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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

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RESISTANCE OF A PLATE IN PARALLEL FLOW AT LOW REYNOLDS NUMBERS

By Zbynek Janour

Translation

"Odpór podélně obtékané desky při malých Reynoldsových číslech."
Letecký Vyzkumný Ústav, Praha, Rep. 2, 1947.

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RESISTANCE OF A PLATE IN PARALLEL FLOW AT LOW REYNOLDS NUMBERS*

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SUMMARY

The frictional resistance of a flat plate in a laminar flow is given by the Blasius formula

$$W = \frac{1.328}{\sqrt{Re}} \, b l \, \frac{\varrho}{2} \, v^2$$

on one side of the plate where l is length, b is width of the plate, and Re is $\frac{vl}{\nu}$. This formula was derived under the supposition of constant static pressure along the plate $\left(\frac{dp}{dx} = 0\right)$ and of large Reynolds numbers. In order to investigate the range of moderate Reynolds numbers, for example, of order of magnitude 10 to 10^3, measurements were carried out by the author in the oil channel of the Göttingen Institute for Fluid Research at the suggestion of Professor Prandtl.

The results of these measurements, which include Reynolds numbers from 12 to 2335, are plotted in figure 10. It is shown that the values of the resistance coefficient are higher than those given by the Blasius formula. In the range of Reynolds numbers from 10 to 10^3, the coefficient can be approximated by the formula

$$c_D = 2.90 \, Re^{-0.601}$$

$$10 < Re < 10^3$$

with a mean error of ±3 percent. For $Re > 10^3$ the curve of the resistance coefficient slowly approaches the line of the Blasius formula.

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The lower limit of Reynolds numbers for the validity of the Blasius formula can be extrapolated to approximately $Re \approx 2.10^4$. The upper limit is given by the transition to turbulent flow in the boundary layer at about $Re = 5.10^5$.

If the plate is not of an infinite span, but has a straight edge parallel to the flow, an additional resistance arises at the edge. In the range of Reynolds numbers from 30 to 2300, its magnitude can be expressed by the formulr

$$D_e = 1.6\ \mu l v$$

with a mean error of ± 6.2 percent.

I. RESISTANCE OF BODIES IN LAMINAR FLOW

1. Introduction

The resistance of a body moving in a gas or liquid or exposed to a medium flowing past it is a very complicated function of the geometric properties of the body and of the physical properties of the medium. The resistance depends on the size of the body, its geometric shape and position, the quality of the surface, and on the velocity and the density, and the viscosity of the medium. A mathematical expression that would make possible the computation of the resistance from all these magnitudes would be hardly constructible and for this reason the problem must be simplified by making certain assumptions:

(1) As regards the surface quality the body is assumed "aerodynamically smooth", that is of such smoothness that the resistance of the body does not change by further improving the surface for example by polishing with finely ground metal, lacquer, and so forth.

(2) The same geometric shape of the body or expressions for geometrically similar bodies are always assumed. The size of the body is expressed by a certain given dimensional characteristic of the body, for example, the radius of a sphere or the profile chord. This investigation is restricted to the case of symmetrical flow where the velocity vector coincides with the axis of symmetry of the body, thereby excluding from consideration buoyant force and the part of the resistance induced by it.

The following relation results:

$$W = f(l, v, \rho, \mu)$$

where

l characteristic dimension of body
v velocity of medium relative to body
ρ density of medium
μ coefficient of internal friction of medium

The analytical expression of this function would be excessively complicated. It is therefore convenient to use for the resistance a certain approximate formula in which the dependence on the variables is expressed by a simple function and to multiply this function by a coefficient which likewise depends on the variables:

$$W = k\ F(l,v,\rho,\mu)$$

The resistance function F must also be the same for bodies of different shapes and by assumption (2), the dependence on the shape is referred to the coefficient of resistance.

If the coefficient is to be nondimensional, that is, independent of the choice of the system of units, the function F must have the same dimensions as the resistance, that is the dimensions of a force. In Newton's formula, which is almost exclusively used in practice:

$$W = c_x F \frac{\rho}{2} v^2 = c_x F q \tag{1}$$

where F is now a characteristic area (for example, frontal) of the body, $q = \rho v^2/2$ is the dynamic pressure (force on unit area), and the product $F \cdot q$ has the dimensions of a force.

If the coefficient k or c_x is to depend on the magnitudes l, v, ρ, and μ and at the same time be nondimensional these magnitudes must appear in the coefficient in the form of nondimensional combinations:

$$l^\alpha\ v^\beta\ \rho^\gamma\ \mu^\delta$$

If the dimensions of the magnitudes are given in the absolute system:

$$\begin{aligned} l &\ \ldots.\ (\mathrm{cm}) \\ v &\ \ldots.\ (\mathrm{cm})(\mathrm{sec}^{-1}) \\ \rho &\ \ldots.\ (\mathrm{g})(\mathrm{cm}^{-3}) \\ \mu &\ \ldots.\ (\mathrm{g})(\mathrm{cm}^{-1})(\mathrm{sec}^{-1}) \end{aligned}$$

the product $\mathrm{cm}^\alpha \cdot \mathrm{cm}^\beta \cdot \mathrm{sec}^{-\beta} \cdot \mathrm{g}^\gamma \cdot \mathrm{cm}^{-3\gamma} \cdot \mathrm{g}^\delta \cdot \mathrm{cm}^{-\delta} \cdot \mathrm{sec}^{-\delta}$ must be nondimensional, or $\alpha+\beta-3\gamma-\delta = 0$, $-\beta-\delta = 0$, and $\gamma+\delta = 0$, from which it follows that $\beta = \gamma = \alpha$ and $\delta = -\alpha$. The nondimensional expressions which may be constructed from the given four variables are then

$$\frac{l^\alpha v^\alpha \rho^\alpha}{\mu^\alpha} = \left(\frac{lv}{\nu}\right)^\alpha$$

($\nu = \mu/\rho$ is the so-called kinematic viscosity). These expressions are powers of the Reynolds number $Re = lv/\nu$. The resistance coefficient is then a function of the Reynolds number, or for constant Reynolds number the coefficient of resistance of geometrically similar bodies has the same value. The dependence of the coefficient of resistance on Re is generally given graphically because of the complexity of its analytical expression.

The resistance formula need not always have the form (1). The Stokes formula for the resistance of a sphere which is valid for very small Reynolds numbers ($Re \ll 1$), is

$$W = 6\pi \mu r v \tag{2}$$

where r is the radius of the sphere. The resistance is proportional to the first power of the velocity. The function $\mu r v$ has the dimensions of a force; and the coefficient 6π is a constant independent of the Reynolds number.

It is, in principle, immaterial which form of the resistance function is selected. If the powers of the velocity are considered, use must be made of the expressions

$$W = k_0(Re) \frac{\mu^2}{\rho} \tag{3}$$

or

$$W = k_1(Re)\ \mu l v \qquad \text{(Stokes)} \tag{3'}$$

or

$$W = k_2(Re)\ l^2 \rho v^2 \qquad \text{(Newton)} \tag{3''}$$

or

$$W = k_3(Re) \frac{l^3 \rho}{\nu} v^3 \tag{3'''}$$

and so on. The most suitable form of the resistance law must be chosen according to the nature of the problem or according to the range of Reynolds numbers. For the resistance of a plate use must be made of the form of Newton's formula in which the area of the plate is substituted in place of l^2. For the resistance of a linear form, for example, the added resistance of the bounding edges (parallel to the stream) of a plate, the expression of Stokes in which the resistance is proportional to the length is convenient. For very low Reynolds numbers the Stokes formula is also generally used. It would be most convenient to use, if possible, in each range of Reynolds numbers, that form of expression for which $k(Re)$ is constant. The most used expression in practice is Newton's formula in form (1) where $c_x = 2k_2$

and the characteristic area of the body F is taken in place of l^2. If Stoke's formula for a sphere (2) in Newton's form is preferred, the coefficient of resistance obtained is

$$c_x = \frac{24}{Re}, \qquad Re = \frac{2rv}{\nu} \tag{2'}$$

2. Theoretical Solution

The possibilities of a theoretical solution of the laminar and steady flow about bodies and the computation of the resistance are briefly examined.

The laminar motion of a viscous fluid is governed by the equation of Navier-Stokes, which in the case where no external force is acting has the form (reference 1),

$$\frac{dw}{dt} = -\frac{1}{\rho}\operatorname{grad} p + \nu \Delta w \tag{4}$$

where p is the static pressure and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. For the components of the two-dimensional problem the equations are:

$$\begin{aligned} \frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \\ \frac{\partial v_y}{\partial t} + v_x\frac{\partial v_y}{\partial x} + v_y\frac{\partial v_y}{\partial y} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \end{aligned} \tag{5}$$

For the case of steady flow the first terms on the left sides $\partial v_x/\partial t$ and $\partial v_y/\partial t$ also drop out. The equations are further supplemented by the condition of continuity, which for an incompressible fluid is $\operatorname{div} w = 0$ or

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{6}$$

and by suitable boundary conditions. These are:

(1) For $x \to \pm\infty$ and $y \to \pm\infty$ the velocity $v = v_0$ = constant and

(2) At the wall $w = 0$, the normal and tangential components of the velocity vanish.

The equations are nonlinear and their general solution is not known because a superposition of particular solutions is impossible. A solution can be obtained only if the equations can be suitably simplified.

(1) If $\nu = 0$ is assumed, from the Navier-Stokes equation the term vanishes that takes into account the internal friction in the fluid. The remaining relation

$$\frac{dw}{dt} + \frac{1}{\rho} \operatorname{grad} p = 0$$

is the Euler equation of motion for an ideal fluid in the simplified case of the absence of an external force (reference 2, p. 101). Its integral along a streamline is the Bernoulli equation, which in the most simple steady state expresses the law of the conservation of energy

$$gz + \frac{v^2}{2} + \frac{p}{\rho} = \text{constant}$$

For computing the velocity field the equation of continuity (6) is employed in which are substituted

$$v_x = \frac{\partial \rho}{\partial x}$$

$$v_y = \frac{\partial \rho}{\partial y}$$

from which is obtained the Laplace equation

$$\Delta \varphi = 0$$

where φ is the so-called velocity potential. These equations must not be subject to the second boundary condition as a whole but the condition that the normal component of the velocity at the wall vanishes. The tangential component is then obtained from the solution of the equation. (This is the so-called Neumann problem (reference 3, pp. 613 and 620).) This condition agrees with the fact that the internal friction in the fluid has been neglected. For a potential flow, frictional resistance does not exist. The body is acted upon only by the pressures given at each point of the surface by the Bernoulli equation. If the body is in a symmetrical flow (the circulation of the velocity about the body is equal to 0) the integral vanishes over the entire surface of the body and the resultant resistance is equal to 0 (D'Allembert paradox).

