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THE STRESS DISTRIBUTION IN SHELL BODIES AND WINGS AS AN EQUILIBRIUM PROBLEM

By H. Wagner

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THE STRESS DISTRIBUTION IN SHELL BODIES AND WINGS AS AN EQUILIBRIUM PROBLEM*

By H. Wagner

SUMMARY

This report treats the stress distribution in shell-shaped airplane components (fuselage, wings) as an equilibrium problem; it includes both cylindrical and non-cylindrical shells. In particular, it treats the stress distribution at the point of stress application and at cut-out points.

I. GENERAL REMARKS ON STRESS DISTRIBUTION

To indicate the angle from which the arguments on shell strength are to proceed, we shall intersperse a few fundamental statements on the stress distribution in statically indeterminate structures.

A. Safety by Any Chosen Stress Distribution

The designer has to form a structural component. Having decided upon a preliminary shape, he ascertains, more or less arbitrarily, the stress distribution in the almost always statically indeterminate component, after which he settles the dimensions of the particular part in such a way that the permissible stress is at no point exceeded. The designer aims to choose the stress distribution consistent with minimum weight.

In view of this, it is important for the designer to control the possibilities for the stress distribution (the equilibrium conditions). Indeed, these equilibrium conditions form the basis of every statically indeterminate

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*"Einiges über schalenförmige Flugzeug-Bauteile." Luftfahrtforschung, vol. 13, no. 9, September 20, 1936, pp. 281-292.

calculation. Quite often the designer foregoes the latter for the following reasons:

In the part which he designed and for which he established the size, the actually produced strain energy - i.e., the strain energy corresponding to the actually produced force flow - is, according to the theorem of minimum strain energy, less than that corresponding to the chosen stress distribution. In other words, the stresses for which he dimensioned the part have, with relation to the total volume of the part, a higher mean value of the stress square than the actually occurring stresses. Thus, having recourse to his arbitrarily chosen stress distribution, the designer is "on the average" on the safe side.

As the actual strain energy is less than the one corresponding to the chosen stress distribution, overstresses can occur only in zones with small volume. The designer must check his structural component carefully for this contingency.

In particular, he must:

Avoid notched places;

Stiffen manifestly weak members (such as free running members of comparatively small section);

Check for the predetermined external loading as to whether this structural part is also able to carry a somewhat altered stress distribution;

Check the strength of the part by changed external load.

All this he would equally have to do in a statically indeterminate calculation.

B. Low Weight and Stiffness as Parallel Requirements

If the designer has chosen the stress distribution with a view to minimum weight, the structural component is on the average (i.e., in mean value of stress square) stiffer than any other identically highly stressed, but heavier design. For, minimum volume by given mean value of stress square means minimum strain energy and consequently, least work and shortest paths of the given external loads. The aim of the designer at lowest possible stresses for given weight, augments the stiffness of the structural part even more. The skillful design is stiff, and at the same time, light.

II. SHELL PROBLEMS

A. Elementary

<u>1. The flat surface shell</u>.- If a hollow space is completely enclosed by statically determinate plane-braced frames without any member passing through the inside of the hollow space, this space framework is statically determinate and, as a rule, stable. It is, according to A. Föppl, a "trellis structure." Substitution of the surface of the hollow space by walls (flat plates) capable of carrying stresses in its plane only, results in a flat surface shell. This body also resists stresses at the corners.

To compute the shell, it is best to establish - for the same external load - the tension forces in a statically determinate framework enclosing the same hollow space, and then apply the tension established on the framework as load. Figure 1 illustrates this for two bars. This method reduces the three-dimensional problem to a number of two-dimensional - although in general statically indeterminate. problems. (The plates or disks are statically indeterminate, reference 1.) Owing to this remaining statical indeterminateness, such shells may be called quasi statically determinate.

Now the determination of the tension in such a framework (lattice structure) is in more cases than not, quite a task. But the number of reports available on oblong structural components of predominantly square cross section (fuselage, wing box (fig. 2)), assure a comprehensive tension determination (reference 2). These reports refer to partially statically indeterminate and partially statically determinate frameworks. The results are transferable to flat surface shells; in fact, some of them are specifically derived for shells.

The substitution of a shell bounded by curved walls by a framework is in general no longer exactly possible. Even an approximate calculation through establishing the tensions in a substitute framework is quite complicated on account of the large number of truss members and their usually slanting direction.

2. Compression, bending, torsion, and flexural torsion.- On application (fig. 3a), at the ends of a thin-walled prismatic bar (tube or channel), of axial stresses linearly distributed over the section, or on application of shearing stresses distributed conformable to linearly distributed axial stresses (that is, the usual bending), the shearing-stress distribution is constant in all sections throughout the length of the bar. The cross sections of the beam do not become twisted. In this case, there is no small problem. This also holds true for shearing stresses in the ends corresponding to St. Venant's torsion.

Under axial stresses corresponding to a distribution of flexural torsion applied at the ends of an open prismatic section, the resultant force is zero. This is a case of stress in the zone of the section end which, although there is no external torsion moment, causes the end to twist and which, as a result of St. Venant's torsional stiffness, cancels out at some distance from the end section. At every cross section of the profile the moment of St. Venant's shearing stresses (fig. 3c) are inversely proportional to the shearing stresses of the flexural torsion (fig. 3d) caused by the change in the axial stresses. In the event that the wall thickness of the section is small, the cancellation does not take place until after a considerable distance. This also is hardly a shell problem, since the distortions of the section are subordinate. The same holds for the case of external torsion moment at the end of the open section, of the type of the shearing stresses accompanying the flexural torsion (fig. 3e).

B. Real Shell Problems

Shell problems arise (fig. 4) when the cross-sectional form of the shell varies throughout its length; when axial or shearing stresses in other than the discussed distribution are applied at the end cross section or at a median section;* when the shell has openings in a curved part of the surface.

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*Applying at a bulkhead in the median zone, of a prismatic shell, for instance, a transverse force through shearing stresses with the usual distribution according to the flexion theory, the unlike shear flows produce on both sides of the bulkhead the tendency toward unlike cross-sectional warping at either side of the bulkhead. But at the bulkhead itself only a definite warping takes place. The bal-
(Continued on p. 5)

Further problems are offered in the buckling of such shells under compression or shear. Thereby the shells may be of smooth sheet, as exemplified in edged dural sections or sheet-steel wing spars (fig. 5). Shells reinforced by longitudinals and uprights, such as used on bodies or wings, are also included (fig. 6).

Most problems present no fundamental difference between shells with straight walls and those with curved surface.

III. METHODS OF CALCULATION

A. Membrane and Flexion Theories

The statics of shells distinguish two methods of treatment (reference 3): the membrane method and the flexion method. In the first - the so-called membrane theory of the shell - the external loads are taken up only by axial stresses and shearing stresses in the shell surface (median area of shell) (fig. 7). Flexural and torsional stresses variable throughout the wall thickness are disregarded. On the strength of this omission a stress distribution, for instance, is then always possible when the shell corresponds to a lattice structure; i.e., when the curved parts of the shell or straight plates form a complete enclosure around a hollow space (fig. 6), and when no concentrated load within the curved part of the shell area is applied perpendicular to the shell surface or in the shell area. This membrane theory is far from simple in halfway general cases. And when the shell does not enclose the hollow space completely, it fails altogether as, for example, with the conventional lattices which have no end bulkhead, or when the curved part of the shell has an opening or cut-out.

In cases of that kind, the flexural stiffness - and in any case, the torsional stiffness - of the shell metal or that of its stiffeners must be resorted to, and one speaks of "bending theory" of shells. This theory allows

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(Continued from p. 4)
ancing of the cross-sectional warping leads (similar to torque application) to secondary stresses in the region of the bulkhead, the determination of which is a shell problem, although they can in most cases be neglected. This problem is disregarded in the present report.

a simple calculation precisely in complicated load cases and by statically complex design.

B. The Negligible Quantities

The membrane design has achieved some remarkable successes of late in the field of superstructures. But the airplane designer's course is prescribed by predetermined, aerodynamically beneficial forms.

