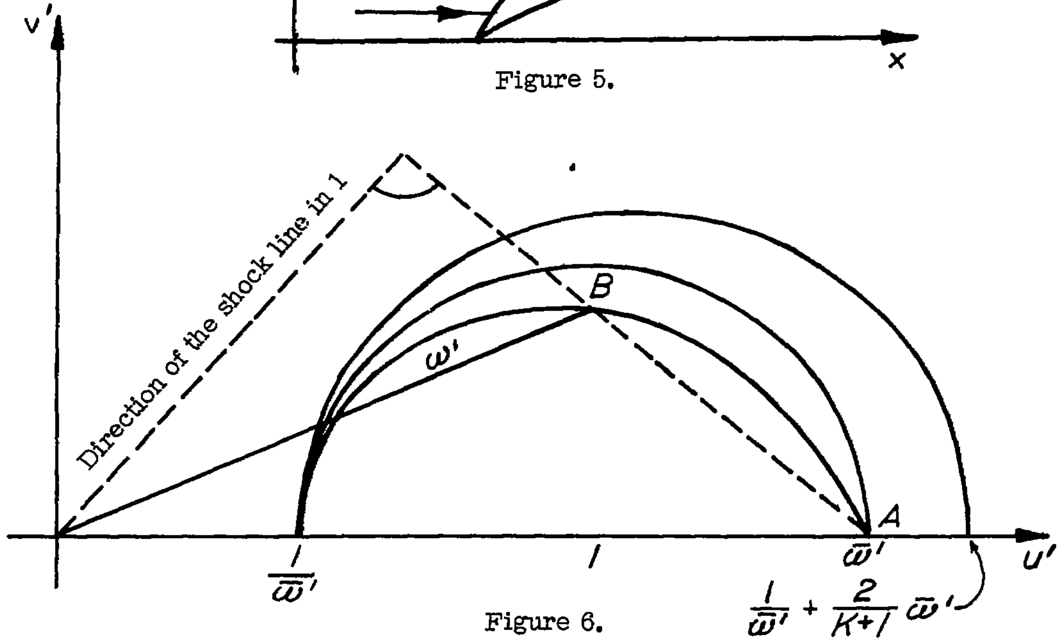

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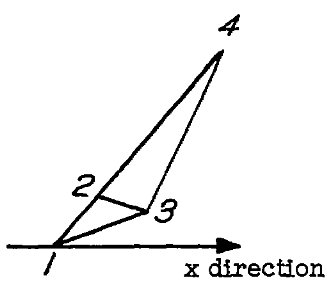

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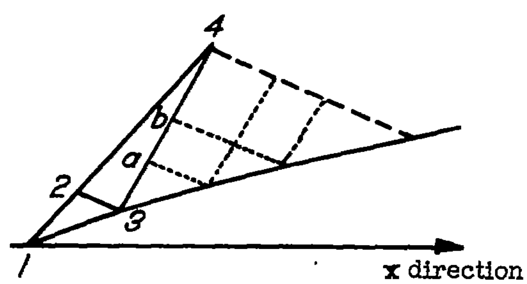

Figure 8.