(2) The nonlinearity of equations (5) and (4) lies in the term on the left side. If this term is neglected, the equation simplifies to the form

$$\mu \Delta w = \text{grad } p$$

The solution of this equation for the flow about a sphere was derived by Stokes (reference 4). The neglected term dw/dt is termed the inertia force in the fluid. These forces may be neglected when compared with the viscous forces $\mu \Delta w$ only if the motions considered are extremely slow or for very low Reynolds numbers. For this reason even at relatively large distances from the body where the effect of viscosity is small the inertia terms cannot be neglected.

(3) Oseen (reference 5) perfected the Stokes theory by replacing the inertia terms $v_x \partial v_x / \partial x$, and so forth by the approximate values $v_{0x} \partial v_x / \partial x$ where v_{0x} is the constant value of the velocity component v_x at infinite distance from the body. Near the body where the values of v_x deviate considerably from v_{0x} the inertia terms are small compared with the viscosity terms so that the Oseen equation becomes the Stokes equation. Oseen obtained an expression for the resistance of a sphere in the form

$$W = \frac{6\pi\mu r v_0}{1 - \frac{3}{8}\frac{r v_0}{\nu}}$$

The coefficient of resistance for the Newtonian form of the resistance law is then

$$c_x = \frac{24}{Re}\left(1 + \frac{3}{16} Re\right)$$

The Oseen method was used by Lamb (reference 6) to compute the resistance of a cylinder at low Reynolds numbers. Lamb's formula is comparatively complicated:

$$W = \frac{4\pi\mu v_0 l}{ln \frac{\nu}{v_0 r} + \frac{1}{2} + 2ln\ 2 - 0.577}$$

(0.577 is the Euler constant = lim $(1+1/2+1/3+ \ldots + l/n - ln\ n)$.)

(4) The solutions of Stokes and Oseen of the Navier-Stokes equation are asymptotic solutions for $\lim Re = 0$. The equations can however be simplified and solved also for the case that $\lim Re = \infty$, as done in the Prandtl theory of the boundary layer (reference 7).

At high Reynolds numbers the thickness of the region in which internal friction occurs is small as compared with the length of the wall, for the viscosity acts only in a thin layer at the wall, the so-called boundary layer. Prandtl made an estimate of the order of magnitude of each term of equations (5) and showed that on the right side the term $\partial^2 v_x/\partial x^2$ in the first equation may be neglected as compared with $\partial^2 v_x/\partial y^2$ and in the second equation the term $\partial^2 v_y/\partial x^2$ may be neglected as compared with $\partial^2 v_x/\partial y^2$. Introducing the so-called stream function ψ by the expressions

$$v_x = \frac{\partial \psi}{\partial y}$$

$$v_y = -\frac{\partial \psi}{\partial x}$$

the equation of the boundary layer is obtained

$$\frac{\partial^2 \psi}{\partial t \partial y} + \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} - \frac{1}{Re}\frac{\partial^3 \psi}{\partial y^3} = -\frac{1}{\rho}\frac{dp}{dx}$$

This equation is nonlinear but its solution in suitable cases simplifies to the solution of ordinary differential equations. This equation has been solved by various methods for the case of the flow about cylindrical bodies; a review of this work has been given by L. Howarth (reference 8). The exact solution for the flow past plates parallel to the stream for $dp/dx = 0$ has been given by Blasius (reference 9). For the coefficient of friction for one side of the plate the following expression is obtained (reference 9):

$$c_x = 1.328 \, Re^{-\frac{1}{2}} \tag{7}$$

(The original value 1.327 of Blasius was corrected by Topfer (reference 10) to 1.328.)

A solution of equations (5) is possible only in the simplified cases when certain terms may be neglected. Solutions also exist for definitely prescribed forms of flow, for example, laminar flow in pipes, the flow at the leading edge of a plate at right angles to the stream, or the solution of several special nonsteady motions (reference 11). For the steady flow about bodies in a free stream, a known solution has been obtained only for very low Reynolds numbers or for the boundary layer for $Re \to \infty$. In the range of Reynolds numbers of "average" magnitude (roughly from 1 to 10^4) no simplifications of equations (5) are possible and therefore an attempted solution in this case is faced with mathematical difficulties which have as yet not been overcome. This range is therefore left entirely to the experimental investigators.

3. Resistance of Bodies

As previously shown, by restricting the investigation to flow without hydrostatic forces, the flow about a body of an ideal fluid is potential and no resistance arises. In an actual medium, however, resistance is always produced by the viscosity of the medium. The resistance may be divided into three different components in accordance with the manner in which the medium acts on the body:

(1) A pressure resistance arises if the stream does not succeed in adhering to the surface of the body and separates from the wall. Behind the body there occurs a dead water or turbulent region in which the pressure is lower than at the forward side of the body. This part of the pressure gives rise to a resistance. The energy required for the motion of the body is dissipated mainly in the turbulent region.

(2) Shear resistance. The internal friction in the fluid requires that the medium in contact with the wall be at rest relative to the wall. At increasing distance from the wall the velocity continuously increases to the full value of the velocity in the outer stream. About the body there is thus formed a frictional or boundary layer which for large Reynolds numbers is relatively thin (compared with the length of the wall). The particles of the medium are retarded in the boundary layer and their momentum is imparted to the wall causing a shear resistance. If μ is the viscosity, v the velocity along the wall, and the coordinate y is at right angles to the wall, the shear force acting on a unit area of the wall is (reference 11, p. 36)

$$\tau = \mu \left(\frac{dv}{dy}\right)_{y=0}$$

Dissipation of energy occurs mainly at the wall where the velocity gradient is greatest (in the direction normal to the wall).

(3) The deformation resistance arises at very small Reynolds numbers. The retarding effect of the walls extends very far into the medium so that the dissipation of energy appears in the entire retarded region about the body, the so-called deformation space. The occurrence of the deformation resistance may be explained as follows: If a body is moved a small distance along its path in a very viscous fluid, the fluid deforms at first like an elastic medium. The individual volume elements in the neighborhood of the body are disturbed in tension or in compression (in transverse contraction or dilatation). The high viscosity permits only slow and delayed equalization of these internal stresses of the volume element to its new shape. In the case of continuous motion of the body, a constant deformation of the volume elements in the fluid occurs.

Spheres and cylinders of infinite length (reference 12) are examples of bodies for which the coefficient of resistance was determined in a wide range of Reynolds numbers. The resistance coefficients of these bodies are plotted in figure 1 on a logarithmic scale, for convenience, in order to give the relative segment $\Delta Re/Re$ as a constant scale unit. Both coefficients, for the sphere and for the cylinder, show a qualitatively constant trend.

The sharp divergence of the coefficient for $Re = 3 \cdot 10^5$ to $5 \cdot 10^5$ is connected with the occurrence of turbulence in the boundary layer and is not included in this investigation, which concerns the range of low Reynolds numbers and laminar flow. For this case a constant value of c_x is found at about $Re = 10^5$. The resistance is caused, as tests show (references 13 and 14), principally by the pressure component, and the constant value of the coefficient shows that the magnitude of the dead water region remains almost constant. As smaller Reynolds numbers are approached the dead water region decreases but at the same time the shear component of the resistance increases. The thickness of the boundary layer and the deformation resistance increase. The coefficient of resistance increases at a constantly greater rate until at very low Reynolds numbers it agrees with the theoretical curves of Stokes and Oseen for the sphere and of Lamb for the cylinder. For $\lim Re \rightarrow 0$ the coefficient c_x increases without limit as does dc_x/dRe. In the logarithmic plot, however, both curves have an asymptote for $Re \rightarrow 0$; that is,

$$\frac{d(\log c_x)}{d(\log Re)} = \frac{dc_x}{c_x} : \frac{dRe}{Re}$$

converges to a constant for $\lim Re \rightarrow 0$.

For a plate in a parallel flow the entire trend of the resistance coefficient curve is not known. For high Reynolds numbers for laminar flow there is only the theoretical formula of Blasius (reference 7). The formula was derived for the two-dimensional problem, that is for the assumption that the plate was of infinite width. The formula is represented in figure 1 by the straight line. The Blasius formula was experimentally confirmed in the region of $Re \sim 10^5$ (reference 15). Measurements of the velocity distribution in the boundary layer which were conducted by B. G. van der Hegge Zijnen (reference 16) and M. Hansen (reference 17) likewise confirm the Blasius theory.

The validity of the Blasius formula is restricted to a certain range of Reynolds numbers, the upper limit depending on the occurrence of turbulence in the boundary layer of a plate, which occurs, depending on the degree of turbulence of the medium, between 10^5 and 10^6. On the average $Re_{crit} = 5 \cdot 10^5$. The lower limit of validity is given by the condition that Re is large enough that the thickness of the boundary layer may be neglected as compared with the length. With decreasing Reynolds number the boundary layer increases and the deformation resistance begins to increase. It has been seen in the case of the sphere and cylinder that this is brought about by the accelerated increase in the resistance coefficient with decreasing Re; it may also be expected that for a plate in a parallel flow the increase in the resistance coefficient will be more rapid than that given by the Blasius formula.

II. MEASUREMENT OF RESISTANCE OF PLATE IN PARALLEL FLOW

1. Assumptions and Method of Measurement

The preceding considerations lead to the following problem: To find the lower limit of the Reynolds numbers for which the Prandtl boundary layer theory and the Blasius formula (7), which is derived from it, for the shear resistance of a plate are valid and how this expression changes for low Reynolds numbers. The difficulties that are encountered in the theoretical solution of the problem have already been mentioned.

In order to determine the variation of the resistance coefficient of a plate in the range of low Reynolds numbers in which deviations may be expected from the Blasius formula, the author in 1935 conducted tests in the oil tunnel for viscous flow of the Wilhelm Institute at Göttingen under the guidance of the director of the Institute, L. Prandtl. It was possible in the measurements to comprise a range of Reynolds numbers from about 10 to 2300.

The measurements are based on the following principle (fig. 3): The plate of thin sheet steel was suspended on a pendulum scale and was partly immersed in the streaming oil. The resistance of the immersed part was determined from the deflection of the scale. The velocity of the oil was measured electrically with the aid of floats because at the velocities used of about 1 to 16 centimeters per second the dynamic pressures were of the order of 10^{-2}, or 1 millimeter column of water, so that measurements with pitot tubes with normal micromanometers were too rough.

The measurements were conducted on a plate of finite width, the longitudinal edges of which were parallel with the flow. The shear resistance on 1 centimeter width are expected to be different near the longitudinal edges from that for a plate of infinite width. Tentative examination was made of the stream changes in the neighborhood of the edges. The conditions were expected to be approximately as shown in figure 2. The loci of constant velocity in a section at right angles to the edges are shown approximately in figure 2. The boundary layer becomes thinner the greater the drop in velocity in the direction normal to the plate $(dv/dy)_{y=0}$. This leads to an increase in the friction near the boundary so that the resistance increases.

