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No. 1215

FLOW PATTERN IN A CONVERGING-DIVERGING NOZZLE

By Kl. Oswatitsch and W. Rothstein

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FLOW PATTERN IN A CONVERGING-DIVERGING NOZZLE*

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The present report describes a new method for the prediction of the flow pattern of a gas in the two-dimensional and axially symmetrical case. It is assumed that the expansion of the gas is adiabatic and the flow stationary. The several assumptions necessary on the nozzle shape effect, in general, no essential limitation on the conventional nozzles. The method is applicable throughout the entire speed range; the velocity of sound itself plays no singular part. The principal weight is placed on the treatment of the flow near the throat of a converging-diverging nozzle. For slender nozzles formulas are derived for the calculation of the velocity components as function of the location.

I. INTRODUCTION

The field of a compressible nozzle flow has been treated repeatedly. Thus, Th. Meyer (reference 1) computed the transition from subsonic to supersonic flow for a given velocity distribution over the nozzle axis. G. I. Taylor (reference 2) calculated the case of subsonic flow which at the throat of the nozzle reaches such speeds that the velocity of sound is exceeded at several points. H. Görtler (reference 3) dealt in particular with the transition from one of these types of flow into the other.

II. DERIVATION OF THE FUNDAMENTAL EQUATIONS

FOR TWO-DIMENSIONAL FLOW

The nozzle axis is indicated with x, the normal to it is y, and the origin of the coordinate system is placed in the center of the narrowest cross section of a Laval nozzle symmetrical about the x-axis, figure 1. The velocity components are u and v, the direction of flow is from left to right. The shape of the nozzle is given by a function $f(x)$, with f denoting half the height

*"Das Strömungsfeld in einer Lavaldüse." Jahrbuch 1942 Luftfahrtforschung, pp. I 91-102.

of the nozzle. Taking $f(0) = 1$ all linear dimensions with half the smallest nozzle height become nondimensional.

In contrast to the earlier reports u and v are expressed by a power series in y, the coefficients of which depend on x. In view of the nozzle shape being symmetrical to the x-axis the flow itself is visualized as symmetrical to the x-axis. The velocity at the x-axis is denoted by the subscript 0. v_0 in this case is zero. Limited to terms of the fourth power of y the velocity components are

$$\left.\begin{aligned} u(x, y) &= u_0 + \frac{1}{2!} a_2 y^2 + \frac{1}{4!} a_4 y^4 + \ldots \\ v(x, y) &= b_1 y + \frac{1}{3!} b_3 y^3 + \ldots \end{aligned}\right\} \tag{1}$$

with u_0, a_2, a_4, b_1 and b_3 as functions of x. The contents of the present report consist in establishing the relationship between these quantities and function $f(x)$. The first task will be to ascertain the relationship of the coefficients a_2, a_4, b_1, and b_3 with the quantities $u_0(x)$ and $f(x)$.

(a) Calculation of the Coefficients a_2, a_4, b_1, and b_3

Since a potential flow is always required, the following equation of irrotational motion

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \tag{2}$$

gives by means of (1)

$$a_2 y + \frac{1}{3!} a_4 y^3 = b_1{}' y + \frac{1}{3!} b_3{}' y^3$$

The prime mark on $b_1{}'$ and $b_3{}'$ is to indicate the derivation with respect to x. Between the coefficients the following relations must therefore prevail:

$$a_2 = b_1{}'; \quad a_4 = b_3{}' \tag{2a}$$

The density ρ is made dimensionless by the "tank density," that is, the density at speed $w = \sqrt{u^2 + v^2} = 0$ and all velocities by the maximum velocity, that is, the velocity at density $\rho = 0$. Thus the energy theorem takes the form

$$w^2 = 1 - \rho^{\kappa-1} \tag{3}$$

κ is the ratio of the specific heats.

Elimination of pressure and density from the Euler equation of the continuity condition and the adiabatic equation, leaves the dynamic-gas equation

$$\left(c^2 - u^2\right) \frac{\partial u}{\partial x} + \left(c^2 - v^2\right) \frac{\partial v}{\partial y} - 2uv \frac{\partial u}{\partial y} = 0$$

c^2 is the squared velocity of sound, which may be written as

$$c^2 = \frac{\kappa - 1}{2} \rho^{\kappa-1} = \frac{\kappa - 1}{2} \left(1 - w^2\right) \tag{4}$$

Elimination of the sonic velocity from the dynamic-gas equation by means of this equation gives

$$\left(1 - u^2 - \frac{\kappa + 1}{\kappa - 1} v^2\right) \frac{\partial v}{\partial y} + \left(1 - \frac{\kappa + 1}{\kappa - 1} u^2 - v^2\right) \frac{\partial u}{\partial x} = \frac{4}{\kappa - 1} uv \frac{\partial u}{\partial y} \tag{5}$$

Forming by (1) the first derivatives of u and v and entering these along with the velocity components themselves in equation (5), gives by comparison of coefficients the relations between u_0, a_2, a_4, b_1, b_3, a_2', a_4', and u_0'. The terms independent of y give a particularly simple result. As is readily apparent

$$b_1 = - \frac{1 - \frac{\kappa + 1}{\kappa - 1} u_0^2}{1 - u_0^2} u_0' \tag{6}$$

With equation (2a) the quantity a_2 in its relation with u_0 and their derivatives can be computed. Furthermore, it is pointed out at this point that the terms $b_1 y$ and $\frac{1}{2}a_2 y^2$ in (1) relative to u_0 are comparatively small for a nozzle flow. Thus for a first approximation of a_2 and b_1 it is sufficient to insert the velocity afforded by the simple flow filament theory and its derivative in equation (6).

With u_s as the flow filament velocity the equation of continuity for the simple flow filament theory reads

$$u_s\left(1 - u_s^2\right)^{\frac{1}{\kappa-1}} f = \text{const.}$$

In compressible flow the function $u_s\left(1 - u_s^2\right)^{\frac{1}{\kappa-1}}$ substitutes for the velocity, for incompressible flow the quantity u_s. A special symbol for this function is found to be practical. We put

$$\theta(u) \equiv u\left(1 - u^2\right)^{\frac{1}{\kappa-1}} \tag{7}$$

The variation of the function is for $\kappa = 1.400$ as follows, the values of the functions are given in table I.

For the derivatives of θ with respect to u the following abbreviations are introduced

$$\frac{1}{\theta}\frac{d\theta}{du} = \frac{\theta_u}{\theta} = \frac{1 - \frac{\kappa + 1}{\kappa - 1}u^2}{u\left(1 - u^2\right)}; \quad \frac{1}{\theta}\frac{d^2\theta}{du^2} = \frac{\theta_{uu}}{\theta} \tag{7a}$$

and

$$\frac{G}{2} = -\frac{\theta_u}{\theta} - u\left[\frac{\theta_{uu}}{\theta} - \left(\frac{\theta_u}{\theta}\right)^2\right] \tag{7b}$$

If the function for the velocity at the axis u_0 or for the flow filament velocity u_s is needed, it is indicated by the subscript o or s. The formulas for b_1 and a_2 can then be written as

$$b_1 f = -\frac{\theta_{uo}}{\theta_0} u_0 u_0{}' f \tag{8a}$$

$$a_2 f^2 = b_1{}' f^2 = -\left[\frac{\theta_{uuo}}{\theta_0} - \left(\frac{\theta_{uo}}{\theta_0}\right)^2\right] u_0 \left(u_0{}' f\right)^2 - \frac{\theta_{uo}}{\theta_0}\left[\left(u_0{}' f\right)^2\right.$$

$$\left. + u_0 u_0{}'' f^2\right] = \frac{G_0}{2}\left(u_0{}' f\right)^2 - \frac{\theta_{uo}}{\theta_0} u_0 u_0{}'' f^2 \tag{8b}$$

The terms $\frac{1}{3!} b_3 y^3$ and $\frac{1}{4!} a_4 y^4$ play a prominent part at the nozzle edge, since they disappear with the third and fourth power of y with approach to the x-axis. Hence it is logical to use these terms to satisfy the boundary conditions. With u_f and v_f designating the velocity components at the nozzle edge, the following boundary conditions must be fulfilled:

$$v_f = u_f f' \tag{9}$$

or also

$$b_1 f + \frac{1}{6} b_3 f^3 = f'\left(u_0 + \frac{1}{2} a_2 f^2 + \frac{1}{24} b_3{}' f^4\right)$$

Hence a differential equation for b_3 which can be written in the form

$$\underline{\frac{1}{6} b_3 f^3 = f'\left(u_0 + \frac{1}{2} a_2 f^2\right)} - b_1 f + \frac{1}{24} b_3{}' f^4 f' \tag{8c}$$

The term not underscored is small compared to the term $f' u_0$, as seen from (1). It probably can be scored as a rule. At least it makes equation (8c) easy to iterate, by scoring the last term at the first step of iteration and then, by graphical differentiation of the b_3 obtained in first approximation, calculating a second approximation of this quantity. With the newly formed derivative this b_3' scarcely differs from the previous one. It should be noted that it is not a matter of the relative variation of b_3', but that the variation of the term $\frac{1}{24} b_3' f^4$ in relation to u_0 is decisive. This iteration does not involve much more paper work, since the coefficient a_4 had been secured. It is recommended to use the following expression which is readily obtained by (2a):

$$\frac{1}{24} a_4 f^4 = \frac{1}{24} b_3' f^4 \tag{8d}$$

Thus with (8a) to (8d) the coefficients of the power series (1) can be computed without excessive paper work, when f and u_0 are given as functions of x. As a rule f is the given quantity, while u_0 is the factor looked for. But before proceeding to the solution of this problem, simpler formulas for the first approximation of the coefficients are indicated.

(b) First Approximation for a_2, a_4, b_1, and b_3

If the principal flow direction is given by the u component of the velocity, as is usually the case for nozzles, and which always holds for the throat of a converging-diverging nozzle particularly, the value of the velocity u_s, obtained by simple flow filament theory, represents a first approximation for the velocity at the axis. Compared to u_0 the terms $b_1 y$, $\frac{1}{2} a_2 y^2$ are small by assumption and a satisfactory first approximation is secured for these quantities when the velocity u_0 and its derivatives in (8) are replaced by the corresponding quantities of u_s.

Bearing in mind that $f = 1$ at the throat, the equation of continuity from which the flow filament speed can be determined, reads

$$\theta_s f = \theta_{max} \tag{10}$$

Differentiation with respect to x readily yields then the velocity increment

$$u_s'f = -\frac{\theta_s}{\theta_{us}} f' \tag{10a}$$

At the throat the equation becomes indeterminate. The application of L'Hopital's law

$$f' = 0; \; u_s f = \frac{1}{\kappa + 1} \sqrt{(\kappa - 1)ff''} \tag{10b}$$

In this equation f could of course be omitted, since it was taken equal to unity, but, in view of the repeated appearance of the expressions $u'f$ and ff'', the form is retained. Quantity ff'' can be interpreted as the ratio of half the nozzle height and curvature radius of the nozzle edge at the throat.

