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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1437

ON POSSIBLE SIMILARITY SOLUTIONS FOR THREE-DIMENSIONAL INCOMPRESSIBLE LAMINAR BOUNDARY-LAYER FLOWS OVER DEVELOPABLE SURFACES AND WITH PROPORTIONAL MAINSTREAM VELOCITY COMPONENTS

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ON POSSIBLE SIMILARITY SOLUTIONS FOR THREE-DIMENSIONAL INCOMPRESSIBLE LAMINAR BOUNDARY-LAYER FLOWS OVER DEVELOPABLE SURFACES AND WITH PROPORTIONAL MAINSTREAM VELOCITY COMPONENTS[1]

By Arthur G. Hansen

SUMMARY

This report presents an analysis of possible similarity solutions of the three-dimensional, laminar, incompressible, boundary-layer equations referred to orthogonal, curvilinear coordinate systems.

Requirements for the existence of similarity solutions are obtained for the following two cases: flow over developable surfaces; flow over nondevelopable surfaces with proportional mainstream velocity components. The analysis obtains permissible forms of mainstream velocity components, the square of differential of arc length on the surface, and the similarity parameter. A basic class of surfaces is found from which all other permissible surfaces may be obtained.

Necessary and sufficient conditions are found for expressing the ordinary differential equations resulting from the similarity transformation in uncoupled form. The analysis shows that uncoupling is possible only when the surface is developable and a surface coordinate system characterized by $(ds)^2 = (dx_1)^2 + (dx_2)^2$ is employed.

INTRODUCTION

Theoretical research on the important problem of three-dimensional boundary-layer flow has been greatly restricted because of the complex

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[1]The information herein was originally presented as part of a thesis entitled "Similarity Solutions of the Laminar Incompressible Three-Dimensional Boundary-Layer Equations" submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at the Case Institute of Technology, May 1958.

nature of the governing equations. However, a degree of success has been achieved in incompressible-flow analysis by searching for exact solutions of the equations for special types of mainstream flows. These exact solutions of the three-dimensional, laminar, incompressible boundary-layer equations have generally been formulated by use of "similarity" techniques (e.g., refs. 1 to 7). In instances where the technique is applicable, the partial differential equations of the boundary layer reduce to a system of ordinary differential equations. The corresponding solutions for the boundary-layer velocity components are such that the velocity profiles differ by, at most, scale factors along coordinate directions.

To date, similarity analyses have been carried out for only very special types of mainstream flows and for special types of coordinate systems. A question has always existed, therefore, as to the most general class of problems amenable to a similarity analysis and as to what limitations are inherent in its use. This question is successfully answered for two-dimensional laminar flows in references 8 to 12. These investigations show that, in general, similarity solutions can only be found for mainstream flows in which the velocity varies as a power of the distance along a surface or as an exponential. Recently, several investigations have attempted to determine conditions under which similarity solutions exist for laminar, incompressible, three-dimensional boundary-layer flows (refs. 13 to 17). In investigations of this kind, the type of coordinate system employed plays an important role because of similarity of velocity profiles in coordinate directions. Consequently, reference 13 considers the special case of a stationary rectangular coordinate system and determines what possible mainstream flows referred to such a system lead to similarity solutions. Reference 14 determines permissible mainstream flows referred to polar coordinates; reference 15 determines mainstream flows confined to regions of small angle variation with respect to polar coordinate systems.

The research presented in reference 16 is more general in scope than that in reference 13 or 14. In reference 16 the coordinate system assumed is an arbitrary orthogonal, curvilinear system. In this respect, the approach employed therein is similar to that which will be used in the present report. However, the complexity of the problem has required the application of additional assumptions, both in reference 16 and in the analysis given here. In reference 16, for example, the mainstream flow is assumed to be irrotational. In the present analysis, two independent assumptions (each different from that of ref. 16) are made and investigated separately.

The first assumption which is applied herein is that the surface over which the flow takes place is developable. Examples of such surfaces are cones, cylinders, and, of course, the plane. Such surfaces are characterized by the geometric property of zero Gaussian curvature. The second assumption pertains to a restriction on the form of the boundary-layer velocity components when the flow takes place over a surface which may be nondevelopable. It will be shown that the assumption is equivalent to specifying proportionality of mainstream velocity components in reference to a particular coordinate system embedded in the flow surface.

The results of the present study are compared in detail with those of reference 16 on page 40. However, it might be well to point out here that there is a major distinction between the philosophy of reference 16 and the present treatment. Here, greater emphasis is given to the geometric aspects of the problem. In reference 16, a great deal more attention is given to establishing necessary and sufficient conditions for the existence of similarity solutions.

In the following sections, general requirements for similarity solutions will first be derived. Then, further developments on the assumption mentioned previously will be presented, and specific conditions will be determined for reducing the boundary-layer equations to ordinary differential equations.

Acknowledgement is made to Dr. Gustav Kuerti of Case Institute of Technology for his interest and advice in the preparation of this work.

CONDITIONS FOR SIMILARITY SOLUTIONS

Consider a surface in space in which an arbitrary orthogonal coordinate system (x_1,x_2) has been embedded. Let y^* be a coordinate normal to the surface (fig. 1). The boundary-layer equations referred to such a system are the following (refs. 18 or 19):

$$\frac{u_1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_1}{\partial x_2} + \frac{v}{\sqrt{\nu}}\frac{\partial u_1}{\partial y} + u_1u_2k_1 - u_2^2k_2 - \frac{\partial^2 u_1}{\partial y^2}$$

$$= \frac{U_1}{h_1}\frac{\partial U_1}{\partial x_1} + \frac{U_2}{h_2}\frac{\partial U_1}{\partial x_2} + U_1U_2k_1 - U_2^2k_2 \tag{1a}$$

$$\frac{u_1}{h_1}\frac{\partial u_2}{\partial x_1} + \frac{u_2}{h_2}\frac{\partial u_2}{\partial x_2} + \frac{v}{\sqrt{\nu}}\frac{\partial u_2}{\partial y} + u_1u_2k_2 - u_1^2k_1 - \frac{\partial^2 u_2}{\partial y^2}$$

$$= \frac{U_1}{h_1}\frac{\partial U_2}{\partial x_1} + \frac{U_2}{h_2}\frac{\partial U_2}{\partial x_2} + U_1U_2k_2 - U_1^2k_1 \tag{1b}$$

$$u_1k_2 + u_2k_1 + \frac{1}{h_1}\frac{\partial u_1}{\partial x_1} + \frac{1}{h_2}\frac{\partial u_2}{\partial x_2} + \frac{1}{\sqrt{\nu}}\frac{\partial v}{\partial y} = 0 \tag{1c}$$

where

u_1, u_2, v boundary-layer velocity components in the x_1-, x_2-, and y^*-directions, respectively

y $y^*/\sqrt{\nu}$

k_1, k_2 geodesic curvatures[a] of coordinate lines x_2 = const. and x_1 = const., respectively, $k_1 = \frac{1}{h_1 h_2} \frac{\partial h_1}{\partial x_2}$; $k_2 = \frac{1}{h_1 h_2} \frac{\partial h_2}{\partial x_1}$

h_1, h_2 metric tensor components related to the differential of arc length on surface by $(ds)^2 = h_1^2 (dx_1)^2 + h_2^2 (dx_2)^2$

U_1, U_2 inviscid main-flow velocity components in vicinity of surface

(All symbols are defined in appendix A.) The boundary conditions are:

$$\text{At } y = 0,\ u_1 = u_2 = v = 0$$

$$\lim_{y \to \infty} u_1 = U_1 \qquad \lim_{y \to \infty} u_2 = U_2$$

Two possible situations might now be noted relative to the mainstream velocity components. The first is when neither U_1 nor U_2 is identically zero. The second is when one of these components is identically zero, and, consequently, one of the coordinate lines corresponds to a streamline of the main flow. For these two cases (when neither U_1 nor U_2 is identically zero) follow the classical approach for obtaining similarity solutions, and assume that u_1 and u_2 are expressible as follows:

$$\frac{u_1}{U_1} = \frac{dF(\eta)}{d\eta} \tag{2}$$

$$\frac{u_2}{U_2} = \frac{dG(\eta)}{d\eta} \tag{3}$$

[a]The geodesic curvature at a point P of a curve embedded in a surface in space is numerically equal to the curvature of the plane curve obtained by projecting the surface curve onto the tangent plane to the surface at P.

where $F(\eta)$ and $G(\eta)$ are as yet undetermined functions of the similarity variable η defined as

$$\eta = \frac{y^*}{\sqrt{\nu}} g(x_1,x_2) = yg(x_1,x_2) \tag{4}$$

and where $g(x_1,x_2)$ is an arbitrary function of x_1 and x_2. (See ref. 13 for a detailed explanation on specification of the form of η.)

For the case of one mainstream velocity component identically zero, it can be assumed without loss of generality (from symmetry considerations) that $U_2 \equiv 0$ and u_1 and u_2 are expressible as

$$\frac{u_1}{U_1} = \frac{dF(\eta)}{d\eta} \tag{5}$$

$$\frac{u_2}{U_2^*} = \frac{dG(\eta)}{d\eta} \tag{6}$$

where U_2^* is a function that will be determined in the analysis.

Transformed Equations for U_1 and U_2 Not Identically Zero

From the continuity equation, equation (1c), and the definitions for u_1 and u_2, it is possible to obtain an expression for the boundary-layer velocity component v. Substituting equations (2) and (3) into equation (1c) and using equation (4) give

$$U_1F'k_2 + \frac{U_1F''}{h_1} y \frac{\partial g}{\partial x_1} + \frac{F'}{h_1}\frac{\partial U_1}{\partial x_1} + U_2G'k_1 + \frac{U_2G''}{h_2} y \frac{\partial g}{\partial x_2} +$$

$$\frac{G'}{h_2}\frac{\partial U_2}{\partial x_2} + \frac{1}{\sqrt{\nu}}\frac{\partial v}{\partial y} = 0 \tag{7}$$

Solving equation (7) for $\partial v/\partial y$ and integrating from 0 to y with respect to y result in

$$v = -\frac{\sqrt{\nu}}{g}\left[F\left(U_1k_2 + \frac{1}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{U_1}{h_1}\frac{\partial \ln g}{\partial x_1}\right) + G\left(U_2k_1 + \frac{1}{h_2}\frac{\partial U_2}{\partial x_2} - \frac{U_2}{h_2}\frac{\partial \ln g}{\partial x_2}\right) + \frac{U_1}{h_1}\frac{\partial \ln g}{\partial x_1}\eta F' + \frac{U_2}{h_2}\frac{\partial \ln g}{\partial x_2}\eta G'\right] + f(x_1,x_2) \tag{8}$$

where

$$f(x_1,x_2) = v(x_1,x_2,0) + \frac{\sqrt{\nu}}{g}\left[F(0)\left(U_1k_2 + \frac{1}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{U_1}{h_1}\frac{\partial \ln g}{\partial x_1}\right) + G(0)\left(U_2k_1 + \frac{1}{h_2}\frac{\partial U_2}{\partial x_2} - \frac{U_2}{h_2}\frac{\partial \ln g}{\partial x_2}\right)\right]$$

At this point, it would be well to discuss restrictions on the functions $F(\eta)$ and $G(\eta)$ resulting from the boundary conditions on u_1, u_2, and v. From equations (2), (3), and (8):

(1) The boundary conditions $u_1 = u_2 = 0$ for $y = 0$ imply

$$F'(0) = G'(0) = 0$$

(2) The conditions $\lim\limits_{y\to\infty} u_1 = U_1$ and $\lim\limits_{y\to\infty} u_2 = U_2$ imply

$$\lim_{\eta\to\infty} F'(\eta) = 1 \quad \text{and} \quad \lim_{\eta\to\infty} G'(\eta) = 1$$

(3) The condition $v = 0$ for $y = 0$ implies

$$\frac{g}{\sqrt{\nu}} f(x_1,x_2) = F(0)\left(U_1k_2 + \frac{1}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{U_1}{h_1}\frac{\partial \ln g}{\partial x_1}\right) + G(0)\left(U_2k_1 + \frac{1}{h_2}\frac{\partial U_2}{\partial x_2} - \frac{U_2}{h_2}\frac{\partial \ln g}{\partial x_2}\right)$$

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Now, it can be shown by a slight extension of an argument presented in reference 13 that there is no loss of generality if it is assumed that $F(0) = 0$ and $G(0) = 0$, which in turn implies $f(x_1,x_2) = 0$. This result will be used in the following development.

