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RECENT RESULTS OF TURBULENCE RESEARCH

By L. Prandtl

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INTRODUCTION

The irregular motions, called turbulence, play a prominent part in all technically important flow phenomena. Turbulence, on the one hand, is the cause of undesirable flow resistance, while, on the other hand, it has the very useful characteristic of increasing the pressure in the currents. The control of these phenomena is very important for the flow specialist. Numerous researches have therefore been recently undertaken for the purpose of discovering the laws of turbulent flow. In the present article an attempt is made to review the most important results of these researches. Relations of immediate practical interest are discussed.

The first two sections treat of two prominent questions, namely the origin of turbulence and the characteristics of turbulent currents. In the third section conclusions are drawn for the flow along a rough wall, whereby an important relation for the velocity distribution is revealed. The principles are also applied to straight rough and smooth tubes. Here it was possible to develop formulas for flow velocity and resistance, which show excellent agreement with the experiments, and which also in contrast with previous purely empirical formulas, hold good for very large Reynolds Numbers for which no experimental data are available. The peculiarities in tubes with fine-grained roughness at moderate Reynolds Numbers are represented by a single curve. Test results with artificially roughened tubes are given and confirm the relationship mentioned.

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[1] "Neuere Ergebnisse der Turbulenzforschung." Zeitschrift des Vereines deutscher Ingenieure, vol. 7, no. 5, February 4, 1933, pp. 105-114.

The results obtained with tubes are applied to the resistance of plates to a longitudinal flow. Moreover, the characteristics of the flow in wide and narrow and curved channels, as likewise the mixture phenomena of fluid currents with surrounding fluids and also the phenomena behind moving bodies are considered. Lastly, newly discovered relations between the turbulent exchange of velocity and heat are considered, and new conclusions are drawn regarding the finer details of turbulent flow.

During the last decade the investigation of the irregular mixing motions, which are called turbulence and which affect all technically important flows, have been especially thorough and fruitful. These mixing motions produce effects, as if the viscosity of the fluid were increased a hundred or ten thousand fold or even more. This circumstance causes the great resistance of fluids in pipes, the frictional resistance of ships and airships and other undesirable resistances, but also the possibility of increased pressure in diffusers or along airplane wings and blower vanes. Without turbulence, separation would occur in these cases, so that there would be only a small recovery of energy in the diffuser and impaired efficiency of wings or vanes.

The investigation consisted of a determination of the numerical data and their systematic arrangement. Generally the investigation was not carried to an actual theory (which is very difficult), but the results help to support theoretical conclusions. Often dimensionsal considerations together with intuitive insight lead to important conclusions. If, e.g. density (i.e. inertia) and viscosity are the only determinative properties of the fluid for the phenomenon, one is led to a Reynods Number = density/viscosity × velocity × length ($Re = vl/\nu$, in which ν is the "kinematic viscosity", i.e. viscosity/density). If Reynolds Number has the same numerical value in two cases, we may expect exactly the same course in both cases, only with a different length and time scale according to circumstances. In individual cases the application of this rule may, however, require consideration as to which velocity and which length is actually determinative for the process.

There are two main questions which were investigated theoretically and experimentally:

1. How and under what immediate conditions does turbulence originate?

2. What can be said regarding turbulent motion, particularly regarding the mean values of the velocities and forces?

The second question is obviously the more important one from the technical viewpoint.

ORIGIN OF TURBULENCE

Regarding the first question I can be quite brief, both because I have recently expressed myself on this subject (reference 5) and because there is here much that is still in doubt. The most important fact is that turbulence always occurs when the velocity profile shows a turning point (fig. 1) and when the viscosity effects are not too great. Any flow with such a velocity profile is unstable in the absence of fluid friction, i. e. small deviations in magnitude and direction increase of themselves and cause a complete reversal of the flow. An originally slight wave in the streamlines leads gradually to the production of turbulence through the toppling over of the waves. These phenomena can be delayed by strong viscosity effects.

This indicates that the tendency to become tubulent depends on the magnitude of the Reynolds Number. Velocity profiles with turning point occur, e.g. in the boundary layers produced by viscosity effects, when the pressure increases in the direction of flow or, in other words, when the flow is retarded. Such points in the fluid therefore have a strong tendency to become turbulent, but even the unaccelerated rectilinear flow along a wall tends to become turbulent at a sufficiently large Reynolds Number. This can be explained by the fact that the inflow is never absolutely undisturbed and that there are always some irregularities in the velocity distribution. Unstable velocity distribution is largely due to only slightly damped turning motions with axes parallel to the direction of flow. Such turning motions direct some portions of the fluid against the wall and other portions away from it, so that, even at low velocity, with the lapse of time, portions having a lower velocity become interspersed with portions having a higher velocity, thus necessarily producing instability.

There is still another cause of turbulence, which was discovered in a theoretical manner (references 2, 3, and 4) and which call for special consideration when there are none of the above-mentioned disturbances. In the flow along a wall, there occur certain slow distrubances which above a certain critical Reynolds Number, increase in strength and thus produce in their retarded zones, after their amplitudes have become great enough, the preliminary condition for turbulence. It is worthy of note that the critical Reynolds Numbers for two different cases, as determined theoretically by Tollmien (reference 4) and Schlichting (reference 6), are in good agreement with the experimental values.

Experiments on the Production of Turbulence

In order to obtain more light on this question, we investigated the production of turbulence by experiments in channels 20 cm (7.87 in.) wide and 6 m (19.68 ft.) long. Though we proceeded with great care, we found it impossible to eliminate all the disturbances, so that here and there nuclei of turbulent motion developed in irregular succession and spread quite rapidly.

Clearer pictures were obtained by purposely initiating a disturbance in the flow, as, e.g. by adding or removing a little water through a small piece of screening inserted in the wall. In the first case, when a small amount of water, not yet participating in the flow, is thrust between the wall and the moving mass, instability is immediately produced and turbulence develops at the point of entrance. The amount of water introduced may be very small. In the second case the greatest disturbance occurred in the portion of the flowing water opposite the screen at the beginning of the removal by suction. Behind this point the thickness of the boundary layer was reduced by the suction, and the inner portion of the water flowing past the boundary layer had therefore to flow over a sort of step from the thinner boundary layer to the thicker layer on the downstream side of the screen. This created enough of a disturbance to cause the disintegration of the boundary layer in a short time. Figure 2 shows this effect and the further development of the turbulent region.