Thus the insertion of the flow filament values in equation (8a) gives the following formula for b_1f (+ sign and several dashes indicate that a first approximation is involved)

$$b_1f = u_sf' + \ldots \tag{11a}$$

By (2a)

$$a_2f^2 = u_sff'' + u_s'ff' - u_sf'^2 + \ldots \tag{11b}$$

The last equation could have been obtained just as well by the insertion of the flow filament values in equation (8b). With the aid of (10), u_s, and $u_s'f$ may be regarded as defined, equation (11) contains no unknown factors. After using the flow filament values for computing these coefficients instead of the axes values, it is logical to employ them also for computing b_3.

The boundary condition (9) is written in first approximation

$$v_f = u_sf' + \ldots$$

which by means of (11a) gives

$$\frac{1}{6} b_3f^3 = u_sf' - b_1f = 0 \tag{11c}$$

hence, owing to (2a)

$$\frac{1}{24} a_4 f^4 = 0 \tag{11d}$$

The last two results are to be expected beforehand. When inserting the flow filament values in the coefficients in (1) it is not to be expected to immediately obtain two terms of the development.

Nothing is said further about (11a). The identical result could be obtained without any theory by an appraisal of v with the aid of (9). Formula (11b) is more instructive. Forming u with its aid, w can be calculated in first approximation.

$$w = u\left[1 + \frac{1}{2}\left(\frac{v}{u}\right)^2 + \ . \ . \ .\right] = u_0\left[1 + \frac{1}{2}\frac{y^2}{f^2}\left(ff'' + \frac{u_s{}'f}{u_s} f'\right)\right] + \ . \ . \ . \tag{11e}$$

Interesting most of all is the last bracketed term; quantity ff'' is always positive at the throat. Assuming subsonic velocity in the converging part of the nozzle and supersonic velocity in the diverging part, the last summand in the subsonic zone is negative, in the supersonic positive. Two effects can be differentiated in first approximation, which cause a deviation in velocity w at a point y from the respective velocity u_0. One of the effects rises as a result of the curvature of the nozzle edge, and is given by the term ff'', the other is caused by the inclination of the nozzle edge and is given by the term $\frac{u_s{}'f}{u_s} f'$. In the subsonic range, curvature and inclination effect carry opposite signs; in the supersonic range the effects are accumulative. So if curves of constant velocity are plotted in the nozzle, the lines in the supersonic range will be curved much more than in the subsonic range. In fact the curves of constant velocity may even run parallel to the y-axis in certain circumstances. This characteristic behavior of curves of constant velocity or constant pressure, which is the same, are continuously observed again.

(c) Derivation of a Differential Equation for $u_0(x)$ Volume Corrections

To complete the solution the function $u_0(x)$ must be found. The character of the profiles is determined by the coefficients. It is clear that an equation for u_0 must be obtained with the aid of the continuity condition, which is written in the form

$$\int_0^b u\left(1 - u^2 - v^2\right)^{\frac{1}{\kappa-1}} dy = M \tag{12}$$

M is a mass flow made dimensionless by the tank density, the maximum velocity and the throat cross section.

An equation for u_0 is obtainable then by developing the integrand at point $u = u_0$ and $v = 0$ up to the terms of the fourth power of y followed by integrating. But a somewhat shorter process is preferred which in addition brings out the importance of the individual terms more clearly.

In analogy to (7)

$$\theta\left(u, v^2\right) = u\left(1 - u^2 - v^2\right)^{\frac{1}{\kappa-1}} \tag{7a}$$

Observing that the development of $u - u_0$ and of v^2 starts with the second power of y, the development for $\theta\left(u, v^2\right)$ can be confined to

$$\theta(u, v^2) = \theta(u_0, 0) + \left(\frac{\partial\theta}{\partial u}\right)_{u_0 0} (u - u_0) + \left(\frac{\partial\theta}{\partial v^2}\right)_{u_0 0} v^2$$

$$+ \frac{1}{2}\left(\frac{\partial^2\theta}{\partial u^2}\right)_{u_0 0} (u - u_0)^2 + \left(\frac{\partial^2\theta}{\partial u \partial v^2}\right)_{u_0 0} (u - u_0) v^2$$

$$+ \frac{1}{2}\left(\frac{\partial^2\theta}{\partial v^2}\right)_{u_0 0} v^4 \tag{7b}$$

It is readily seen that this equation can be expressed differently when function (7) is introduced everywhere:

$$\theta(u, v^2) = \theta_0 + \theta_{uo} (u - u_0)$$

$$+ \frac{1}{2u_0}\left(\theta_{uo} - \frac{\theta_0}{u_0}\right) v^2 + \frac{1}{2} \theta_{uuo} (u - u_0)^2$$

$$+ \frac{1}{2u_0}\left(\theta_{uuo} - \frac{1}{u_0}\theta_{uo} + \frac{1}{u_0^2}\theta_0\right)(u - u_0) v^2$$

$$+ \frac{1}{8u_0^2}\left(\theta_{uuo} - 3\frac{\theta_{uo}}{u_0} + 3\frac{\theta_0}{u_0^2}\right) v^4$$

Introducing in this expression the development of $u - u_0$ and of v^2, where only terms up to including the fourth power of y are considered, and performing the integral in (12), the application of (8a), (8b), and (8c) gives

$$\frac{M}{f} = \underline{\theta_0} + \theta_{uo}\left\{\underline{\frac{1}{6} a_2 f^2} + \underline{\frac{1}{120} a_4 f^4} + \frac{1}{6u_0}\left(b_1 f + u_0{}'f\right)\left[\underline{f'u_0} + \frac{1}{2} a_2 f^2 \left(f' - \frac{6}{5}\frac{1}{u_0} b_1 f - \frac{3}{5}\frac{1}{u_0} a_2 f^2 b_1 f\right) + \frac{1}{24} a_4 f^4 f' + \frac{1}{30} b_3 f^3\right]\right\}$$

$$+ \theta_{uuo}\left(\underline{\frac{1}{40} a_2^2 f^4 + \frac{1}{20}\frac{1}{u_0} a_2 f^2 b_1^2 f^2}\right)$$

$$+ \frac{1}{40 u_0^2}\left(\theta_{uuo} - 3\frac{\theta_{uo}}{u_0} + 3\frac{\theta_0}{u_0^2}\right) b_1^4 f^4$$

This equation can be considerably simplified when certain limiting assumptions are made. Bearing in mind that the use of still other terms in equation (1) besides $\frac{1}{120} a_4 f^4$ would also involve a term of the form $\frac{1}{7!} a_6 f^6$ it is apparent that there is no sense in carrying summands amounting to a mere tenth part of $\frac{1}{120} a_4 f^4$ or $\frac{1}{30} b_3 f^3$. The same holds when they are smaller than the hundredth of $\frac{1}{6} a_2 f^2$.

To effect a substantial simplification consistent with accurate results the task is restricted to an area near the throat section. For this area it is assumed that

$$|f'| \leq 0.10 \tag{13}$$

and

$$ff'' \leq 0.50 \tag{14}$$

With the aid of (10b) and (11a) it immediately affords the following estimates for $\kappa = 1.400$, that is, say, for air

$$b_1 f \cong 0.04; \quad u_0{}'f \cong 0.20$$

Since u_0 has at the throat, in general, the value of 0.40, we get

$$\frac{1}{6u_0}\left(b_1 f + u_0{}'f\right) \cong 0.1$$

$$\frac{1}{2} a_2 f^2 \left(f' - \frac{6}{5}\frac{1}{u_0} b_1 f\right) \cong 0.01\ a_2 f^2$$

All terms in the expression in the brackets can be voided except the underscored term; the terms are small enough so that there is no danger that added up they could contribute a substantial amount. In the last bracketed expression the greatest term is given by θ_{uuo}, as seen in table I. This term cannot exceed the value 15. Since $b_1 f$ occurs in the fourth power there is no hesitation to omit the last term also. The result is the following equation to which on the assumption of (13) and (14) very substantial accuracy is attached:

$$\frac{M}{f} = \theta_0 + \theta_{uo}\left[\frac{1}{6} a_2 f^2 + \frac{1}{120} a_4 f^4 + \frac{1}{6} f'\left(b_1 f + u_0{}'f\right)\right] + \theta_{uuo}\left[\frac{1}{40}\left(a_2 f^2\right)^2 + \frac{1}{20u_0} a_2 f^2 \left(b_1 f\right)^2\right] \tag{12a}$$

Equation (12a) may be written in the form

$$\begin{aligned}\frac{M}{f} = \theta_0 &+ \theta_{uo}\left[\frac{1}{6} a_2 f^2 + \frac{1}{120} a_4 f^4 + \frac{1}{6} f'\left(b_1 f + u_0{}'f\right)\right] \\ &+ \frac{1}{2}\theta_{uuo}\left[\frac{1}{6} a_2 f^2 + \frac{1}{120} a_4 f^4 + \frac{1}{6} f'\left(b_1 f + u_0{}'f\right)\right]^2 \\ &+ \theta_{uuo}\left[\frac{1}{90}\left(a_2 f^2\right)^2 - \frac{1}{720} a_2 f^2 a_4 f^4\right. \\ &\left. + \frac{1}{20u_0} a_2 f^2 \left(b_1 f\right)^2 - \frac{1}{36} a_2 f^2 \left(b_1 f + u_0{}'f\right) f' + \ldots\right]\end{aligned}$$

The development of θ was restricted to terms of the second derivative at the most, which is ample for the small velocity differences. In the last equation the first three terms can equally be regarded as development for the value

$$u_0' + du = u_0 + \frac{1}{6} a_2 f^2 + \frac{1}{120} a_4 f^4 + \frac{1}{6} f' \left(b_1 f + u_0' f\right) \tag{15}$$

at point $u = u_0$, hence write the equation in the form

$$M = f\theta\left(u_0 + du\right) + f\theta_{uuo}\left[\frac{1}{90}\left(a_2 f^2\right)^2 - \frac{1}{720} a_2 f^2 a_4 f^4 + \ldots\right] \tag{12b}$$

The brackets for θ contain the value of the function argument. The second summand contains only the terms related to the throat. The principal term is the first one, as seen from (1).