Substitution of equations (2), (3), and (8) into equations (1a) and (1b) yields, respectively,

$$\frac{1}{h_1}\frac{\partial U_1}{\partial x_1}F'^2 - \left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{U_1}{2h_1}\frac{\partial \ln g^2}{\partial x_1} + U_1k_2\right)FF'' - g^2F''' +$$
$$\left(\frac{U_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2k_1\right)G'F' - \left(\frac{1}{h_2}\frac{\partial U_2}{\partial x_2} - \frac{U_2}{2h_2}\frac{\partial \ln g^2}{\partial x_2} + U_2k_1\right)GF'' -$$
$$\frac{U_2^2}{U_1}k_2G'^2 - \left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} + \frac{U_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2k_1 - \frac{U_2^2}{U_1}k_2\right) = 0 \qquad (9)$$

$$\frac{1}{h_2}\frac{\partial U_2}{\partial x_2}G'^2 - \left(\frac{1}{h_2}\frac{\partial U_2}{\partial x_2} + U_2k_1 - \frac{U_2}{2h_2}\frac{\partial \ln g^2}{\partial x_2}\right)GG'' - g^2G''' +$$
$$\left(\frac{U_1}{h_1}\frac{\partial \ln U_2}{\partial x_1} + U_1k_2\right)G'F' - \left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{U_1}{2h_1}\frac{\partial \ln g^2}{\partial x_1} + U_1k_2\right)FG'' -$$
$$\frac{U_1^2}{U_2}k_1F'^2 - \left(\frac{U_1}{h_1}\frac{\partial \ln U_2}{\partial x_1} + \frac{1}{h_2}\frac{\partial U_2}{\partial x_2} + U_1k_2 - \frac{U_1^2}{U_2}k_1\right) = 0 \qquad (10)$$

Equations (9) and (10) will be termed the "transformed equations," and the principal problem of concern herein will be to find the necessary and sufficient conditions for reducing the transformed equations to a system of ordinary differential equations. Before this step is undertaken, however, the transformed equations will be determined for the case when one of the mainstream velocity components is identically zero.

Transformed Equations for One Mainstream Velocity Component Identically Zero

Now consider the case $U_2 \equiv 0$ and define u_1 and u_2 by equations (5) and (6). Again, the boundary-layer velocity component v can be obtained. The form for v will be identical to that given in equation (8), with U_2 replaced by U_2^*.

Substitution of the expressions for u_1, u_2, and v into equations (1a) and (1b), respectively, then gives

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$$\frac{1}{h_1}\frac{\partial U_1}{\partial x_1}F'^2 - \left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} + U_1k_2 - \frac{U_1}{2h_1}\frac{\partial \ln g^2}{\partial x_1}\right)FF'' - g^2F''' +$$
$$\left(\frac{U_2^*}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2^*k_1\right)G'F' - \left(\frac{1}{h_2}\frac{\partial U_2^*}{\partial x_2} - \frac{U_2}{2h_2}\frac{\partial \ln g^2}{\partial x_2} + U_2^*k_1\right)GF'' -$$
$$\frac{U_2^{*2}}{U_1}k_2G' - \frac{1}{h_1}\frac{\partial U_1}{\partial x_1} = 0 \tag{11}$$

$$\frac{1}{h_2}\frac{\partial U_2^*}{\partial x_2}G'^2 - \left(\frac{1}{h_2}\frac{\partial U_2^*}{\partial x_2} + U_2^*k_1 - \frac{U_2^*}{2h_2}\frac{\partial \ln g^2}{\partial x_2}\right)GG'' - g^2G''' +$$
$$\left(\frac{U_1}{h_2}\frac{\partial \ln U_2^*}{\partial x_1} + U_1k_2\right)G'F' - \left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{U_1}{2h_1}\frac{\partial \ln g^2}{\partial x_1} + U_1k_2\right)FG'' -$$
$$\frac{U_1^2}{U_2^*}k_1F'^2 - \frac{U_1^2}{U_2^*}k_1 = 0 \tag{12}$$

Equations (11) and (12) are of exactly the same form as equations (9) and (10) except for the absence of several terms which are not coefficients of terms involving F, G, or their derivatives. This fact will be important in determining the general conditions necessary for reducing the transformed equations to ordinary differential equations.

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It should be stated that the boundary conditions for $G'(\eta)$ for this case are given by

$$G'(0) = 0$$

$$\lim_{\eta \to \infty} G'(\eta) = 0$$

The boundary conditions on $F'(\eta)$ are the same as for the previous case.

Necessary Conditions for Obtaining Similarity Solutions

It is obvious that equations (9) and (10) or equations (11) and (12) become ordinary differential equations if the coefficients that depend on x_1 and x_2 are proportional. This assumption will now be introduced (hereafter called assumption A). The assumption of proportionality is a sufficient condition for the solution of the system of equations to depend only on η. Whether this is a necessary condition will be discussed on page 40. At present, proportionality of the coefficient is assumed simply as an additional hypothesis under which to solve the systems. First, examine the implications of this assumption relative to equations (9) and (10). As the function g^2 serves as a coefficient in both equations, the general requirements are as follows:

$$g^2 = a_1 \frac{1}{h_1}\frac{\partial U_1}{\partial x_1} = a_2\left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} - \frac{U_1}{2h_1}\frac{\partial \ln g^2}{\partial x_1} + U_1 k_2\right)$$

$$= a_3\left(\frac{U_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2 k_1\right) = a_4\left(\frac{1}{h_2}\frac{\partial U_2}{\partial x_2} - \frac{U_2}{2h_2}\frac{\partial \ln g^2}{\partial x_2} + U_2 k_1\right) = a_5 \frac{U_2^2}{U_1} k_2$$

$$= a_6\left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} + \frac{U_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2 k_1 - \frac{U_2^2}{U_1} k_2\right) = a_7 \frac{1}{h_2}\frac{\partial U_2}{\partial x_2}$$

$$= a_8\left(\frac{U_1}{h_2}\frac{\partial \ln U_2}{\partial x_1} + U_1 k_2\right) = a_9 \frac{U_1^2}{U_2} k_1$$

$$= a_{10}\left(\frac{U_1}{h_1}\frac{\partial \ln U_2}{\partial x_1} + \frac{1}{h_2}\frac{\partial U_2}{\partial x_2} + U_1 k_2 - \frac{U_1^2}{U_2} k_1\right) \qquad (13)$$

Now, the following observations are made. First of all, g^2 cannot be identically zero (see eq. (4)). Therefore, if the individual terms $\frac{1}{h_1}\frac{\partial U_1}{\partial x_1}$, $\frac{U_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2 k_1$, and $\frac{U_2^2}{U_1} k_2$ are each proportional to g^2 (or identically zero), the term $\left(\frac{1}{h_1}\frac{\partial U_1}{\partial x_1} + \frac{U_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2 k_1 - \frac{U_2^2}{U_1} k_2\right)$ will be proportional to g^2 (or identically zero). Similarly, if $\left(\frac{U_1}{h_2}\frac{\partial \ln U_2}{\partial x_1} + U_1 k_2\right)$, $\frac{1}{h_2}\frac{\partial U_2}{\partial x_2}$, and $\frac{U_1^2}{U_2} k_1$ are each proportional to g^2 (or identically zero), the term $\left(\frac{U_1}{h_1}\frac{\partial \ln U_2}{\partial x_1} + \frac{1}{h_2}\frac{\partial U_2}{\partial x_1} + U_1 k_2 - \frac{U_1^2}{U_2} k_1\right)$ will be proportional to g^2 (or identically zero). The conditions imposed by equation (13) therefore reduce to the requirement that nine coefficients be proportional. Furthermore, various terms in certain of these coefficients can be deleted when these terms are required to be individually proportional to g^2. This leaves the following list of nine terms that must be mutually proportional (or identically zero) if similarity solutions of equations (9) and (10) are to be obtained:

$$\left.\begin{array}{ll}
(1) & \frac{1}{h_1}\frac{\partial U_1}{\partial x_1} \\[2ex]
(2) & U_1 k_2 - \frac{U_1}{2h_1}\frac{\partial \ln g^2}{\partial x_1} \\[2ex]
(3) & g^2 \\[2ex]
(4) & \frac{U_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + U_2 k_1 \\[2ex]
(5) & U_2 k_1 - \frac{U_2}{2h_2}\frac{\partial \ln g^2}{\partial x_2} \\[2ex]
(6) & \frac{U_2^2}{U_1} k_2 \\[2ex]
(7) & \frac{1}{h_2}\frac{\partial U_2}{\partial x_2} \\[2ex]
(8) & \frac{U_1}{h_1}\frac{\partial \ln U_2}{\partial x_1} + U_1 k_2 \\[2ex]
(9) & \frac{U_1^2}{U_2} k_1
\end{array}\right\} \qquad (14)$$

Proportionality among the preceding terms is equivalent to specifying a system of partial differential equations. Solutions of the equations determine mainstream velocity components U_1 and U_2 and the components h_1 and h_2 of the metric tensor associated with the orthogonal coordinate system. As solutions of these equations will result in equations (9) and (10) being reduced to ordinary differential equations, a stated proportionality between any two terms will be called "an ordinary differential equation condition" and will be abbreviated "o.d.e. condition."

If o.d.e. conditions are set up for equations (11) and (12), it will follow that these conditions will be exactly the same as those given for equations (9) and (10), except for the replacement of U_2 by U_2^*. Hence, solutions for various functions in one case will correspond exactly to those in the other case. In setting up the equations for the o.d.e. conditions, $\overline{U}_2$ is therefore used to denote either U_2 or U_2^*.

A general analysis would now involve finding solutions of the system of equations derived from set (14) without further restricting assumptions on the nature of the unknown quantities. As mentioned in the INTRODUCTION, this most general problem has not as yet been analyzed because of inherent complexity. Consequently, the problem will first be attacked under the assumption that the flow surface is developable. This is equivalent to specifying that the Gaussian curvature K of surface is identically zero. In turn, this assumption leads to an additional equation involving h_1 and h_2.

Following the analysis for developable surfaces, the problem will be solved under the assumption that $U_1/\overline{U}_2$ is constant with no restriction on surface geometry. It will be shown later that a consequence of this assumption (for $\overline{U}_2 \equiv U_2$) is that streamlines of the main-flow cross coordinates lines at a constant angle.

The requirements for similarity solutions will be derived according to the following topical scheme. (No attempt will be made here to solve the corresponding ordinary differential equations that result.)

(1) Hypothesis: $K = 0$ (pp. 12-24)

(a) $k_1^2 + k_2^2 \neq 0$

(b) $k_1 = 0$, $k_2 = 0$

(2) Hypothesis: $U_1 = c\overline{U}_2$ (pp. 25-38)

(a) $k_1 k_2 \neq 0$

1. k_1 and k_2 nonconstant
2. k_1 and k_2 constant

($k_1 k_2 \neq 0$ and only one choice of k_1 constant will be shown to be impossible)

(b) $k_1 k_2 = 0$

ANALYSIS OF POSSIBLE SIMILARITY SOLUTIONS FOR FLOW OVER SURFACES ISOMETRIC WITH THE EUCLIDEAN PLANE

In the following sections it is assumed that the flow takes place over a developable surface (i.e., a surface isometric with the Euclidean plane). Relative to this assumption, solutions for h_1, h_2, k_1, k_2, U_1, U_2, and g^2 will be obtained from the o.d.e. conditions. Before this can be carried out, however, certain relations are needed from the intrinsic geometry of surfaces. These relations will now be developed.