The flow was rendered visible by scattering aluminum dust on the surface of the water. A slowly operating motion-picture camera was mounted on a car which kept pace with the flow so that the same group of vortices remained in the field of the camera. In the top picture the oblique streamlinesat the left show the location of the suction point, while the formation of the first vortex in the middle indicates the location of the "step." Other vortices developed on the upstream side. In the last picture the original vortex is shown at the extreme right. It is evident that it carried water from the boundary layer (which was purposely strewn more thickly with aluminum dust) far into the interior of the flow.

CHARACTERISTICS OF TURBULENT CURRENTS

We will now consider the laws of fully developed turbulence. The method of presentation which I shall employ, does not follow the historical development, but is intended to show the present status all the more palinly. I shall begin with a statement, regarding the behavior of an ideal fluid without viscosity. In reality there is no such fluid, but it is of advantage for many considerations to know what would occur in such an ideal fluid, because the laws of the ideal fluid (due to the absence of viscosity) are simpler than those of an actual fluid.

According to our previous statements the tendency to creases or, in other words, as the viscosity decreases (under otherwise like conditions). At the zero limit of viscosity the Reynolds Number obviously becomes infinite, necessitating the conclusion that the flow of an ideal fluid would generally be turbulent. If it is also assumed that the bodies or walls, past which the fluid flows, are mathematically smooth, the surface friction would also be zero and we would thus obtain the theoretical behavior of the ideal fluid, as stated in old textbooks on hydrodynamics. If however, the surfaces are rough, it may be assumed that an area of separation develops at each individual point of roughness, however slight.[2] The flow thus acquires

[2]In slightly viscous fluids, a regular separation of the flow occurs on projecting parts of the wall. In the limiting transition to vanishing viscosity, Helmholtz separation surfaces are developed with finite velocity jumps.

a turbulent character from the mutual effect of the various small areas of separation which are unstable in themselves and have a disturbing effect on one another. At each rough spot a pressure difference develops between its upstream and downstream sides, thus producing a resistance which is proportional to the square of the velocity.

From this consideration it may be assumed that it is permissible to make theoretical assumptions regarding the laws of turbulence, in which the viscosity of the fluid is put at zero. The following considerations clearly show that we are thus on the right track and that, as a matter of fact, the turbulent resistance in the interior of the flow is practically independent of the viscosity. In a thin layer near the wall, however, the effect of the viscosity persists, provided it is not concealed by the effect of great roughness.

We will briefly explain a conception which has been found useful for the more accurate investigation of the turbulent mixing processes. This is the so-called "mixing path," which plays a similar role in turbulent mixing processes to that played by the mean free path in the molecular diffusion of gases. In both these precesses shearing stresses (or apparent shearing stresses) are developed by the continuous interchange of energy between fluid layers flowing parallel to one another at different velocities. The following simplified representation can be made of these really quite complex processes.

It is assumed that any particle, which, by collision with neighboring particles, acquires a motion crosswise to the flow, has, in the direction of flow, the mean momentum of the layer from which it came, and that it now traverses a distance l crosswise to the flow, before it collides with other particles or mingles with them. Such exchanges occur in both directions, and thus the faster layer receives particles from the slower layer, which naturally retard the former, and, conversely, the slower layer receives particles from the faster layer with an accelerating effect on the former.

The effect of the two fluid layers on each other is therefore the same as if there were friction between them. The difference between the molecular processes and the turbulent processes is due only to the fact that, in one case, the individual molecules, and, in the other case, whole groups of molecules participate in the exchange. If u is the velocity of the flow and y the coordinate in the direction at right angles to the flow in which the change in velocity occurs, the difference between the velocities of the two layers, separated by the distance l, is $l\,(du/dy)$. This, according to what precedes, is also the velocity difference of a particle which, coming from the other layer, mingles anew with its present environment.

In order to determine the magnitude of the frictional force or, more accurately stated, the shearing stress between the two layers, we must know the magnitude of the mass exchanged per second. This, as referred to the unit area, can be expressed by the product of the density $\rho (=\gamma/g)$ and an exchange velocity v'. In the case of the molecular motion, this velocity is proportional to the velocity of heat transfer. Since the latter is one third each along the x, y and z axes and since, in our example, we can put, in first approximation, $v' = c/3$, where c is the mean velocity of the heat transfer. Hence the shearing stress[3]

$$\tau = \frac{1}{3}\,\rho\; c\; l\,\frac{du}{dy} = \eta\frac{du}{dy} \tag{1}$$

In the case of the turbulent exchange of masses, the velocity v' should naturally be taken of the same order of magnitude as the difference in the velocities of the two layers at the distance l from each other, since the fluid masses collide at velocities of this order of magnitude (references 8, 9, and 10). On eliminating the unknown numerical factor v', we thus obtain the shearing stress

$$\tau = \rho\left(l\,\frac{du}{dy}\right)^2 \tag{2}$$

The elimination of the numerical factor only denotes a somewhat different definition of l. In this way we obtain, for the simple viscosity effect, shearing stresses proportional to du/dy and, for the turbulent exchange (whereby the effect of viscosity is disregarded), shearing stresses proportional to $(du/dy)^2$, which is in good agreement with the hydraulic resistances proportional to the

[3] By a more accurate calculation, Boltzmann found, for the viscosity η, the value $\eta = 0.3503\,\rho\,c\,l$, which differs but little from that in equation 1.

square of the velocity.

With formula 2 the problem of the hydraulic flow resistances is brought back to the other problem of the distribution of the mixing path l in the flow. So long as we have no rational theory of turbulent flow which deduces the laws of turbulent phenomena from hydrodynamic differential equations, we have to obtain the data regarding the distribution of the mixing path by experimentation, so that only one unknown quantity is thus replaced by another. Nevertheless, considerable progress has been made, since it has been found, at least for the larger Reynolds Numbers (from about 10^5 up) that the mixing path is practically independent of the magnitude of the velocity and is, moreover, subject to quite simple rules for its distribution in space.