The assumptions to be made for airplane shells must be governed by the outer form of the shell, by its proper structural execution, and its types of loading. Probably all airplane shells vary but little from the prismatic shape. Unless the conditions are unusual, the torsional strength of the sheet and that of the stiffeners can be ignored on such a shell, along with the flexural stiffness of the stringers, thus reducing the problem to the membrane strength of the shell area and the flexural strength of the ring sections (bulkheads).

In an analysis of the membrane strength of the shell area and of the flexural strength of the rings, probably all shells constitute statically indeterminate components; that is, consistently different possibilities exist in such parts for stress distribution. But this very statical indeterminateness makes it possible to have recourse to particularly simple stress distributions in the design and, indeed, in the smooth part of the shell as well as at load-application points and by cut-outs.

The foregoing line of reasoning regarding airplane shells is predicated on the existence of bending-resistant rings. The rings of shell bodies or wings (ribs, partitions, bulkheads, frames) are probably always able to carry the almost always small bending moments without special stiffeners. In sections fabricated from smooth sheet, the section sheet itself takes up the section deflecting stresses.

In support of the omission of flexural stiffness of the longitudinal stiffeners and of the torsional stiffness of the shell, I chiefly depend upon the fact that the effect of these stiffeners in the wrinkling theory, dealing with prismatic shells, recedes in the face of the effect of the flexural stiffness of the ring sections.

C. Sample Problem: Bending of an Airplane Body

The body consists of longitudinal stiffeners, bulkheads (rings), and metal skin (fig. 8). In cross section, the body may be slightly elliptical, as in figure 8, although this particular shape is of no consequence in the following argument. The distribution of the longitudinal stiffeners over the periphery of the body cross section and the cross-sectional area of the individual stiffeners is arbitrary - with the proviso, however, that the cross sections of all longitudinal stiffeners running lengthwise along the body, change proportionately.

The body is clamped at the left end, while at the right a vertically upward transverse load Q is applied. The longitudinal stiffeners of the shell take up the bending moment, the sheet being visualized as being wholly or in part, supporting. The lines of action of the axial loads to be carried by the longitudinal stiffeners run linearly between two bulkheads (rings), and so join the course of the stiffeners that the area enclosed by stiffener and line of action is zero. At the bulkhead itself, every line of action experiences a break.

Figure 8 (bottom) shows the body contour with exaggerated curvature, along with the lines of action of the axial loads for the uppermost stiffener for a section A at the left and for a section B at the right of the particular bulkhead. We extend these lines of action to that of the applied transverse load Q. Then we apply the same procedure to the lowest stiffener of the body and designate the length cut off by the lines of action of the highest and lowest stiffener by h_A for section A, and by h_B for section B, the height of the bulkhead itself being h. It is further assumed that the lines of action of every longitudinal stiffener at either side of the bulkhead, lie on a conical surface. Owing to the break in the line of action at the bulkhead, each section (tension X) deposits a load on the bulkhead, the vertical component of which is V. Assuming linear distribution of bending stress, the total load deposited at the bulkhead is:

$$\Delta Q_x = \Sigma V = Q \frac{h_B - h_A}{h}$$

In the section A directly to the left of the particular bulkhead, load Q is in part taken up by the longitudinal stiffeners (as a result of their sloping toward the body axis); this share Q_{x_A} amounts to

$$Q_{x_A} = Q\left(1 - \frac{h_A}{h}\right)$$

For the other share

$$Q_{S_A} = Q - Q_{x_A} = Q\,\frac{h_A}{h}$$

the transvere load is taken up by the shear in the shell elements. The difference

$$Q_{S_A} - Q_{S_B} = \Delta\, Q_S = Q\,\frac{h_A - h_B}{h}$$

of web transverse loads at the sections A and B to both sides of the bulkhead is lodged at the bulkhead and exactly balances the bulkhead load due to the longitidinal stiffeners. The distribution of these shear loads over the ring periphery corresponds exactly to the shear stress distribution in a prismatic girder of identical cross-sectional form and identical stiffener spacing. The calculation of the stresses in the bulkhead ring explains all other stresses at that point.

This example had recourse to the flexural stiffness of the longitudinal stiffeners only to the extent necessary to transmit the axial loads between the individual bulkheads.

IV. PRACTICAL CALCULATION

A. Resultant of Constant Shear Flow

If a constant shear flow $q = \tau s$ in a straight section I, II (fig. 9) acts through a sheet of constant or variable wall thickness s, the resultant of this shear flow has the direction from I toward II and the quantity $q\,a$ (a = I, II). The shear flow component in any direction is equal to $q\,a'$, a' being the component of length a toward this direction.

The moment of shear flow M_P about any point P is $M_P = q\,2f_p$, where f_p is the area enclosed by both radii vectors PI and PII and the shear flow. The resultant R of the shear flow is at distance 2H from I, II when

H = mean height of the area f enclosed by shear flow and length I, II $\left(H = \frac{f}{a}\right)$.

B. Prismatic Shells with Few Flanges

Figure 10 gives the cross sections of several prismatic shells. They have few individual flanges. The cross-sectional area of the webs is small compared to the flange cross section. It is assumed that the flanges alone take up any created axial loads. Then it follows that in each web bounded by two flanges (in every section of the beam) the shear flow q is constant.

No flange - one web (tube).- Such a cross section can only take up torsional moments.

One flange - one web (tube).- This section can only take up a torsional moment and a longitudinal force in the flange.

Two flanges - one web (channel).- The beam can take up longitudinal forces which lie in the plane of both flanges; it can also take up transverse loads Q_o (fig. 10a) which act parallel to the plane of the flange and which are at distance $2H$ from the plane of the flange. The shear flow created by Q_o is

$$q = \frac{Q_o}{a}$$

a to be measured from center to center of flange.

Two flanges - two webs (tube) (fig. 10b).- Such a structural component can take up axial loads lying in the plane of the two flanges; it can also take up transverse loads Q lying parallel to the plane of the flange and acting at any point.

To establish the shear flow, define first the line of action of the shear flow for each one of the two webs; decompose the external Q_o according to these two lines of action parallel to it, for example:

$$Q_2 = Q_o \frac{d}{2H}$$

which gives the shear flow in the left curved web at

$$q_2 = \frac{Q_2}{a}$$

Three flanges - two webs (channels) (fig. 10c).- This beam can take up any axial load; it can also take up any load (Q) from any direction which passes through the shear center S. The latter is defined as the intersection of the shear resultants of the two webs (at distance 2H from the plane of the flange).

Three flanges - three webs (tube) (fig. 10d).- Six (internal) loads are applied at one sectional plane (cross section) through this structural component: three flange loads and the shear resultants of the three webs. The decomposition of a load according to six lines of action in space being clear, this component can (in distinction to the preceding examples) take up and transmit any external load.

The calculation of the shear flows following an external transverse load Q_0 evolves on the the decomposition of Q_0 according to the three shear resultants Q_1, Q_2, and Q_3. Then shear flow q_1, for example, is:

$$q_1 = \frac{Q_1}{a_1}$$

Four flanges - three webs (channels) (fig. 10e).- This structural component is already statically indeterminate. Assume it to be clamped at one end while unrestricted against warping at the other places. The flange stress due to axial load depends on the type of load application. Stressed under a transverse load Q_0, we decompose this in the cross section at which it is applied into the shear resultants of the three webs, after which each one of the two flanges is treated in regard to its share of Q_0, as if it existed by itself as beam (according to fig. 10a). This affords the flange loads. In the flanges which simultaneously bound two webs, the final flange load appears as the sum of the obtained partial flange loads.

Multi-flanges - multi-webs.- For such statically indeterminate beams, it is usually assumed that the axial stresses in a cross section are linearly distributed; this beam is computed as a conventional beam. If it pertains to open cross-sectional forms, the stresses due to torsional moments are computed conformable to the theory of flexural distortion.

In the two preceding examples, the treatment according to the bending theory (determination of center of gravity, principal axes of inertia, etc.) would be more complicated than the discussed method.

C. Conical Shells with Few Flanges

The discussion may be limited to a few examples.