The resistance of a strip of unit width is normally computed as

$$W_1 = c_x \, l \, \frac{\rho}{2} v^2$$

where $c_x(\mathrm{Re})$ is the coefficient referred to the case of a plate of infinite width. If the plate is bounded, an additional resistance exists at the edges and the total resistance of the plate immersed to a depth b is

$$W = W_1 b + W_h \tag{8}$$

The resistance was therefore measured so that the plate was gradually immersed and the deflections of the scale read. A curve of the dependence of the resistance on the immersed depth yields the straight line corresponding to equation (8) whose slope is W_1 and whose intersection on the coordinate axis is W_h (fig. 11).

In this way, the problem of the coefficient of resistance of a plate and the problem of the resistance of its edges were solved simultaneously.

It may be expected, on the basis of the considerations of the preceding section, that the values of the coefficient of resistance of the plate in the range of Reynolds numbers of the measurements are higher than those obtained from the Blasius formula (7), for the Blasius formula was derived on the assumption that the thickness of the boundary layer was small compared with the length of the plate. The Prandtl and Blasius theory permits computation of the thickness of the boundary layer (for example, reference 11, p. 89). The so-called displacement thickness, that is, the distance at which the external potential flow is affected by the boundary layer is

$$\begin{gathered} \delta^* = 1.73\sqrt{\frac{\nu x}{v}} = \frac{1.73}{\sqrt{\mathrm{Re}}}\, x \\ \mathrm{Re} = \frac{vx}{\nu} \end{gathered} \tag{9}$$

where x is the distance from the leading edge of the plate. For the actual thickness, that is, the distance up to which the effect of the retardation of the liquid or gas extends, the rough value given by the following equation is assumed:

$$\delta = \frac{5.2}{\sqrt{Re}} x \qquad (9')$$

For example, for $Re = 10^4$ the thickness of the boundary layer is equal to about 5 percent of the length x. If the lowest Reynolds numbers for which the Blasius formula is valid are to be determined from the measurements (within the limits of accuracy of the measurements), it is necessary to consider the maximum thicknesses that can be acquired by the boundary layers in order to be assumed thin enough to satisfy the condition of the Prandtl theory.

2. Measuring Apparatus

Following are the parts of the measuring apparatus and their functions:

(1) Oil tunnel. - The oil tunnel was rectangular with circulating flow. At each of the longer sides was a narrowed work section 179 millimeters deep by 148 millimeters wide and about 60 centimeters long. On the opposite side was a rotary pump driven by an electric motor. Ahead of the pump either a cooling coil in which cold water was circulated or an electric heater could be immersed in the oil. In this way the viscosity of the oil could be changed within wide limits and the range of Reynolds numbers extended. It was possible to vary the temperature (in the summer) from 15° to 35°. The velocity was regulated by the driving motor resistance and its constancy controlled by a voltmeter giving impulses to the motor.

In the work section the tunnel was contracted from 251 millimeters ahead of the section to 148 millimeters. The oil then accelerated and its velocity in the transverse direction equalized out and was constant over the entire width. The velocity profile at the start of the work section was then rectangular as at the tunnel entrance. The development of the velocity profile with increasing distance from the entrance could, at not too large a distance, be considered such that at the walls the oil is retarded and a boundary layer formed which increased in thickness with distance from the entrance. The free stream, that is, the part unretarded by the walls, then constantly contracts and accelerates. The velocity was constant at right angles to the stream but increased in the direction of the stream. Figure 4 shows the longitudinal velocity profile across the tunnel for certain velocities at the surface and beneath the surface. The transverse profile at about 3/4 of the distance through the work section (the measured mean velocity in the section 32 to 54 cm from the entrance) is shown in figure 5.

It would be necessary to measure the values of the resistance of the plate in a uniform unaccelerated stream in order to obtain for the relation sought those ideal conditions for which the Blasius formula is valid. Use of the measurements causes certain errors the magnitude of which is evaluated in the section on the accuracy and corrections of the measurements.

The problem arises as to which part of the work section of the tunnel is most suitable for conducting the measurements. The longitudinal non-homogeneity of the velocity of the free stream is somewhat greater at the mouth and decreases with increasing distance. In addition, the effect of the measured plate on the stream must be taken into account. The immersed plate retards the part of the liquid in its boundary layer and leads to a further acceleration of the stream about the plate. As the velocity is measured at the location of the plate before its immersion and after its removal the resistance is measured at higher velocities than those measured and a correction must be made on the measurements. This correction will be smaller the smaller the cross-sectional area of the liquid retarded by the plate as compared with that of the free stream, that is, the wider the free stream. Because the free stream is wider at the entrance to the work section, the measurements were conducted at this place. It is also possible to use the concept of the boundary layer at the walls for determining the width of the free stream.

(2) Scale. - The support of T shape was suspended over the tunnel in a horizontal position. The longitudinal beam (parallel to the direction of the stream) was double and the plate was clasped between the two parts. The scale was made of a sheet of 0.5 millimeter thickness reduced in weight by holes and stiffened by curved edges. At the ends of the cross beam were mounted damping blades moving tightly in troughs with oil. Damping of the swings was necessary especially at large velocities of the oil so that reliable readings of the deflections would be possible. This was shown by an indicator fastened to one arm of the balance and moving over a graduated rule which could be read to about 0.1 millimeter.

The support was suspended at three points and at two of these points the supporting fibers formed a V to prevent rotation of the scale. The fibers were of untwisted silk and so dimensioned that their elongations with change in weight could be neglected.

Length of suspension	L = 202.9 centimeters
Weight of support	13.67 grams
1/2 weight of fibers	0.04 grams
	13.71 grams

The effective weight is M

$$Mg = 13.7 \text{ grams (weight of support)} + \text{weight of plate}$$
$$-0.5 \text{ grams (buoyancy of dampers)} - \text{buoyancy of plate}$$

If x is the deflection of the scale in centimeters, the resistance is

$$0 = Mg \frac{x}{L} = 4.835 \text{ Mx dynes}$$

(3) Plates. - The plates were of sheet steel 0.5 millimeter thick sharpened at the edges and polished. The following measurements were computed:

Plate	Length (cm)	Width (cm)	Weight (gram)	Buoyant force (gram)
I	10.0	19.2	25.6	0.44·b
II	2.15	19.95	5.49	.09·b
III	10.0	15.0	19.93	.44·b
IV	15.0	10.0	19.93	.66·b
V	0.95	21.8	2.75	.022·b

In the expression for the buoyancy, b is the immersed depth and the corresponding coefficient is the product of a volume of 1 centimeter width of the plate by the specific weight of the oil.

(4) Oil. - The oil was Russian of the so-called free-flowing type. The dependence of the specific weight on the temperature was measured by Mohr weights in a water bath at a determined temperature level. There was measured

$$\rho = 0.8880 - 0.000666 \ T \pm 0.00009$$

The viscosity as a function of the temperature was measured by a capillar viscosimeter and is plotted in figure 6.

(5) Electrical measurement of velocity. - The velocity was measured by floats of the form shown in figure 7. The small ball below was weighted with mercury and its flasks could be raised to different depths or to the surface. Observation showed that they acquire the velocity of the oil very well and are not retarded as suspended particles are.

Two bridges were placed across the tunnel which permitted a gap in the middle bounded at the sides by two knife edges connected to the poles of a battery through a recording electrical apparatus. A thin wire was placed across the end of the knife edges to connect the current. When the float threw off the wire the current was interrupted and this break was recorded by the motion of a marker on a cylinder. Another marker of the apparatus recorded the second impulses of a chronometer. The time between the disconnecting of the two wires was then computed by measurement of the record.

Except for the shortest plates, the distance of the wires was chosen equal to the length of the plate so that the mean velocity of the free stream was measured. For short plates, measurements were made at the section 5 centimeters. Inasmuch as it was impossible to measure the velocity and resistance simultaneously a series of five measurements of the velocity was always conducted before and after the measurements of the resistance. During the measurements, the impulses on the motor were held constant. In measuring the resistance the immersion of the plate was varied by 1/2 or 1 centimeter. "One measurement" denotes a series of resistance readings (10 to 12, less for the small plates) and both series of readings of the velocity (before and after the measurement of the resistance).

3. Calculation of Results of Measurements

The arithmetical average was taken from all readings of the velocity in one measurement. The probable error of the arithmetical mean of the velocity fluctuates in the individual measurements between 0.27 and 1.7 percent (average, 0.63 percent). Three measurements with too large an error (about 3 percent), where the velocity changed during the measurements and the values read before the measurement of the resistance differed considerably from the values read after it, were omitted. From the arithmetical mean of the velocities, the length of the plate, and the values ρ and μ corresponding to the temperature of the oil, the Reynolds number of a measurement $Re = l\, v/\nu$ was obtained.

The values of the resistance for various depths of immersion must satisfy equation (8). The values of W_l and W_h were obtained from the measured values by the method of least squares. From the value of W_l and the corresponding velocity (arithmetical mean) the resistance coefficient v was computed. Although it is usual to compute the shear resistance only for one side of the plate, in this investigation it was measured from both sides and the coefficient is

$$c_x = \frac{W_l}{F\rho v^2} = \frac{W_l}{l\rho v^2}$$

$(F = b\, l,\ b = 1)$. A pair of values Re and c_x is then one point of the curve $c_x(Re)$.

As previously mentioned, a correction of the measurements must be made in order to take account of the change in velocity produced by the immersion of the plate. It is therefore necessary to know the approximate dependence of the resistance on the velocity and the uncorrected values must first be computed.

The measurements are given in table I. If the values of $c_x(Re)$ (eighth column of the table) are plotted in a logarithmic diagram, the relation $\log c_x(Re)$ is approximately linear. The relation is therefore approximated by a straight line giving an analytical expression convenient for making corrections. The method of least squares gives the preliminary result

$$\log c_x = 0.466 - 0.59 \log Re \pm 0.016$$

or

$$c_x = 2.93\ Re^{-0.59} \tag{10}$$

with a mean error of ± 3.9 percent for the range $Re = 10$ to $2.3 \cdot 10^3$. This relation was assumed as the basis for corrections.

4. Correction and Accuracy of Measurements

Before making corrections on the measured values, the extent to which use of the measurements for deriving the law of resistance is justified must be determined. The purpose is to express the coefficient of resistance of a plate in a flowing medium of infinite extent which everywhere, except for the effect of the plate itself, has a constant velocity. It has already been pointed out that the stream in the tunnel is accelerated in the longitudinal direction because the liquid is retarded at the walls of the tunnel and forms a boundary layer whereby the free stream, that is the part not retarded by the walls, is contracted and accelerates.