Dimensional considerations often furnish useful indications. For example, in considering the flow near a more or less smooth flat wall, on the assumption that neither the viscosity nor the roughness of the wall has any appreciable effect at the point under consideration in the interior of the fluid, we are in a position to make a statement regarding the distribution along the mixing path. For a point at the distance y from the wall there is no other characteristic length than this distance y. The mixing path l is also a length, so that there is no other possibility than to put the mixing path proportional to the distance from the wall:

$$l = \kappa y.$$

Here κ is a universal numerical coefficient, which can be determined experimentally. If we assume a state of flow in which the shearing stress τ is constant, we obtain

$$\frac{du}{dy} = \frac{1}{\kappa y} \sqrt{\frac{\tau}{\rho}}$$

according to equation 2, and therefore

$$u = \frac{1}{\kappa}\sqrt{\frac{\tau}{\rho}}\,(\ln y + \text{const.}) \qquad (3)$$

Such a velocity curve, dependent on the distance from the wall, is quite like the one actually observed (fig. 3).

Comparison with the experimental results yields the number 0.4 as the approximate value of κ.

Karman's Theory

Von Karman (reference 12) assumed that the turbulent mixing processes are the same in all cases, so that only variations in the length and time scales occur from case to case and from place to place in the flow. Under these circumstances the effects of viscosity are regarded as negligible in comparison with the effects of turbulence. Conclusions are now drawn from Euler's equations regarding these two scales, the first of which obviously agrees in principle with our mixing path l. The velocity u of the basic flow, which is assumed to be a function of y alone, is determined from a Taylor series interrupted after the quadratic term. The mean forward velocity of the particle under consideration has no immediate effect on its inner motion. Of the given quantities therefore, only du/dy and d^2u/dy^2 need to be considered here. We first have a time

$$T \sim 1/\frac{du}{dy}$$

as the time criterion for the period of the mixing process. For dimensional reasons, the interference velocities u' in the X direction and v' in the Y direction are therefore proportional to l/T, i.e.

$$u' \sim v' \sim l\frac{du}{dy}$$

which agrees with the previous formulas. For the longitudinal scale of the mixing process, Von Karman finds the relation

$$l = \kappa' \frac{du}{dy}\Big/\frac{d^2u}{dy^2}$$

in which κ' is a constant determined experimentally. This expression of Karman's theory goes beyond previous expressions, because it furnishes a method for calculating the magnitude of the mixing path independently of the distance from a wall. If this expression is introduced into equation 2 and integrated on the assumption of a constant shearing stress in the region under consideration, we obtain

$$u = \frac{1}{\kappa'}\sqrt{\frac{\tau}{\rho}}\,[\ln(y + C_1) + C_2] \qquad (4)$$

i.e. practically equation 3 again. The required agreement with the experimental results obviously leads to putting $\kappa' = \kappa$. Hence both formulas yield the same velocity distribution in case of constant shearing stress.

There is no longer any agreement regarding the shearing stress in the other assumptions. Moreover, the formula $l = \kappa y$ is without any valid basis, since, due to the variability of the shearing stress, a still further length $\tau / \frac{d\tau}{dy}$ is available; but even Karman's formula

$$l = \kappa \frac{du}{dy} \Big/ \frac{d^2u}{dy^2}$$

here means only another estimated approximation, since it was obtained by disregarding the effect of d^3u/dy^3 and higher terms in the series development for u. In the case τ = constant, the two solutions coincide, because the velocity distribution, according to equation 3, is transferred by changing the integration constant, in case the shearing stress τ remains unaltered, so that there is here also a pronounced similarity with the basic flow.

From equation 3 it is easily seen that the quantity $\sqrt{\tau/\rho}$ is a velocity. This velocity is very valuable for various similarity considerations in what follows. We will therefore designate it by v_* and call it "shearing-stress velocity." The formula $\tau = \rho v_*^2$ is of similar form to that for the dynamic pressure

$$p_d = \frac{1}{2} \rho u^2,$$

which is comprehensible for dimensional reasons, since the shearing stress is also a force per unit area. The apparent shearing stress τ of the turbulence is generally very small as compared with the dynamic pressure. Hence in v_* we are also dealing with a velocity which is relatively small as compared with the flow velocity u. Comparison with equation 2 shows, moreover, that

$$v_* = l \frac{du}{dy}$$

Hence v_* is of the order of magnitude of the mixing velocities u' and v'.

FLOW ALONG A ROUGH WALL

From our standpoint the flow along a rough wall is simpler than along a smooth wall, because the viscosity plays a preponderant role in the latter case, but not in the former. It is therefore better to consider the flow along a rough wall first. If k is a length indicating the roughness of the wall, it follows, from a simple similarity consideration on the basis of the ideal fluid, that the velocity distributions near the wall, with geometrically similar roughnesses, are also geometrically similar, so that the size of the grain k furnishes the criterion for it. The formulated expression of this relationship is that the velocity at the distance y is a function of the ratio y/k. If this velocity distribution is based on equation 3, which, according to what has preceded, is at least advisable for the regions farther in the interior of the fluid, it is found that the integration constant of equation 3 = constant - ln k.

A hitherto unpublished series of experiments by Nikuradse with tubes of various diameters, which were given different degrees of roughness by gluing to them sifted sand with a suitable varnish, showed that the new constant = 3.4 = ln 30, k being the mean diameter of the grains of sand used to produce the roughness. With $1/\kappa = 2.5$, we obtain the formula

$$u = 2.5\ v_* \ln\left(\frac{30\ y}{k}\right)$$

By a shifting of the coordinates by the amount of $k/30$, it is also possible to obtain $u = 0$ for $y = 0$.[4] Hence,

$$u = 2.5\ v_* \ln\left(1 + \frac{30\ y}{k}\right) \tag{5}$$

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[4]It is still an open, though not very important question as to the exact location of the axis of the coordinates between the protuberances of the roughness.

or, if the natural logarithm is replaced by a common logarithm,

$$u = 5.75\ v_* \log\left(1 + \frac{30\ y}{k}\right) \tag{5a}$$

Equations 5 and 5a therefore show a fixed relation between the velocity distribution, shearing-stress velocity, distance from the wall and the degree of roughness k. This first holds good for the kinds of roughness used in the experiments. For other forms of surface roughness, moreover, there is probably another number instead of 30, also dependent on the manner of defining the roughness scale. Preparations for tests in this connection are being made in Göttingen.