Coordinate Curvature Relations for a Surface of Zero Gaussian Curvature

Two surfaces are said to be isometric if it is possible to find a coordinate system embedded in one surface which has the same surface metric tensor components as a coordinate system embedded in the other surface. Furthermore, if two surfaces are isometric, an invariant exists called the "total" or "Gaussian" curvature, which will be the same for both surfaces. For a system of orthogonal coordinates it can be shown that the expression for the Gaussian curvature is

$$K = -\frac{1}{2h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{h_1h_2}\frac{\partial h_2^2}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{h_1h_2}\frac{\partial h_1^2}{\partial x_2}\right)\right] \tag{15}$$

(See ref. 20, p. 169.)

Substituting the expressions for coordinate geodesic curvature into equation (15) gives

$$K = -\left(\frac{1}{h_1}\frac{\partial k_2}{\partial x_1} + \frac{1}{h_2}\frac{\partial k_1}{\partial x_2}\right) - \left(k_1^2 + k_2^2\right) \tag{16}$$

Now employ a theorem from the geometry of surfaces which states that a necessary and sufficient condition that a surface be isometric with the Euclidean plane is that the Gaussian curvature be zero (ref. 20, p. 168). Hence, for the case under consideration, from equation (16),

$$k_1^2 + k_2^2 = -\left(\frac{1}{h_1}\frac{\partial k_2}{\partial x_1} + \frac{1}{h_2}\frac{\partial k_1}{\partial x_2}\right) \tag{17}$$

With this condition relating the coordinate curvatures and metric tensor components, and with the o.d.e. conditions referred to in the previous section, it is possible to determine the permissible coordinate systems and mainstream flows leading to similarity solutions of the boundary-layer equations. Equation (17) clearly shows that k_1 and k_2 cannot be nonzero constants.

In general, it will be necessary to make certain initial assumptions regarding the numbered terms appearing in set (14) before a unique set of o.d.e. conditions can be considered for analysis. This follows from the fact that an equation expressing proportionality between two terms can only make sense if the terms are not identically zero. Consideration is first given to coordinate systems in which $k_1 \not\equiv 0$ and $k_2 \not\equiv 0$.

Reduction of O.D.E. Conditions for $k_1 \not\equiv 0$, $k_2 \not\equiv 0$

Under the assumption that $k_1 \not\equiv 0$ and $k_2 \not\equiv 0$, the terms numbered 3, 6, and 9 in set (14) cannot be identically zero. Hence, the following o.d.e. conditions prevail:

$$\frac{\bar{U}_2^2}{U_1} k_2 = a_1 g^2 \qquad a_1 \neq 0 \tag{18}$$

$$\frac{U_1^2}{\bar{U}_2} k_1 = a_2 g^2 \qquad a_2 \neq 0 \tag{19}$$

Differentiating equation (18) with respect to x_1 gives

$$\frac{\partial k_2}{\partial x_1} = a_1 \left(\frac{U_1}{\overline{U}_2^2} \frac{\partial g^2}{\partial x_1} + \frac{g^2}{\overline{U}_2^2} \frac{\partial U_1}{\partial x_1} - 2 \frac{U_1 g^2}{\overline{U}_2^3} \frac{\partial \overline{U}_2}{\partial x_1} \right)$$

$$= a_2 \frac{U_1}{\overline{U}_2^2} g_1^2 \left(\frac{\partial \ln g^2}{\partial x_1} + \frac{\partial \ln U_1}{\partial x_1} - 2 \frac{\partial \ln \overline{U}_2}{\partial x_1} \right)$$

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$$= k_2 \left(\frac{\partial \ln g^2}{\partial x_1} + \frac{\partial \ln U_1}{\partial x_1} - 2 \frac{\partial \ln \overline{U}_2}{\partial x_1} \right) \tag{20}$$

Similarly, by differentiating equation (19) with respect to x_2 it follows that

$$\frac{\partial k_1}{\partial x_2} = k_1 \left(\frac{\partial \ln g^2}{\partial x_2} + \frac{\partial \ln \overline{U}_2}{\partial x_2} - 2 \frac{\partial \ln U_1}{\partial x_2} \right) \tag{21}$$

It will now be shown that the right sides of equations (20) and (21) can be simplified in form. Following the simplification, equations (20) and (21) will be substituted into equation (17), and a relation between U_1 and $\overline{U}_2$ will be established.

The assumed proportionality between the coefficients listed in set (14) leads to the following set of statements:

(1) From the o.d.e. condition for ② and ③,

$$\frac{\partial \ln g^2}{\partial x_1} = \left(k_2 + \frac{a_3 g^2}{U_1} \right) 2h_1$$

(2) Either $\dfrac{\partial \ln U_1}{\partial x_1} \equiv 0$, or from the o.d.e. condition for ① and ③,

$$\frac{\partial \ln U_1}{\partial x_1} = a_4 \frac{g^2 h_1}{U_1}$$

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(3) From the o.d.e. condition for ③ and ⑧,

$$-2 \frac{\partial \ln \overline{U}_2}{\partial x_1} = \left(k_2 + \frac{a_5 g^2}{U_1}\right) 2h_1$$

(4) From the o.d.e. condition for ③ and ⑤,

$$\frac{\partial \ln g^2}{\partial x_2} = \left(k_1 + \frac{a_6 g^2}{\overline{U}_2}\right) 2h_2$$

(5) Either $\frac{\partial \ln \overline{U}_2}{\partial x} \equiv 0$, or, as an o.d.e. condition for ③ and ⑦,

$$\frac{\partial \ln \overline{U}_2}{\partial x_2} = a_7 \frac{g^2 h_2}{\overline{U}_2}$$

(6) From the o.d.e. condition for ③ and ④,

$$-2 \frac{\partial \ln U_1}{\partial x_2} = \left(k_1 + \frac{a_8 g^2}{\overline{U}_2}\right) 2h_2$$

From these observations it follows that equations (20) and (21) must be expressible as follows:

$$\frac{\partial k_2}{\partial x_1} = 2k_1 h_2 \left(nk_2 + a_9 \frac{g^2}{U_1}\right) \tag{22}$$

$$\frac{\partial k_1}{\partial x_2} = 2k_1 h_2 \left(mk_1 + a_{10} \frac{g^2}{\overline{U}_2}\right) \tag{23}$$

where

$$n = 0,\ 1,\ \text{or}\ 2$$

$$m = 0,\ 1,\ \text{or}\ 2$$

As a consequence of the hypothesis $k_1 \not\equiv 0$, $k_2 \not\equiv 0$, at least one of the quantities n, m, a_9, or a_{10} must be nonzero. Now, by using equations (22), (23), (18), and (19), equation (17) can be rewritten as

$$k_1^2\left(1 + 2m + a_{10}a_2^{-1}\frac{U_1^2}{\overline{U}_2^2}\right) + k_2^2\left(1 + 2n + a_1^{-1}a_9\frac{\overline{U}_2^2}{U_1^2}\right) = 0 \tag{24}$$

An o.d.e. condition between ⑥ and ⑨ is 4975

$$\frac{\overline{U}_2^2}{U_1}k_2 = a_{11}\frac{U_1^2}{\overline{U}_2}k_1$$

$$\therefore \left(\frac{\overline{U}_2}{U_1}\right)^3 = a_{11}\frac{k_1}{k_2} \tag{25}$$

Substituting equation (25) into equation (24) results in

$$\frac{1}{a_{11}^2}H^3\left(1 + 2m + a_{10}a_2^{-1}H^{-1}\right) + 1 + 2n + a_1^{-1}a_9 H = 0 \tag{26}$$

where

$$H = \left(\frac{\overline{U}_2}{U_1}\right)^2$$

From equation (26) and the definition of H, it follows that H must be a positive constant. Therefore, the following lemma is established.

<u>Lemma 1</u>. - Given an orthogonal coordinate system in which neither k_1 nor k_2 is identically zero, a necessary condition for equations (9) and (10) or equations (11) and (12) to possess similarity solutions under assumption A is that

$$U_1 = c_1\overline{U}_2 \quad (c_1 \text{ a constant})$$

where $\overline{U}_2 = U_2$ when considering equations (9) and (10) and $\overline{U}_2 = U_2^*$ when considering equations (11) and (12).

Furthermore, from lemma 1 and equation (25) there follows lemma 2.

Lemma 2. - Given an orthogonal coordinate system in which neither k_1 nor k_2 is identically zero, a necessary condition for equations (9) and (10) and equations (11) and (12) to possess similarity solutions under assumption A is that

$$k_1 = c_2 k_2 \quad (c_2 \text{ a constant})$$

With the establishment of lemmas 1 and 2, it now becomes possible to determine explicit forms for the coordinate-line curvature k_1 and k_2 and the functions h_1 and h_2 through solutions of o.d.e. conditions from set (14). The details of the analysis are presented in appendix B. The results obtained can be summarized in the following theorem.

Theorem 1. - Given an orthogonal, curvilinear coordinate system embedded in a surface of zero Gaussian curvature and in which neither k_1 nor k_2 is identically zero and neither U_1 nor $\overline{U}_2$ is identically zero. Then, necessary and sufficient conditions for equations (9) and (10) or equations (11) and (12) to be reduced to ordinary differential equations under a similarity transformation and assumption A are that

(1) $$h_1 = ap_1'(x_1)q_2(x_2) \qquad (a \neq 0) \qquad p_1'(x_1) \not\equiv 0$$

$$h_2 = bp_1(x_1)q_2'(x_2) \qquad (b \neq 0) \qquad q_2'(x_2) \not\equiv 0$$

(2) $$k_1 = \frac{1}{bp_1(x_1)q_2(x_2)}$$

$$k_2 = \frac{1}{ap_1(x_1)q_2(x_2)}$$

(3) $$g^2 = C_1 p_1^{n-1}(x_1) q_2^{m-1}(x_2) \qquad C_1 \neq 0$$

(4) $$U_1 = C_2 p_1^n(x_1) q_2^m(x_2) \qquad C_2 \neq 0$$

$$\overline{U}_2 = C_3 p_1^n(x_1) q_2^m(x_2) \qquad C_3 \neq 0$$

Equations (9) and (10) become, respectively,

(5) $$A_1 F'^2 - A_2 FF'' - A_3 F''' + A_4 G'F' - A_5 GF'' - A_6 G'^2 - (A_1 + A_4 - A_6) = 0$$

$$B_1 G'^2 - A_5 GG'' - A_3 G''' + B_2 G'F' - A_2 FG'' - B_3 F'^2 - (B_1 + B_2 + B_3) = 0$$

where

$$A_1 = \frac{C_2 n}{a} \qquad A_6 = \frac{C_3^2}{C_2 a}$$

$$A_2 = \frac{(n+3)C_2}{2a} \qquad B_1 = \frac{C_3 m}{b}$$

$$A_3 = C_1 \qquad B_2 = \frac{C_2(n+1)}{a}$$

$$A_4 = \frac{C_3}{b}(m+1) \qquad B_3 = \frac{C_2^2}{bC_3}$$

$$A_5 = \frac{C_3}{2b}(m+3)$$

Equations (11) and (12) reduce to, respectively, the same forms as those given in (5), except that the first equation has only A_6 as a constant term and the second has only B_3 as a constant term.

Coordinate Systems for Which $k_1 \not\equiv 0$, $k_2 \not\equiv 0$

The following two questions are now posed:

(1) Is there a unique coordinate system corresponding to the metric components h_1 and h_2 defined by condition (1) of Theorem 1 ?

(2) What is the nature of coordinate systems defined by these equations ?

The approach to the previous questions can be simplified somewhat by carrying out the following transformation of coordinates. Let

$$\left.\begin{aligned} X_1 &= p_1(x_1) \\ X_2 &= q_2(x_2) \end{aligned}\right\} \tag{27}$$

The square of the differential of arc length in the (X_1,X_2) system is then given by

$$(ds)^2 = a^2X_2^2(dX_1)^2 + b^2X_1^2(dX_2)^2 \tag{28}$$

The coordinate lines X_1 = constant will coincide with x_1 = constant, and the lines X_2 = constant will coincide with x_2 = constant. The metric components are given by

$$\left.\begin{aligned} h_1^* &= aX_2 \\ h_2^* &= bX_1 \end{aligned}\right\} \qquad (29)$$

The principal difference between the (X_1,X_2) system and the (x_1,x_2) system will be in the measurement of length along the coordinate lines. A transformation of the previous type will hereafter be called a "change of scale" transformation.