The resistance which is measured in this nonhomogeneous stream at the plate would arise on the same plate in a homogeneous stream for a certain "ideal" velocity v_i. This ideal velocity is assumed equal to the measured average velocity at the location of the plate and is denoted by $\bar{v}$. The error from this assumption is then to be estimated.

The plate is situated in an accelerated stream with its leading edge at a distance x_0 from the mouth; the length of the plate is l (fig. 8). The minimum velocity of the stream at the plate is at the mouth, $v_{min} = v(x_0)$ and the maximum is at the other end of the plate, $v_{max} = v(x_0 + l)$. The ideal value v_i is certainly between these two values, that is $v(x_0) \leq v_i \leq v(x_0 + l)$. The error in determining the velocity is then $\bar{v} - v_i$ and the relative error is

$$\vartheta = \frac{\bar{v}-v_i}{\bar{v}} = 1 - \frac{v_i}{\bar{v}}$$

The maximum possible relative error is then equal to the larger of the numbers $\left|1 - v(x_0)/\bar{v}\right|$ and $\left|1 - v(x_0 + l)/\bar{v}\right|$.

In order to evaluate the error the variation of the velocity in the tunnel must be known. The velocity is directly proportional to the cross section of the free stream, $Q\,v$ = constant. The free stream contracts from the entrance to the tunnel by the thickness of the boundary layer (according to expression (9)) for which for simplicity it is assumed that it is formed at the walls under the effect of an outside stream of constant velocity $\bar{v}$. (Actually the velocity outside the boundary layer is variable so that the thickness would have to be corrected. However, inasmuch as the thickness gives a correction of the first order for the velocity, the correction of the thickness would give a correction of the second order for the velocity. This is neglected because only the limit of the errors is of interest.)

If x is the distance from the entrance, the width s is 14.8 centimeters, and the depth of the tunnel h is 17.9 centimeters, the initial section $Q_0 = s\,h$ contracts to

$$Q(x) = s\,h - (2h+s)\,\delta^*(x) = Q_0\left[1-0.191\,\delta^*(x)\right] = Q_0(1-k\sqrt{x}) \qquad (11)$$

where $k = 0.191 \cdot 1.73 \sqrt{\nu/\bar{v}}$.

The velocity is then

$$v(x) = \frac{v_0 Q_0}{Q(x)} = v_0 \frac{1}{1-0.191\,\delta^*(x)} = \frac{v_0}{1-k\sqrt{x}}$$

The average cross section from x_0 to $x_0 + l$, that is the section through which the liquid would flow with velocity $\bar{v}$, is

$$\bar{Q} = \frac{1}{l}\int_{x_0}^{x_0+l} Q(x)dx = \frac{Q_0}{l}\int_{x_0}^{x_0+l}(1-k\sqrt{x})dx =$$

$$= Q_0\left\{1 - \frac{2}{3}k\sqrt{l}\left[\left(\frac{x_0}{l}+1\right)^{\frac{3}{2}} - \left(\frac{x_0}{l}\right)^{\frac{3}{2}}\right]\right\} \qquad (12)$$

$$= Q_0\left\{1 - \frac{0.220}{\sqrt{Re}}\quad l\,M\right\}$$

where $M = (x_0/l + 1)^{3/2} - (x_0/l)^{3/2}$

From a comparison of equations (11) and (12) the average thickness of the boundary layer over the distance x_0 to $x_0 + l$ is obtained:

$$\overline{\delta^*} = \frac{2}{3}\, 1.73 \sqrt{\frac{\nu l}{\bar{v}}}\, M = 1.73 \sqrt{\frac{\nu}{\bar{v}}} \sqrt{\bar{x}} \equiv \delta^*(\bar{x}) \tag{13}$$

where $\bar{x}$ is the distance from the entrance at which the boundary layer of this average thickness is actually formed

$$\sqrt{\bar{x}} = \frac{2}{3} \sqrt{l}\, M$$

$$\bar{x} = \frac{4}{9} l\, M^2$$

At this distance the velocity of the stream actually has the value $\bar{v}$.

The thickness of the boundary layer at another place is obtained from the equation

$$\frac{\delta^*(x)}{\delta^*(\bar{x})} = \sqrt{\frac{x}{\bar{x}}}$$

Computing the maximum error requires the equations

$$\frac{v(x_0)}{\bar{v}} = \frac{\bar{Q}}{Q(x_0)} = \frac{1-0.191\,\delta^*(\bar{x})}{1-0.191\,\delta^*(x_0)} = \frac{1-0.191\,\delta^*(\bar{x})}{1-0.191\,\delta^*(\bar{x})\sqrt{\frac{x_0}{\bar{x}}}}$$

Adjusting equation (13) yields

$$\delta^*(x) = 1.15\, M \sqrt{\frac{\nu l}{\bar{v}}} = \frac{1.15}{\sqrt{Re}}\, M\, l$$

and this gives

$$\frac{v(x_0)}{\bar{v}} = \frac{1 - \frac{0.220}{\sqrt{Re}}\, M\, l}{1 - \frac{0.220}{\sqrt{Re}}\, M\, l \sqrt{\frac{x_0}{\bar{x}}}} = \frac{\sqrt{Re} - 0.220\, M\, l}{\sqrt{Re} - 0.220\, M\, l \sqrt{\frac{x_0}{\bar{x}}}}$$

Similarly,

Similarly,

$$\frac{v(x_0+l)}{\bar{\bar{v}}} = \frac{\sqrt{Re} - 0.220\ M\ l}{\sqrt{Re} - 0.220\ M\ l\sqrt{\frac{x_{0+l}}{\bar{x}}}}$$

From the equations it is seen that the maximum possible error is greater for lower Reynolds numbers and depends on the length of the plate. Its boundary values, that is, its values for the extreme Reynolds numbers corresponding to each plate are computed. In the measurements x_0 had the value 3.3 centimeters. The results are given in the following table:

Plate	I and III	II	IV	V
l	10 cm	2.15 cm	15 cm	0.95 cm
M	1.35	2.14	1.21_5	2.99
$\bar{x}$	8.11	4.37	9.83	3.77

For the measurements at the minimum Re of the plate,

$\sqrt{Re}$	14.8	5.45	26.9	3.46
$\frac{v(x_0)}{\bar{\bar{v}}}$	0.918	0.972	0.932	0.907
$\frac{v(x_0+l)}{\bar{\bar{v}}}$	1.076	1.027	1.070	1.014
$\vartheta_{max}(\nu)$	±8.2 percent	±2.8 percent	±7.0 percent	±1.4 percent

For the measurements at the maximum Re of the plate,

$\sqrt{Re}$	33.7	15.7	48.3	5.63
$\frac{v(x_0)}{\bar{\bar{v}}}$	0.967	0.992	0.963	0.992
$\frac{v(x_0+l)}{\bar{\bar{v}}}$	1.027	1.008	1.035	1.008
$\vartheta_{max}(\nu)$	±3.3 percent	±0.8 percent	±3.7 percent	±0.8 percent

For the long plates I, III, and IV, the maximum possible error, particularly for smaller Reynolds number, is rather large, as was to be expected, because the velocities at the beginning and end of the plate are relatively larger. However, the resistance varies continuously with change in velocity at any point of the plate, and in a homogeneous stream (in the range of measured Reynolds numbers) it varies approximately with the velocity to a power less than the second, $W \sim v^{1.4}$. In the curve of W against v (fig. 12) a parabola of this power is shown. The measured resistance $\bar{W}$ then certainly lies between the values which the plate would give in a homogeneous stream for the extreme values of the velocity $W(v = v(x_0))$ and $W(v = v(x_0 + l))$ and it must be assumed that it does not lie too close to these extreme values. If the resistance were near the minimal value, for example, the value for $v(x_0)$, it would mean that the principal part of the resistance arises at the forward part of the plate (where this is the actual velocity) so that the prolongation of the plate would change the resistance very little. If, on the other hand, $\bar{W}$ were near the value for $v(x_0 + l)$ it would mean that the larger part of the resistance arises at the end part of the plate and its forward part contributes very little to the resistance. Both assumptions, however, contradict experience because, according to the Blasius formula,

$$W = \frac{1.328}{\sqrt{\frac{vl}{\nu}}} bl \frac{\rho}{2} v^2$$

$W \sim l^{0.5}$, according to our preliminary results $W \sim l^{0.41}$. The value of $\bar{W}$ and therefore of v_i cannot be too near the extreme values of the resistance or the velocity. It follows that the actual error arising from the nonhomogeneity of the stream $\bar{v} - v_i$ must be considerably lower, equal only to a fraction of the estimated maximum possible error.

The computed maximum possible errors $\vartheta_{max}(v)$ are errors in the determination of the velocity which correspond to the measured value of the resistance $\bar{W}$. However, the dependence of the resistance coefficient on the Reynolds number is considered and the error in the resistance coefficient which arises from the non-homogeneity of the stream must be determined. If use is made of the preliminary result (10), approximately by differentiation

$$\frac{\Delta c_x}{c_x} = -0.6 \frac{\Delta v}{v}$$

or

$$\vartheta(c_x) = -0.6 \vartheta(v)$$

is obtained and for the maximum possible error in absolute value

$$\vartheta_{max}(c_x) = 0.6\,\vartheta_{max}(v)$$

The maximum possible error for each plate then lies between the limits

I and III	4.9 - 2.0 percent, mean	3.5 percent
II.	1.7 - 0.5 percent,	1.1 percent
IV.	4.2 - 2.2 percent,	3.2 percent
V	0.8 - 0.5 percent,	0.7 percent

The first value always corresponds to the least Reynolds number at which the resistance of the plate was measured and the second value corresponds to the maximum Re of the plate. The actual error produced by the non-homogeneity of the stream is considerably lower as has already been emphasized. This error is systematic so that the measured values of c_x do not lie on an ideal curve $c_x(\mathrm{Re})$ but deviate from the curve, for each plate separately, the deviation being greater for smaller Re and less for higher Re. Hence if the relation $c_x(\mathrm{Re})$ is plotted from the measurements on all the plates it will deviate from the ideal curve only by the average values of the errors on each plate and the change in the errors with Reynolds number appears in the increased scatter of the points about the curve rather than as an increased mean error of the results.

The preceding considerations on the errors which can arise from the fact that the resistance is measured in an accelerated stream instead of a stream of everywhere constant velocity may be reduced to the following considerations: The test curve of the dependence of the resistance coefficient on the Reynolds number must deviate from the theoretical values in the region of high measured Reynolds numbers ($\mathrm{Re} \sim 500$ to 2000) by a maximum of about 3.5 percent; in the range of $\mathrm{Re} \sim 100$ by at most 1.1 percent; and in the range of lowest measured Reynolds numbers ($\mathrm{Re} \sim 20$) by at most 0.7 percent. The actual deviations however must be considerably lower, that is equal to a fraction of these values. Furthermore, the nonhomogeneity of the stream must give rise to an increase in the mean errors of the results.