Equation 5 immediately affords us the opportunity to check the above statement regarding the behavior of the ideal fluid. Represent the velocity at the distance $y = h$ by $u = u_1$. With this assumption v_* can be eliminated from equation 5a.

$$v_* = \frac{u_1}{5.75 \log\left(1 + 30\ \frac{h}{k}\right)}$$

and consequently

$$u = u_1\ \frac{\log\left(1 + 30\ \frac{y}{k}\right)}{\log\left(1 + 30\ \frac{h}{k}\right)} \tag{6}$$

The corresponding shearing stress is

$$\tau = \rho\ v_*^{\ 2} = \frac{\rho\ u_1^{\ 2}}{33\left[\ \log\left(1 + 30\ \frac{h}{k}\right)\right]^2} \tag{7}$$

from which it follows that the shearing stress is proportional to the square of the flow velocity u_1. The effect of the roughness of the wall is likewise shown by equation 7.

If we pass to the mathematically smooth wall, i.e., to $k = o$, then, according to equation 6, $u = u_1$ and $\tau = o$ for all values of y constant, as stated in the classical hydrodynamics on the ideal fluid. It is also obvious that even a submicroscopic roughness with a k of the order of magnitude of the diameter of an atom would still show considerable deviations from the ideal behavior. Our formulas can no longer be used for such cases. The relations are here considerably altered by the viscosity, as will be shown in what follows.

THE FLOW IN TUBES

It is an important discovery that, in a straight tube, the relative motion of the fluid particles at moderately large Reynolds Numbers depends on the fall in pressure and not at all on the character of the wall, so that therefore, with constant fall in pressure, the velocity-distribution curves in tubes of greater and less wall roughness can be brought into conformity by shifting along the velocity axis (of course aside from a layer in immediate contact with the wall, where the velocity increase is naturally greater on a smoother surface than on a rougher one). This relation was discovered by Darcy (reference 14) 75 years ago in his researches on resistance in pipes and was then emphasized, but was subsequently forgotten. Fritsch discovered it anew by direct observation in his experiments with rough channels at Aachen (reference 17). From our standpoint this discovery is identical with the fact that the distribution of the mixing path along the inside of the tube is practically independent of the nature of the wall. In connection with our earlier discoveries it is natural to surmise that the formula

$$l = r\ f_1\left(\frac{y}{r}\right)$$

can be written for the mixing path, where y is the distance from the wall and r the radius of the tube. Since the distribution of the shearing stress along the tube is known when the pressure fall is given, the distribution of l can be verified by measuring the velocity distribution with the aid of equation 2. It is found that the above statement is confirmed, at least for the higher Reynolds Numbers. Figure 4 gives the result in nondimensional form and consequently shows the course of the function f_1. In the function $f_1(y/r)$, l is the mixing path; r, radius of tube; y, distance from wall; and k, mean longitudinal dimension of roughness.

Conversely, on the basis of this function and with the aid of equation 2, we can calculate du/dy, from which, by an integration, an expression for the velocity itself can be obtained. On the introduction of the shearing stress velocity v_* this expression takes the form

$$u_{max} - u = v_* f_2\left(\frac{y}{r}\right) \tag{8}$$

This equation, which was first developed by Von Karman (reference 12), has also been experimentally confirmed, as shown by figure 5, in which the test points are given for smooth tubes and for various rough tubes. In function $f_2(y/r)$, u_{max} is the maximum flow velocity; u, flow velocity at the point y; v_*, shearing-stress distribution $\sqrt{\tau/\rho}$; τ, shearing stress; ρ, density.

We can now pass from the velocity u at any distance y from the wall to the mean velocity $\overline{u}$. We thus obtain from equation 8 an expression of the form

$$u_{max} - \overline{u} = v_* \times \text{coefficient} \tag{9}$$

Nikuradse's Göttingen experiments yielded 4.07 as the value of this coefficient. It was a piece of good luck that our equation 3 or the special form for a rough wall (equation 5) yielded[5], up to the middle of the tube, a useful approximation for the function $f_2(y/r)$, namely[6]

[5]For more accurate calculations, a small supplementary term would have to be added, which will be included later, at least in the final result.

[6]Darcy (reference 14) deduces from his experiments

$$u_{max} - u = 11.3 \frac{\sqrt{i}}{r} (r - y)^{3/2}$$

(i is the gradient and therefore $= -\frac{1}{g\rho}\frac{dp}{dx}$; the meter is the unit of length). This equation can be put in the form of equation 8 and thus becomes

$$f_2 \frac{y}{r} = 5.08 \left(1 - \frac{y}{r}\right)^{3/2}$$

which, with the exception of the wall vicinity where Darcy made no measurements, agrees very well with modern results (fig. 5).

$$f_2\left(\frac{y}{r}\right) = 2.5 \ln \frac{r}{y} = 5.75 \log \frac{r}{y} \tag{10}$$

We now have all that is needed to calculate the resistance of a rough tube for a given quantity. We will first write the customary expression for the drag coefficient λ:

$$-\frac{dp}{dx} = \frac{\lambda}{d}\frac{\rho \bar{u}^2}{2} \tag{11}$$

From the equilibrium of a water cylinder of radius $r = d/2$, we obtain, for the shearing stress τ_o of the wall, the expression

$$-\pi r^2 \frac{dp}{dx} = 2 \pi r \tau$$

and accordingly

$$-\frac{dp}{dx} = \frac{2\tau}{r} = \frac{2 \rho v_*^2}{r} \tag{12}$$

The comparison of equations 11 and 12 yields, with the tube diameter $d = 2r$,

$$v_*^2 = \frac{\lambda}{8} \bar{u}^2 \tag{13}$$

By the use of equation 5a at the middle of the tube $(y = r)$ we obtain, when, under the logarithm, we disregard 1 in comparison with the very great value $30\, r/k$ and put $\log 30 = 1.477$,

$$u_{max} = v_*(5.75 \log \frac{r}{k} + 8.5) \tag{14}$$

On the other hand, according to equation 9

$$\bar{u} = u_{max} - 4.07\, v_* = v_*(5.75 \log \frac{r}{k} + 4.43)$$

Taking equation 13 into consideration, we now have

$$\lambda = \frac{8\, v_*^2}{\bar{u}^2} = \frac{8}{(5.75 \log \frac{r}{k} + 4.43)^2} \approx \frac{1}{(2.0 \log \frac{r}{k} + 1.57)^2} \tag{15}$$