The surface in which our coordinate system is embedded is assumed to be isometric with the Euclidean plane. Hence, the question of whether or not a unique coordinate system exists with metric components defined by equations (29) is equivalent to asking whether or not a unique transformation of variables exists of the form

$$\left.\begin{aligned} Y_1 &= Y_1(X_1,X_2) \\ Y_2 &= Y_2(X_1,X_2) \end{aligned}\right\} \qquad (30)$$

where Y_1 and Y_2 are Cartesian coordinates; that is,

$$(ds)^2 = (dY_1)^2 + (dY_2)^2$$

This question is discussed in reference 20. It is shown in this reference that a necessary and sufficient condition for a suitable transformation of the form (30) to exist is that the Riemann-Christoffel tensor formed from the h_1^* be a zero tensor. For a surface having zero Gaussian curvature, this condition is satisfied. Furthermore, the transformation will define a unique rectangular coordinate system except for possible translations and rotations.

The second question on the nature of the coordinate system requires a more detailed investigation, which is presented in appendix C. It is shown there that the coordinate lines become logarithmic spirals when referred to the Euclidean plane as a developable surface. If ρ and θ denote polar coordinates, the equations for the coordinate lines $X_1 = X_1^o$ = constant and $X_2 = X_2^o$ = constant are given, respectively, by

$$\rho = c(X_1^o)^{d^2} e^{-\theta d} \qquad (C8)$$

$$\rho = c(X_2^o)^{1/d^2} e^{\theta/d} \qquad (C9)$$

where

$$c^2 = \frac{a^2b^2}{a^2 + b^2}$$

$$d^2 = \frac{a^2}{b^2}$$

A typical network of such coordinate lines is shown in figure 2. 4975

Permissible main-flow streamline shapes in spiral coordinate system ($k_1 \neq 0$, $k_2 \neq 0$). - The equations for main-flow streamlines in the (x_1,x_2) coordinate system or equivalently in the (X_1,X_2) system are solutions of the equation

$$\frac{U_1}{U_2} = \frac{h_1}{h_2}\frac{dx_1}{dx_2} = \frac{h_1^*}{h_2^*}\frac{dX_1}{dX_2} = \frac{aX_2}{bX_1}\frac{dX_1}{dX_2} \tag{31}$$

From lemma 1, $U_1/\overline{U}_2 = c_1$. Hence,

$$\frac{aX_2}{bX_1}\frac{dX_1}{dX_2} = c_1 \tag{32}$$

Equation (32) has the solution

$$X_1 = (\text{const.})\ X_2^r \tag{33}$$

where

$$r = \frac{c_1 b}{a} \neq 0$$

This, then, is the defining equation for the permissible main-flow streamlines in the (X_1,X_2) coordinate system. Once again, a more familiar form for equation (33) can be obtained by expressing the equation in terms of (ρ,θ) coordinates. The transformation between the (ρ,θ) and (X_1,X_2) coordinate systems is given in appendix C by the equations

$$\left.\begin{aligned} \rho &= cX_1X_2 \\ \theta &= d \ln X_1 - \frac{\ln X_2}{d} \end{aligned}\right\} \tag{C5}$$

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Substituting equation (33) into (C5) gives

$$\left.\begin{aligned} \rho &= \rho_0 X_2^{r+1} \quad (\rho_0 \ \text{ a constant}) \\ \theta &= \left(dr - \frac{1}{d}\right) \ln X_2 + \theta_0 \quad (\theta_0 \ \text{ a constant}) \end{aligned}\right\} \tag{34}$$

If X_2 is eliminated from equations (34),

$$\frac{\rho}{\rho_0} = e^{m(\theta - \theta_0)} \tag{35}$$

where

$$m = \frac{d(r + 1)}{d^2 r - 1} \qquad (d^2 r \neq 1)$$

If $m \neq 0$, equation (35) is the equation for families of logarithmic spirals in a planar polar coordinate system. If $m = 0$, there are circles in such a system. Finally, if $d^2 r = 1$, equation (34) gives θ = constant, that is, radial lines in a planar polar coordinate system.

Transformation of coordinates to a basic system. - Before leaving the analysis of this system, both the form for $(ds)^2$ and the equation of the main-flow streamlines will be put in a particular form for later reference. First, a scale transformation is introduced:

$$X_1 = e^{\overline{X}_1/a}$$

$$X_2 = e^{\overline{X}_2/b}$$

Equation (28) then becomes

$$(ds)^2 = e^{2\left(\frac{\overline{X}_1}{a} + \frac{\overline{X}_2}{b}\right)} \left[(d\overline{X}_1)^2 + (d\overline{X}_2)^2\right] \tag{36}$$

The equation for $U_1 = c_2 \overline{U}_2$ becomes

$$U_1 = e^{n'\overline{X}_1 + m'\overline{X}_2} \tag{37}$$

In the $(\overline{X}_1,\overline{X}_2)$ coordinate system the equation of the main-flow streamlines assumes a particularly simple form. As the metric tensor components are identical, the governing differential equation is

$$\frac{d\overline{X}_2}{d\overline{X}_1} = \frac{\overline{U}_2}{\overline{U}_1} = \text{const.}$$

Hence, the streamline equation is 4975

$$a'\overline{X}_1 + b'\overline{X}_2 + c' = 0$$

It is of interest to note that the streamlines in the various coordinate systems previously employed cross the coordinate lines at constant angles. This follows from an examination of the formula for the cosine of the angle θ between a streamline and a coordinate line. For flows in which the mainstream velocity component $U_2 \not\equiv 0$ and $U_1 = c_1 U_2$, $(h_1\ dx_1)/(h_2\ dx_2) = c_1$ on a streamline in a coordinate system (x_1,x_2). The cosine of the angle between the streamline and an x_1 coordinate line is then (ref. 20, p. 150)

$$\cos\ \theta = h_1^2 \frac{dx_1}{(h_1\ dx_1)} \cdot \frac{dx_1}{(h_1\ dx_1)(1 + c_2^2)} = \frac{1}{1 + c_2^2}$$

$$\therefore\ \theta = \text{constant}$$

A similar result holds for the angle between the streamline and an x_2-coordinate line.

Reduction of O.D.E. Conditions When One Coordinate Curvature is Identically Zero

Now solutions of the o.d.e. conditions are considered when the curvature of one set of coordinate lines vanishes while the other curvature does not identically vanish. Initially, it is assumed that $k_1 \equiv 0$ and $k_2 \not\equiv 0$. The case $k_2 \equiv 0$, $k_1 \not\equiv 0$ will follow directly from symmetry considerations.

The analysis based on the o.d.e. conditions and $K = 0$ leads directly to the following theorem, which is proved in appendix D for $k_1 \equiv 0$, $k_2 \not\equiv 0$.

Theorem 2. - Let (x_1,x_2) be a curvilinear orthogonal coordinate system embedded in a surface of zero Gaussian curvature and in which

k_2 is zero and k_1 is not identically zero. Then, a necessary condition for equations (9) and (10) and equations (11) and (12) to possess similarity solutions under assumption A is that $h_1 = \frac{p(x_1)}{q(x_2)}$ and $h_2 = -\frac{q'(x_2)}{q^2(x_2)}$ where $p(x_1)$ and $q'(x_2)$ are not identically zero. (Indices may also be interchanged uniformally in the previous statements.)

The form of k_1 and k_2 indicates that the coordinate system (x_1,x_2) is a modified polar coordinate system when referred to a plane. That is, the system (x_1,x_2) differs from the usual polar coordinate system only in scale variations.

The analysis of similarity solutions for polar-type coordinate systems is completely documented in reference 14. The coordinate system of this reference can be defined by $h_1 = x_2$ and $h_2 = 1$. The analysis presented in reference 14 shows that only one form for $U_1 = c_2\overline{U}_2$ is possible. This form is

$$U_1 = ax_2^n e^{mx_1}$$

This case is investigated further by introducing a scale change transformation

$$x_1 = X_1, \ x_2 = e^{X_2}$$

The equation for $(ds)^2$ on the surface then becomes

$$(ds)^2 = x_2^2(dx_1)^2 + (dx_2)^2 = e^{2X_2}\left[(dX_1)^2 + (dX_2)^2\right]$$

Finally, by an orthogonal transformation,

$$X_1 = \sin\alpha\,\overline{X}_1 - \cos\alpha\,\overline{X}_2 \quad (\alpha \text{ a constant})$$

$$X_2 = \cos\alpha\,\overline{X}_1 + \sin\alpha\,\overline{X}_2$$

The equation for $(ds)^2$ can be written

$$(ds)^2 = e^{2(\sin\alpha\,\overline{X}_1 + \cos\alpha\,\overline{X}_2)}\left[(d\overline{X}_1)^2 + (d\overline{X}_2)^2\right]$$

This is basically the same expression as equation (36). The equation for $U_1 = c_2\overline{U}_2$ becomes

$$U_1 = (\text{const.})e^{n'\overline{X}_1 + m'\overline{X}_2}$$

which corresponds to equation (37). Therefore, the following important theorem can be stated.

Theorem 3. - The boundary-layer flow problems that can be solved by employing the coordinate system and associated mainstream flows of Theorem 1 are identical to those that can be solved by employing the coordinate system and associated mainstream flows of Theorem 2. 4975

Reduction of O.D.E. Conditions When Both

Coordinate Curvatures Vanish

The final case to be considered is $k_1 \equiv 0$ and $k_2 \equiv 0$. The coordinate lines will therefore be geodesics of the surface. In the plane such a system simply becomes a rectangular, Cartesian coordinate system. This particular problem is discussed in reference 13 and, hence, details will be omitted. The results of this analysis with $h_1 = h_2 = 1$ give the following principal forms for U_1, U_2, and g^2 (note: U_2 here corresponds to an actual velocity component):

(1) $U_1 = ae^{nx_1}x_2^{m-1}$ $\qquad g^2 = \text{const. } U_1$

$U_2 = be^{nx_1}x_2^{m}$

(2) $U_1 = ax_1^{n}x_2^{m-1}$ $\qquad g^2 = \text{const. } \dfrac{U_1}{x_1}$

$U_2 = bx_1^{n-1}x_2^{m}$

(3) $U_1 = ax_1^{n}$ $\qquad g^2 = \text{const. } \dfrac{U_1}{x_1}$

$U_2 = bx_1^{m}$

(4) $U_1 = ae^{nx_1}$ $\qquad g^2 = \text{const. } U_1$

$U_2 = be^{mx_1}$

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CZ-4

ANALYSIS OF POSSIBLE SIMILARITY SOLUTIONS FOR $U_1/\overline{U}_2$ = CONSTANT

Throughout the previous main section it has been assumed that the Gaussian curvature of the surface K was identically zero. As a consequence of this definition, it was determined that $U_1 = c_1\overline{U}_2$ in every case where one of the coordinate curvatures was nonzero. In the present section it will be assumed at the outset that $U_1 = c_1\overline{U}_2$, and no assumptions regarding the nature of K will be made.

Reduction of O.D.E. Conditions for $k_1 \not\equiv 0$, $k_2 \not\equiv 0$

If it is assumed that $k_1 \not\equiv 0$ and $k_2 \not\equiv 0$ under the basic assumption that $U_1 = c_1\overline{U}_2$, the o.d.e. condition for ⑥ and ⑨ yields

$$k_1 = c_2 k_2$$

Two possibilities now exist. One possibility is that k_1 and k_2 are constant. (This was not allowed in the previous analysis except for $k_1 = k_2 = 0$.) The other possibility is that k_1 and k_2 are nonconstant. If k_1 and k_2 are assumed nonconstant, the analysis from the beginning of appendix B to equation (B13) will apply since the analysis used only o.d.e. conditions and the results of lemmas 1 and 2. No recourse was made to $K = 0$ in that section. Hence, by substituting the expressions from equations (B1), (B4), and (B13) into equation (16), the following expression for K is obtained:

$$K = -\left(k_1^2 + k_2^2\right)\left(1 + \frac{1}{d_1}\right)$$

or

$$K = -k_2^2\left(c_2^2 + 1\right)\left(1 + \frac{1}{d_1}\right) \tag{38}$$

On the other hand, if $k_1 = c_2 k_2$ = const., there results from equation (16):

$$K = -k_2^2\left(c_2^2 + 1\right) \tag{39}$$

Permissible forms for h_1, h_2, k_1, k_2, and $\overline{U}_2$. - In attempting to determine permissible forms, we will initially distinguish between the case where k_1 and k_2 are nonconstant and the case where both are constant. First consider the nonconstant case.