<u>Correction of measurements.</u> - In the preceding considerations the character of the stream without the immersed plate has been discussed. Because the immersed plate reduces the cross-sectional area of the free stream, the stream accelerates and the non-homogeneity increases at a rate increasing with the depth of immersion of the plate and the measured value for the velocity of the stream as it was before the immersion of the plate must be corrected. The correction can be carried out only

approximately so that the thickness of the boundary layer is again expressed by equation (9), which properly holds only for a homogeneous stream, and in place of the acceleration at each point of the plate the average increase in the velocity along the plate is used.

If the plate is immersed to a depth b centimeters, the liquid about the plate is then retarded in the average section

$$\frac{1}{l}\int_0^l 2b \int^{*}(x)dx = 3.46\, b\sqrt{\frac{\nu}{\bar{v}}}\,\frac{1}{l}\int_0^l \sqrt{x}\, dx = \frac{2.30}{\sqrt{Re}}\, bl$$

This is the average value of the section of the boundary layer along the plate for which, however, the change in thickness at the longitudinal edge is neglected, see figure 2.

The average cross section of the free stream at the location of the plate, that is, from x_0 to $x_0 + l$ according to equation (12) is $\bar{Q} = Q_0(1-0.220\, l\, M/\sqrt{Re})$ and the average cross section contracted by the plate is

$$\bar{Q}' = \bar{Q} - \frac{2.30}{\sqrt{Re}}\, bl$$

The ratio of the two areas is

$$\frac{\bar{Q}}{\bar{Q}'} = \frac{1}{1 - \frac{2.30}{\bar{Q}\sqrt{Re}}\, bl} \doteq 1 + \frac{2.30}{\bar{Q}\sqrt{Re}}\, bl$$

If $\bar{v}$ is the average velocity without the plate and $\bar{v}'$ the average velocity with the plate,

$$\bar{v}' = \bar{v}\,\frac{\bar{Q}}{\bar{Q}'}$$

$$\Delta v = \bar{v}' - \bar{v} = \bar{v}\left(\frac{\bar{Q}}{\bar{Q}'} - 1\right) = \frac{2.30}{\bar{Q}\sqrt{Re}}\, bl\bar{v}$$

If the velocity is increased by Δv, the initial resistance $\bar{W}$ increases to $\bar{W}' = \bar{W} + (dW/dv)\Delta v$, which is the value that is actually measured. The value of the resistance is then reduced to

$$\bar{W} = \bar{W}' - \frac{dW}{dv}\,\Delta v \qquad (14)$$

Computing dW/dv from the expression for the resistance of a plate (for both sides) $W = c_x \rho b l v^2$ using the results of the measurements (10) gives

$$\frac{dW}{dv} = 2c_x \rho b l v - \rho b l v^2 \; 2.9 \cdot 0.59 \; Re^{-1.59} \frac{l}{\nu}$$

$$= \frac{2W}{v} - 0.59 \; \rho b l v \; 2.9 \cdot Re^{-0.59} \approx 1.41 \frac{W}{v}$$

where v is written in place of $\bar{v}$. Then, approximately,

$$\bar{W} = \bar{W}' \left(1 - 1.41 \frac{\Delta v}{\bar{v}}\right) = \bar{W}' \left(1 - \frac{3.25}{\bar{Q} \; Re} b l\right) = \bar{W}' \; (1 - \varepsilon b)$$

where

$$\varepsilon = \frac{3.25}{\bar{Q} \sqrt{Re}} l = \frac{3.25 \; l}{Q_0 (\sqrt{Re} - 0.220 \; lM)}$$

Using the preceding values for M and l (see preceding table) and $Q_0 = 265 \text{ cm}^2$ yields for each plate

I, III	$\varepsilon_I = \dfrac{0.123}{\sqrt{Re} - 2.97}$
II	$\varepsilon_{II} = \dfrac{0.0264}{\sqrt{Re} - 1.01}$
IV	$\varepsilon_{IV} = \dfrac{0.184}{\sqrt{Re} - 4.01}$
V	$\varepsilon_V = \dfrac{0.0116}{\sqrt{Re} - 0.63}$

These values as functions of the Reynolds number are plotted in figure 9.

Correction (15) must be made for each individual reading of the resistance (for different immersions at the same velocity of the oil). The straight line of the dependence of the resistance on the immersed depth is plotted and the corrected values W_l and W_h are obtained (the dotted line in fig. 11) from which are obtained the corrected resistance coefficients for given Reynolds numbers. These coefficients are given in column 11 of table I and are plotted in figure 10.

As has already been pointed out, the correction made is only approximate. No account is taken of the effect of the change in velocity on the boundary layer in expressing the thickness of the boundary layer by equation (9). In the range of these measurements equation (9) holds only approximately because it is derived from the Blasius and Prandtl theory for high Reynolds numbers and the form of the boundary layer in a non-homogeneous stream differs from the form which the boundary layer has in in a homogeneous stream. More accurate corrections would however be too complicated (if not impossible) to carry out because of the lack of theoretical basis for a more accurate computation of the boundary layer in this range of Reynolds number and in a non-homogeneous stream; that is, for the variation of the velocity behind the entrance to the tunnel.

III. RESULTS OF MEASUREMENTS

1. Resistance coefficient. - The measured and corrected values of the resistance coefficient are plotted in figure 10 as a function of Reynolds number. These values do not agree with the theoretical values of the Blasius formula (7), which are indicated in the figure by the dotted straight line. The actual values are higher and the deviation from the theoretical values increases with decrease in Reynolds number, that is, with increase in thickness of the boundary layer and deformation component of the resistance.

In the range of Re from 10 to 10^3, the curve of the resistance coefficient in the logarithmic plot is very close to a straight line and its analytic approximation by a straight line is then applicable. Computation by the method of least squares gives the expression

$$\log c_x = 0.463 - 0.60 \log \mathrm{Re} \pm 0.013$$

or the exponential law

$$c_x = 2.90\ \mathrm{Re}^{-0.601} \quad \text{for } 10 < \mathrm{Re} < 10^3 \qquad (16)$$

with a mean error ± 3.0 percent.

This is the approximate expression in this range of Reynolds numbers. For $\mathrm{Re} > 10^3$ the curve of the coefficient slowly bends in such way as to asymptotically approach the Blasius straight line.

If the result (16) is compared with the preliminary result (10) it is seen that the correction was only slightly changed. It is clear that the approximate correction as carried out is entirely sufficient within the limits of accuracy of the measurements. The mean error of the results decreased as compared with the mean error of the preliminary results because the measurements were partly equalized by the corrections and because the interval of Reynolds numbers in which the values of the coefficient are approximated by a straight line was shortened.

In the preceding section the error was considered which arises when the resistance is measured in a non-homogeneous accelerated stream in place of a uniform stream. Its maximum possible limits were determined and the fact emphasized that the actual error must be considerably less. It should then be expected that the actual errors will lie entirely within the limits of the mean error ± 3 percent of the approximate expression, in which the errors of non-homogeneity are already partially taken into account. The computed mean error of the result ± 3 percent will then correctly express the accuracy of the measurements.

2. Limit of validity of Blasius formula. - It was impossible to make measurements in the oil tunnel at high enough Reynolds numbers to have the resistance coefficient agree with the Blasius formula. Figure 10 shows only that such agreement starts at $Re > 10^4$. If the test curve is extrapolated for the lower limit of validity of the Blasius law, the approximate value of about $Re \approx 2 \cdot 10^4$ is obtained.

It was stated previously that from this value there can be computed the maximum thickness of the boundary layer for which the Prandtl boundary-layer theory still holds or the thickness which must still be assumed small as compared with the length of the plate. From expression (9') for the thickness of the boundary layer it follows that for $Re = 2 \cdot 10^4$, $\delta \approx 0.04\ x$. This means that at the end of the plate the thickness of the boundary layer must be at most 4 percent of the length of the plate in order that the Prandtl and Blasius theory may be used for computing the boundary layer and resistance.

3. Edge resistance. - If the parallel flow about the plate is not of infinite width but is bounded by an edge parallel with the direction of the stream the resistance at the edge is increased. The normal resistance (from the assumption of an infinite plate) for a width (immersed depth) b is

$$W = 2c_x b l \frac{\rho}{2} v^2$$

and to this must be added the edge resistance W_h.

The measured and corrected values of the edge resistance are given in column 10 of table I. The edge resistances are small as compared with the total resistance of the plate and their relative errors are then greater than the errors of the resistance W_1 (that is the resistance of a strip of unit width of a plate of infinite width). In the case of the shortest plate V of length 0.95 centimeter the edge resistance is of the order of 1 dyne, that is, the resistance corresponding to the deflection of the scale 0.1- 0.2 millimeter. The edge resistance was then of the order of the measurements and for this plate was not taken into account. Only the measurements on plates I, II, III, and IV were used except the value $W_h = 65$ dynes for Re = 403, which evidently is in large error probably on account of a shift of the scale or erroneous reading of the zero point.

If it is assumed that the edge resistance is proportional to the length of the edge and to the velocity of the medium, the Stokes formula (3') is used for expressing the resistance:

$$W_h = k_h \mu 2 v$$

The computed values k_h (column 13 of table I) fluctuate in the entire range of the measured Reynolds numbers about the same value so that within the limits of accuracy of these measurements k_h must be assumed constant. The arithmetical mean of the measured values is

$$k_h = 1.62 \pm 0.10$$

In the range of Reynolds numbers 30 to 2300, then,

$$W_h = 1.6\ \mu l v$$

with a mean error of ±6.2 percent.

IV. CONCLUSIONS

The frictional resistance of a plate in a laminar parallel flow is given by the Blasius formula

$$W = \frac{1.328}{\sqrt{Re}} bl \frac{\rho}{2} v^2$$

for one side of the plate (l = length, b = width of plate, $Re = vl/\nu$), which holds for high Reynolds numbers provided that the flow at the plate remains laminar. This condition is satisfied, as measurements have shown, for Reynolds numbers above a certain lower limit, which is approximately $2 \cdot 10^4$; the upper limit is given by the occurrence of turbulence in the boundary layer and is found to be approximately $5 \cdot 10^5$. For Reynolds numbers lower than $2 \cdot 10^4$ the resistance of the plate is not governed by the Blasius law but, as shown in the present paper, has higher values.