This is very well confirmed by experiment, with only the slight difference that 1.74 is better than 1.57 in the denominator. This difference is connected with the suppressed auxiliary term in equation 10. The experimental confirmation of the formula is best accomplished by plotting $1/\sqrt{\lambda}$ against $\log r/k$. According to the foregoing

$$1/\sqrt{\lambda} = 2.0 \log \frac{r}{k} + 1.74 \tag{16}$$

The plotting must therefore yield a straight line. Figure 6 shows this line for six rough tubes according to measurements by Nikuradse. (See also figure 9.) The general form of equation 14, as likewise an equation analogous to equation 16 for a coefficient of resistance based on u_{max}, was first developed by Von Karman. He also made the rectilinear graph.

Effect of Viscosity (smooth tube)

It has already been mentioned that the effect of viscosity is greater when the roughness is less, but of course only on the boundary-layer phenomena. The rough places are here more or less covered by a slower-moving layer of fluid and are thus rendered ineffective as regards resistance. Progress can also be made here with a dimensional consideration. The shearing stress is responsible for what takes place on the wall and consequently the velocity v_* based on this shearing stress, and also the criterion of roughness k. A wall characteristic $v_* k/\nu$ can be developed from these two with the kinematic viscosity by analogy with the Reynolds Number. Since, with fixed v_*, the state of flow in the interior remains unaltered, the only remaining problem is to adapt the integration constant of equation 3 to the new relations. This is accomplished by introducing a modified roughness criterion,

$$k' = k\, f_3\left(\frac{v_* k}{\nu}\right)$$

instead of k, into equations 5 to 7 and 14 to 16. Regarding the course of the function f_3, it follows from the foregoing that it must be equal to 1 for large values of the wall characteristic, in order to restore the previous relations. An immediate conclusion can, however, be drawn as to what form the function f_3 must assume for small values of $v_* k/\nu$. The observations show that, for slight but still appreciable roughness, the rough tube does not differ practically from a perfectly smooth tube, provided the Reynolds Number is not unusually high. Such a condition is obtained when

$$f_3\left(\frac{v_* k}{\nu}\right) = \text{coefficient} \times \frac{\nu}{v_* k}$$

since k is thus removed from the foregoing formulas and is replaced by coefficient $\times \frac{\nu}{v_*}$. The experiments confirm this result and show, with respect to the coefficient which leaves the dimensional consideration still open, that our previous value of $k/30$ must be replaced by $\nu/9\ v_*$. Instead of equation 5a, we now obtain the formula for the velocity distribution in the tube

$$u = v_* \left(5.75 \log \frac{v_* y}{\nu} + 5.5\right) \tag{17}$$

On plotting u/v_* against $\log\ v_* y/\nu$, we obtain a straight line which must contain all the points near the wall for the velocity profiles of all smooth pipes. An exception is formed only by the values at very small nondimensional distances from the wall $v_* y/\nu$, at which the turbulence is still affected by the viscosity. Up to the previously mentioned supplementary function, equation 17 is also valid to the middle of the tube. The experimental points in figure 7 actually contain not only the parts near the wall, but extend almost to the middle of the tube. One can therefore note small systematic deviations from the straight line, which of course have to be considered in a more accurate theory (reference 18).

For comparison figure 7 also shows, by a dash line, the velocity-distribution law

$$\frac{u}{v_*} = 8.7\left(\frac{v_* y}{\nu}\right)^{1/7} \tag{18}$$

as determined on the basis of the Blasius formula for the friction of the tube. It is found that, in a central region for which alone data were formerly available, it practically coincides with the straight line of equation 17, but deviates considerably above and below this region. In fact it was long since discovered that, at higher Reynolds

Numbers, the seventh root is replaced by the eighth and ninth roots, etc. The reason for this behavior is manifest, since the law of the seventh root now appears to be only an approximation formula for the real law, which is represented by equation 17, whereby the particular numerical values of the approximation formula naturally still depend on the region in which they should agree with the accurate formula[7].

For the coefficient of resistance, we obtain from equation 16 by the same modification

$$\frac{1}{\sqrt{\lambda}} = 2.0 \log \frac{v_* r}{\nu} + 0.5$$

Taking equation 13 into consideration, we can put

$$\frac{v_* r}{\nu} = \frac{\bar{u} r}{\nu} \frac{v_*}{\bar{u}} = \frac{\bar{u} d}{\nu} \sqrt{\lambda} \frac{1}{2\sqrt{8}}$$

With $\bar{u} d/\nu = Re$, we obtain

$$1/\sqrt{\lambda} = 2.0 \log (Re\sqrt{\lambda}) - 1.0 \qquad (19)$$

This formula was verified experimentally be Nikuradse (reference 20) up to the Reynolds Number 3.4×10^6. It must be changed only by the consideration of the previously mentioned supplementary function of the numerical value from -1.0 to -0.8. The final formula for the resistance coefficient is then

$$1/\sqrt{\lambda} = 2.0 \log (Re\sqrt{\lambda}) - 0.8 \qquad (20)$$

The calculation of the resistance coefficient corresponding to any given value of Reynolds Number encounters no particular difficulties, although $\sqrt{\lambda}$ occurs once more on the right side. One can, for example, assume provisionally any value for $\sqrt{\lambda}$ on the right side and calculate $1\sqrt{\lambda}$ and then repeat the process, if the discrepancy is too great. In figure 8 the course of λ is plotted with respect to Re according to equation 20 together with the experimental val-

[7]Below $\log v_* y/\nu = 2$, the straight line of equation 17 shows appreciable deviations from the test points. This is due to the influence of the viscosity. If the smallest supercritical Reynolds Numbers are disregarded, this deviation occurs only in a very thin layer near the wall of the tube.

ues. By especially good luck this formula agrees with the experiments down to the smallest supercritical Reynolds Numbers.