Two flanges - one web (channel).- The beam (fig. 11a) can take up axial loads lying in the plane of the two flanges and passing through the point of intersection of both flanges. The beam can also take up transverse loads Q_o, which (fig. 11a) act parallel to the plane of the flange,* and which at the section at which they are applied, are at distance $2H$ from the plane of the flange. At the point of application of Q_o the shear flow produced by Q_o amounts to

$$q = \frac{Q_o}{a_o}$$

At another point x part of the external transverse load is taken up by the two flange loads forming an angle; the remainder $Q_o \frac{a_o}{a}$ creates at this point x the shear flow

$$q = Q_o \frac{a_o}{a^2}$$

Three flanges - three webs (tube) (fig. 11b).- To compute the internal loads set up at a section point by an external load P_o, it is advisable to divide P_o into a load S passing through the tip of the cone and into a load Q lying in one of the cross-sectional planes. The load S gives the three flange loads; it does not stress the webs. Load Q is divided in the three shear resultants. The latter change in magnitude inversely proportional to the web heights along the beam.

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*This does not define the direction of the transverse load quite definitely. The cross-sectional plane of the beam in which the load acts could, for instance, be placed perpendicular to the top flange or, say, perpendicular to the bottom flange. According to this uncertainty, the web reveals axial stresses due to the transverse load. However, these are small and insignificant compared to the shearing stresses, so long as the transverse load acts in a plane halfway perpendicular to the course of the beam.

D. Example: Stress in End Bulkhead Due to Applied Transverse Load

On a beam consisting of three flanges and three webs (fig. 12) the end bulkhead is subjected to a transverse load Q_0. What is the stress of the end bulkhead?

Establish the shear resultants Q_1, Q_2, Q_3 in the three webs and the three shear flows q_1, q_2, q_3, say, at $q_1 = \frac{Q_1}{d_1}$. The shear resultants are applied as external loads on the assumedly isolated bulkhead.

To define the panel-point loads, it is assumed that no bending moments are transmitted at the joints (hinges). The shear flows are distributed over the web portions lying between the individual joints; for example:

$$q_1 \; d_1 = q_1 \; e_1 \; +\!\!\longrightarrow q_1 \; e_2$$

As $d_1 = e_1 +\!\!\longrightarrow e_2$, such partial loads of the shear flow always conform to the relative geometrical dimensions of the bulkhead. That is, the geometrical dimensions are treated as loads. Then these partial loads are applied at the joints; for example:

$$q_1 \; e_1 = q_1 \; g_1 \; +\!\!\longrightarrow q_1 \; g_2 \quad \text{(note that } e_1 = g_1 +\!\!\longrightarrow g_2\text{)}$$

This subdivision of itself is simply statically indeterminate; the lines g_1 and g_2 are chosen so as to hug the curved bars as closely as possible. The result is two applied loads at every truss joint. To the extent that these loads arise from the same shear flow, they are readily collected; for instance:

$$q_1 \; h = q_1 \; g_2 \; +\!\!\longrightarrow q_1 \; g_3 \quad \text{(observe that } h = g_2 +\!\!\longrightarrow g_3\text{)}$$

Now the tension is established by substituting the straight line connecting the two joints for the curved bars. Lastly, the bending moments in the curved bars are determined. Member 1 is no truss member, hence possesses no truss tension. On it the two forces $q_1 \; g_1$ and $q_1 \; g_2$ and the shear flow q_1 are applied as external load. The bending moment at a point I is:

$$M_b = q_1 \; (g_1 \; p - 2f)$$

At truss member 2 we apply, apart from the corresponding loads, tension S as external load.

A lower stress in the curved members is obtained when the lines of action of the truss tensions are made to hug the curved members more closely (fig. 13). This brings the hinge joints, which at the same time are intersection points of the truss members, outside of the curved members. This constructive measure in nowise renders the calculation complicated.

E. The Membrane Equilibrium of the Unwarped Shell Element of Finite Size

The designer must be just as conversant with the equilibrium conditions of the shell elements as with the tensions in a truss. Figure 14 shows two views of a developable, very (infinitely) thin shell element of finite dimensions. The straight lines representing in their entirety the developable area, are called "generatrix." The element is bounded at two sides by straight intersection lines in the direction of the generatrix; these sections are called longitudinal sections. The other two, in general curved lines of intersections, are to lie in two parallel planes at distance l. These sections are called ring sections; the sectional planes are called ring or bulkhead planes.

In the following, the equilibrium and stress distributions in such shell elements are analyzed.

Figure 15 is a straight, cylindrical shell element; the ring sections are perpendicular to the generatrices. The three possible stress equilibriums are illustrated at the left; the corresponding load equilibriums which the membrane element is able to take up without becoming deflected, are given at the right. The element can transmit axial loads in the direction of its generatrix; it can take up shearing stresses; and it can transfer an axial load applied at a ring section as shear, to a longitudinal section. The general equilibrium condition presents a superposition of these three individual cases. The one important fact to remember, is to replace in the shear condition (fig. 15, center) the load or stress applied at one of the sections by the three loads holding the equilibrium on the other sections.

Longitudinal stresses applied at the ring section may also vary along the section. Be it particularly noted that a shell cannot take up a concentrated axial load acting in the plane of the shell, unless a member is specifically provided for taking up the entire axial load.*

Figure 17 depicts an oblique cylindrical shell element. The two ring sections are no longer perpendicular to the generatrix. The general membrane-stress condition is preferably divided in pure tension (fig. 17, left), in a "shear attitude " (fig. 17, center), and in the change from axial load to shear (fig. 17, right).

In "shear attitude" the applied shear flow q is the same along all sectional planes. This stress condition is no "pure shear" in the sense of the strength theory, but rather a superposition of pure shear (with shear flow q) and pure tension in direction of the generatrix (fig. 18). Since each one of these two stress conditions is a possible membrane-stress condition of the shell, the "shear attitude" q on the oblique cylinder itself is a membrane-stress condition.

Be it also noted that the resultant of the shear flow applied at section a has perpendicular to the plane of the ring the component $L = q \cdot l$.

Figure 19 depicts as the most general case of a non-twisted shell a conical shell element formed of straight generatrices.** By "shear condition" the shear flow is variable along the generatrix. With L again denoting the component of the shear load Q applied at a generatrix perpendicular to the plane of the wing, the following relations hold:***

$$\frac{L}{l} = \frac{Q}{d} = \frac{Q_a}{b} = \frac{Q_b}{a} = q_a \frac{a}{b} = q_b \frac{b}{a} \quad \text{(fig. 19)}$$

The shell designer must become conversant with this equilibrium.

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*But a concentrated axial load may be applied in a shell when this load acts somewhat within the shell area (fig. 16). Then the particular ring must be provided with a bending-resistant plate relative to loads perpendicular to the plane of the ring, which distributes this concentrated load over a number of longitudinal stiffeners of the shell.

**A conical shell element with any base is, apart from special cases, always oblique.

***This relation is closely related to those of the "Tension Method" in Zeitschrift für Flugtechnik und Motorluftschifahrt, vol. 18, 1928.

F. The Twisted Shell Element

Figure 20 shows a square surface element with side length 1. The surface is twisted, and the amount of twist expressed in angle α. Loading this surface element in shear flow q, the four shear loads q applied at the sides are not in equilibrium, but rather have a resultant $2q\alpha$ in direction perpendicular to the surface. If this surface element is crossed by a stiffening section, this resultant $2q\alpha$ loads the stiffener in bending (transverse load perpendicular to sheet surface).

Figure 21 is a small isolated shell element lying between two parallel bulkheads (distance l). As the curvature of the element (in ring direction) is of no moment in this argument, the element is shown as straight. The element has a stiffener; the lateral edges of the element are to run midway between this and the adjacent stiffener.