Since θ_{uuo} is invariably negative, the last term represents a small negative quantity, which is easily estimated. At $ff'' = 0.20$ it corresponds to a 1:5 ratio of half the nozzle height to curvature radius at the nozzle edge of the throat; a first approximation by (11b) gives $a_2 f^2 = 0.08$; $\frac{\theta_{uu}}{\theta}$ is less than 15. The second summand of (12b) is in this instance, not quite 0.001 part of the first. A later accurate calculation gives it at 0.5 percent of the first. The through flow volume is nothing else but $\int_0^f \theta \, dy$ at the throat, but being near the maximum of function θ, the value of the integral of $f\theta_{max}$ is but very little different. Consequently, the through flow volume of a Laval nozzle should approach, as a rule, the value from flow filament theory very closely. So, if the portion of the through flow volume by which this is smaller than by the simple flow filament theory is designated as volume correction dM, equation (12b) can be written with the aid of (10) as

$$\theta_s f + dM = f\theta\left(u_0 + du\right) + f\theta_{uuo}\left[\frac{1}{90}\left(a_2 f^2\right)^2 - \frac{1}{720} a_2 f^2 a_4 f^4 + \ldots\right] \tag{12c}$$

The principal terms here are the first term at the right- and the left-hand side. Since the last term is very small, its variation with progressing x near the throat is very small, too, hence a solution by equating the two principal terms. From

$$\theta_s = \theta\left(u_0 + du\right)$$

the equality of the arguments can, of course, not be deduced, since the reverse function of θ is ambiguous. However, it is obvious that to a subsonic u_s a subsonic value $u_0 + du$ must correspond and the same holds for supersonic value, else the difference of u_s and $u_0 + du$ would continue to increase with increasing distance from $x = 0$, whereas the two values are certainly near each other. Hence by (15)

$$u_s = u_0 + \frac{1}{6}\left[a_2 f^2 + \frac{1}{20} a_4 f^4 + f'\left(b_1 f + u_0{}'f\right)\right] \tag{16}$$

The relative volume correction is then

$$\frac{dM}{M_s} = \frac{\theta_{uuo}}{\theta_{max}}\left[\frac{1}{90}\left(a_2 f^2\right)^2 - \frac{1}{720} a_2 f^2 a_4 f^4\right] \tag{17}$$

the values at the throat to be inserted at the right-hand side. It is of insignificant influence when the following approximation formula is used:

$$\frac{\theta_{uuo}}{\theta_{max}} = \frac{\theta_{uus}}{\theta_{max}} + \ . \ . \ . = -\frac{(\kappa + 1)^2}{\kappa - 1} + \ . \ . \ .$$

Equation (16) has a special characteristic. Select a place at the nozzle where $f' = 0$. It need not be the narrowest place of the nozzle, and average the u component of the velocity over the cross section. This average may even be equated to the average velocity w. Although v is in no ways equal to zero within the limits of accuracy for $f' = 0$, v still is small enough that it plays no part for the formation of w, where it enters squared. Hence for $\underline{f' = 0}$

$$\bar{w} = \bar{u} = u_0 + \frac{1}{6} a_2 f^2 + \frac{1}{120} a_4 f^4$$

On considering equation (16) it is seen that at the throat or any other place with parallel walls the velocity averaged over the section is equal to the flow filament velocity u_s.

For the derivation of (16) the assumption had been made that the relation

$$dM = f\theta_{uuo}\left[\frac{1}{90}\left(a_2f^2\right)^2 - \frac{1}{720}a_2f^2a_4f^4 + \ldots\right]$$

is satisfied; but this is complied with exactly only at the throat where the correctness of the equation was simply demanded.

To ascertain the extent to which the fact that this equation does not hold at other points of the nozzle may affect the equations (16), we put

$$dM - f\theta_{uuo}\left[\frac{1}{90}\left(a_2f^2\right)^2 - \frac{1}{720}a_2f^2a_4f^4 + \ldots\right] = \epsilon f$$

$u_o + du$ is then not exactly equal to u_s and differs, say by du_s, from this quantity.

$$u_0 + du = u_s + du_s$$

Introduction of ϵf in (12c) followed by development gives

$$\theta_s + \epsilon = \theta\left(u_s + du_s\right) = \theta_s + \theta_{us}du_s + \frac{1}{2}\theta_{uus}\left(du_s\right)^2;$$

that is a quadratic equation for du_s with the result

$$du_s = \frac{\theta_{us}}{\theta_{uus}}\left(\left|\sqrt{1 + \frac{2\epsilon\theta_{uus}}{\theta_{us}^2}}\right| - 1\right)$$

But the practical calculation shows that the second summand under the root is small even very close to the throat with respect to unity. Therefore the development gives

$$f' \neq 0; \quad du_s = \frac{\epsilon}{\theta_{us}}$$

For a nozzle with the value $ff'' = 0.20$ at the throat the following relationship between the error du_s and the value f' was ascertained: The bracketed expression contains a few more terms that were omitted in the equation of ϵf. Besides, the value was computed also within a region of f' which no longer satisfies (13). Figure 2 indicates that the error is entirely insignificant. It amounts to less than 0.5 percent of the velocity. So, within the limits of error of the theory, equation (16) can be regarded as exact solution of (12) and (12a).

(d) First Approximation of u_0 and of the Volume Correction,

Range of Velocity of the First Approximation

As in the foregoing a first approximation for u_0 is secured by the insertion of the first approximation of a_2f^2, a_4f^4, and b_1f in equation (16). Quantity u_0' is, of course, replaced by u_s'

$$u_0 = u_s\left[1 - \frac{1}{6}\left(ff'' + 2\,\frac{u_s'f}{u_s}\,f'\right)\right] + \ldots \tag{16a}$$

Thus the velocity distribution in a nozzle can be computed by (11a) and (16a) when its principal flow direction is given by the velocity component x; u_s and $u_s'f$ are obtained by (10), (10a), and (10b). Equations (16) and (16a) are significant by themselves. In many cases the pressure distribution is measured in the nozzle axis. To compare it with the theoretical results quantity u_0 can be computed by (16) or (16a) depending upon the degree of accuracy required and the pressure at the axis readily obtained. If satisfied with a first approximation, a direct relation between the pressure at the axis and the pressure of the flow filament theory is simply derived according to (16a).

Equation (16a) resembles (11e) very much. Forming, for example, the average velocity in first approximation, we can, because the expression

$$\frac{1}{2}\,\frac{y^2}{f^2}\left(ff'' + \frac{u_s'f}{u_s}\,f'\right)$$

must be small relative to unity, write by (11e)

$$u_0 = \overline{w}\left[1 - \frac{1}{6}\left(ff'' + \frac{u_s{}'f}{u_s} f'\right)\right]$$

where, of course, $u_s = \overline{w} = \overline{u}$ for $f' \doteq 0$. On the other hand it is seen that this equation is not valid for $f' \neq 0$ since the term with f' in (16a) contains the factor 2, but the qualitative statements regarding curvature and inclination effect remain applicable as before. While in the supersonic range u_0 must always be smaller than u_s, u_0 can be $>$ or $\leq u_s$.

Equation (16a) also affords a means for the range of validity of the first approximation. In the coefficients a_2f^2 and b_1f the quantity u_0 can be replaced with more justification by u_s as the slope f' of the nozzle and the dimensionless curvature ff'' become less. Forming with (16a) the first approximation of $u_0{}'f$ gives

$$u'_0f = u_s{}'f \pm \frac{1}{6}\Big(u_sf^2 f''' + 3\,u_s{}'f\,ff'' + u_sf'\,ff''$$
$$+ 2\,u_s{}'f\,f'^2 + 2\,u_s{}''\,f^2f'\Big) + \ldots \qquad (18)$$

From this equation the question of substitution $u_0{}'f$ by $u_s{}'f$ is seen to be much more difficult to answer. At any rate it may be stated that this can be done with less justification at the throat of nozzles symmetrical about the y-axis than for u_0 and u_s, for by (18)

$$u_0 = u_s\left(1 - \frac{1}{6}\,ff''\right) + \ldots; \quad u_0{}'f = u_s{}'f\left(1 - \frac{1}{2}\,ff''\right) + \ldots$$

where $(u_0{}'f)^2$ also enters in the formation of a_2f^2, and in it the main reason for a persistent failure of the first approximation is to be found. It must be reckoned with that for ff'' small with respect to unity the error of the first approximation near $f' = 0$ is of the order of magnitude of ff'' itself, which is, that the first approximation differs too much from the flow filament theory. Therefore, the applicability of the first approximation is predicated on the assumption that in the development of $1 + ff''$ with respect to ff'' the series can be broken off after the first term.

Equation (17) affords the relative volume correction factor, that is, the difference of actual through flow volume from that computed by flow filament theory. Putting in the equation the first approximation results in

$$\frac{dM}{M_s} = - \frac{\kappa + 1}{90} (ff'')^2 + \ldots \tag{17a}$$

Considerable inaccuracy must be presumed for greater ff'', because the same statement made about the calculation of u_0, where $a_2 f^2$ was employed, applies in a considerably greater measure here, where this quantity appears squared and $a_4 f^4$ appears. Subsequent examples will show that this formula exhibits very substantial errors, where the other first approximations still give very good service. But it is practical as estimation formula. The volume correction for nozzles to which the present theory is applied, is unimportant.

(e) Solution of the Equation for u_0

What is the best solution of (16)? Since $a_2 f^2$, $b_1 f$, and $a_4 f^4$ are in the final analysis dependent upon u_0 and its derivatives, an ordinary differential equation for u_0 is involved. By (8a) and (8b)

$$u_0 = u_s - \frac{1}{120} a_4 f^4 - \frac{1}{6}\left[\frac{G_0}{2}\left(u_0'f\right)^2 + u_0'f\, f' - u_0 \frac{\theta_{u0}}{\theta_0} f \frac{d}{dx}\left(u_0'f\right)\right] \tag{16b}$$

$a_4 f^4$ has been left, for the rest the equation contains only functions of u_0, such as G_0 and $\frac{\theta_{u0}}{\theta_u}$, derivatives of u_0 and functions of x, such as f and u_s. The greatest terms are u_0 and u_s, while $\frac{1}{120} a_4 f^4$ represents the smallest term, when the bracketed expression is regarded as one term.

Bearing in mind that $\frac{1}{120} a_4 f^4$ is only one-fifth of the value which the last term in the development of u (equation (1)) can assume, it is clear that the quantity u_1, given by

$$u_1 = u_0 + \frac{1}{120} a_4 f^4 \tag{19}$$

differs very little from u_0. Quantity u_1 is naturally a much better solution of u_0 than what was called "first approximation" of u_0. The derivatives of u_0 are therefore very closely reproduced by the derivitives of u_1 and for the calculation of $a_2 f^2$ and $b_1 f$, which themselves represent only portions of u_0, quantity u_1 is certainly sufficient. So if this quantity is used for computing those two coefficients equation (16) can be written as follows:

$$6\left(u_s - u_1\right) = \frac{G_1}{2}\left(u_1{}'f\right)^2 + u_1{}'f\,f' - \frac{\theta_{u1}}{\theta_1} u_1 f \frac{d}{dx}\left(u_1{}'f\right) \tag{16c}$$

It is a differential equation of the second degree and second order for u_1 with very characteristical properties, as will be shown. Quantity u_1 can now be determined by an iteration process, approximate values for u_1 and its derivatives being entered at the right-hand side of the equation and a new $u_1(x)$ calculated. This method however does not always give a very quick result. It is less disturbing to make a double graphical differentiation of u_1 because of the last term. In view of the smallness of the particular term near sonic velocity the error introduced here is, in general, unimportant. Much more disturbing is the fact that the error in $u_1{}'f$ may substantially exceed that in u_1. Naturally the iteration is best started with the insertion of the approximation by (16a) and (18), which can be just as well regarded as first approximations for u_0 and $u_0{}'f$ as for u_1 and $u_1{}'f$. Figure 3 shows the convergence of this method for the nozzle a = 0.20 (described in section IVc). Its value at the throat x = 0 is $ff'' = 0.20$.