We now turn once more to the general problem of the rough tube. On the basis of measurements by Nikuradse (now being prepared for publication) the course of the resistance coefficient is plotted in figure 9 against the Reynolds Number for tubes of different relative roughness k/r. The curves in figure 9 are based on experiments with tubes of well-defined roughness produced by gluing grains of sand of definite and different sizes (k) to the inside of tubes. The conditions to the left of the critical Reynolds Number represent the laminar condition of smooth flow. It is evident that there is here very little difference between the smooth and rough tubes. The curves diverge greatly, however, as soon as the turbulence begins, i.e. above Re_{crit}. The curves for the lesser roughness first follow the curve for the smooth tube and then separate from the latter in order.

The foregoing considerations indicate a way to find a law for the turbulent portion. We will take the wall characteristic $v_* k/\nu$ or its logarithm as the abscissa and a quantity which is constant according to the laws of the fully developed roughness flow as the ordinate. For example, we can take the quantity

$$1/\sqrt{\lambda} - 2.0 \log \frac{r}{k}$$

or, if we want the corresponding law for the velocity distribution, the quantity

$$\frac{u}{v_*} - 5.75 \log \frac{y}{k}.$$

The plotting of these two quantities on the basis of the experimental results brings in fact the test points measured with very different roughnesses approximately on a single curve. The two curves agree with each other up to the scale corresponding to the relations here reprensented. The whole problem thus finds a very comprehensive solution on the basis of combining a few experimental values with theoretical conclusions. What remains to be done is to find the curves for other forms of roughness in addition to the curve of figure 10, which we have thus far determined only for the special sandpaper form of roughness. Preparations are now being made for such experiments.

APPLICATION TO OTHER CASES

Plate Resistance - Accelerated and Retarded Flows

From the behavior of the flow in tubes, when the Blasius law of resistance

$$\lambda = 0.316 \left(\frac{\bar{u}\ d}{\nu}\right)^{-1/4}$$

dominated the field, conclusions had already been drawn regarding the frictional resistance of plates subjected to flow along their surface (references 15 and 16). According to the momentum theory, the decrease in the momentum of the flow due to the friction was represented by a formula in terms of the exposed length of the plate in accord with the laws for the velocity distribution. This decrease in momentum per unit length along the plate was expressed as equal to the frictional force per unit length. The resulting formula for the coefficient of frictional resistance c_f (resistance divided by the surface area and dynamic pressure),

$$c_f = 0.074 \left(\frac{v\ l}{\nu}\right)^{-1/5} \qquad (21)$$

(l = length of plate, v = velocity of plate), showed similar discrepancies, in comparison with the experimental results, to those shown in the resistance of tubes. The obvious thing to do now was to apply the improved law of tubular flow also to plates. The calculations are here rather troublesome. They were first made by Von Karman (references 13 and 21). A new calculation in a somewhat different way was made by the writer (reference 18), who compiled a numerical table the values of which agree very satisfactorily with Kempf's measurements. The values in the table were obtained by the following approximation formula of H. Schlichting, which though it is only an interpolation formula, can be used throughout the whole practical region of turbulent flow.

$$c_f = \frac{0.455}{\left(\log \frac{v\ l}{\nu}\right)^{2.58}} \qquad (22)$$

For the rough plate, a corresponding calculation was made on the basis of the law of roughness represented in figure 10 (reference 19).

The behavior of the turbulent friction layer in an accelerated or retarded flow is of greater importance. An important special case, the flow in a widened or narrowed channel with flat side walls, was investigated by Dönch for air (reference 22) and by Nikuradse for water (reference 23). Buri's work at Zurich should be metioned here, as also Cuno's experiments on an airplane wing at Hannover (reference 27).

Buri and Gruschwitz have now made, in somewhat different manner, the very important attempt to develop purely mathematical methods for calculating the course of the phenomena in the frictional layer. Buri's method is simpler, while that of Gruschwitz is more complete. Lack of space forbids further consideration here of these rather complicated calculations. With these methods it is possible to predict the course of the frictional layer for any given pressure distribution and, under some circumstances, even to make the important determination as to whether this flow will adhere to the wall, as assumed, or will separate at some point. A further attempt is now being made to predict in this way the actual characteristics of an airplane wing including the profile drag and maximum lift. Should the results show a satisfactory agreement with experimental results, this method would constitute a very considerable advance.

FURTHER PROBLEMS

The investigation of currents in strongly curved channels (references 30 and 31) shows that, aside from the "secondary currents" on the side walls as already described by earlier writers , even the real nature of the turbulence is here substantially altered. The two kinds of phenomena are related in that the faster portions of the fluid along the curved wall develop stronger centrifugal forces than the slower portions. The faster portions therefore tend to displace the slower portions on the outer wall. However, since the portions in immediate contact with the wall are continually retarded by friction, a materially accelerated exchange is produced on the outer side of the channel by the displacement of these retarded portions. On the contrary the slower portions tend toward the inner side and the exchange is considerably retarded.

The phenomena are very similar to those in the flow of a fluid over a heated or cooled bottom surface. In the former case the heated and simultaneously retarded portions tend to rise from the bottom, while in the latter case the cooled portions, because of their greater density, tend to remain near the bottom (references 11 and 32), so that the turbulent friction is increased in the former case and decreased in the latter case. Since both groups of phenomena have been or are being investigated in Göttingen, numerical expressions for these influences may be expected.

Another important kind of phenomena is involved in the turbulent spreading of fluid jets and the wakes of moving bodies. The outer portions of a jet emerging, e.g. from a larger orifice (nozzle, etc.) are very unstable and develop into a more or less irregular vortex system. Even for this kind of phenomena the conception of the mixing path held good, and it was possible, with the aid of the simple assumption that the mixing path in a cross section is constant and proportional to the width of the mixing zone at that point, to predict the form of the mixing zone and the velocity distribution in it in a very satisfactory manner, whereby only the ratio of the mixing path to the mixing zone had to be taken from the experiments (references 9, 10, 28, 29, 36).