From equations (B7) and (B13), the following relation occurs between h_1 and h_2:

$$h_2 = h_1\left[\frac{f_2(x_1)}{f_1(x_2)}\right]^{d_2/c_2}$$

This equation can be rewritten as

$$h_2 = h_1 f_{10}(x_1) f_{11}(x_2) \tag{40}$$

As $k_1 = c_2 k_2$, equation (B10) is valid:

$$\frac{\partial h_1}{\partial x_2} = c_2 \frac{\partial h_2}{\partial x_1} \tag{B10}$$

Substitution of equation (40) into (B10) then gives

$$\frac{\partial h_1}{\partial x_2} = c_2 f_{11}(x_2)\left[f_{10}(x_1)\frac{\partial h_1}{\partial x_1} + h_1 f_{10}'(x_1)\right]$$

which in turn can be written as

$$\frac{\partial \ln h_1}{\partial x_2} - c_2 f_{11}(x_2) f_{10}(x_1)\frac{\partial \ln h_1}{\partial x_1} = c_2 f_{11}(x_2) f_{10}'(x_1) \tag{41}$$

Equation (41) is a first-order linear partial differential equation that can be solved by classical methods. In particular, if the method of Lagrange is applied, a general solution for h_1 is

$$h_1 = c_{10} f_{12}'(x_1) e^{\varphi(\tilde{u})} \tag{42}$$

where $\varphi(\tilde{u})$ is an arbitrary function of

$$\tilde{u} = f_{12}(x_1) + c_2 f_{13}(x_2) + \text{const.}$$

The functions $f_{12}(x_1)$ and $f_{13}(x_2)$ are related to $f_{10}(x_1)$ and $f_{11}(x_2)$ by

$$f_{10}(x_1) = \frac{1}{f'_{12}(x_1)} \neq 0$$

$$f_{11}(x_2) = f'_{13}(x_2) \neq 0$$

The corresponding expression for h_2 can be determined from equations (40) and (42) and is given by

$$h_2 = c_{10} f'_{13}(x_2) e^{\varphi(\tilde{u})} \tag{43}$$

From equations (42) and (43) the following expressions for k_1 and k_2 can be obtained:

$$\left.\begin{aligned} k_1 &= \frac{c_2}{c_{10}} \varphi' e^{-\varphi} \\ k_2 &= \frac{1}{c_{10}} \varphi' e^{-\varphi} \end{aligned}\right\} \tag{44}$$

If k_1 and k_2 are now considered constant,

$$\frac{\partial \ln h_1}{\partial x_2} = h_2 k_1$$

Hence,

$$\frac{\partial^2 \ln h_1}{\partial x_1 \, \partial x_2} = k_1 \frac{\partial h_2}{\partial x_1} = h_1 h_2 k_1 k_2$$

Similarly,

$$\frac{\partial^2 \ln h_2}{\partial x_1 \, \partial x_2} = h_1 h_2 k_1 k_2$$

Therefore,

$$\frac{\partial^2 \ln h_2}{\partial x_1 \, \partial x_2} = \frac{\partial^2 \ln h_1}{\partial x_1 \, \partial x_2}$$

or

$$\frac{\partial^2 \ln \frac{h_2}{h_1}}{\partial x_1 \, \partial x_2} = 0$$

The solution to the previous equation leads directly to an equation of the form of equation (40), and the analysis that follows is applicable. However, with constant k_1 and k_2, the solutions of equations (44) can be written at once as

$$\varphi = \ln \left(\tilde{u} \cdot \text{const.} + \text{const.}\right)^{-1}$$

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By means of the o.d.e. conditions it is now possible to determine all possible forms of the function φ and the remaining unknowns. Before the conditions are applied, however, the numbered coefficients in set (14) will be replaced by an equivalent set obtained from the original list after substituting $U_1 = c_1\overline{U}_2$, $k_1 = c_2k_2$, and $g^2 = (\text{const.})\,\frac{U_1^2}{\overline{U}_2}\,k_1$, following from ① and ⑨, into the various terms. This new set is the following:

$$\left.\begin{array}{l}
①^* \ \dfrac{1}{h_1}\dfrac{\partial \ln \overline{U}_2}{\partial x_1} \\[2ex]
②^* \ \dfrac{1}{h_1}\dfrac{\partial \ln \left(h_2^2/\overline{U}_2k_2\right)}{\partial x_1} \\[2ex]
③^* \ \dfrac{1}{h_2}\dfrac{\partial \ln \overline{U}_2h_1}{\partial x_2} \\[2ex]
④^* \ \dfrac{1}{h_2}\dfrac{\partial \ln \left(h_1^2/\overline{U}_2k_2\right)}{\partial x_2} \\[2ex]
⑤^* \ \dfrac{1}{h_1}\dfrac{\partial \ln h_2}{\partial x_1} \\[2ex]
⑥^* \ \dfrac{1}{h_2}\dfrac{\partial \ln \overline{U}_2}{\partial x_2} \\[2ex]
⑦^* \ \dfrac{1}{h_1}\dfrac{\partial \ln \overline{U}_2h_2}{\partial x_1} \\[2ex]
⑧^* \ \dfrac{1}{h_2}\dfrac{\partial \ln h_1}{\partial x_2}
\end{array}\right\} \quad (45)$$

Proportionality between any two of these expressions will constitute an o.d.e. condition.

As in the past, it will be necessary to make certain assumptions regarding the nature of $\overline{U}_2$ before a unique set of o.d.e. conditions can be obtained. Four cases are again distinguished. (Note: $U_1 = c_1\overline{U}_2$).

(1) Case A: $\overline{U}_2 = \text{const.}$

(2) Case B: $\dfrac{\partial\overline{U}_2}{\partial x_1} \not\equiv 0;\ \dfrac{\partial\overline{U}_2}{\partial x_2} \equiv 0$

(3) Case C: $\dfrac{\partial\overline{U}_2}{\partial x_1} \equiv 0;\ \dfrac{\partial\overline{U}_2}{\partial x_2} \not\equiv 0$

(4) Case D: $\dfrac{\partial\overline{U}_2}{\partial x_1} \not\equiv 0;\ \dfrac{\partial\overline{U}_2}{\partial x_2} \not\equiv 0$

The analysis of these four cases is presented in appendix E. The results can be summarized in the following theorem.

Theorem 4. - Let (x_1,x_2) be an orthogonal curvilinear coordinate system for which $k_1 \not\equiv 0$ and $k_2 \not\equiv 0$. Further, assume that $U_1 = (\text{const.})\overline{U}_2$. Then, necessary and sufficient conditions for equations (9) and (10) and equations (11) and (12) to be reduced to ordinary differential equations under assumption A and the given similarity transformation are that

(1) $h_1 = c_1 p'(x_1) e^{\varphi(\tilde{u})}$

$h_2 = c_1 q'(x_2) e^{\varphi(\tilde{u})}$

where $\tilde{u} = p(x_1) + c_2 q(x_2) + c_3$. (The symbols c_1, c_2, and c_3 are not to be confused with previous cases where these constants were employed.)

(2) The following sets of conditions hold for $\overline{U}_2$ and φ:

(a) $\overline{U}_2 = \text{constant}$: $\varphi = \ln(a\tilde{u} + b)^n$ or $\varphi = a\tilde{u} + b$

(b) $\overline{U}_2 = re^{sp(x_1)}$: $\varphi = a\tilde{u} + b$

(c) $\overline{U}_2 = re^{sq(x_2)}: \ \varphi = a\tilde{u} + b$

(d) $\overline{U}_2 = re^{s\varphi}: \ \varphi = a\tilde{u} + b$ or $\varphi = \ln(a\tilde{u} + b)^n$

(e) $\overline{U}_2 = re^{sp(x_1)+tq(x_2)}: \ \varphi = a\tilde{u} + b$

(3) $g^2 = (\text{const.})\overline{U}_2 k_1$

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where s and t are constants.

At this point, it is of interest to determine the cases that are distinct from those referred to in Theorem 1. At the outset, note that in all cases where $\varphi = a\tilde{u} + b$ the equations for h_1 and h_2 can be written as

$$h_1 = (\text{const.})p'(x_1)e^{ap(x_1)}e^{ac_2q(x_2)}$$

$$= (\text{const.})e^{ac_2q(x_2)} \frac{d}{dx}\left[e^{ap(x_1)}\right]$$

$$h_2 = (\text{const.})q'(x_2)e^{ap(x_1)}e^{ac_2q(x_2)}$$

$$= (\text{const.})e^{ap(x_1)} \frac{d}{dx}\left[e^{ac_2q(x_2)}\right]$$

By denoting $e^{ap(x_1)}$ by $P(x_1)$ and $e^{ac_2q(x_2)}$ by $Q(x_2)$, the previous expressions can be written as

$$h_1 = (\text{const.})P'(x_1)Q(x_2)$$

$$h_2 = (\text{const.})Q'(x_2)P(x_1)$$

These expressions are the same as those used to define h_1 and h_2 in Theorem 1. It follows, therefore, that $K = 0$, and the solutions resulting from $\varphi = a\tilde{u} + b$ duplicate previous solutions. However, it is readily verified that solutions different from those given in Theorem 1 result when $\varphi = \ln\,(a\tilde{u} + b)^n$. For $n \neq 0$, $K \neq 0$, the corresponding surfaces are nondevelopable. These observations lead to the following theorem.

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Theorem 5. - The forms for h_1, h_2, and $\overline{U}_2$ given in Theorem 4 which are associated with nondevelopable surfaces are characterized by $\varphi = \ln(a\tilde{u} + b)^n$. The expressions for h_1, h_2, and $\overline{U}_2$ can be written

$$h_1 = (\text{const.})p'(x_1)\left[a'p(x_1) + b'q(x_2) + c'\right]^n$$

$$h_2 = (\text{const.})q'(x_2)\left[a'p(x_1) + b'q(x_2) + c'\right]^n \qquad n \neq 0$$

$$\overline{U}_2 = (\text{const.})\left[a'p(x_1) + b'q(x_2)\right]^m$$

Geometric considerations. - The specific expressions for h_1, h_2, and $\overline{U}_2$ given in Theorem 5 make possible a rather general analysis of the geometric aspects of the problem. In order to simplify the analysis, $p(x_1)$, $q(x_2)$, and the constants that appear in the expression for h_1 and h_2 are chosen in such a way that

$$h_1 = (ax_1 + bx_2)^n$$

$$h_2 = (ax_1 + bx_2)^n$$

Given the corresponding differential quadratic form,

$$(ds)^2 = (ax_1 + bx_2)^{2n}\left[(dx_1)^2 + (dx_2)^2\right] \tag{46}$$

the existence of an associated class of surfaces is assured (ref. 21, p. 122). This class of surfaces will now be studied more closely. Consider an orthogonal transformation of coordinates defined by

$$\begin{aligned} x_1 &= X_1^* \sin\alpha - X_2^* \cos\alpha \\ x_2 &= X_1^* \cos\alpha + X_2^* \sin\alpha \end{aligned} \qquad (\alpha \text{ a constant})$$

Under this transformation, equation (46) becomes

$$(ds)^2 = \left[(a \sin\alpha + b \cos\alpha)X_1^* + (b \sin\alpha - a \cos\alpha)X_2^*\right]^{2n}\left[(dX_1^*)^2 + (dX_2^*)^2\right]$$

If α is now chosen such that

$$a \sin\alpha + b \cos\alpha = 0$$

there results

$$(ds)^2 = c^2(X_2^*)^{2n}\left[(dX_1^*)^2 + (dX_2^*)^2\right] \tag{47}$$

where

$$c = b \sin \alpha - a \cos \alpha$$

Finally, by choosing

$$X_2 = \frac{c}{n + 1} (X_2^*)^{n+1} \qquad (n \neq -1)$$

$$X_1 = c\left(\frac{n + 1}{c}\right)^{\frac{n}{n+1}} X_1^*$$

equation (47) can be written

$$(ds)^2 = X_2^{2m}(dX_1)^2 + (dX_2)^2 \tag{48}$$

where

$$m = \frac{n}{n + 1}$$

It follows from the above equation that m cannot be equal to 1.