The element is stressed in "shear," with L as shear component perpendicular to the ring plane applied at the lateral edges. Now it can be readily proved that equilibrium exists between all loads acting on the element, when every load applied on a ring section is in direction of the other ring section and possesses the magnitude

$$\frac{L}{l}\,b \quad \text{and} \quad \frac{L}{l}\,a \quad \text{(given in fig. 21)}$$

This equilibrium is now divided into two parts. On the left equilibrium all loads act at the edges in edge direction, at the same time as a load p acts perpendicular to the sheet surface. This equilibrium is not accompanied by bending moments in the sheet (in longitudinal direction), when p has at every point the magnitude

$$p = 2\,\frac{L}{l}\,\frac{\alpha}{l}\,\frac{a^2\,b^2}{c^3}$$

Here α is the fairly small assumed angle between both ring sections of the element (fig. 21, plan view), c is the width of the element at the particular point for p.

The second equilibrium comprises load p and the two opposing loads

$$\frac{L}{l}\,b\,\alpha \quad \text{and} \quad \frac{L}{l}\,a\,\alpha$$

which at each bulkhead section are proportional to the length of the other bulkhead section, and which are to be deposited on the bulkhead. This equilibrium bends the stiffener. The distributed load p has the resultant

$$\frac{L}{l}\,\alpha\,(a + b)$$

As concerns the calculation of the bending moment, p may be assumed as evenly distributed without appreciable error.

This bending moment in the longitudinal stiffener is as a rule small. It compares, for example, with the loading of stiffeners in a curved and untwisted tension field. The deflection of the tension in the tension field on the stiffener causes a distributed loading directed perpendicular to the shell surface and amounting to about $\beta\,\frac{L}{l}$, β being the angle formed by two stiffeners (fig. 22). Suppose, for example, that the length l of the shell element is twice the mean distance $\frac{a + b}{2}$ of the stiffeners; then the bending moment in the stiffener due to the twisting is only as great as the moment in a tension field when the angle of twist α is of magnitude β.

The loads in the twisted shell transmitted at the bulkheads perpendicular to the shell surface

$$\frac{L}{l}\,b\,\alpha \quad \text{and} \quad \frac{L}{l}\,a\,\alpha$$

are additive to the loads deposited by the longitudinally adjacent shell elements, when the twisting of the shell is uniform longitudinally. The then ensuing bending moments in the bulkhead may reach appreciable values.

G. Shell Bounded by Two Parallel, Flexurally Stiff Bulkheads

1. The cylindrical shell (sample problems).- The following simple problem is of importance for the shell calculation. Two curved bars bending-resistant in plane are joined by a cylindrical web plate. The spacing of the rings is l. This so fashioned cylindrical piece is now fixed at one end (at a longitudinal section) and a load applied at the other end (fig. 23). St. Venant's torsional stiffness is disregarded.

The membrane equilibrium of the web under a transverse load Q_o is given at the right-hand side of figure 23. Between web and both edge stiffeners a shear flow

$$q = \frac{Q_o}{l}$$

is transmitted, causing the edge stiffeners to be stressed in their plane in tension, transverse load and bending. Tension and transverse load have at a section I in quantity and direction, given by a_I, the resultant*

$$R_I = a_I \, q$$

These two loads R_I in conjunction with the transverse load $Q = Q_o$ transmitted by the web, hold the external Q_o in equilibrium.

Further, the shear flow q creates in each ring a bending moment M_i of magnitude

$$M_i = q \, 2f \quad \text{(fig. 23, right side)}$$

This moment M_i is inversely equivalent in the two edge stiffeners - that is, gives no resultant. This moment M_i cannot be simply computed from the equilibrium with the external loading; rather, it is dependent on the course of the beam between the applied external load Q_o and the particular section point I. The stresses created by M_i are called flexural distortion stresses.

If, in addition, torsional and bending moments are applied at the end of the cylindrical piece, the resulting total moment at the cross section is divided into two inversely equivalent loads applied at the top and bottom edge stiffener, after which the bending moments M_i in

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*This shear flow is assumed constant over the beam height. On a flat beam this premise holds only when the cross-sectional area of the web is small relative to the cross-sectional area of both flanges. In our curved beam the shear flow is constant over the beam height so long as the flexural stiffness of the web is small compared to that of the bulkhead rings. In this case no axial loads can occur in the web in ring direction, regardless of the size of the web area compared to the cross-sectional area of the rings.

the edge stiffeners due to this external load can be readily obtained.

If this moment loading M_o (fig. 23, bottom) is applied, together with a transverse load Q_o, the total resultant is formed; this consists of a transverse load Q_{res} of magnitude Q_o applied at a certain point. Then the bending moment M_i in the edge stiffeners at a point I (fig. 23, bottom) is:

$$M_i = \frac{Q_o}{l} 2f = q \ 2f$$

The area f is enclosed by the radii vectors plotted from Q_{res} to the initial cross section; i.e., to the section at which Q_{res} is applied and to point I and to the cylindrical piece (in plan form).

Example: Application of Axial Loads in Cylindrical Shells by Means of Two Bulkheads

A bending moment is to be introduced at the end of the shell in an airplane fuselage. The bending moment is applied in the form of four concentrated loads (fig. 24). In this case it is expedient to fit a bending-resistant ring (bulkhead) at the shell end itself and one each a certain distance away from the end of the shell.

With a view to the determination of the stress distribution, this piece of the shell is isolated from the rest and the four concentrated loads applied at the left end of the piece and the linearly distributed bending stress at the right end. From this the load Q transmitted by each cross section c of the shell and consequently, the shear flow $q = Q/l$ can be determined. This shear flow q is shown at the lower left of figure 24 over the rolling up from I to II. The smooth curve is the result of the distributed loads applied at the right bulkhead. At the point of application of a load L the entire shear flow undergoes a break to the amount of L/l.

Now the shear loads q exerted by the shell on the rings are known (lower right of fig. 24) and the ring stresses become calculable. They are, in general, not very great; they become so much smaller as the bulkhead spacing is chosen wider. But the longitudinal stiffeners which introduce the concentrated loads in the shell must in any case go from one bulkhead to the other; their bulk may decrease the starting distance from the end cross section.

If the fuselage space permits the use of two trussed bulkheads, they themselves probably will be capable of carrying the loads set up by the shear flow q, so that a more exact calculation of these stresses may be foregone, and the work confined to a check on the shearing strength of the shell plate.

Example: Cylindrical Shell with Cut-out

The stress in the cut-out zone can be treated in similar fashion. To illustrate: Suppose it concerns a torsionally stressed fuselage with a cockpit opening or cut-out. It is expedient to install four rings, one each at either end of the cut-out, and one each at a certain distance from it. In addition, two longitudinal flanges are fitted - one at either side of the cut-out - the flanges extending as far as the outside rings (fig. 25, top).

Isolate the part directly outside of the two outer rings and apply the shear flow q_o of the torsion moment at both ends. Then divide this equilibrium of the external loads by applying the shear flow q_o at the cut-out into one which produces everywhere constant shear flow and in a second, the actual disturbance due to the cut-out.

The stress due to the latter forces is now analyzed. Make a cut m through the center of the shell. This cut has to transmit a transverse load $q_o a$ lying in the plane of the cut (a = width of cut).

There are three webs between the four longitudinal stiffeners; we determine the three shear flows q_o, q_1, q_2 in these webs which to the resultant have the cited load $q_o a$. Next isolate the two rings to the right of the cut along with the intermediate shell piece and plot it twice: once, giving in the ring area the applied shear flows q_o, q_1, q_2 which balance each other and merely stress the ring adjacent to the cut-out; and a second time, showing the axial loads $(q_o + q_1) l_{mc}$ and $(q_1 + q_2) l_{mc}$. The calculation of the shear flows and ring stresses in this antisymmetrically loaded shell piece cd presents no difficulties.

2. The conical shell (with sample problem).- A shell having the shape of a truncated cone is bounded by two bending-resistant bulkheads; the latter lie in two parallel planes at distance l. The structural component is fixed at one generatrix and loaded at the other end by a transverse load Q_o in the direction of the generatrix (fig. 26). L is the component of Q_o perpendicular to the bulkhead plane.