The heat exchange is quite closely related to the turbulent velocity exchange. Insofar as it concerns the flow along a wall, as shown by the experiments of Elias (reference 33), the exchange factor has exactly the same value, so that the curve of the temperature distribution agrees with the velocity distribution. For the phenomena in the wake of moving bodies, Taylor (reference 34) has recently shown that here the heat exchange is twice as great as the velocity exchange, so that the temperature and velocity curves differ appreciably[8]. Taylor could also

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[8]Taylor demonstrates that in this case the rotational force of the main motion is exchanged in the same manner as the heat. The exchange factor is $\rho l^2 (du/dy)$; the rotational force in parallel motion, however, is du/dy; and the fall of the rotational strength in the direction y is therefore d^2u/dy^2. Taylor shows that then
(Concluded at bottom of page 23)

show that theoretically the former condition (like form of these curves) is to be expected when the vortex axes of the interference motion are parallel to the streamlines of the main motion, but the latter (unlike) when they are perpendicular to them. The unpublished Göttingen experiments of P. Ruden show that the Taylor law of exchange is also valid for the spreading of jets.

It follows therefore that, on closer inspection, there are two kinds of turbulence to be distinguished, which differ in their nature. We may call one "wall turbulence" and the other "jet turbulence." In the former (according to Elias) the vortices parallel to the streamlines obviously predominate. This rather important discovery will perhaps once more indicate the way to a real theory of the phenomena. So long as this is not discovered, we must be staisfied with half-empirical considerations of the kind here described.

Translation by Dwight M. Miner,
National Advisory Committee
for Aeronautics.

$\frac{\partial \tau}{\partial y} = \rho \, l^2 \frac{du}{dy} \frac{d^2u}{dy^2}$ which can be integrated to

$\tau = \frac{1}{2} \rho \, l^2 \left(\frac{du}{dy}\right)^2$ for l constant in a cross section. The factor $\frac{1}{2}$ in this formula differentiates it from our equation 2.

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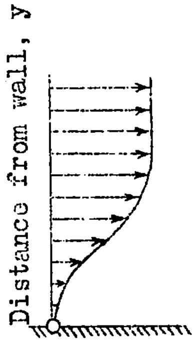

Figure 1.-Velocity profile with turning point.

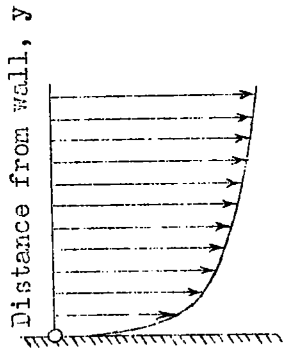

Figure 3.-Velocity profile with turning point.

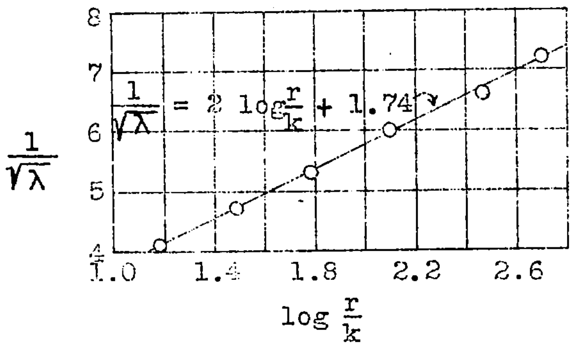

Figure 6.-Resistance curve for a rough tube.

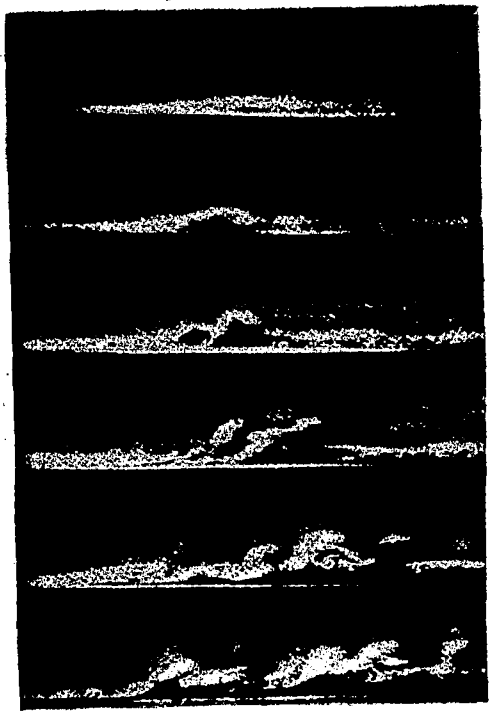

Figure 2.-Development of turbulence from an initial disturbance.

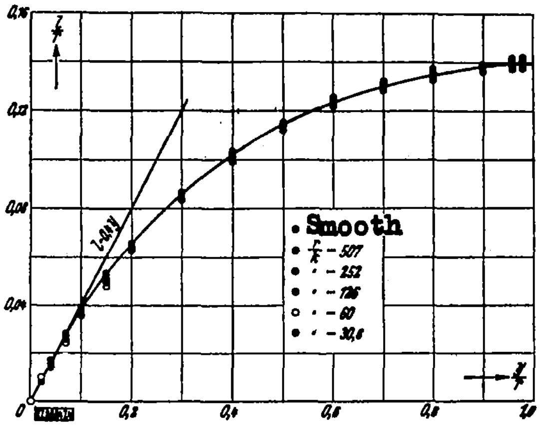

Figure 4.- Distribution of mixing path in tube at large Reynolds Numbers.