The resistance curve was measured in the interval of Reynolds numbers 12 to 2335 and is plotted in figure 10. In the interval $Re = 10$ to 10^3 the resistance can be expressed by the formula

$$c_x = 2.90 \ Re^{-0.60_1}$$

The change in the resistance coefficient for still lower Reynolds numbers may be estimated as follows: If the Reynolds number is decreased by shortening the length of the plate then for $\lim Re = 0$ the plate (of infinite width) becomes a straight line at right angles to the flow (or becomes a point in the two-dimensional problem). The same condition is true for a cylinder if its diameter is decreased. The resistances of a plate and of a cylinder for $\lim Re = 0$ then converge to the same value. (The resistance of the plate must be measured on both sides of the plate.) It follows that the coefficients of resistance for very low Reynolds number must be constant because it must not be asserted that the characteristic lengths used in forming the Reynolds number, that is, the length of the plate and the diameter of the cylinder, are equivalent. These lengths can only be assumed proportional; therefore, the curves for the coefficient are expected to have, in the logarithmic plots, parallel asymptotes for $\lim(\log Re) \rightarrow -\infty$. Such behavior of the resistance coefficient must be expected not only for a plate and cylinder but for all 'two-dimensional bodies', that is, bodies of infinite width at right angles to the flow.

A similar consideration shows that the resistance coefficient of three-dimensional bodies must also for $\log Re \rightarrow -\infty$ have asymptotes parallel to the Stokes straight line for the sphere. This may be explained by the concept of the occurrence of resistance as follows: The geometrical shape of the body has an effect on the shape of the deformation region only at distances comparable with the dimensions of the body. At distances which are large compared with the dimensions of the body, as long as the effect of the body extends for sufficiently low Reynolds number, the shape of the deformation region is independent of the particular shape of the body (for example whether it is curved or sharp edged). If, therefore, the dimensions of the body may be neglected compared with the dimensions of the deformation region the resistance is independent of the shape of the body.

The measurement of the resistance was carried out on a plate of finite width although the resistance coefficient refers to a plate of infinite width. The measurements show that the effect of the longitudinal edges of the plate appears in the increase of the resistance at the edge. This additional resistance of the edge, within the limits of accuracy of these measurements, may be expressed by the formula

$$W_h = 1.6 \mu l v$$

in the range of Reynolds numbers from 30 to 2300.

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TABLE I - MEASUREMENTS

	Re	v (cm/sec)	ν (cm^2/sec)	ρ (g/cm^3)	W_l (dyn)	W_h (dyn)	c_x	$W_{l,corr.}$ (dyne)	$W_{h,corr.}$ (dyne)	$c_{x,corr.}$	$\mu l v$	k_h
						Plate V						
1	11.93	2.56	0.2037	0.8775	3.81		0.698	3.64		0.668		
2	15.83	3.39_5	.2037	.8775	5.34		.556	5.13		.534		
3	15.83	3.39_5	.2037	.8775	5.85		.609	5.41		.564		
4	20.00	4.29	.238	.8775	7.9		.515	7.65		.500		
5	23.88	5.13	.2040	.8775	9.92		.452	9.63		.440		
6	28.02	6.01	.2037	.8775	11.42		.380	11.1		.369		
7	31.67	6.80	.2037	.8775	13.8		.359	13.44		.348		
						Plate II						
8	29.7	2.63	0.1898	0.8764	5.2_1	2.0_4	0.400	4.88	2.63	0.375	0.943	2.79
9	34.9	3.25	.200	.877	7.72	.63	.388	7.24	1.57	.364	1.224	1.28
10	36.62	3.475	.2040	.8775	7.70	2.0	.339	7.28	2.62	.321	1.336	1.96
11	40.08	3.42	.1834	.876	6.95	1.67	$.316_5$	6.61	2.14	.300	1.185	1.81
12	43.33	4.05	.2008	.877	10.3	1.7	.334	9.96	1.98	.323	1.533	1.29
13	47.6	3.55	.1604	.8735	7.25	.56	$.306_5$	6.89	1.17	.291	1.070	1.09
14	50.7	3.44	.146	.872	6.23	2.14	.281	5.94	2.60	.268	.941	2.76
15	53.4	4.98	.2008	.877	13.1	1.8	.280	12.5	3.0	.268	1.883	1.59
16	60.3	5.13	.183	.876	12.5	1.9	.253	12.0	2.6	.243	1.767	1.47
17	75.6	6.73	.1915	.8765	19.3	2.1	.226	18.6	3.3	.218	2.425	1.36
18	92.9	7.11	.1645	.874	18.7	2.1	.197	18.1	3.4	.191	2.200	1.55
19	102.5	6.92	.143	.872	16.4	.2	.183	15.9	1.0	.177	1.853	.54
20	115.5	8.81	.164	.874	25.2	1.7	.173	24.5	2.9	.168	2.715	1.07
21	138.4	9.97	.1549	.873	29.2	6.0	.157	28.5	6.9	.153	2.900	2.38
22	165.6	11.56	.1501	.8725	34.3	7.2	.137	33.6	8.0	.134	3.255	2.46
23	227.3	14.8	.1400	.8713	46.0	6.2	.1124	45.6	6.7	.111	3.880	1.73
24	247.2	16.1	.1400	.8713	52.8	4.6	.1088	52.4	4.96	.108	4.220	1.18
						Plate III						
25	219.1	3.55	0.1620	0.8737	13.6	5.85	0.124	11.8	18.8	0.107_3	5.02_5	3.75
26	234.5	3.44	.1467	.872	12.3	4.5_2	$.119_3$	11.0	6.6	.1068	4.400	1.50
27	318.1	5.17	.1627	.874	24.0	1.9	.1025	21.3	6.57	.0912	7.35	.89
28	340.0	5.18	.1524	.8725	22.3	6.45	.0953	20.4	9.43	.0872	6.90	1.37
29	403	6.0	.149	.872	27.9	(60.5)	.0890	25.2	(65.0)	.0806	7.80	
30	432	7.11	.1645	.874	36.4	8.0	.0823	33.7	12.2	.0764	10.22	1.19
31	483	6.90	.143	.8715	31.0	10.2	.0748	29.1	12.6	.0702	8.60	1.47
32	516	8.67	.168	.8745	46.9	14.5	.0714	44.6	17.2	.0678	12.73	1.35
33	640	9.97	.1558	.873	55.9	17.4	.0644	53.8	19.5	.0620	13.56	1.44
34	993	13.4	.135	.8705	75.7	15.2	.0485	71.8	22.4	.0460	15.7_3	1.42
35	1050	14.8	.141	.8715	88.6	24.9	.0464	87.3	26.4	.0458	18.2	1.45
36	1134	16.10	.142	.8715	100	34.8	.0443	98.6	35.6	.0438	19.9	1.79
						Plate IV						
37	724	6.85	0.142	0.8715	35.3	18.2	0.0575	33.6	20.3	0.0547	12.7	1.60
38	785	8.73	.1668	.874	53.0	30.5	.0530	52.6	26.8	.0528	19.08	1.40
39	808	8.81	.1636	.874	56.0	21.3	.0551	53.0	24.7	.0520	18.90	1.31
40	1143	8.335	.1094	.867	42.25	15.6	.0469	40.6	17.3	.0450	11.85	1.46
41	1336	9.74	.1094	.867	52.9	18.5	.0429	51.3	20.0	.0417	13.85	1.44
42	1542	11.25	.1094	.867	66.45	20.0	.0403	64.4	22.1	.0391	16.00	1.38
43	1680	11.1	.0992	.865	56.9	27.4	.0352	55.2	29.1	.0345	14.3	2.03
44	1921	12.7	.0992	.865	69.5	26.8	.0332	68.0	27.9	.0326	16.35	1.71
45	2133	14.11	.0992	.865	82.7	33.5	.0321	80.7	35.0	.0313	18.16	1.93
46	2262	14.0	.0932	.863	79.7	27.0	.0314	77.7	29.3	.0305	16.93	1.73
47	2335	15.44	.0987	.865	96.3	27.2	.0312	94.0	29.3	.0304	19.88	1.47

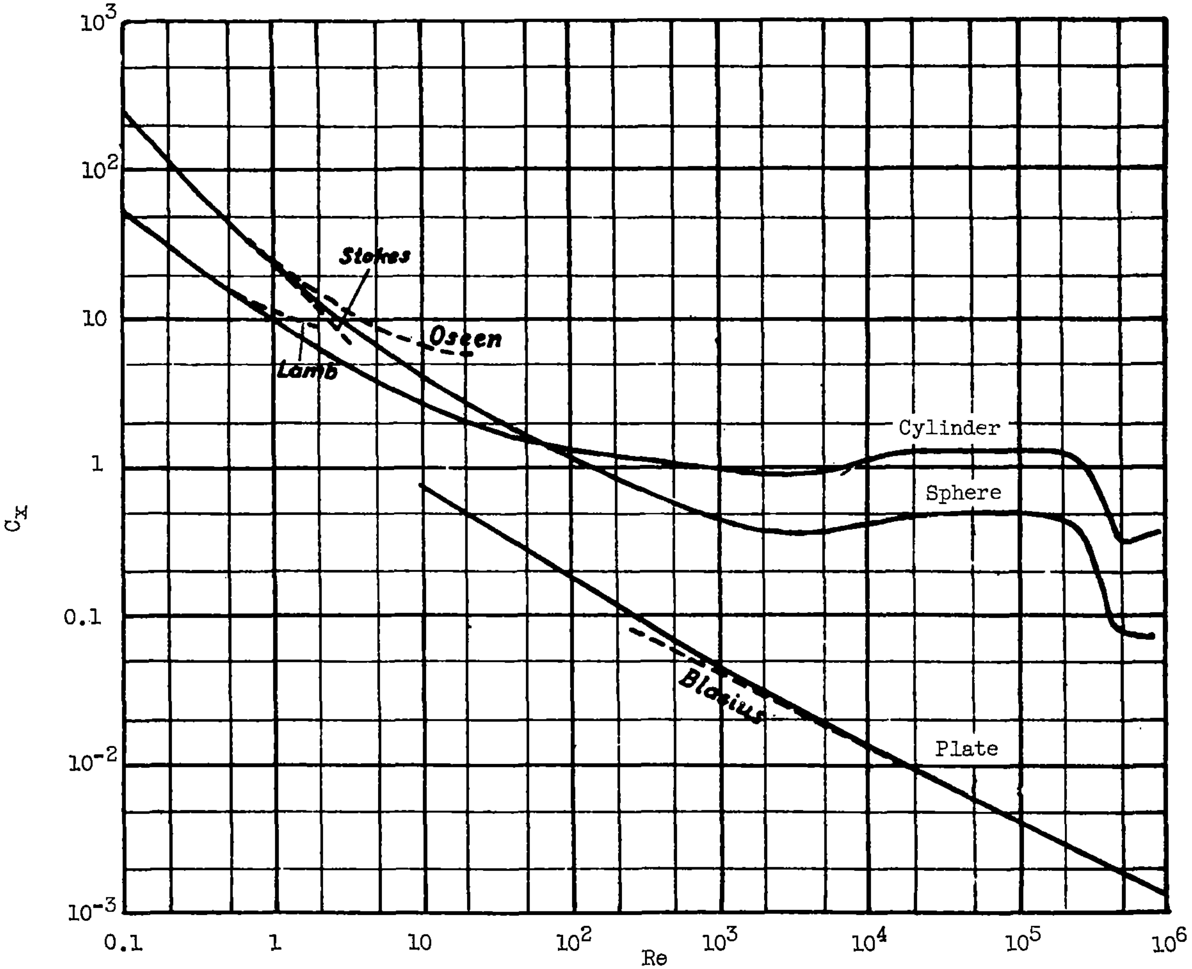

Figure 1. - Dependence of resistance coefficient on Reynolds number for sphere, cylinder, and plate.