The x-axis represents the result of the flow filament theory, a subscript added for u_1 indicates the number of iterations; u' is the first approximation, obtained direct from the formulas.

The second approximation is not as yet quite satisfactory, but even so it lies in vicinity of the throat $x = 0$ very close to the solid curve which represents the final solution. This was obtained by means of a method which yielded several solutions in the subsonic range $x < 0$. The solution plotted here has the property of having a common intersection point about equal to -1.2 with all approximations for x. Three to four steps are necessary to reach a result corresponding to the accuracy of the method. The convergence of this iteration is likely to be largely dependent upon the nozzle shape. In many cases where the first approximation is still a little uncertain, the second should give ample service.

Another method of solving (16c) gives quicker results. As mentioned in the foregoing, it is seen that solutions are lost in the subsonic zone with the iteration method. We shall deal with these cases where the flow in vicinity of the throat rises to supersonic speeds. After reaching sonic velocity $\theta_{u1} = 0$ the last term plays a subordinate part in the neighborhood of the throat. It is advisable to first establish the first approximation of u_1 and $u_1'f$ which are nothing else but the first approximation of u_0 and $u_0'f$, by means of (16a) and (18). The magnitude of the last term can then be easily estimated by a second differentiation, since the functions G and $u \frac{\theta_u}{\theta}$ are tabulated. In the neighborhood in which the last term is only about 10 percent of the right hand of the equation the equation is best regarded as differential equation of the first order in u_1 and then solved with respect to $u_1'f$.

$$u_1'f = \frac{-f' + \left|\sqrt{2G_1\left[6\left(u_s - u_1\right) + f\frac{d}{dx}\left(u_1'f\right)u_1\frac{\theta_{u1}}{\theta}\right] + f'^2}\right|}{G_1} \qquad (16e)$$

The characteristics of this equation are best seen when the small terms f' and $u_1 \frac{\theta_{u1}}{\theta_1} f \frac{d}{dx}\left(u_1'f\right)$ are regarded as nonexistant. If u_1 is chosen too small, the value for u_1' will be too high. Hence in a progressive integration in positive x direction, that is, in flow direction, all integral curves in throat vicinity lead asymptotically to the same end curve, independent of the initial value. If the size of the steps is not chosen small enough a strongly damped oscillation of the values result instead of a one-sided approach to the asymptote. Therefore a solution of (16e) is secured by inserting close approximation values under the root and then defining u_1 by progressive integration in flow direction. While for u_1 only an initial value must be assumed, $\frac{d}{dx}(u_1{}'f)$ must be taken from the first approximation until the curve of u'_1f is so steady that this curve can also be differentiated. Of course, the differential quotient of $u_1{}'f$ can never be computed at the exact place where $u_1{}'f$ itself is looked for, but this is unimportant at the smallness of the particular term. It is practical to plot the departure from the stream filament velocity $u_s - u_1$ and $u_1{}'f$ against x in the interpretation of (16e) and to plot the curves of the stream filament theory, and of the first approximation before starting the calculation. In the first case the flow filament theory naturally gives the x-axis.

If the initial value of u_1 is chosen incorrectly, which is evidenced by considerable fluctuation of $u_1{}'f$ during many steps, it is better to start all over again, in order to obtain a smooth curve before the throat as soon as possible. Equation (16e) is very appropriate even in the entire supersonic range.

The described behavior of (16e) thus indicates that values of u_1 and $u_1{}'f$ are obtained at the throat which are practically independent of the chosen initial conditions. Physically it means that there is always a tendency toward a well defined flow attitude at the throat.

In the range in which the last term of (16e) already plays an essential part, that is, exceeds about 10 percent of the right-hand side; it is better to apply a formula obtained by integrating the just cited equation.

$$u_1{}'f = \int_{x_0}^{x} \frac{1}{f}\frac{\theta_1}{\theta_{u1}u_1}\left[\frac{1}{2}G_1\left(u_1{}'f\right)^2 + f'u_1{}'f - 6\left(u_s - u_1\right)\right]dx + \left(u_1{}'f\right)_{x_0} \quad (16d)$$

Having defined by (16e) the piece of the curve of $u_1{}'f$ and u_1 near the throat, the whole curve in the subsonic and supersonic range can be computed by (16d); x_0 is the point at which the last calculation is started. Approximate values of $u_1{}'f$ therefore substitute for x in the integral, as obtained by extrapolating the $u_1{}'f$ curve. The determination of $u_1{}'f$ proceeds in the subsonic range in the direction toward the floor and it is found that the integral curves in this direction spread out and after several steps lead to perceptibly different results. It results in deviations from the curves obtained by iteration, in both directions, while in the supersonic range and in the transitional zone all the computed curves are coincident (fig. 8). The next probably surprising fact of spreading of the integral curves in the subsonic range has a very good sense physically. Flows, which in the subsonic zone exhibit marked differences, still give the same flow in the narrowest section of the nozzle. This agrees with the experimentally known fact that the flow in supersonic nozzles is almost independent of the shape of the nozzle entry. It probably always tends toward the state of flow for maximum through flow volume.

Attempts to start the integration in flow direction for arbitrarily specified initial values u_1 and u_1f in the subsonic range do not lead to the desired result. Even a slight variation of u_1 at greater distance from the throat for constant $u_1'f$ produces profound deviations before the throat. Figure 4 illustrates two examples (the same nozzle is in fig. 3).

The solid curve is the solution for general passage through the sonic velocity near the throat. The choice of a slightly too high value of u_1 (dashed curve) at $x = -1.5$ results in a climb of $u_1'f$ beyond all limits. The flow cannot be continued at all through the narrowest cross section. The choice of a little too small a value of u_1 (dash-dots) leads to solutions at which the velocity of sound is not reached at all. Near the throat $u_1'f = 0$ and is ultimately negative.

The previously cited asymptotic movement of the values to one and the same end curve therefore refers only to a very narrow range near $f' = 0$.

It is seen that it is now possible to define u_1 and $u_1'f$ as function of x. The coefficients a_2, a_4, b_1, and b_3 can be computed by (8) where u_1 is utilized instead of u_0 with sufficient accuracy. Equation (19) then yeilds u_0 and equation (1) the velocity components u and v.

III. DERIVATION OF THE FIRST APPROXIMATION FOR THE CASE OF AXIALLY SYMMETRICAL FLOW

In general the slope and curvature of the walls in axially symmetrical nozzles is not as great as for the two-dimensional problem, hence the first approximation is here of greater interest.

Rather than repeat the previous derivations, a new method is employed. With x denoting the nozzle axis, y the radial coordinate and f the nozzle radius the equation of continuity is written in the form

$$\frac{\partial}{\partial y}(\rho v y) = -y\frac{\partial}{\partial x}(\rho u) \tag{20}$$

where u and v are the velocity components in x and y direction; and, as before, $w^2 = u^2 + v^2$.

Now, in order to obtain an approximate value for v it must suffice in the above equation to insert an approximate value for ρu, since v is to be small only relative to u. Restricted to the value of the stream filament, designated by $\rho_s u_s$, its equation of continuity is

$$f^2 \pi \rho_s u_s = M \tag{20a}$$

As before f is made nondimensional by the radius at the throat of the Laval nozzle, ρ by the chamber density, ρ_0 and u by the maximum speed, so that M is a through flow volume which is made nondimensional in exactly the same way as in (12).

By logarithmic differentiation of (20a)

$$\frac{1}{\rho_s u_s}\frac{d(\rho_s u_s)}{dx} = -2\frac{f'}{f} = \frac{\theta_{us}}{\theta_s}u_s' \tag{20b}$$

It should be noted that (10a) and (10b) are valid only when f is proportional to the cross section. If the cross section is proportional to the quantity f^2 as in the axially symmetrical case the factor 2 is additive to f' in (10a) and to ff'' in (10b), as equation (20b) also indicates.

By means of the last equation

$$\frac{\partial}{\partial y}(\rho v y) = 2y\rho_s u_s \frac{f'}{f} + \ldots$$

This is, of course, an approximate equation, because the relationship of product ρu and y is not considered at all, but an approximate value introduced for it. The right-hand side of the equation merely contains functions of x, aside from y, hence integration with due regard to the boundary condition yields

$$y = 0; \quad v = 0; \quad v = b_1 y = u_s f' \frac{y}{f} + \ldots \tag{21a}$$

with $\frac{\rho}{\rho_s} = 1$ on the right-hand side according to the earlier omissions. With observance of the freedom from rotation we immediately get

$$u = u_0 + \frac{1}{2} a_2 y^2 = u_0 + \frac{1}{2}\left(u_s f f'' + u_s' f f' - u_s f'^2\right)\frac{y^2}{f^2} + \ldots \tag{21b}$$

The formula for u and v agrees in first approximation with the formulas for the velocity components in the two-dimensional case. Quantity u_0 is again computed by the continuity equation in the form

$$M = 2\pi \int_0^f yu\left(1 - u^2 - v^2\right)^{\frac{1}{\kappa-1}} dy = 2\pi \int_0^f y\theta\left(u, v^2\right) dy \tag{22}$$

For $\theta\left(u, v^2\right)$ the same development as before is used (17b), but the result after integration is slightly different, namely,

$$\frac{M}{f^2\pi} = \theta_0 + \theta_{uo}\left(\frac{1}{4} a_2 f^2 + \frac{1}{4}\frac{1}{u_0} b_1{}^2 f^2 + \frac{1}{8}\frac{u_0{}'f}{u_0} b_1 f + \ldots\right)$$

$$+ \theta_{uuo}\left[\frac{1}{24}\left(a_2 f^2\right)^2 + \ldots\right]$$

$$= \theta\left(u_0 + \frac{1}{4} a_2 f^2 + \frac{1}{4u_0} b_1{}^2 f^2 + \frac{1}{8u_0} u_0{}'fb_1 f\right)$$

$$+ \theta_{uuo}\left[\frac{1}{96}\left(a_2 f^2\right)^2 + \ldots\right]$$

The omitted terms play no part in the first approximation. In the last equation the brackets for θ contain the argument; this is no product! The solution is found the same as before

$$u_s - u_0 = \frac{1}{4}\left(a_2 f^2 + \frac{1}{u_0} b_1{}^2 f^2 + \frac{1}{2u_0} u_0{}'fb_1 f\right) + \ldots$$

$$= \frac{1}{4}\left(u_s f f'' + \frac{3}{2} u_s{}'f\ f'\right) + \ldots \tag{23}$$

This equation therefore differs in several factors from (16a), the corresponding first approximation of the two-dimensional problem. The first approximation for the relative volume correction follows as

$$\frac{dM}{M_s} = - \frac{\kappa + 1}{96} (ff'')^2 + \ldots \tag{24}$$

Its difference from (17a) is very little.