If $n = -1$,

$$X_2 = c' \ln X_2^*$$

$$X_1 = c'X_1^*$$

The differential quadratic form (47) becomes

$$(ds)^2 = e^{-2X_2/c'}(dX_1)^2 + (dX_2)^2 \tag{49}$$

Now consider surfaces in space in which it is possible to embed coordinate systems having differential forms corresponding to equations (48) and (49). First note that h_1 in equations (48) and (49) is a function of X_2 alone. Now for a $(ds)^2$ of the type

$$(ds)^2 = f^2(X_2)(dX_1)^2 + (dX_2)^2$$

a specific class of surfaces can be associated (see ref. 21, p. 206). These surfaces are surfaces of revolution given by (Y_i Cartesian coordinates)

$$Y_1 = af(X_2)\cos\frac{X_1}{a}$$

$$Y_2 = af(X_2)\sin\frac{X_1}{a}$$

$$Y_3 = \int\sqrt{1 - (a)^2(f')^2}\,dX_2 + \text{const.}$$

In relation to the present problem this report will first investigate surfaces for which $f(X_2) = X_2^m$. Choosing $X_1 = 0$ and eliminating the remaining parameter show that the surface of revolution is generated by rotating the curve:

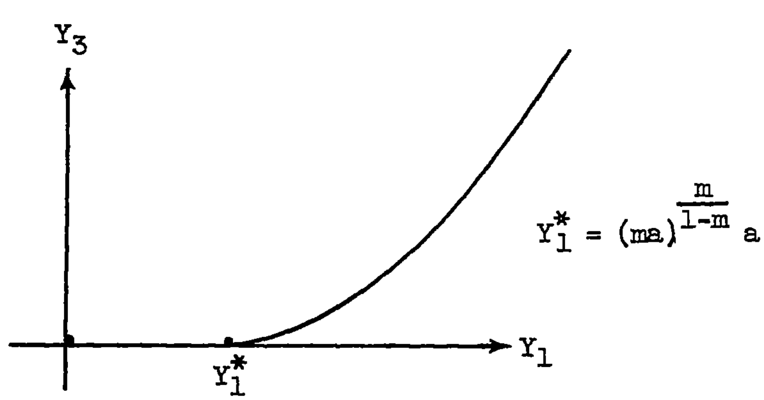
$$Y_3 = \int\sqrt{\frac{1}{(am)^2}\left(\frac{Y_1}{a}\right)^{\frac{2(1-m)}{m}} - 1}\;dY_1 \tag{50}$$

about the Y_3-axis. This is also the class of surfaces investigated by Geis in reference 7.

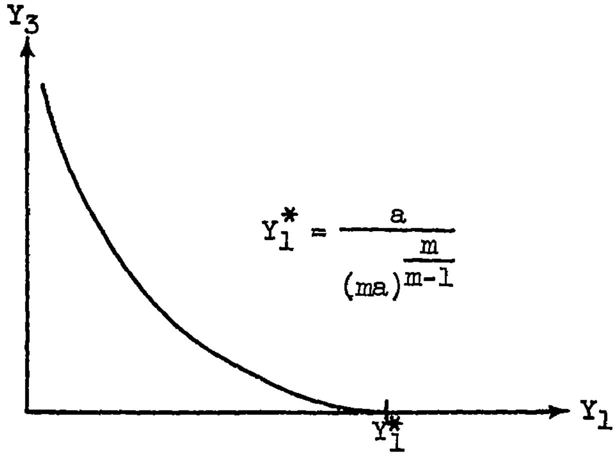
The nature of the curves can be qualitatively determined from an examination of the derivative

$$\frac{dY_3}{dY_1} = \sqrt{\frac{1}{(ma)^2}\left(\frac{Y_1}{a}\right)^{\frac{2(1-m)}{m}} - 1}$$

For $0 < m < 1$ the curve should have the following general shape:



For $m > 1$ or $m < 0$, the curve has the general shape



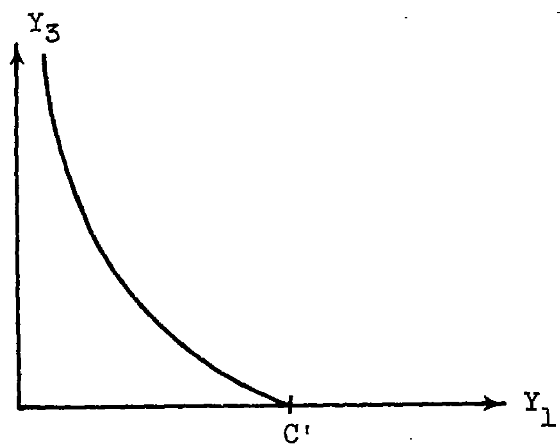
Finally, investigate the surface of revolution determined by $h_1 = e^{-X_2/c'}$ (see eq. (49)). The parametric equations can be written

$$Y_1 = ae^{-X_2/c'} \cos \frac{X_1}{a}$$

$$Y_2 = ae^{-X_2/c'} \sin \frac{X_1}{a}$$

$$Y_3 = \int \sqrt{1 - \frac{a^2}{(c')^2} e^{-2X_2/c'}} \, dX_2 + \text{const.}$$

For this particular case, it is known (ref. 21, p. 207) that the surface of revolution is obtained from revolving the tractrix shown below about the Y_3-axis:



As indicated in reference 7, the characteristics of the surface are such that the boundary-layer equations will probably apply over limited regions. It is, of course, necessary to keep in mind that, in any flow problem over curved surfaces, the boundary-layer equations are valid only in regions where the minimum value of the principal radii of curvature does not exceed the boundary-layer thickness.

While a special class of surfaces has been exhibited with coordinate systems satisfying the expressions for $(ds)^2$ in equations (48) and (49), it is well to consider what other surfaces might be admissible. The admissible surfaces are surfaces that are "applicable" to the surfaces of revolution illustrated previously (ref. 21, pp. 172-174). This means the class of surfaces obtained from the surfaces of revolution by bending, without stretching, compression, or tearing.

The streamlines of the mainstream flows will be lines that make constant angles with the coordinate lines. In systems where one set of coordinate lines corresponds to meridian curves on a surface of revolution, the streamlines are the loxodromes of the surface.

Reduction of O.D.E. Conditions When One Coordinate Curvature Is Identically Zero

Solutions of the o.d.e. conditions are now to be obtained under the assumptions that one of the coordinate curvatures vanishes identically and that $U_1 = c_1\overline{U}_2$.

Initially, consider the case of $k_1 \equiv 0$. (The case for $k_2 \equiv 0$ then can be readily determined from symmetry considerations.) Once again, a set of o.d.e. conditions is obtained by assuming various possible forms for $\overline{U}_2$.

The analysis is presented in appendix F and can be summarized in the following theorem, which is stated for $k_1 \not\equiv 0$ and $k_2 \equiv 0$.

<u>Theorem 6</u>. - Let (x_1,x_2) be an orthogonal coordinate system for which $k_1 \not\equiv 0$ and $k_2 \equiv 0$. Further assume that $U_1 = (\text{const.})\overline{U}_2$. Then necessary and sufficient conditions for equations (9) and (10) and (11) and (12) to be reduced to ordinary differential equations under assumption A are

(1) $\overline{U}_2 = \text{const.}$, and

$$h_1 = p(x_1)q^n(x_2)$$

$$h_2 = \frac{q'(x_2)}{q(x_2)}$$

or

$$h_1 = p(x_1)q^n(x_2)$$ 4975

$$h_2 = n\,\frac{q'(x_2)}{q^2(x_2)} \qquad \text{where} \quad n \neq 0$$

(2)

$$\overline{U}_2 = (\text{const.})e^{\int p(x_1)dx_1}$$

$$h_1 = \frac{p(x_1)}{q(x_2)}$$

$$h_2 = \frac{q'(x_2)}{q^2(x_2)}$$

(3) $\overline{U}_2 = (\text{const.})q^m(x_2)$, and h_1 and h_2 are expressed by either the first or second set of relations in condition (1)

(4) $\overline{U}_2 = (\text{const.})q^n(x_2)e^{\int p(x_1)dx_1}$, and h_1 and h_2 are expressed as the set of relations in condition (2)

The function g^2 is determined by

$$g^2 = (\text{const.})\overline{U}_2 k_1$$

(Indices may be uniformally interchanged in the previous statements.)

Geometric considerations. - If the Gaussian curvature is calculated from the values of h_1 and h_2 in condition (2) of Theorem 6, it can be shown that $K \equiv 0$. Consequently, cases (2) and (4) are completely covered by previous analyses.

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The first set of relations for h_1 and h_2 in condition (1) of Theorem 6 can be written

$$h_1 = e^{nx_2}$$

$$h_2 = 1$$

by choosing $p_1(x_1) = 1$, $q_1(x_2) = e^{x_2}$.

The quadratic differential form then becomes

$$(ds)^2 = e^{2nx_2}(dx_1)^2 + (dx_2)^2$$

This form is identical to equation (49) with $c' = -\frac{1}{n}$.

The second set of relations for h_1 and h_2 in condition (1) of Theorem 6 can be written

$$h_1 = x_2^{-n}$$

$$h_2 = 1$$

by choosing

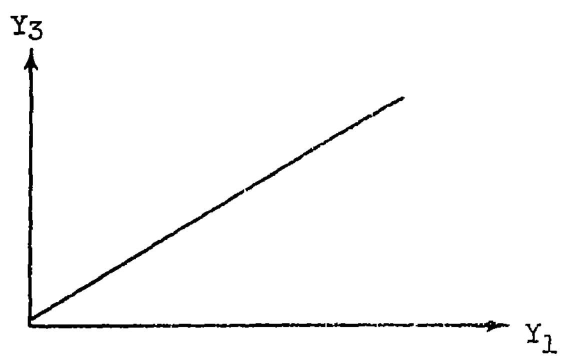
$$q_1(x_2) = -nx_2^{-1}$$

$$p_1(x_1) = (-n)^{-n}$$

For this case,

$$(ds)^2 = (x_2)^{-2n}(dx_1)^2 + (dx_2)^2$$

This result can be compared with equation (48) with $m = -n$. However, in this equation it is possible to choose $n = -1$, which leads to a developable surface. (The value $m = 1$ was not allowed in eq. (49).) Referring to equation (50) shows that, for $m = 1$ and $\frac{1}{a^2} \geqq 1$, $\frac{dY_3}{dY_1} = \sqrt{\frac{1}{a^2} - 1}$ and the corresponding surface of revolution is a cone $\left(\frac{1}{a^2} > 1\right)$ (see following sketch) or a circular cylinder $\left(\frac{1}{a^2} = 1\right)$.



Meridian curve for $m = 1,\ \frac{1}{a_2} > 1$

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As would be expected, the expressions for $U_1 = c_1\overline{U}_2$ determined from Theorem 6 can readily be shown to be identical to expressions for U_1 determined from Theorem 5 after the form for $(ds)^2$ in Theorem 5 is transformed into the form determined by Theorem 6.

This section concludes by observing that further analysis considering $k_1 \equiv 0$ and $k_2 \equiv 0$ is not necessary. In this instance $K \equiv 0$, and this case has been completely analyzed.