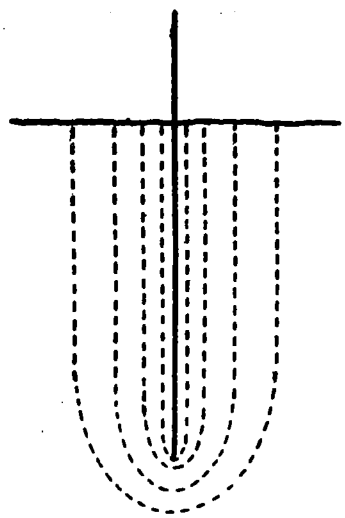

Figure 2. - Isotachs (curves of equal velocity) about an edge.

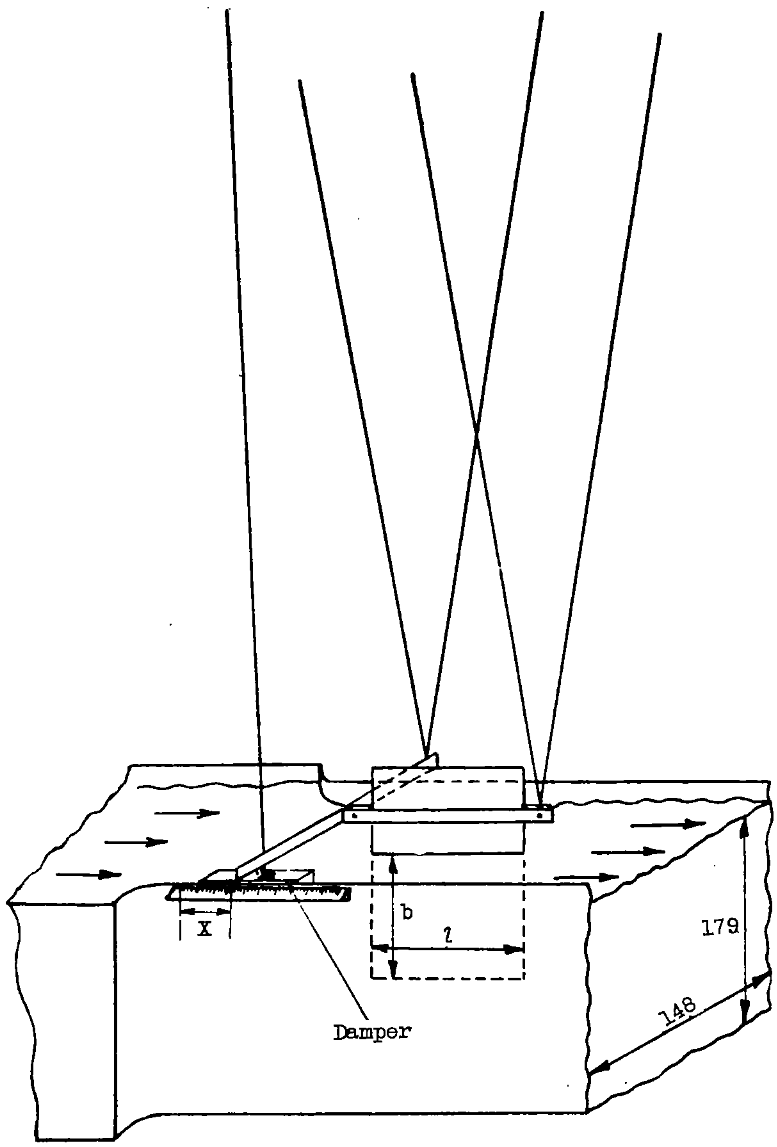

Figure 3. - Scheme of tunnel and scale.

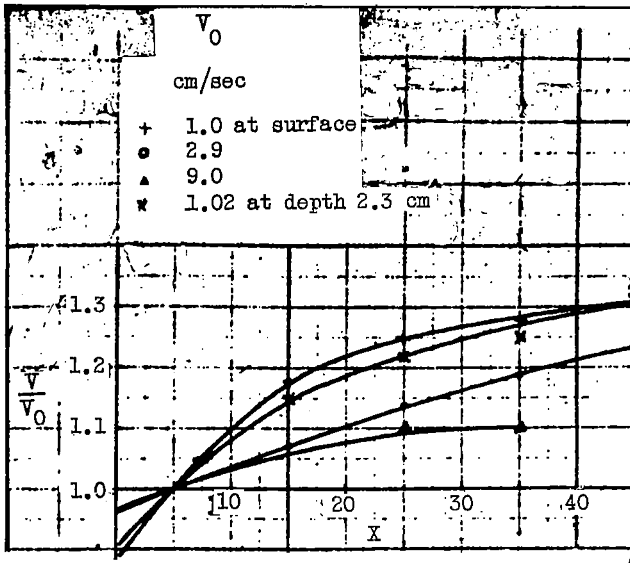

Figure 4. - Longitudinal velocity profile across tunnel.

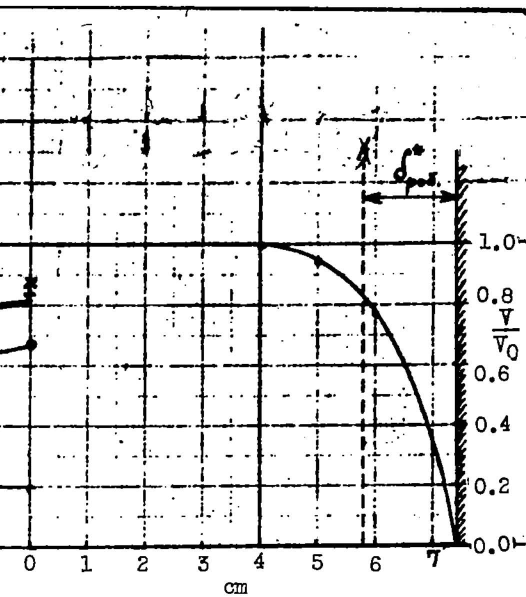

Figure 5. - Transverse velocity profile. V_o, 7.7 centimeters per second; at distance 32 to 54 centimeters.

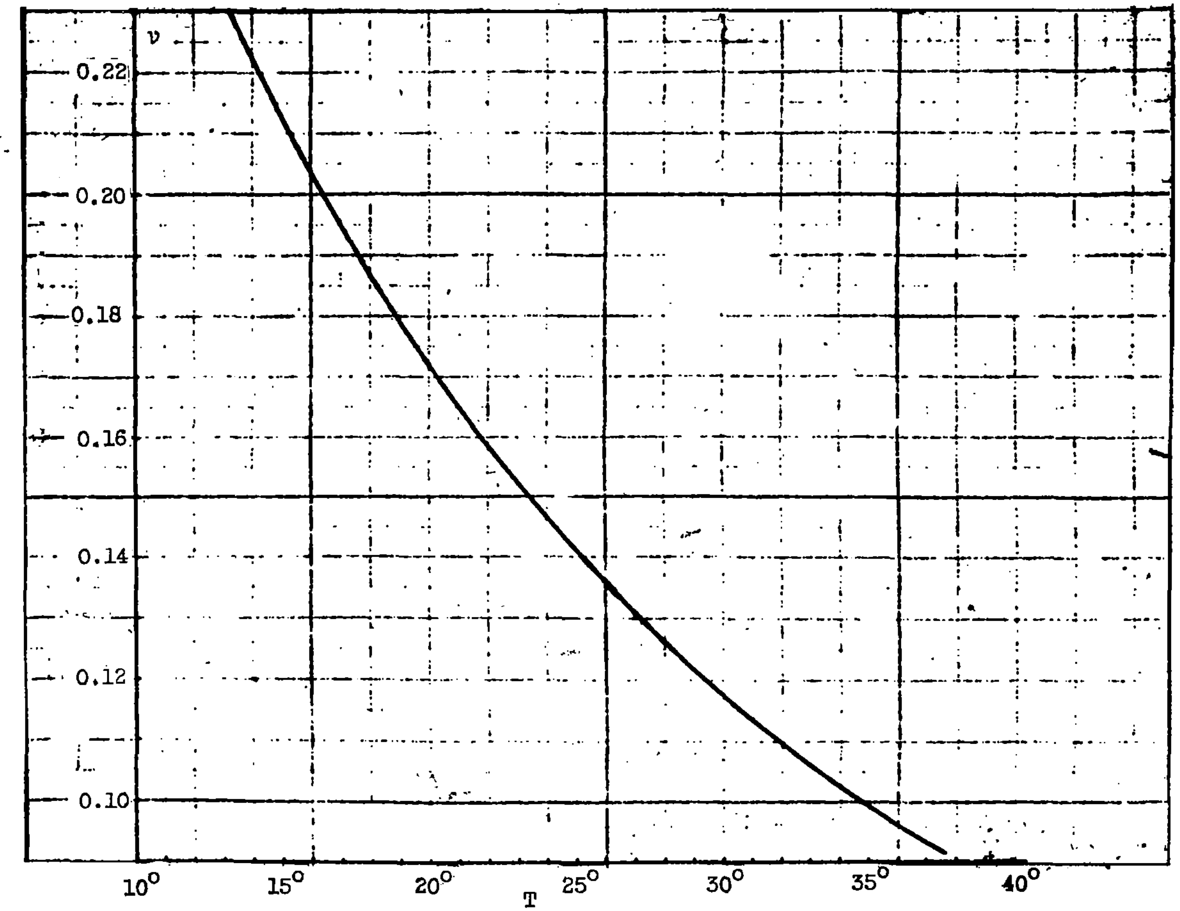

Figure 6. - Dependence of viscosity on temperature.

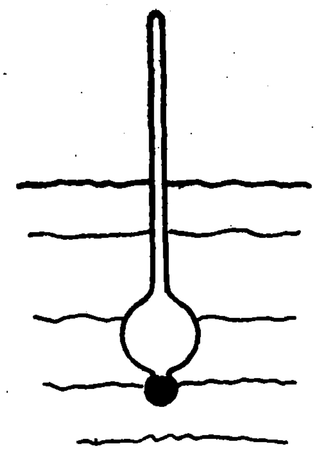

Figure 7. - Float

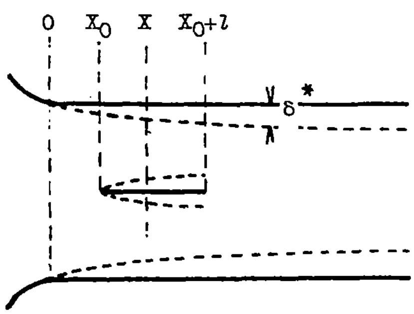

Figure 8. - Contraction of stream by plate at walls.