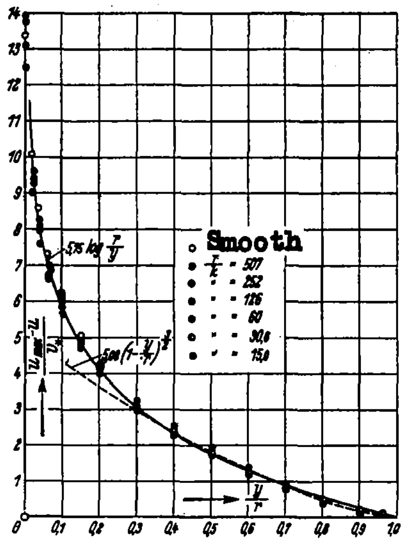

Figure 5.-Course u_{max}-u in tube at large Reynolds Numbers.

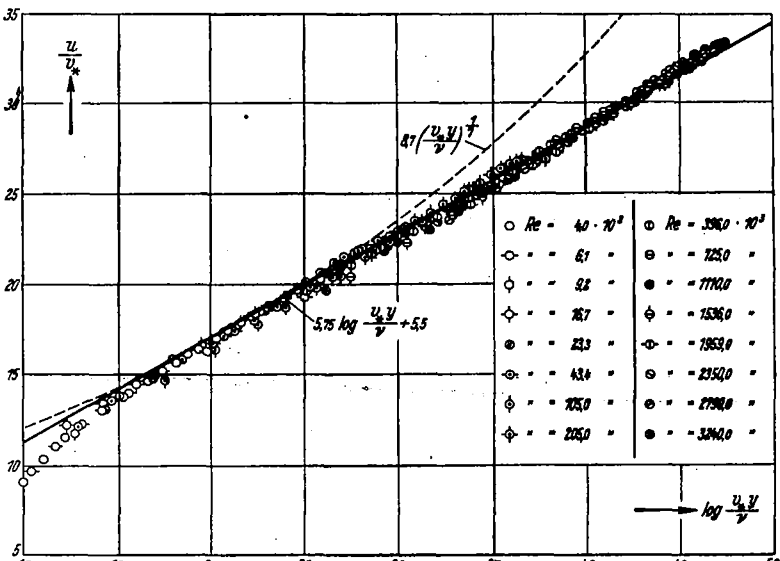

Figure 7.- Velocity distribution in a smooth tube.

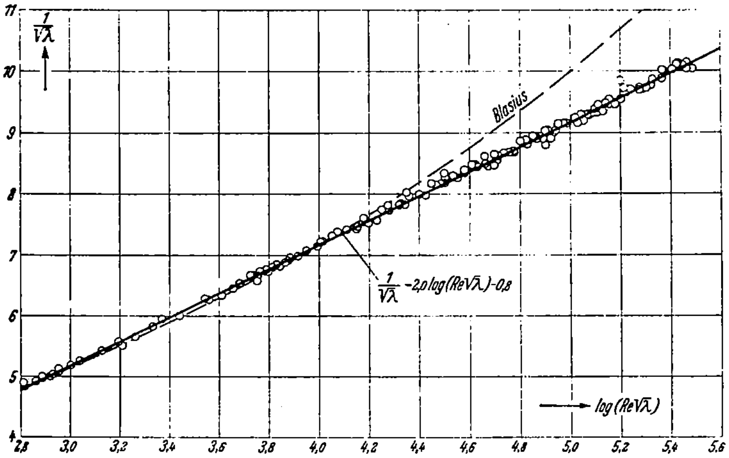

Figure 8.-Resistance in smooth tube according to eq.20 with test values.

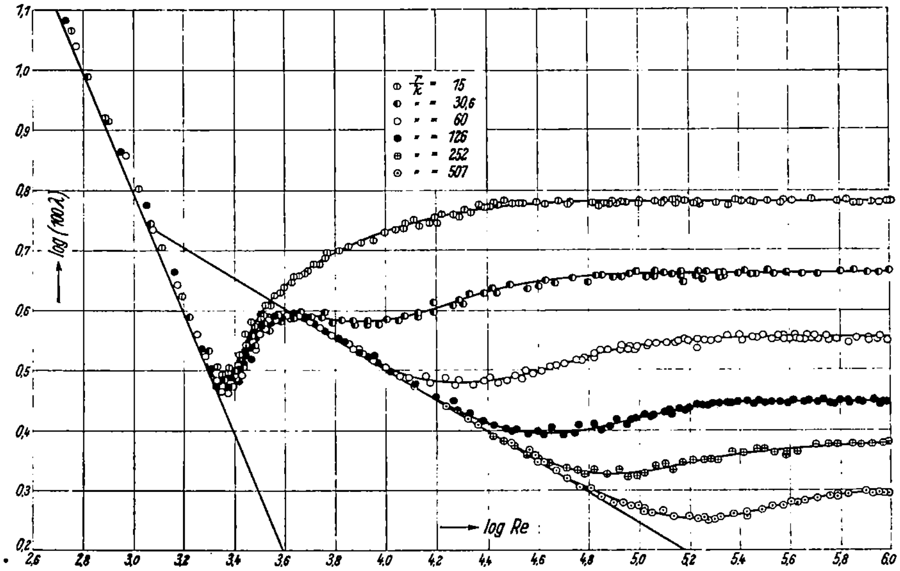

Figure 9.-Resistance coefficients λ of rough tubes.

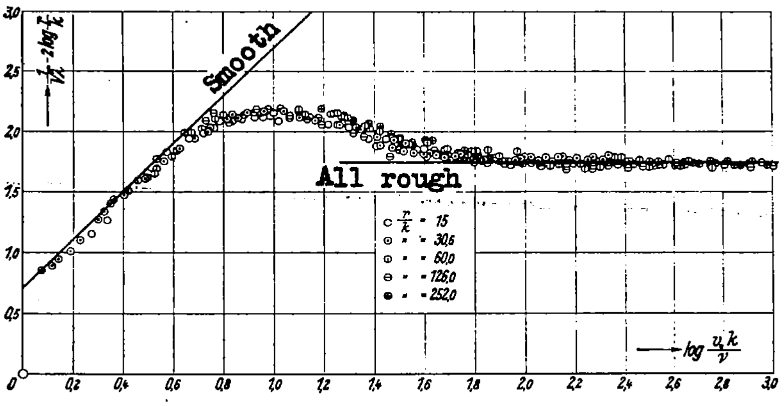

Figure 10.-Roughness function.