IV. APPLICATIONS AND MODEL PROBLEMS

(a) The Two-Dimensional Source for Compressible Flow

The formulas of the first approximation for two-dimensional flow are checked against an exact solution. Chosen is the example of compressible source flow, by which a flow between two flat walls sloped at angle 2α is given. The problem can be solved generally, when assuming that angle α is small. Consider the flow at point of the "nozzle" of cross section f. The point of intersection of this cross section with the x-axis is distant x_0 from the origin. In addition $r^2 = f^2 + x^2$ (fig. 5).

Therefore $f' = \tan \alpha = \frac{f}{x_0}$; $ff'' = 0$. The angle α is measured in radians, so that the stream filament velocity at the point of f is given by

$$f\theta\left(u_s\right) = \alpha x_0 \theta\left(u_0\right)$$

This equation can be developed. Putting $\alpha = \tan \alpha - \frac{1}{3} \tan^3\alpha$ gives

$$\left[\theta_s + \theta_{us}\left(u_0 - u_s\right)\right]\left(\tan \alpha - \frac{1}{3} \tan^3\alpha\right) = \tan \alpha\theta_s$$

With observance of (10a) and omission of terms of higher order

$$u_0 - u_s - \frac{1}{3}\frac{\theta_s}{\theta_{us}} f'^2 = u_0 - u_s + \frac{1}{3} u_s' ff' = 0$$

or exactly the equation (16a) for $ff'' = 0$

The velocity w_f in the point with the coordinates $x = x_0$ and $y = f$ can be expressed in simple manner. First, it must be equal to the velocity on the x-axis at distance r from the origin, developed it gives

$$w_f = u_0 + u_0'\left(r - x_0\right)$$

Second, w_f can be expressed by the x component of the velocity in this point u_f and the cosine of the angle,

$$w_f = u_f \frac{r}{x_0}$$

Development of r gives

$$\frac{r}{x_0} = 1 + \frac{1}{2} f'^2 \quad \text{and} \quad r - x_0 = \frac{1}{2} ff'$$

which written in the last two equations gives after eliminating w_f

$$u_f - u_0 = - \frac{1}{2} u_s f'^2 + \frac{1}{2} u_s' ff'$$

The same result had been obtained for $ff'' = 0$ by (11b) and (1).

(b) Comparison of Theory and Experiment on an

Axially Symmetrical Nozzle

T. E. Stanton (reference 4) measured the velocity distribution on the axis and along a line parallel to the axis whose distance from the wall at the point of minimum section amounted to about 2.5 millimeters, where $ff'' = 0.210$. As explained in the two-dimensional case at such a high value of ff'' no complete agreement of the first approximation with the actual flow is to be expected. Our theoretical values on the axis are a little too low and on the wall a little too high. Stanton measured at various chamber pressures (fig. 6, c^* is the velocity of sound at $M = 1$). In the first three cases A, B, and C the chamber (or tank) pressure is so low that the maximum velocity is not attained at the throat of the nozzle, with increasing section the velocity drops again. In proximity of the wall the velocity of sound in case C is even exceeded in some areas. Thus the problem here involves a solution which is symmetrical to the minimum section of the nozzle. This symmetry is, of course, somewhat disturbed by boundary-layer effects. The symmetrical solutions are predicated upon some quantity in order to be able to determine the theoretical curves. In the present instance the axial velocity at throat was taken as specified, in consequence of which theory and test are nearly in complete accord on the axis. The computed velocity in proximity of the wall is, as in case E - F, generally too low, but this marked deviation is at any event, attributable to the method of measurement.

In the case E - F the tank (or chamber) pressure is already high enough so that a general transition to supersonic speeds in proximity of the throat is involved. There the calculation of the speed necessitates no further assumption. The theoretical values of the speed at the axis are a little too low, exactly as expected in the first approximation. The experimental wall velocities in the supersonic range are peculiarly high. The cause of velocity drop in the neighborhood of $x = 0.15$ is undoubtedly introduced by a compressibility shock. Stanton's test series D is not reproduced, since it cannot be interpreted without including boundary layer effects.

(c) Flow Through Hyperbolic Nozzles, Two-Dimensional Problem

Figure 7 represents the transition from subsonic to supersonic velocity on three nozzles. The nozzle edge is given by the function

$$f = \sqrt{1 + ax^2}$$

This is the equation of a hyperbola. The half nozzle height at the throat is taken as equal to unity. The reciprocal curvature radius for $f' = 0$ is $f'' = a$. The nozzle forms represented by the above function were preferred over those of Görtler and Taylor, since they, like the nozzles used in practice, have the property of decreasing side curvature with increasing distance from the throat. Besides on this nozzle form a great curvature of the nozzle edge for small x values goes hand in hand with a strong slope of the nozzle edges for great x values. Accordingly the smaller a is the greater the accuracy of the theory.

In conformity with the entire theory the results are velocity profiles rather than lines of constant velocity. But it is an easy matter to change to lines of constant velocity. They are preferable for the representation.

The three nozzles have the following values of a:

$$a = 0.10; \qquad a = 0.20; \qquad a = 0.30$$

The lines of constant velocity of the final solution are shown as solid lines, those of the first approximation in dashes and those of the stream filament solution, which are straight, of course, in dashes and dots. The related velocities, expressed in multiples of the velocity of sound c^* at $M = 1$, are entered at the intersection point with the related line of constant velocity.

As is seen the first approximation for all three nozzles still provides fairly practical value, and gives a fairly accurate account of the deviations even in the case of $a = 0.30$. The error in the final solution within the region which satisfies the condition (13) $|f'| < 0.10$, is about 10 percent of the difference of this solution from the first approximation. The error of the first approximation at the axis is, as already appraised for $f' = 0$, about a times the difference from the stream filament value at this point. The result in the subsonic zone is not definite. Figure 7 shows the solutions obtained for u_0 by iteration. This always gives only one solution.

Figures 8(a), (b), and (c) indicate this ambiguity for the nozzle $a = 0.20$ in the subsonic zone. In figure 8(a), $\frac{u_s - u_0}{u_s}$ is plotted against x; figure 8(b) shows the corresponding curve for $u_0 'f$; figure 8(c) represents the lines of constant velocity for the computed extreme cases. Thus the solid and the dashed curves represent exact solutions within our accuracy.

Figure 9(a) is the first approximation for the nozzle $a = 0.20$ for the case of maximum velocity at the throat and subsequent decrease. The value i denotes the Mach number at the throat based on the stream filament solution. Thus for $i = 1.0$ it gives solutions which are symmetrical to the throat and asymmetrical. At values of i near unity the Mach number 1 is exceeded in wall proximity. Figure 9(b) represents the practically not so important cases of symmetrical supersonic flow. It attains its lowest velocities in the least section. The i values are here so chosen that in one case each for $i < 1.0$ and $i > 1.0$ the same value of θ, that is, the same through flow volume, prevails. The result is that this series of figures represents essentially all possible cases from the lowest to the highest velocities. The figures are arranged so that flows with equal through flow volumes have correspondingly equal places. The case for $i = 0.98$ in the supersonic zone is not shown. The lines of constant velocity in the supersonic zone are fundamentally different from those in the subsonic zone. Interesting is the fact that at very high velocities the velocity change is largely transverse to the flow direction, and small velocity variations are already accompanied by appreciable changes in density.

(d) Flow Through Hyperbolic Nozzles, Axially Symmetrical Flow

The same function chosen for the two-dimensional problem was used for the distance of the nozzle edge from the axis f. The nozzle parameter was indicated by a_R. Merely the first approximation, whose lines of constant velocity are shown as solid lines, was computed. Figure 10 represents the three cases

$$a_R = 0.10; \qquad a_R = 0.20 \qquad \text{and} \qquad a_R = 0.30$$

As the velocity changes with the square of f, hence is much greater than in the two-dimensional case, the lines of constant velocity are closer together. A velocity change in proximity of the axis does not amount to as much as one in wall proximity, since the flow volumes are substantially greater. In accord with it, it is seen that in the axially symmetrical case a departure of the velocity from the stream filament value at the wall is compensated by a much higher velocity variation at the axis. For the rest, the great velocity differences on a section are less than in the case of two-dimensional flow.

(e) Correction of Volume for Hyperbolic Nozzles

With (17) and (17a) the volume correction can be computed in first approximation from the hyperbolic nozzles described in section c. With $\left(\frac{dM}{M_s}\right)_1$ denoting the first approximation the results are as follows:

a	$\frac{dM}{M_s}$	$\left(\frac{dM}{M_s}\right)_1$
0.10	-0.00018	-0.00027
.20	-.00053	-.00105
.30	-.00083	-.0024

The volume correction is extremely small. In addition it is seen that formula (17) is merely practical for estimating. At $a = 0.30$ the value obtained in first approximation is three times too high.

(f) The Significance of the Wall Curvature at Throat in the Construction of Supersonic Nozzles with Parallel Jet

The construction of supersonic nozzles with parallel flow for supersonic wind tunnels by the Prandtl-Busemann method generally proceeds from the assumption that in this case with sufficient accuracy at the throat of the nozzle the Mach number over the entire section can be put equal to unity. To analyze the error introduced by this assumption two constructions of a supersonic nozzle with a parallel jet with a Mach number of about equal to 2 were effected, one on the basis of an assumed constant velocity over the section at the throat, the other on the basis of a velocity distribution attained by our method. At the throat ff'' was taken as $ff'' = 0.30$. Minimum directional changes of one radian were involved. The volume correction of the nozzle amounts to about 0.8 percent. The end sections obtained with the two constructions should therefore differ by no more than 0.8 percent but the difference of the end sections did amount to 2.4 percent and for that reason this quantity can be regarded as measure for the accuracy of the construction. The greatest sectional difference in the two constructions amounted to 3.3 percent so that the two curves may be said to show no difference within the accuracy of construction. The justification of proceeding with the construction of supersonic nozzles with parallel jet from the assumption $Ma = 1$ in the narrowest section of the nozzle is therefore confirmed.