Through shear flows the web plate is set in equilibrium* (fig. 26):

$$q_a = \frac{L}{l}\frac{b}{a} \qquad q_b = \frac{L}{L}\frac{a}{b}$$

We further obtain the flange loads:

$$G_a = \frac{L}{l}\, b \qquad G_b = \frac{L}{l}\, a$$

acting on a cut I (say, at the point of fixation) and preserving the equilibrium of the external load Q_o. The bending moments M_i in the two flanges are inversely equivalent:

$$M_i = q_a\, 2f_a = \frac{L}{l}\frac{b}{a}\, 2f_a = q_b\, 2f_b = \frac{L}{l}\frac{a}{b}\, 2f_b$$

Example: Cockpit Cut-out in Fuselage

We repeat the example of the cylindrical fuselage without, however, touching upon the fundamental problems. The contour of the fuselage is arbitrary, but all bulkheads must be geometrically similar.

We immediately start with the partial loading corresponding to the disturbance (fig. 27, top). The longitudinal forces in the flanges are assumed to be zero at a cut m, and the pieces of the fuselage between m and the end bulkheads a and d as being conical.

The shear flow at the rear edge of the cut corresponding to the torsion moment, is assumedly given $q_o = \frac{Q}{c}$. We forthwith determine the shear loads applied in cut c (fig. 25, bottom). From the moment equilibrium of the

---

* a/b is the ratio of any two mutually corresponding linear dimensions of the two rings.

shear flows about point P follows:

$$q_0 \, c \, f = q_2 \, c_2 \, (e + f)$$

Furthermore, the sum of the horizontal components of the transverse loads gives:

$$q_1 \, (c_2 - c) = q_0 \, c - q_2 \, c_2$$

These shear flows q_0, q_1, q_2 only load the bulkhead ring lying in cut c.

Next we determine the axial loads to be transmitted in cut c, proceeding immediately to defining the component L perpendicular to the bulkhead plane. The basic relation is (fig. 19):

$$\frac{L}{l} = q_a \frac{a}{b}, \quad \text{that is,} \quad L = l \, q_a \frac{a}{b}$$

It affords the two loads:

$$L_I = l_{mc} \, \frac{c}{m} \, (q_0 + q_1)$$

$$L_{II} = l_{mc} \, \frac{c}{m} \, (q_1 + q_2)$$

Here c/m denotes the ratio of the linear cross-sectional dimensions of cut c and m. The four axial loads L_I and L_{II} load the shell fraction cd antisymmetrically. The calculation of the shear flows and ring stresses offers no difficulty.

H. Cylindrical Sheet Metal Wall Bounded by Two Parallel Ring Cuts

Consider a straight cylindrical shell of length l, the shell consisting of sheet metal of wall thickness s; the free end of this shell being loaded by a transverse load Q_0 and a bending moment M_0, whose resultant is a transverse load Q_{res} (fig. 28); the other end of the shell is fixed.

Following the loading, the shell piece develops bending stresses whose moment balances the moment of the external loads, while in addition, the torsion produces variable flexural distortion stresses σ_{bd} over the wall thickness, which in every cross section I reaches its peak

$$\sigma_{bd} = \pm \frac{Q_{res}\,2f}{\frac{s^2\,l^2}{36}}$$

in the four corners of the cross section. (f is the shaded area in fig. 28.) St. Venant's torsional stiffness is ignored. This formula is readily derived according to the data given in IV G 1.

If the sheet is regularly fitted with individual stiffeners in ring direction (fig. 29), it is:

$$\sigma_{bd} = \frac{Q_{res}\,2f}{\frac{l^2}{6}\,W_\varphi}$$

whereby $W_\varphi = W/t_\varphi$, W being the section modulus of a ring section and t_φ the spacing of the sections. Stiffeners running longitudinally have no influence on this problem.

This result is equally important for the calculation of sleeves or sockets (as, for instance, on landing-gear strut fittings) and on rings which form a frame around a cut-out in the covering of an airplane.

Mr. Tintea (in his thesis, Berlin, 1934) has extended these results to include circular cylinders with consideration of St. Venant's torsional stiffness.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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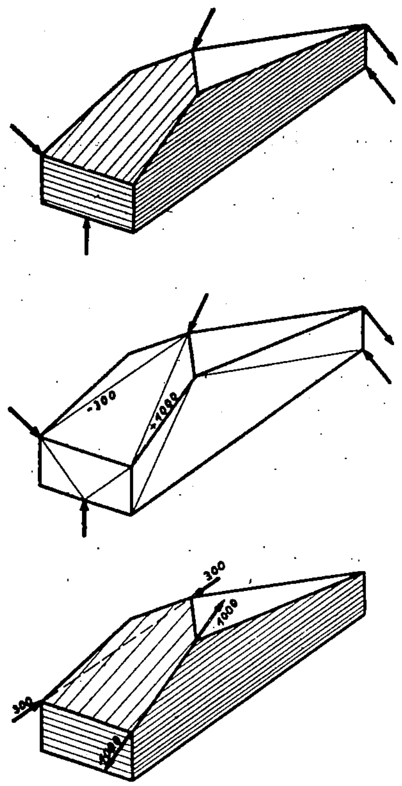


Figure 1.- For computing plane surface shells.

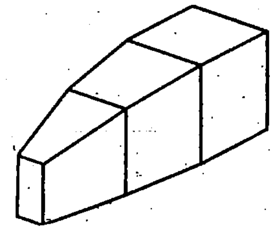


Figure 2.- Box-shaped airplane body .

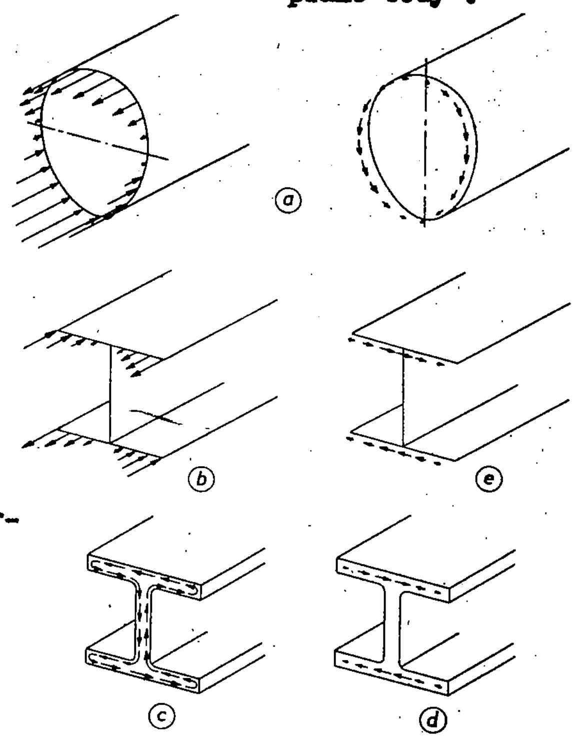


Figure 3.- Simple load cases.

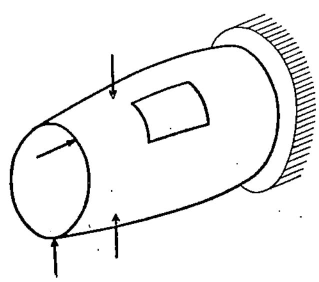


Figure 4.- Advanced shell problem.

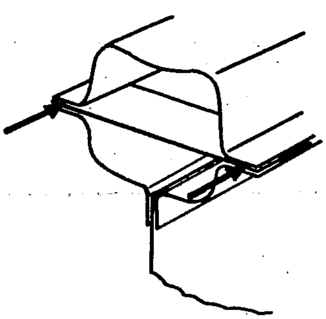


Figure 5.- Wing spar.

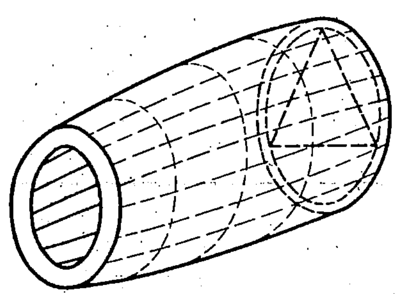


Figure 6.- Shell body.

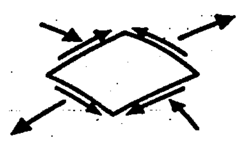


Figure 7.- Membrane equilibrium.