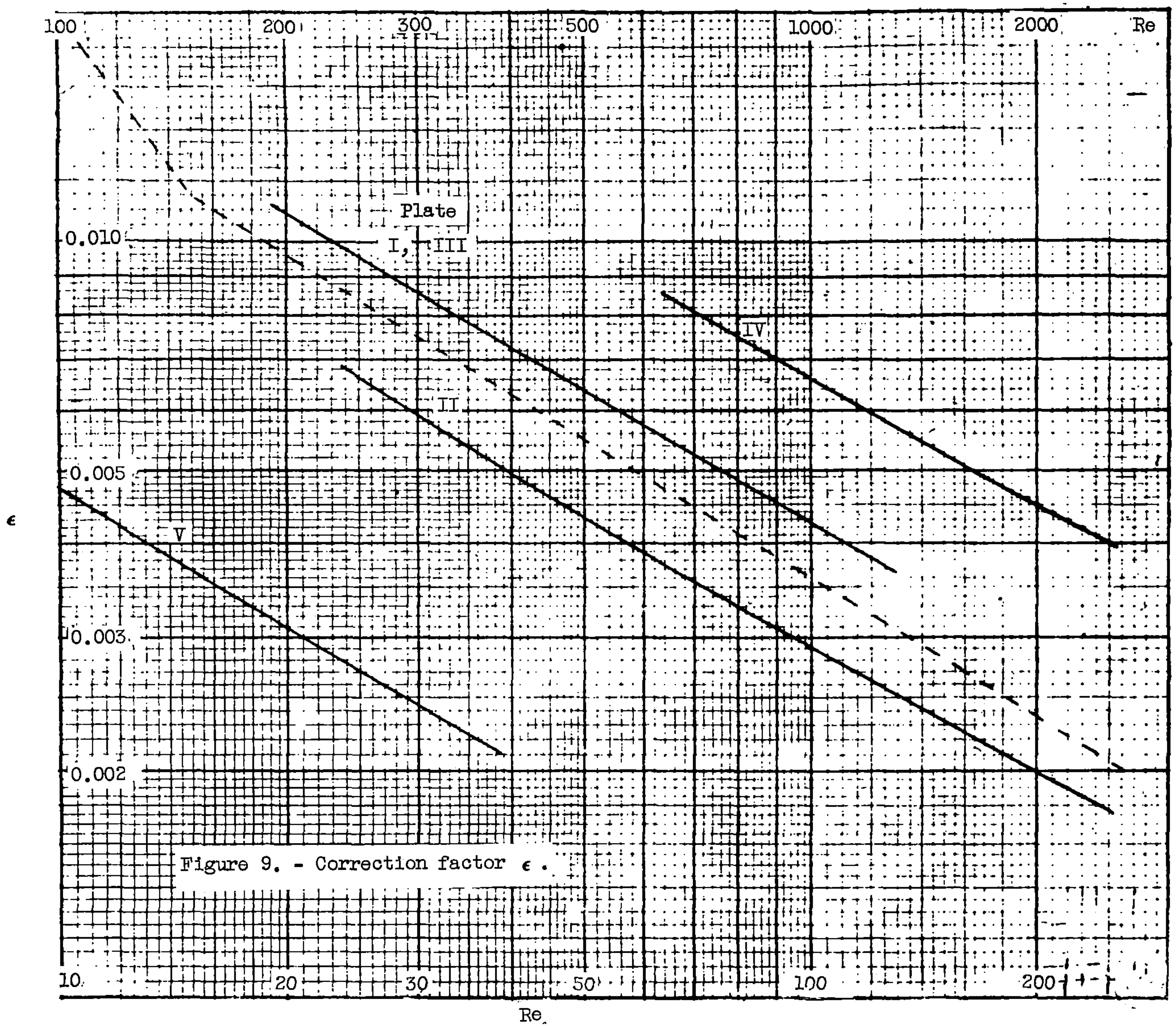

Figure 9. - Correction factor ε.

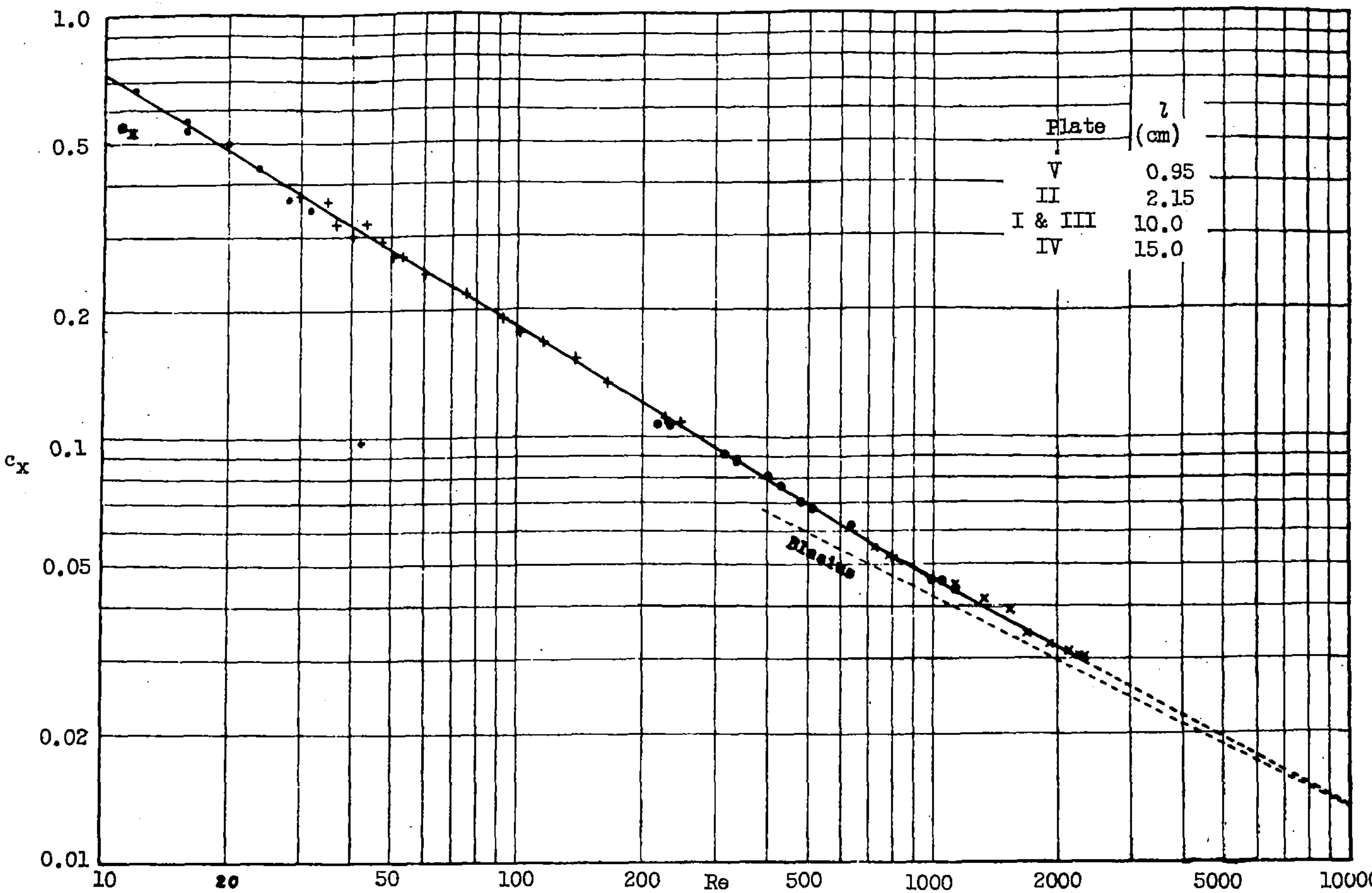

Figure 10. - Resistance coefficient of a plate in a parallel flow.

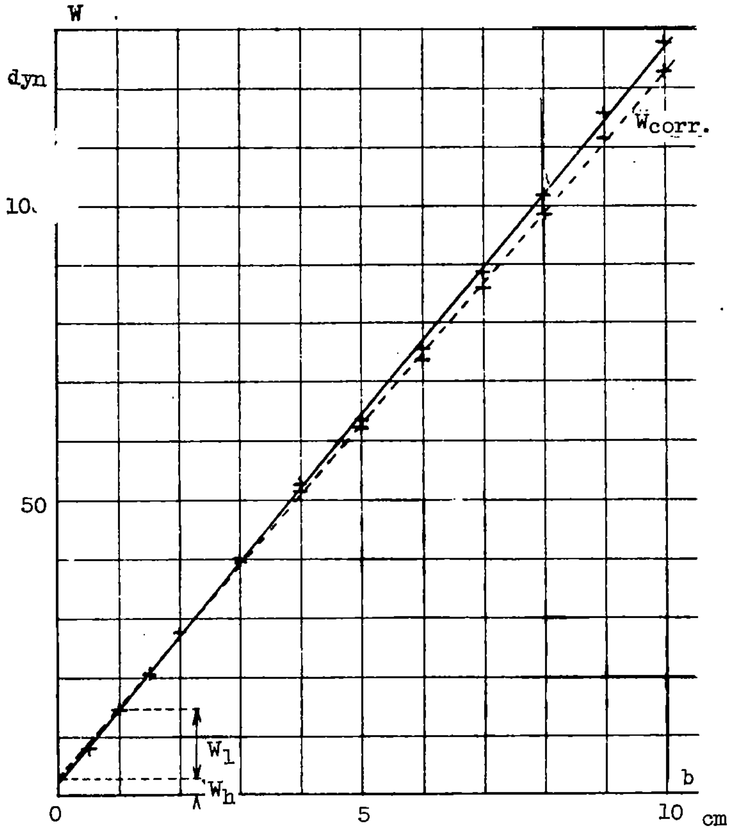

Figure 11. - Dependence of resistance on depth of immersion of plate for 16 measurements. (Plate II., Re = 60.3)

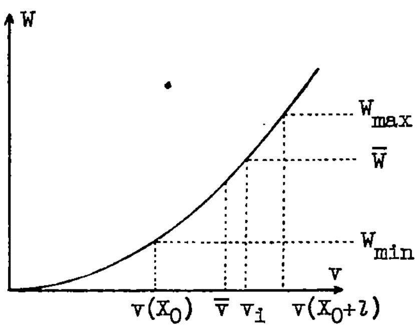

Figure 12. - Scheme of dependence of resistance on velocity.