Translation by J. Vanier
National Advisory Committee
for Aeronautics

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TABLE I

$\frac{u}{c^*}$	u	$\theta(u)$	$\frac{\theta_u}{\theta}$	$\frac{\theta_{uu}}{\theta}$	G	$u\frac{\theta_u}{\theta}$
0.10	0.04083	0.04065	24.290	-15.00	0.82	0.992
.20	.08165	.08030	11.836	-15.00	1.67	.966
.30	.12247	.11795	7.544	-15.00	2.52	.924
.40	.16330	.15263	5.285	-14.99	3.45	.863
.50	.20412	.18352	3.834	-14.97	4.45	.783
.55	.22454	.19729	3.271	-14.96	4.98	.735
.60	.24495	.20985	2.780	-14.94	5.54	.681
.65	.26536	.22108	2.341	-14.91	6.14	.621
.70	.28577	.23094	1.943	-14.88	6.78	.555
.75	.30619	.23939	1.577	-14.84	7.46	.483
.80	.32660	.24637	1.234	-14.79	8.19	.403
.85	.34701	.25177	.909	-14.72	8.97	.315
.90	.36742	.25569	.598	-14.63	9.82	.220
.95	.38784	.25802	.296	-14.53	10.75	.115
1.00	.40825	.25880	.000	-14.40	11.76	.000
1.05	.42866	.25803	-.293	-14.24	12.87	-.126
1.10	.44907	.25572	-.586	-14.04	14.09	-.263
1.15	.46949	.25192	-.881	-13.80	15.45	-.414
1.20	.48990	.24677	-1.182	-13.50	16.96	-.579
1.25	.51031	.24004	-1.490	-13.14	18.66	-.761
1.30	.53072	.23209	-1.810	-12.69	20.56	-.961
1.35	.55114	.22292	-2.144	-12.14	22.74	-1.181
1.40	.57155	.21262	-2.495	-11.47	25.21	-1.426
1.45	.59196	.20130	-2.867	-10.63	28.06	-1.697
1.50	.61237	.18910	-3.266	-9.60	31.35	-2.000
1.60	.65320	.16260	-4.166	-6.69	39.74	-2.721
1.70	.69402	.13424	-5.247	-2.05	51.50	-3.642
1.80	.73485	.10541	-6.627	5.67	69.46	-4.870
1.90	.77567	.07767	-8.447	19.22	97.77	-6.554
2.00	.81650	.05238	-11.023	45.02	146.95	-9.000
2.10	.85732	.03100	-15.009	100.39	244.16	-12.868
2.20	.89815	.01476	-22.115	246.16	480.60	-19.863
2.30	.93897	.00453	-38.609	817.68	1341.06	-36.253
2.40	.97980	.00031	-121.479	8630.54	12248.82	-119.025
2.45	1.00000	.00000	$-\infty$	∞	∞	$-\infty$

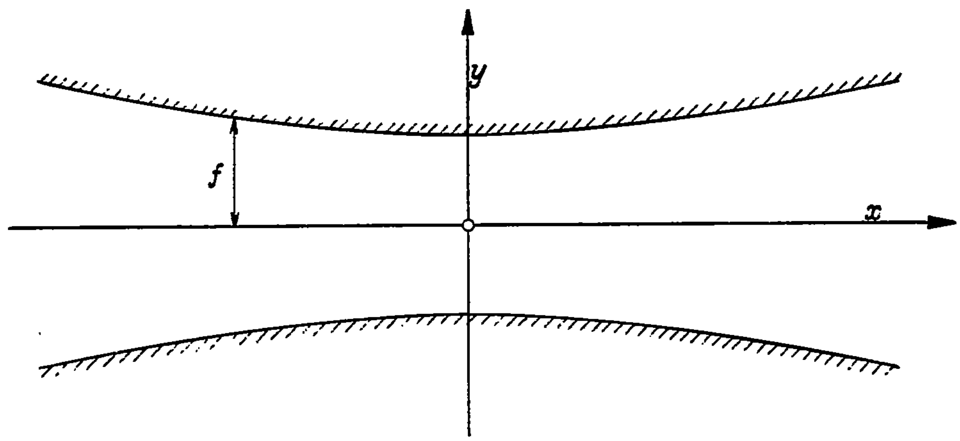
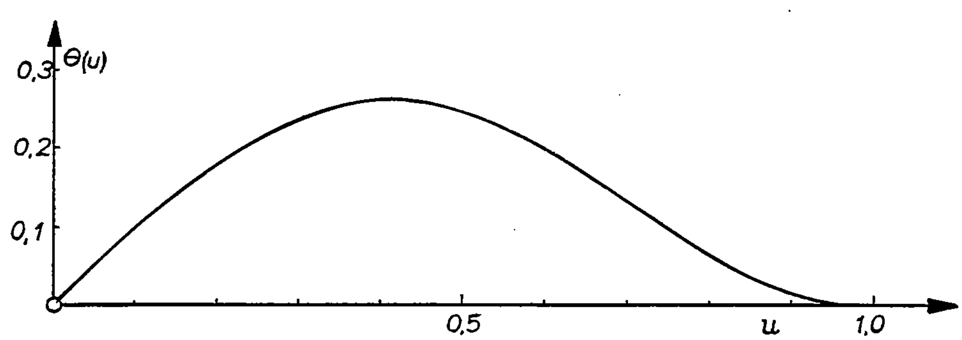


Figure 1.- Section of converging-diverging nozzle.



Sketch: The function θ (u) for κ = 1.400.

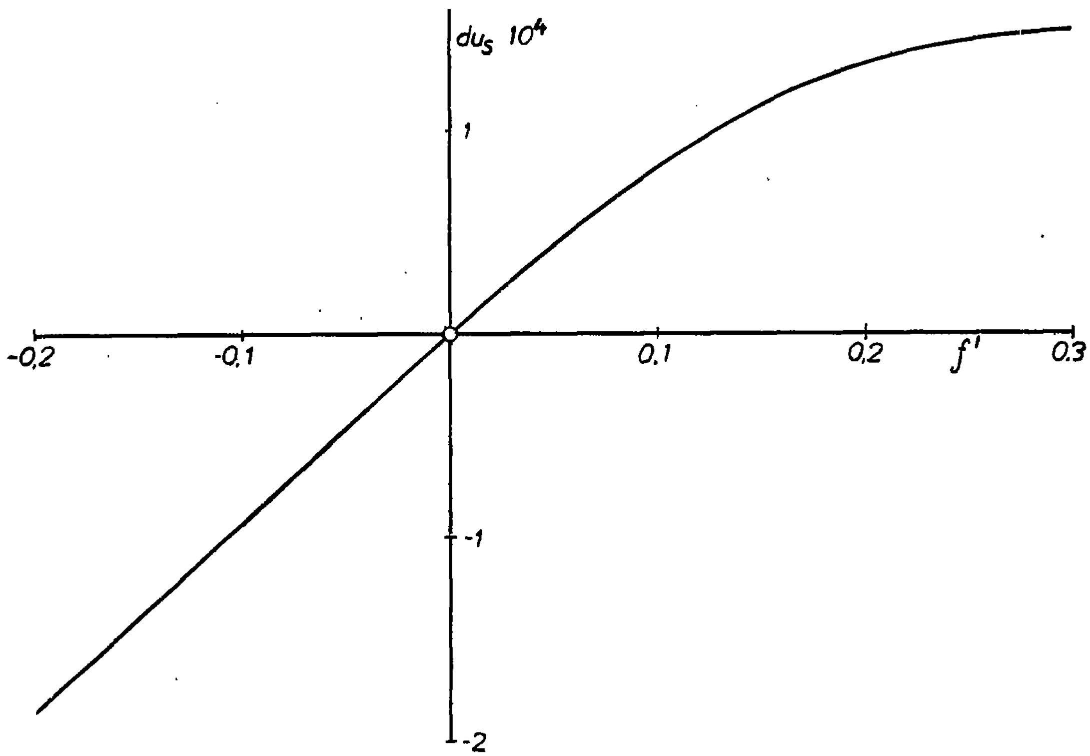


Figure 2.- The error du_s as a function of the wall slope f'.

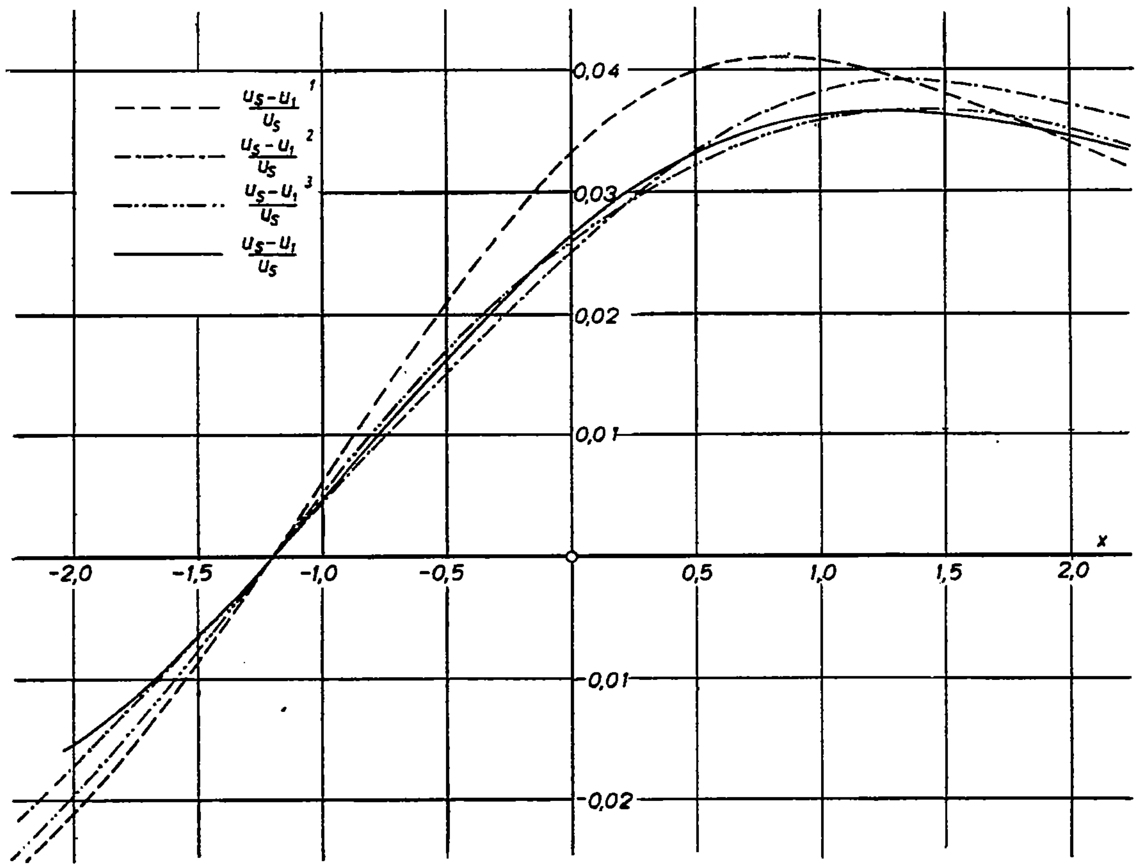


Figure 3.- Convergence of the iteration method.