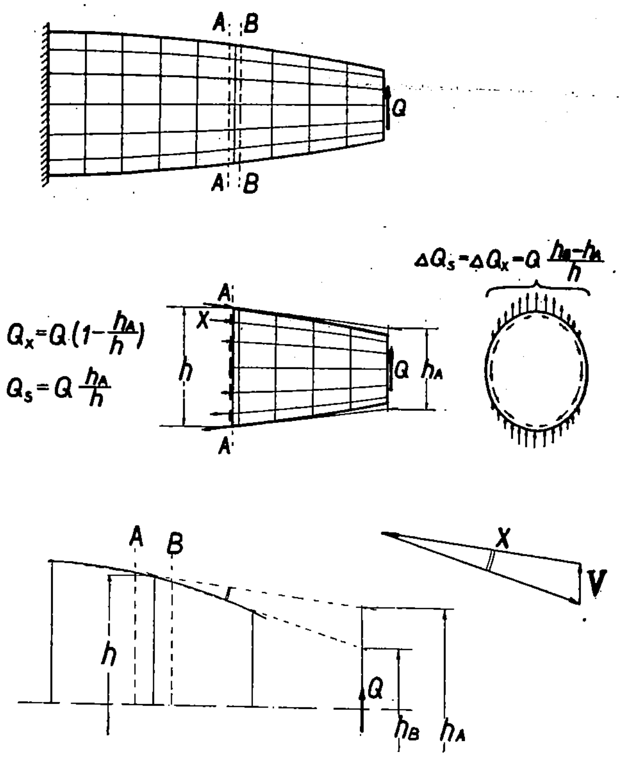
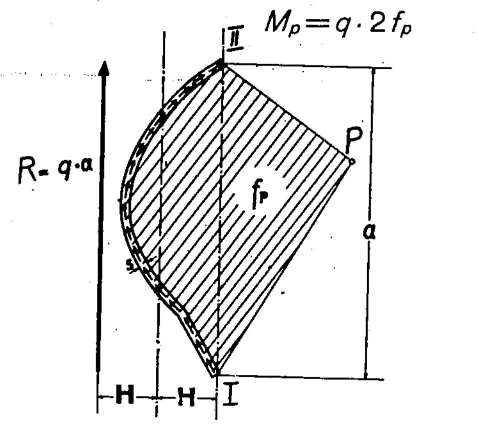


Figure 8.- Bending in a shell body.



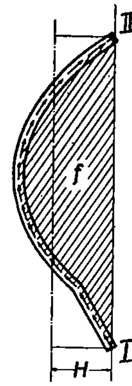
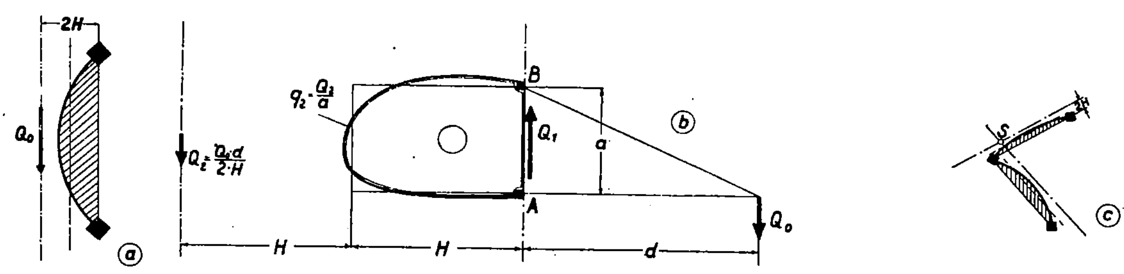


Figure 9.- The resultant of a constant shear flow.



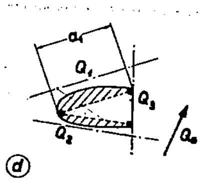
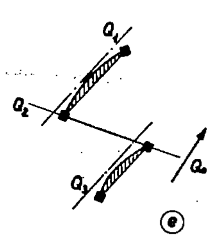


Figure 10.- Prismatic beam having few flanges.



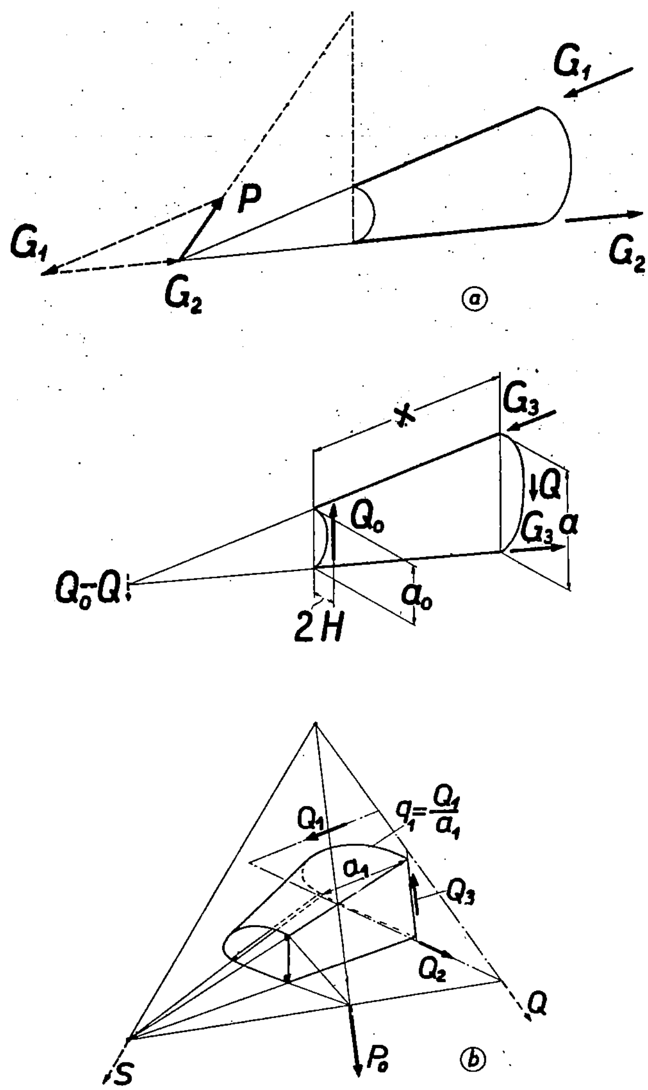


Figure 11.- Conical shell having few flanges.

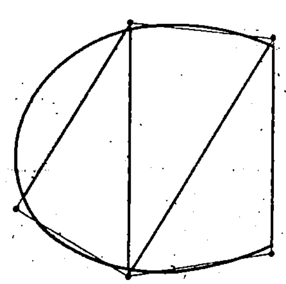


Figure 13.- Improved end bulkhead design.