GENERAL SUMMARY OF REQUIREMENTS FOR THE EXISTENCE OF SIMILARITY SOLUTIONS

The various results obtained for flows over developable surfaces and for flows over nondevelopable surfaces characterized by $U_1 = (\text{const.})\overline{U}_2$ can be summarized as follows. For surfaces of nonconstant Gaussian curvature, constant curvature, and zero curvature, a suitable choice of Gaussian coordinates can be made in each case that reduces the corresponding forms of h_1, h_2, U_1, $\overline{U}_2$, and g^2 to a basic form given later. Associated with each basic form there can be chosen a characteristic surface of revolution. The meridian curves on the surface and their orthogonal trajectories constitute the coordinate lines of the basic system. All other permissible coordinate systems are obtained from the basic system by a scale transformation and an orthogonal transformation. All permissible flow surfaces for a given value of K are the class of surfaces applicable to the characteristic surface of revolution.

The basic forms for h_1, h_2, U_1, $\overline{U}_2$, g^2, and the equation for the associated surface of revolution are as follows:

(1) $K \neq$ constant

$$h_1 = x_2^n \qquad n \neq 0,1$$

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$h_2 = 1$

$U_1 = (\text{const.})\overline{U}_2 = x_2^m$

$g^2 = (\text{const.})\ \frac{U_1}{x_2}$

Surface of revolution: obtained by revolving the curve

$$Y_3 = \int \sqrt{\frac{1}{a^2n^2}\left(\frac{Y_1}{a}\right)^{\frac{2(1-n)}{n}} - 1}\ dY_1 + \text{const.}$$

about the Y_3-axis.

(2) K = constant (nonzero)

$h_1 = e^{nx_2}$

$h_2 = 1$

$U_1 = (\text{const.})\overline{U}_2 = e^{mx_2}$

$g^2 = (\text{const.})U_1$

Surface of revolution: obtained by revolving the tractrix

$$Y_3 = \frac{1}{a} \int \sqrt{(nY_1)^{-2} - 1}\ dY_1 + \text{const.}$$

about the Y_3-axis.

(3) $K = 0$

(a) $k_1^2 + k_2^2 \neq 0$

$h_1 = x_2$

$h_2 = 1$

$U_1 = (\text{const.})\overline{U}_2 = x_2^n e^{mx_1}$

$g^2 = (\text{const.})\ \frac{U_1}{x_2}$

Surface of revolution: a cone

(b) $k_1 = k_2 = 0$

$h_1 = h_2 = 1$

1. $U_1 = ae^{nx_1}x_2^{m-1}$

$U_2 = be^{nx_1}x_2^m$

$g^2 = (\text{const.})U_1$

2. $U_1 = ax_1^n x_2^{m-1}$

$U_2 = bx_1^{n-1}x_2^m$

$g^2 = (\text{const.})\ \dfrac{U_1}{x_1}$

3. $U_1 = ax_1^n$

$U_2 = bx_1^m$

$g^2 = (\text{const.})\ \dfrac{U_1}{x_1}$

4. $U_1 = ae^{nx_1}$

$U_2 = be^{mx_1}$

$g^2 = (\text{const.})U_1$

Surface of revolution: circular cylinder

METHOD AND RESULTS OF REFERENCE 16

The assumption of proportionality among coefficients in equations (9) and (10) and (11) and (12) (assumption A) was used in preceding sections as a sufficient condition for reducing these equations to ordinary differential equations. Geis (ref. 16) has shown in the meantime that such proportionality is also a necessary condition for the <u>complete boundary value problem</u>. He actually shows that the previously mentioned coefficients divided by g^2 are constant. The proof is based on a detailed study of these coefficients when the basic equations are subject to the boundary conditions on $F(\eta)$ and $G(\eta)$. This proof is the content of section 4 of Geis' paper.

The search after permissible forms of the functions h_1, h_2 and U_1, $\overline{U}_2$ proceeds in the present report along lines that are quite different from what is done in the corresponding section 5 of reference 16. Geis discusses a subcase of the case $K = 0$, namely, $k_1 = k_2 = 0$. The rest of his discussion uses the alternate assumption that

$$\frac{\partial(h_1U_1)}{\partial x_2} - \frac{\partial}{\partial x_1}(h_2U_2) = 0$$

which is satisfied if the main flow is irrotational. This hypothesis stands in contradistinction to the assumptions of developable surfaces or $U_1 = (\text{const.})\overline{U}_2$ made here. It is remarkable that all but two of the solutions obtained by Geis for U_1, $\overline{U}_2$, h_1, h_2, and g^2 are also found in the present analysis. In two exceptional cases Geis gives implicit solutions that were not obtained here. The first of these solutions is characterized by

$$U_1 = 1 \qquad \overline{U}_2 = \frac{A}{\sqrt{1 - \varphi'}}$$

$$h_1 = 1 \qquad h_2 = \frac{1}{\overline{U}_2} \qquad g^2 = \frac{1}{2\varphi}$$

where $\varphi = \varphi(x_1)$ is a solution different from $(x_1 + \text{const.})$ of

$$\varphi\varphi'' + (\varphi' - 1)^2 = 0$$

The second of these solutions is characterized by

$$U_1 = \varphi(x_1) \qquad \overline{U}_2 = A\sqrt{U_1^2 + B}$$

$$g^2 = -\varphi'$$

$$h_1 = 1 \qquad h_2 = \frac{1}{\overline{U}_2}$$

where φ is a nonconstant solution of

$$(\varphi^2 + B)\varphi'' + \left(\frac{1}{2}\varphi^3 - C\varphi^2 + \frac{1}{2}B\varphi - BC\right)\varphi'^2 = 0$$

Geis notices, however, that his explicit solutions are also valid when his condition of irrotationality is not satisfied.

It might finally be added that certain other solutions given implicitly in reference 16 are found in explicit form in the present paper.

It might be mentioned again that the emphasis here has been to examine the geometric aspects of the problem, that is, to determine the nature of coordinate systems, main-flow streamlines, and flow surfaces from the solutions found for U_1, $\overline{U}_2$, h_1, and h_2. This would appear to be a necessary requirement for fully evaluating the physical significance of the results.

UNCOUPLING OF EQUATIONS

The principal advantage of employing similarity techniques in an analysis of boundary-layer flows is the reduction of the partial differential equations for the flow to ordinary differential equations. However, the solution of the system of ordinary differential equations is generally difficult to obtain. Inspection of equations (9) and (10) or equations (11) and (12) discloses that the equations are of order three and nonlinear. Solution of systems in the past has usually required application of numerical techniques. If a wide range of flows is to be studied, this can be laborious and time-consuming. An additional difficulty that arises is that in a large number of cases the equations are coupled, which means that the functions $F(\eta)$ and $G(\eta)$ appear in both equations. If the equations are uncoupled, and one equation is expressible in terms of a single function, that equation can be solved quite readily, and the values obtained can be used in the solution of the remaining equation. It is of interest to know, therefore, for what classes of flow problems this is possible.

With this concept in mind the present section will determine the restrictions on h_1, h_2, U_1, and $\overline{U}_2$ that lead to the uncoupling of equations (9) and (10) and (11) and (12).

Uncoupled equations in the trivial sense of one equation identically vanishing will not be considered. (This could come about in certain classes of flow problems in which $U_1 \equiv 0$ or $U_2 \equiv 0$ and the main-flow streamlines have zero geodesic curvature.)

The requirement for uncoupling is simply that the coefficients of terms involving both F and G should vanish in either equation (9) or (10) (or eqs. (11) and (12)). Without loss of generality the requirement for uncoupling will be imposed on equation (9) (results hold also for (11)). Examination of equations (9) and (11) discloses that the following equations must be satisfied:

$$\frac{\overline{U}_2}{h_2}\frac{\partial \ln U_1}{\partial x_2} + \overline{U}_2 k_1 \equiv 0 \tag{51}$$

$$\frac{1}{h_2}\frac{\partial \overline{U}_2}{\partial x_2} - \frac{\overline{U}_2}{2h_2}\frac{\partial \ln g^2}{\partial x_2} + \overline{U}_2 k_1 \equiv 0 \tag{52}$$

$$\frac{\overline{U}_2^2}{U_1} k_2 \equiv 0 \tag{53}$$

Equation (53) gives at once the following lemma.

Lemma 5

The ordinary differential equations (9) and (10) (or (11) and (12)) are uncoupled only if one of the coordinate curvatures vanishes identically, that is, only if one set of coordinate lines is a system of geodesics on the flow surface.

Attention is first restricted to flows over developable surfaces. First, note from equation (51) that, if $\partial \ln U_1/\partial x_2 \equiv 0$, then $k_1 \equiv 0$ and, hence, both sets of coordinate lines are geodesics. If it is now assumed that $\partial \ln U_1/\partial x_2 \not\equiv 0$ and $k_1 \not\equiv 0$, then (see appendix D)

$$U_1 = c_1 \overline{U}_2$$

Hence, from o.d.e. conditions 3 and 9 in set (14),

$$g^2 = c_2 U_1 k_1$$

$$\therefore \frac{\partial \ln g^2}{\partial x_2} = \frac{\partial \ln U_1}{\partial x_2} + \frac{\partial \ln k_1}{\partial x_2}$$

But from equations (51), (52), and $U_1 = c_1\overline{U}_2$,

$$\frac{\partial \ln g^2}{\partial x_2} \equiv 0$$

$$\therefore \frac{\partial \ln U_1}{\partial x_2} = - \frac{\partial \ln k_1}{\partial x_2} \tag{54}$$

Examination of equation (17) shows that for $K \equiv 0$

$$\frac{1}{h_2}\frac{\partial \ln k_1}{\partial x_2} = - k_1$$

Substituting this result in equation (54) gives

$$\frac{1}{h_2}\frac{\partial \ln U_1}{\partial x_2} = k_1$$

If this expression is in turn substituted in equation (51),

$$2k_1 = 0$$

which contradicts the assumption that $k_1 \not\equiv 0$. The following lemma now results.

Lemma 6

For flow over developable surfaces, equations (9) and (10) are uncoupled only if $k_1 \equiv k_2 \equiv 0$, that is, both sets of coordinate lines are geodesics.

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If $k_1 \equiv k_2 \equiv 0$, it follows at once from equation (51) that $U_1 = U_1(x_1)$. Furthermore, examination of possible forms for U_1, U_2, and g^2 for this case (see p. 40) shows that in every instance $g^2 = (\text{const.})\frac{U_1}{x_1}$ or $g^2 = (\text{const.})U_1$. Hence, with $U_1 = U_1(x_1)$ equation (52) becomes (now it is required that $\overline{U}_2 \equiv U_2$)

$$\frac{\partial U_2}{\partial x_2} = 0$$

Therefore, the permissible flows can be reduced to the following two cases (see again p. 40):

$$U_1 = ae^{nx_1}$$

$$U_2 = be^{mx_1}$$

or

$$U_1 = ax_1^n$$

$$U_2 = bx_1^m$$

Now, consider flows over nondevelopable surfaces with $K \neq$ const. As $k_2 \equiv 0$, the permissible solutions for h_1, h_2, g^2, and U_1 can be taken as those given in case (1) (p. 38):

$$h_1 = x_2^n$$

$$h_2 = 1$$

$$U_1 = (\text{const.})\overline{U}_2 = x_2^m$$

$$g^2 = (\text{const.})U_1/x_2$$

Now, for $U_1 = (\text{const.})\overline{U}_2$ it was previously shown that

$$\frac{\partial \ln g^2}{\partial x_2} = 0$$

However, $g^2 = (\text{const.})x_2^{m-1}$, and, therefore, it is necessary that $m = 1$. Substituting the permissible forms for U_1, h_1, and h_2 into equation (51) gives

$$\frac{1}{x_2} + \frac{n}{x_2} = 0$$

Hence, $n = -1$.

While the previous results appear to lead to a valid case of uncoupled equations, substitution of the actual forms into equation (9) shows that this is not so. As may be verified, substitution leads to the equation

$$F'''(\eta) = 0$$

The function $F(\eta)$ cannot, therefore, fulfill the imposed boundary conditions, and this case must be ruled out.

Finally, consider K = constant (nonzero). The solutions for h_1, h_2, g^2, and U_1 of case (2) (p. 39) then apply.