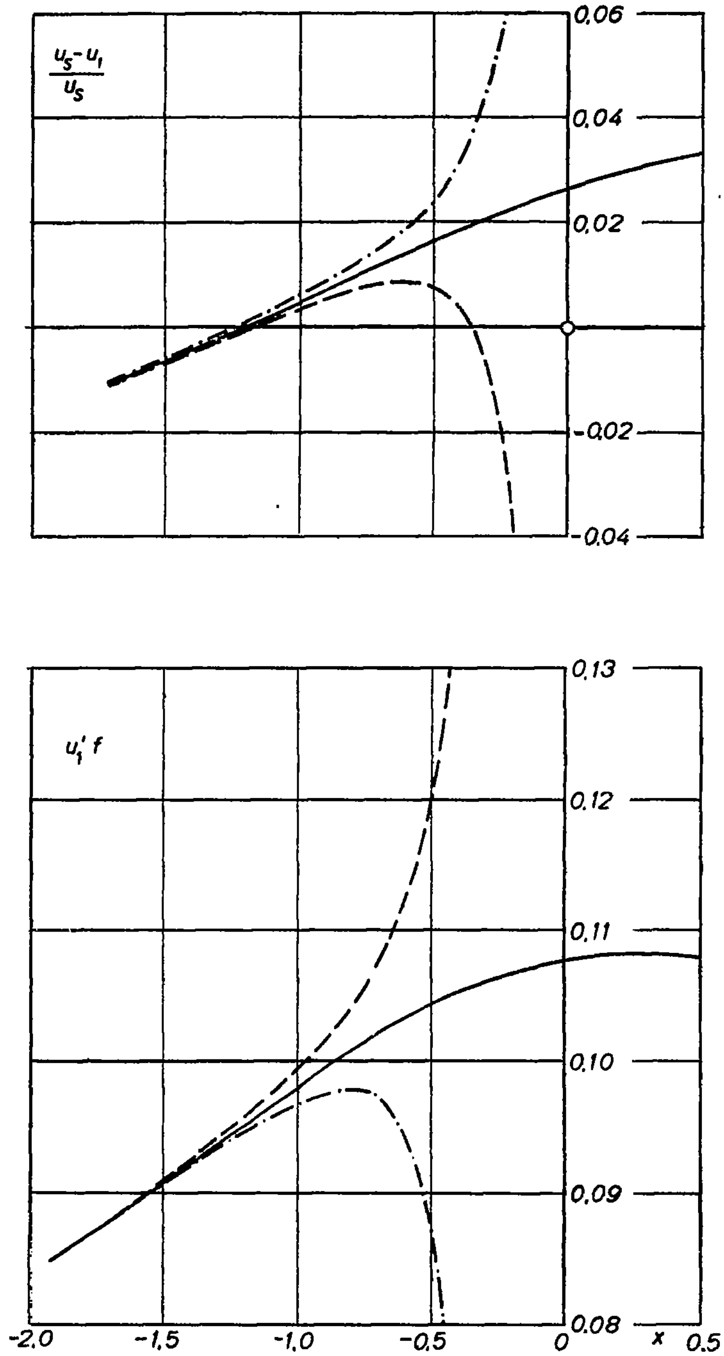


Figure 4.- Integration in flow direction for given u_1 and $u_1'f$ at greater distance from the throat of the nozzle.

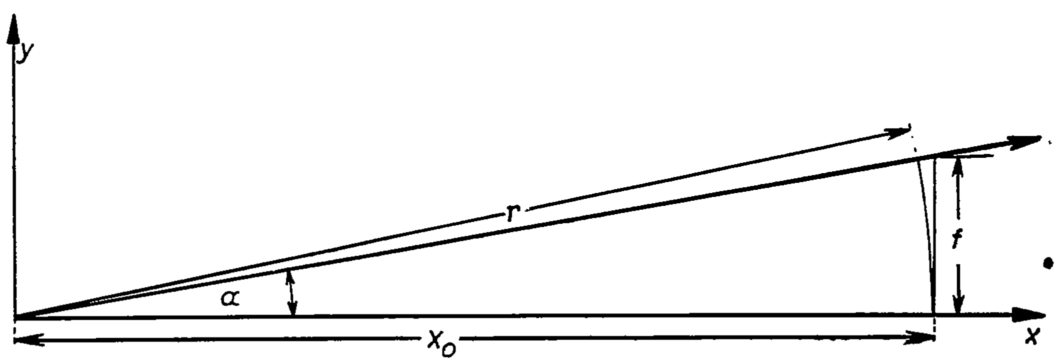


Figure 5.- Notation for the two-dimensional source.

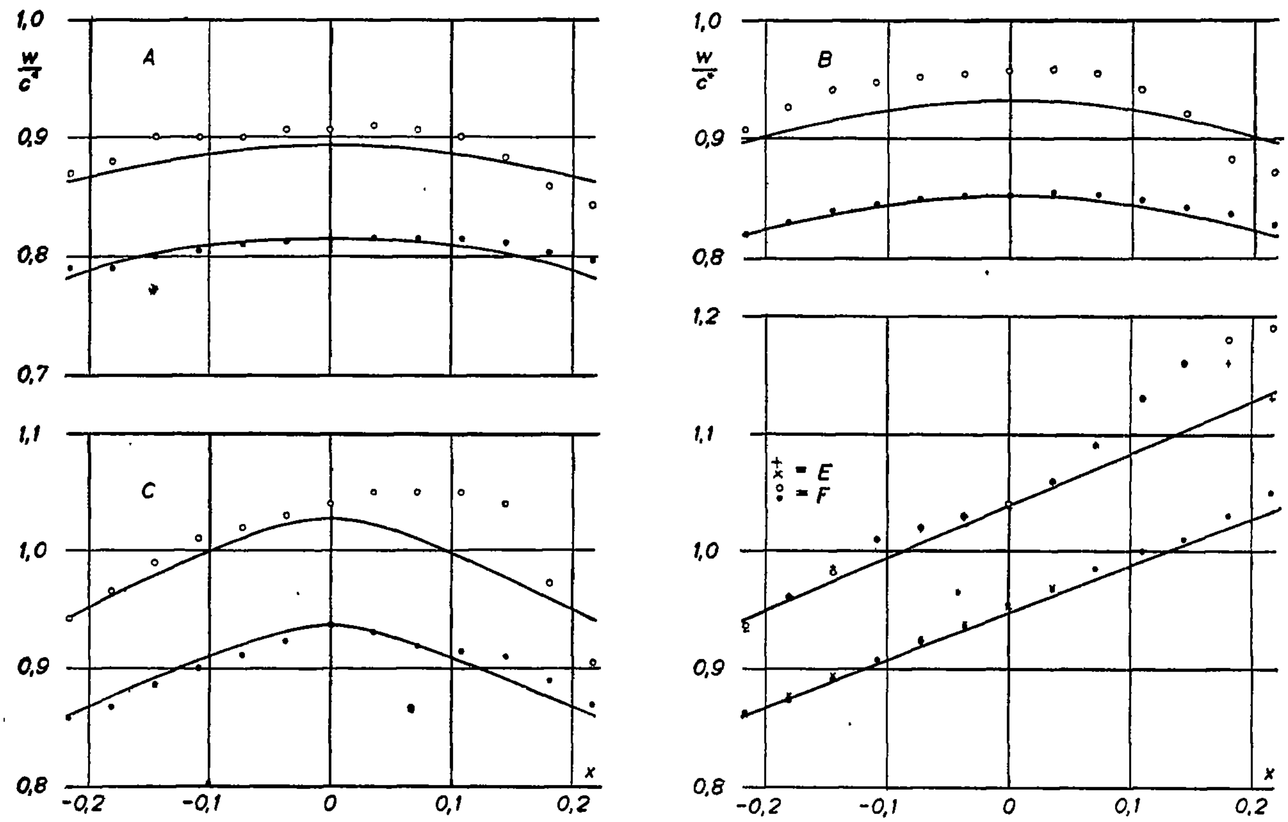


Figure 6.- Comparison of theory and test (first approximation).

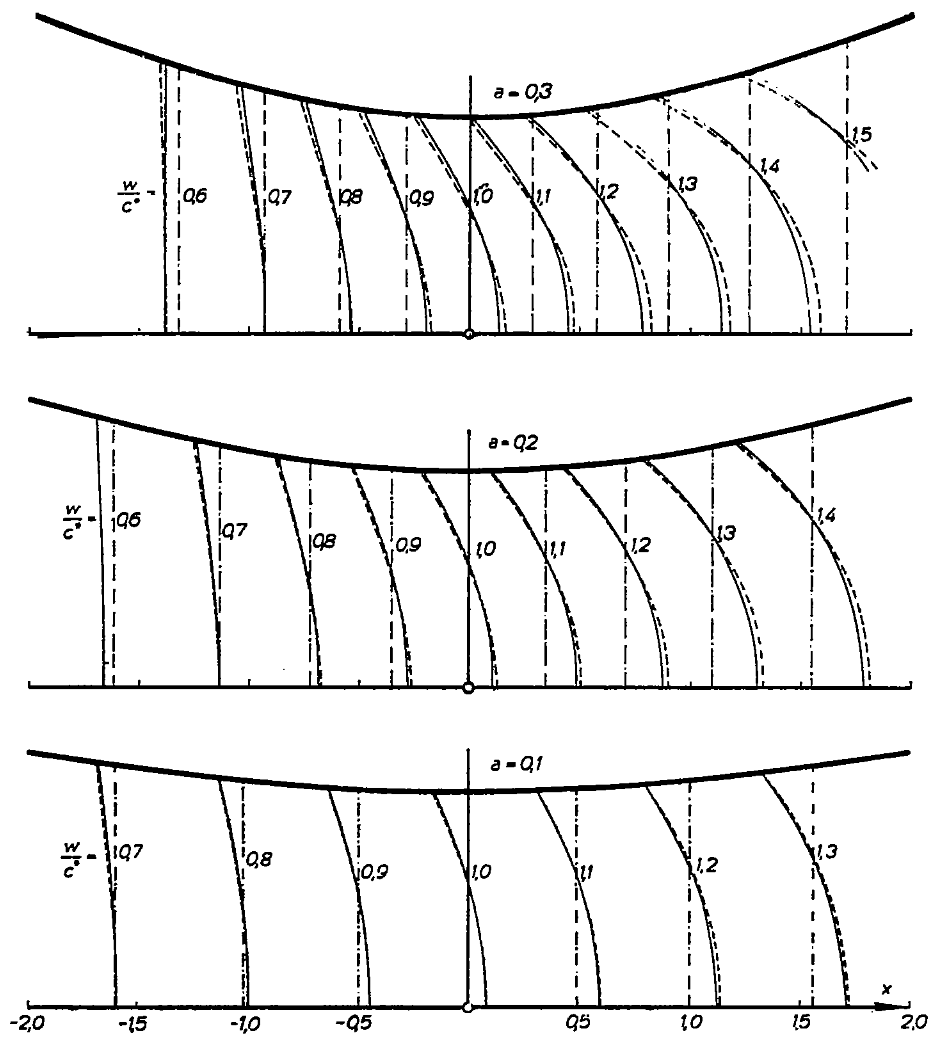


Figure 7.- Lines of constant velocity of the stream filament theory, of the first approximation and the final solution (two-dimensional problem).