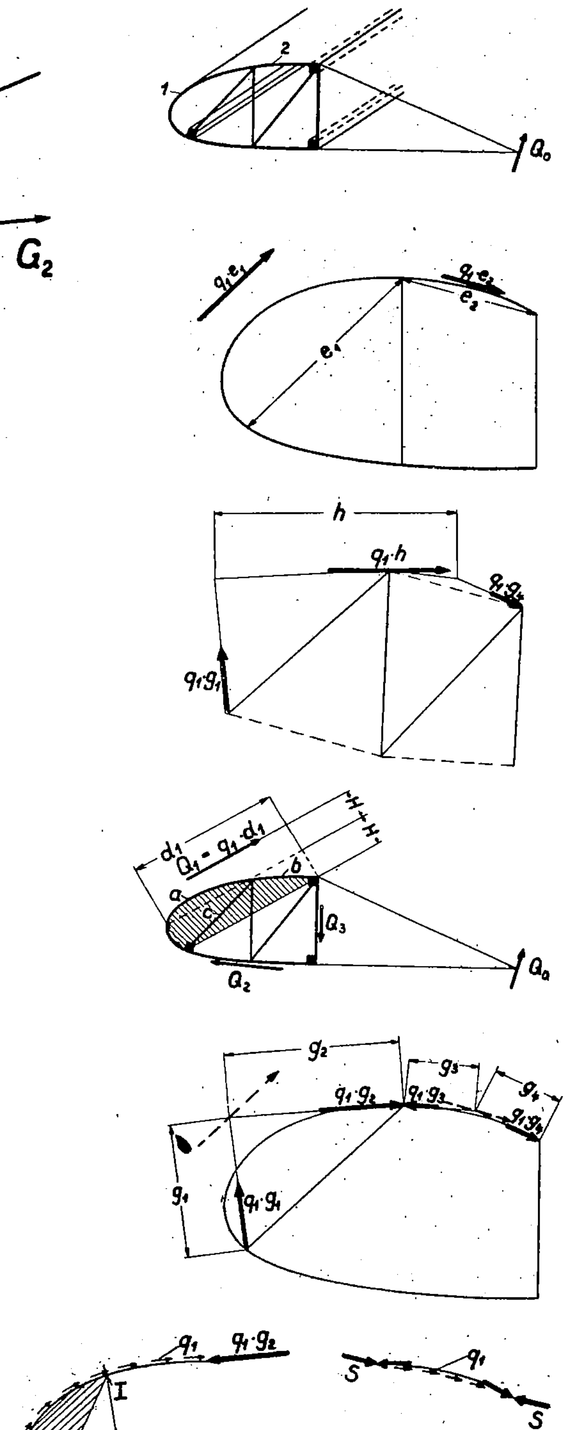
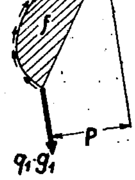



Figure 12.- Stress in end bulkhead due to applied transverse load.

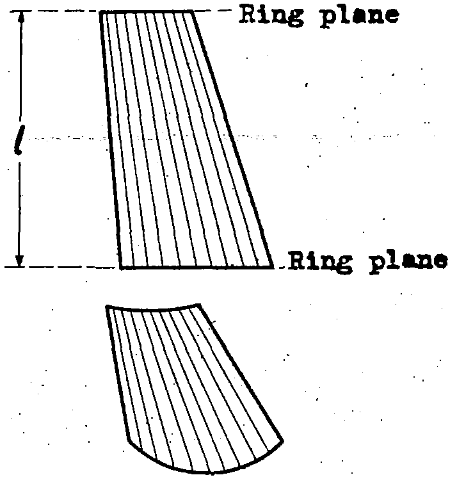


Figure 14.- Shell element between parallel ring planes.

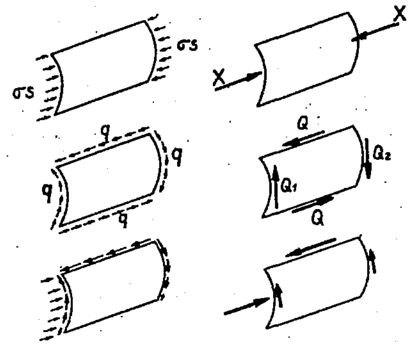


Figure 15.- Membrane equilibrium of untwisted shell element.

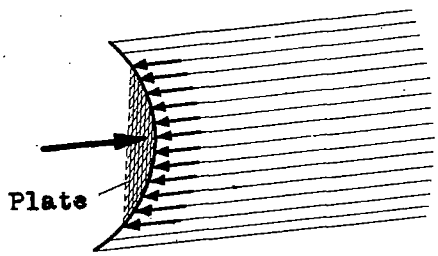


Figure 16.- Application of an axial load.

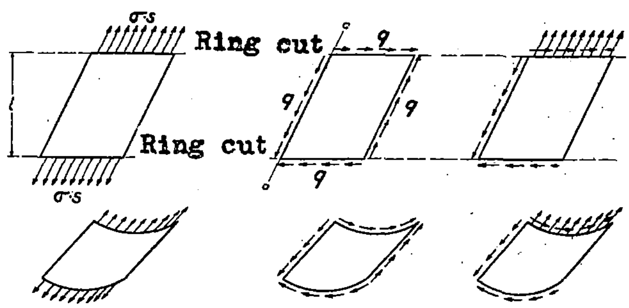


Figure 17.- Oblique cylindrical shell element.

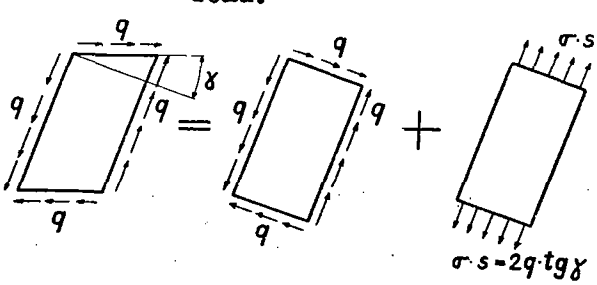
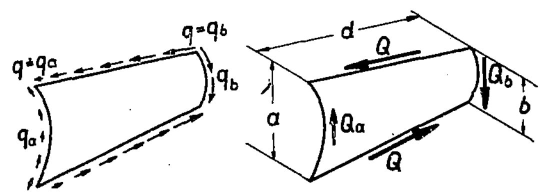


Figure 18.- The "shear attitude".



$$q_a \cdot a^2 = q_b \cdot b^2$$

Figure 19.- The "shear attitude" in the conical shell element.

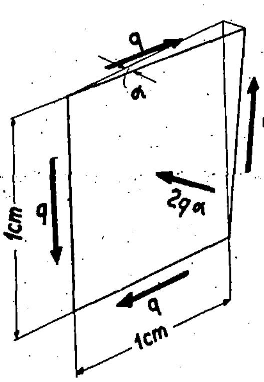


Figure 20.- Twisted shell element under shear.

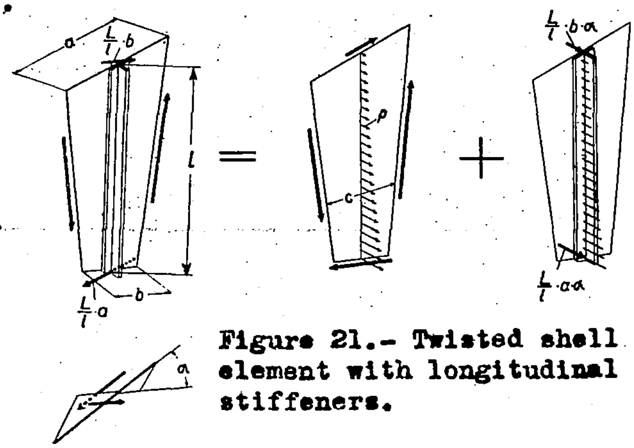


Figure 21.- Twisted shell element with longitudinal stiffeners.

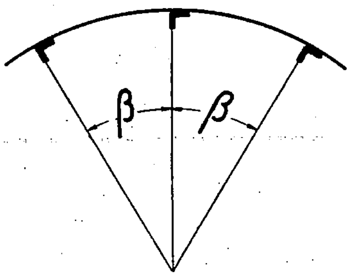


Figure 22.- The comparative curved tension field.