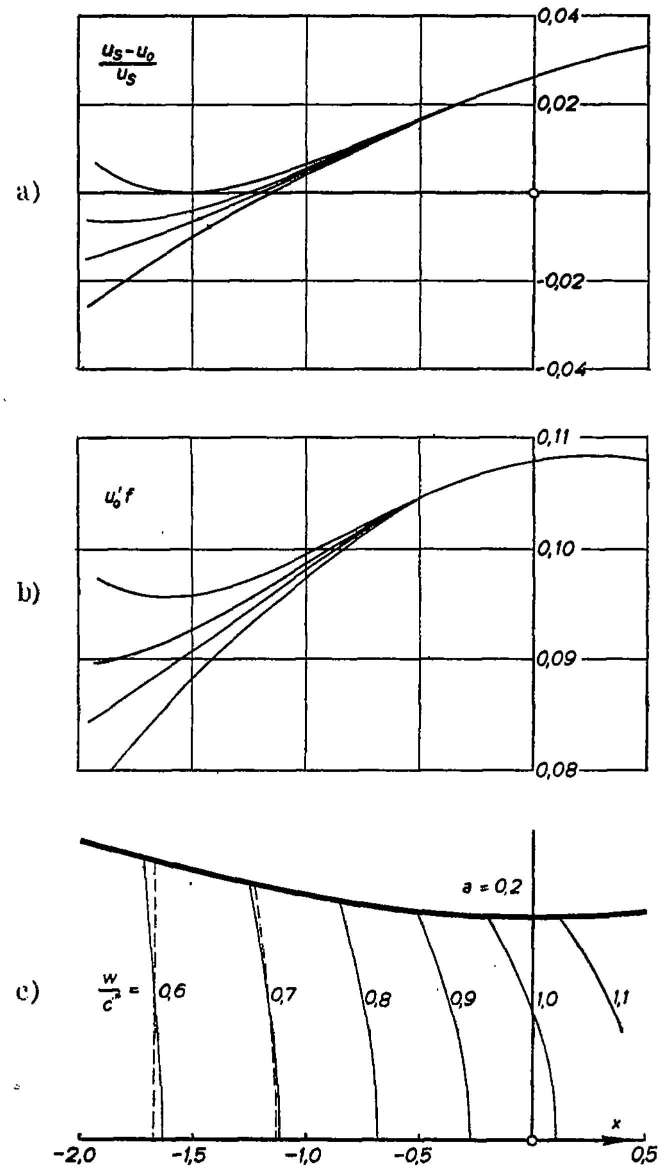


Figure 8.- Ambiguity of the solution in the subsonic zone.

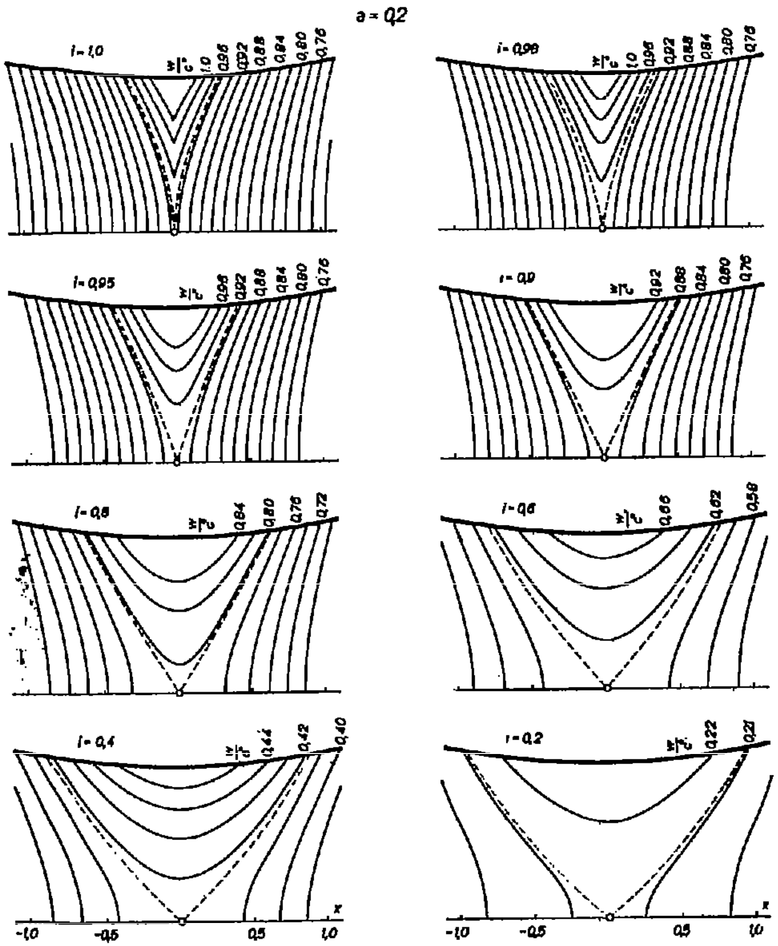


Figure 9a.- Lines of constant velocity (1. approximation) for symmetrical subsonic flow.

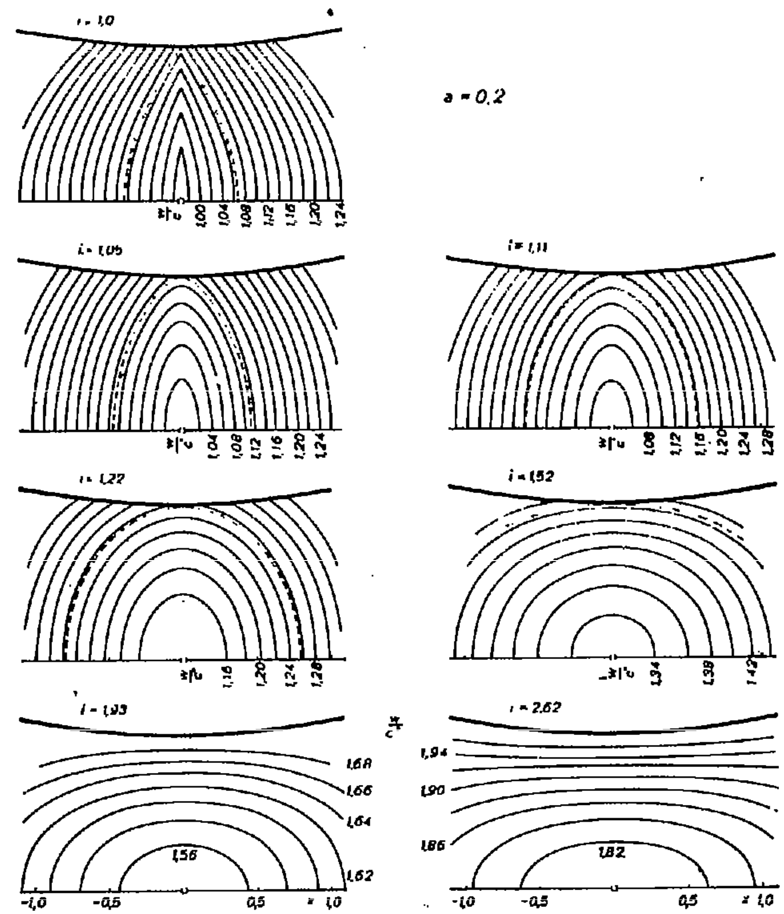


Figure 9b.- Lines of constant velocity (1. approximation) for symmetrical supersonic flow.

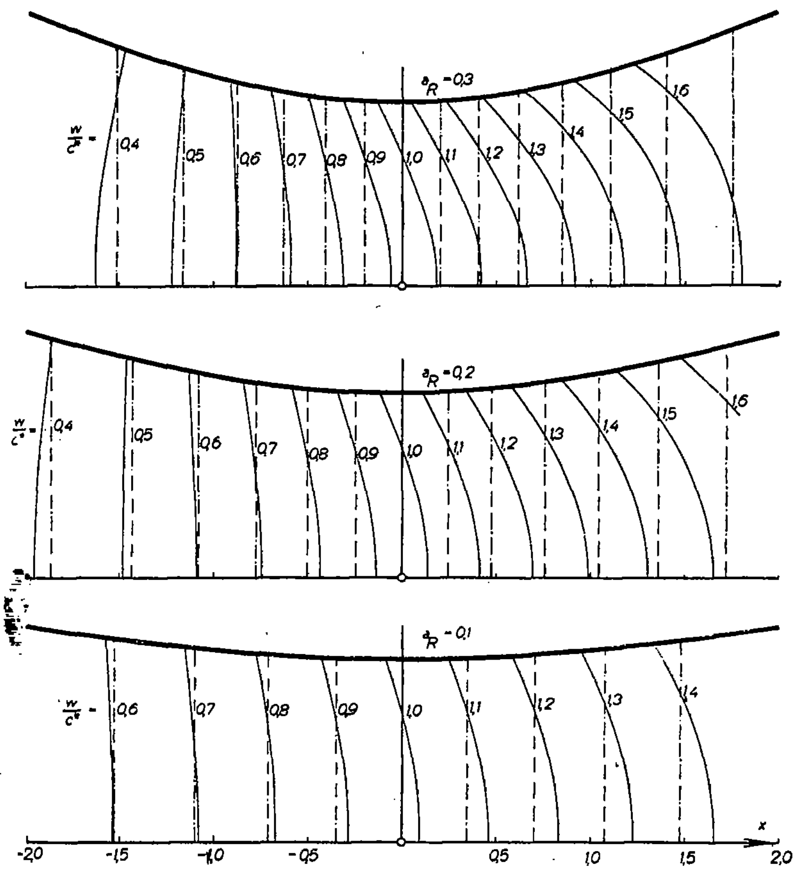


Figure 10.- Lines of constant velocity of the stream filament theory and the 1. approximation (axially symmetrical problem).