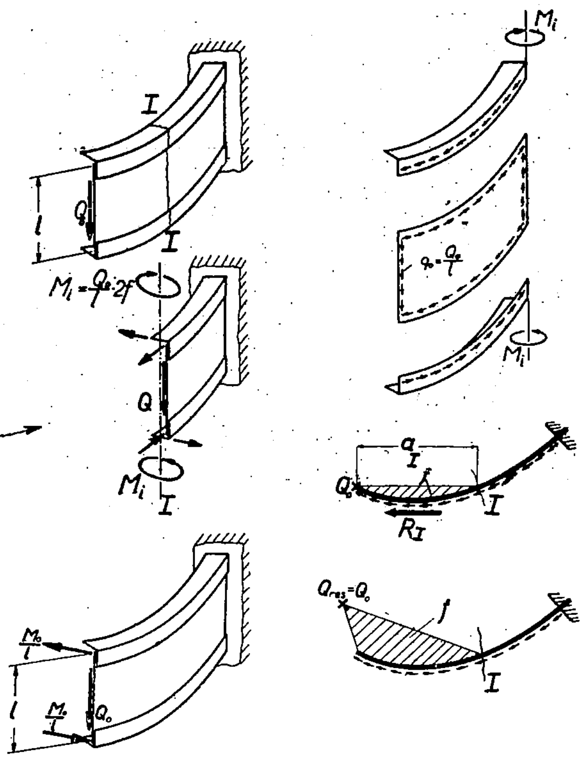
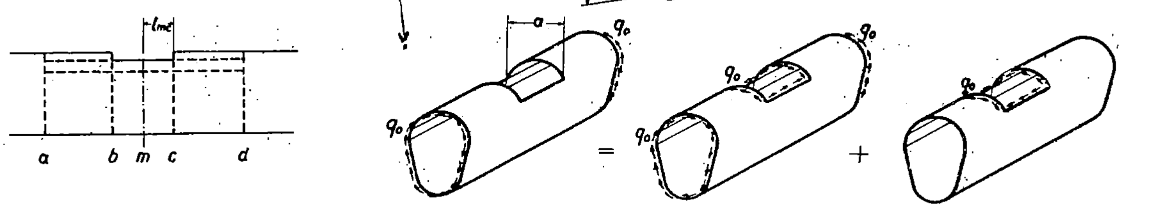
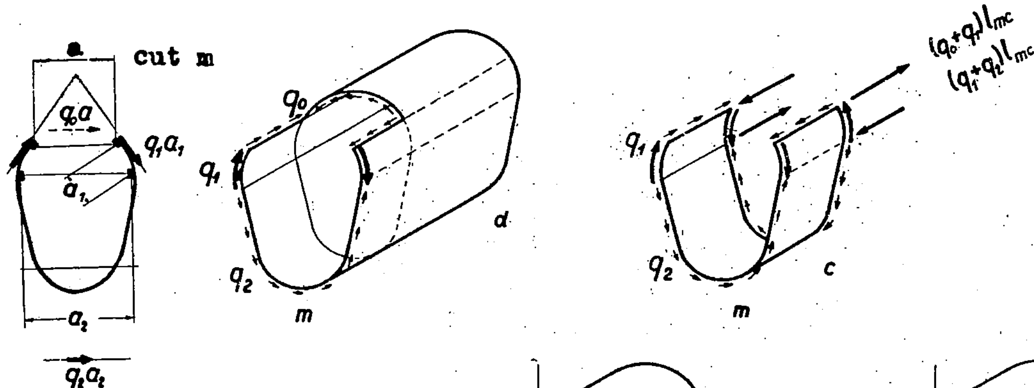


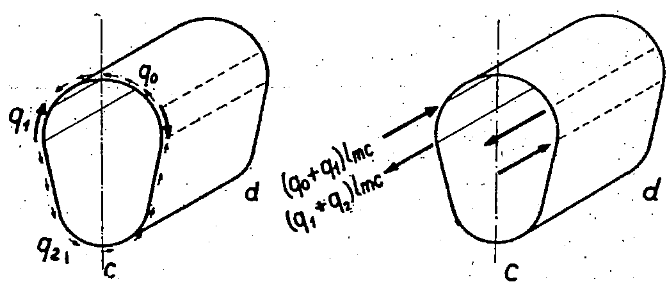
Figure 23.- Stress in cylindrical shell between two rings.

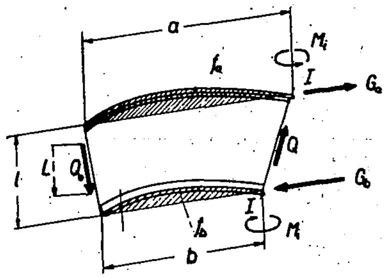
Figure 25.- Cutout in torsionally stressed cylindrical shell.





Load q_1 a_1 is not plotted accurately; it should run parallel to the transverse line connecting the flanges, as readily seen in fig. 27.





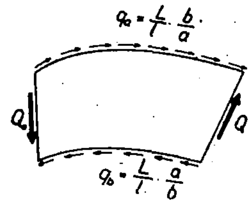
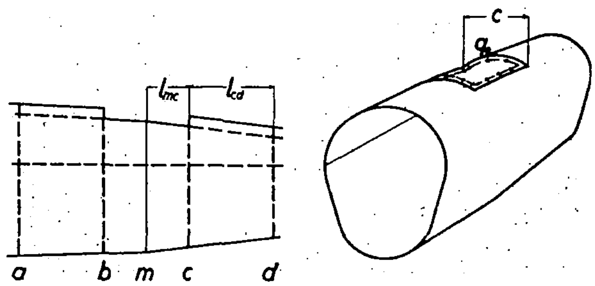


Figure 26.- Conical shell between two rings.



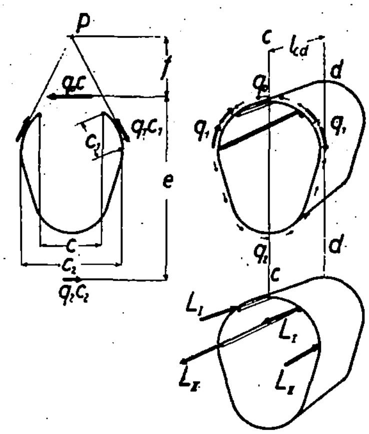


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Figure 27.- Cockpit cut-out in fuselage.

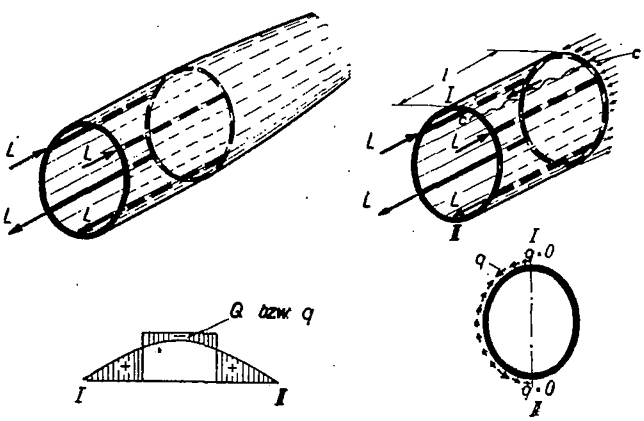

Figure 24.- Introduction of axial loads through two bulkheads.

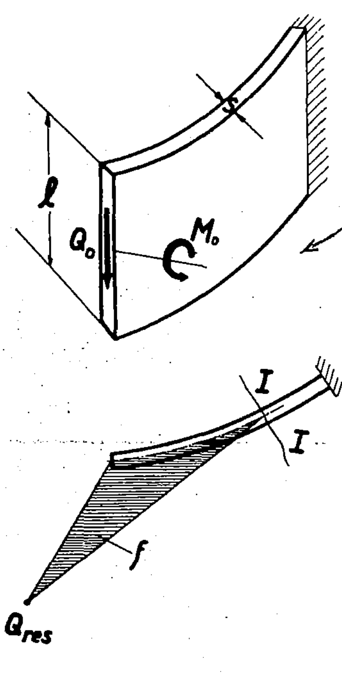

Figure 28.- Stress in unstiffened cylindrical shell

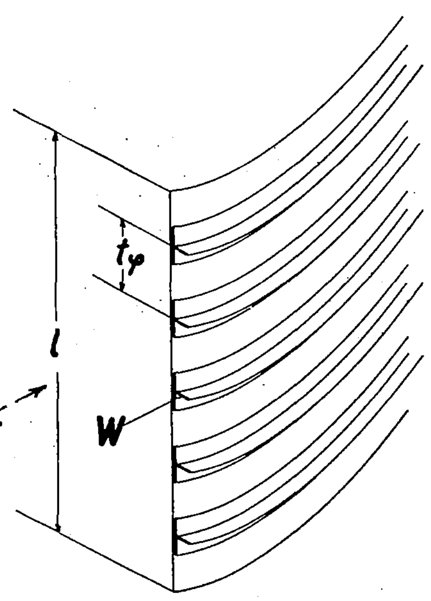

Figure 29.- Cylindrical shell stiffened in ring direction.