As

$$g^2 = (\text{const.})U_1$$

and, as once again

$$\frac{\partial \ln g^2}{\partial x_2} = 0$$

there results

$$\frac{\partial \ln U_1}{\partial x_2} = 0$$

This result substituted in equation (51) yields $k_1 = 0$. This, however, would require (with $k_2 = 0$) that the surface be developable and would therefore contradict the hypothesis.

The results are summarized in the following theorem.

Theorem 7

The ordinary differential equations (9) and (10) are uncoupled if and only if

(1) Flow surface is developable, and both sets of coordinate lines are geodesics.

(2) U_1, U_2, and g^2 have one of the two following forms (except for change in indices):

$$(A)\quad \begin{cases} U_1 = ae^{nx_1} \\ U_2 = be^{mx_1} \\ g^2 = cU_1 \end{cases}$$

$$(B)\quad \begin{cases} U_1 = ax_1^n \\ U_2 = bx_1^m \\ g^2 = c\,\dfrac{U_1}{x_1} \end{cases}$$

Equations (9) and (10) have, respectively, the following forms if (A) is employed:

$$n\left[(F')^2 - \frac{FF''}{2} - 1\right] - cF''' = 0 \tag{55}$$

$$m(F'G' - 1) - \frac{nG''F}{2} - cG''' = 0 \tag{56}$$

If the forms of (B) are employed, there result, respectively, for equations (9) and (10)

$$n\left[(F')^2 - 1\right] - (n + 1)\frac{FF''}{2} - cF''' = 0 \tag{57}$$

$$m(F'G' - 1) - (n + 1)\frac{G''F}{2} - cG''' = 0 \tag{58}$$

For both sets of equations the boundary conditions are

$$F(0) = F'(0) = G(0) = G'(0) = 0$$

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$$\lim_{\eta \to \infty} F'(\eta) = 1 \qquad \lim_{\eta \to \infty} G'(\eta) = 1$$

It is not necessary to consider equations (11) and (12) as $U_2 = 0$ leads to straight-line flows in the present case.

PRACTICAL APPLICATIONS OF THEORY

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From a practical standpoint, similarity solutions might be applied to the study of boundary-layer flows over such aerodynamic configurations as wings, missiles, fuselage forms, or channel flows. From the standpoint of analyzing flows over wings or in channels, the only type of analysis that seems promising is the one employing rectangular coordinates (i.e., coordinate lines are geodesics). The principal reason is that in such configurations the boundary layer is generally initiated along a straight line on the surface (e.g., leading edge). This physical case can only be approximated when an analysis allows a boundary layer of zero thickness to exist along such a line. This obviously can only come about in a rectangular coordinate system analysis. On the other hand, polar and spiral coordinate systems may have particular application to flow over such configurations as missiles where the boundary layer develops from a point (e.g., nose of the missile).

In any of these cases, however, it should be kept in mind that a similarity analysis will generally predict only qualitative behavior of the flow. The restrictions imposed on the main-flow velocity components will, in general, be too severe to conform to a specified flow configuration, and, at best, an approximation to this flow can be constructed. Nevertheless, experimental verification of certain aspects of flow behavior predicted by theory has been very encouraging in at least one instance. The investigation presented in reference 1 shows that limiting-flow deflection on a channel surface can be accurately predicted.

The calculation of boundary-layer velocity profiles from a similarity analysis can also serve as a guide in setting up approximate analysis of boundary-layer flows using so-called momentum-integral methods. This method generally requires an a priori specification of the velocity profile shapes, which are then approximated by an analytic expression. Lack of information on the forms of three-dimensional velocity profiles has seriously hampered application of this technique.

CONCLUDING REMARKS

The following conclusions can be drawn from the analysis presented:

1. The requirements for a similarity analysis of the boundary-layer equations for flow over developable surfaces can be completely determined if the equations are referred to orthogonal coordinates. Two basic sets of solutions are sufficient for analyzing all permissible flows.

2. For mainstream flows over nondevelopable surfaces with $U_1 = c_1\overline{U}_2$, two basic sets of solutions are sufficient for all permissible flows. One set applies to surfaces with $K \neq$ const., while the other applies to K = nonzero const. All permissible flow surfaces are surfaces geometrically applicable to two basic classes of surfaces of revolution.

3. Uncoupled systems of ordinary differential equations resulting from a similarity analysis occur only when coordinate lines are geodesics (rectangular system in the plane).

In conclusion, the following problems pertaining to the analysis of similarity solutions seem worthwhile for continued investigation:

1. A general analysis for all possible types of similarity solutions has not yet been evolved in the sense that investigations to date (ref. 16 and the present paper) have employed certain assumptions to reduce the complexity of the problem.

2. Solutions of the ordinary differential equations arising in a similarity analysis are few in number. Little work has been done on coupled equations. It would be of interest to study variations in solutions for a range of parameter values either through a program on high-speed computing equipment or by developing suitable approximation techniques.

3. Extensions of the present theory for incompressible laminar-boundary-layer flow to compressible laminar-boundary-layer flow would be of interest. Some investigations of this kind have been carried out (e.g., ref. 22).

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
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APPENDIX A

SYMBOLS

A_i	constants
a,a_i	constants
B,B_i	constants
b,b_i	constants
C_i	constants
c,c_i	constants
d,d_i	constants
e	constant
$F,F(\eta)$	functions of η
f,f_i	arbitrary functions
$G,G(\eta)$	functions of η
g_{ij}	metric tensor for three-dimensional coordinate system
$g,g(x_1,x_2)$	arbitrary function occurring as a factor of η
h_1,h_2	square roots of metric-tensor components in orthogonal coordinate system
h_1^*,h_2^*	specific forms for h_1 and h_2 defined by eq. (29)
K	Gaussian curvature of surface
k_1,k_2	geodesic curvature of coordinate lines of x_1 and x_2, respectively
m	constant
n	constant

p_i	arbitrary functions
q_i	arbitrary functions
r	constant
s	arc length
t	constant
U_1,U_2	mainstream velocity components in x_1- and x_2-directions
U_2^*	function of x_1,x_2, eq. (6)
$\overline{U}_2$	common designation for either U_2 or U_2^*
$\tilde{u}$	special designation for function of x_1 and x_2 occurring in eq. (42)
u_i	boundary-layer velocity components in x_i-direction
v	transformed boundary-layer velocity component normal to surface
X_1,X_2	coordinates defined by eq. (27)
X_1^*,X_2^*	curvilinear coordinates
$\overline{X}_1,\overline{X}_2$	coordinates
x_i	curvilinear coordinates
Y_1,Y_2,Y_3	Cartesian coordinates
y	$y^*/\sqrt{\nu}$
y^*	physical coordinate normal to flow surface
η	similarity parameter
θ,ρ	coordinates defined by eq. (C5)
ν	coefficient of kinematic viscosity
ξ_i	function of X_1,X_2, eq. (C1)
$\varphi(\tilde{u})$	arbitrary function, eq. (42)

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Subscripts:

i,j,k,l,m,r,s denote index numbers

Superscripts:

Primes denote differentiation

APPENDIX B

ANALYSIS: $K = 0$; $k_1 \not\equiv 0$, $k_2 \not\equiv 0$. PROOF OF THEOREM 1

The solutions for the various unknowns in set (14) on page 10 are obtained here for the case $K = 0$; $k_1 \not\equiv 0$, $k_2 \not\equiv 0$.

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Permissible Forms for k_1, k_2, h_1, and h_2

Before k_1, k_2, h_1, and h_2 can be determined, certain other general results are needed. In this regard, it will first be shown that, under the assumption of neither k_1 nor k_2 being identically zero, it must follow that k_1 and k_2 each possess nonvanishing first derivatives with respect to both x_1 and x_2.

Assume that a derivative of either k_1 or k_2 with respect to one coordinate is identically equal to zero, while the derivative with respect to the other coordinate is not.

Specifically, assume $\dfrac{\partial k_1}{\partial x_1} \equiv 0$ and $\dfrac{\partial k_1}{\partial x_2} \not\equiv 0$. It will first be shown that the assumption $\dfrac{\partial k_1}{\partial x_2} \not\equiv 0$ leads to the equation

$$k_1 = \frac{d_1}{h_2} \frac{\partial \ln k_1}{\partial x_2} \tag{B1}$$

From lemmas 1 and 2 and equation (19),

$$g^2 = \frac{c_1^2}{a_2} \overline{U}_2 k_1$$

Hence,

$$\frac{\partial \ln g^2}{\partial x_2} = \frac{\partial \ln \overline{U}_2}{\partial x_2} + \frac{\partial \ln k_1}{\partial x_2} \tag{B2}$$

Now, both $\partial \ln g^2/\partial x_2$ and $\partial \ln \overline{U}_2/\partial x_2$ cannot be zero, for then $\dfrac{\partial \ln k_1}{\partial x_2} = 0$, which would violate the initial assumption. However, if $\dfrac{\partial \ln g^2}{\partial x_2} = 0$ and $\dfrac{\partial \ln \overline{U}_2}{\partial x_2} \not\equiv 0$, from equation (B2), o.d.e. condition for

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⑥ and ⑦, and lemma 1 there results

$$\frac{\partial \ln k_1}{\partial x_2} = - \frac{\partial \ln \overline{U}_2}{\partial x_2} = b_1 h_2 k_1$$

It therefore follows that equation (B1) holds.

If $\frac{\partial \ln g^2}{\partial x_2} \not\equiv 0$ and $\frac{\partial \ln \overline{U}_2}{\partial x_2} \equiv 0$, then from equation (B2), o.d.e. condition for ⑤ and ⑥, and lemmas 1 and 2,

$$\frac{\partial \ln k_1}{\partial x_2} = \frac{\partial \ln g^2}{\partial x_2} = b_2 h_2 k_1$$

and, again, equation (B1) holds.

Finally, if both $\partial \ln g^2/\partial x_2$ and $\partial \ln \overline{U}_2/\partial x_2$ are not identically zero, equation (B2), the o.d.e. conditions for ⑤, ⑥, and ⑦, and lemmas 1 and 2 yield

$$\frac{\partial \ln k_1}{\partial x_2} = b_3 h_2 k_1$$

and equation (B1) holds.

Now, from equation (B1) and the assumption that $\frac{\partial k_1}{\partial x_1} \equiv 0$,

$$k_1 \frac{\partial h_2}{\partial x_1} = (\text{const.}) \frac{\partial^2 \ln k_1}{\partial x_1 \, \partial x_2} \equiv 0$$

However, $k_1 \frac{\partial h_2}{\partial x_1} = h_1 h_2 k_1 k_2 = 0$ contradicts the basic assumption that neither k_1 nor k_2 vanish identically. Similarly, it can be shown that the case $\frac{\partial k_1}{\partial x_1} \not\equiv 0$ and $\frac{\partial k_1}{\partial x_2} \equiv 0$ is incompatible with that assumption.

Similar statements hold for k_2. Hence, the following lemma can now be stated.

Lemma 3. - If both k_1 and k_2 are different from zero and neither reduces to a constant, then both coordinate curvatures k_1 and k_2 must possess nonvanishing first partial derivatives with respect to both x_1 and x_2.

By employing a procedure similar to the one used in establishing equation (B1), it is also possible to show that

$$k_1 = \frac{d_2}{h_1} \frac{\partial \ln k_1}{\partial x_1} \tag{B3}$$

or, by using lemma 2,

$$k_2 = \frac{d_2}{c_2 h_1} \frac{\partial \ln k_2}{\partial x_1} \tag{B4}$$

Hence, from equation (B4) and the definition of k_2,

$$\frac{d_2}{c_2} \frac{\partial \ln k_2}{\partial x_1} = \frac{\partial \ln h_2}{\partial x_1} \ (= h_1 k_2)$$

Therefore,

$$\frac{\partial}{\partial x_1} \ln \left(k_2 h_2^{-c_2/d_2} \right) = 0$$

$$\therefore k_2 = h_2^{c_2/d_2} f_1(x_2) \tag{B5}$$

where $f_1(x_2)$ is arbitrary.

From the definition of k_1, equation (B1), and lemma 2,

$$h_2 k_1 = \frac{\partial \ln h_1}{\partial x_2} = d_1 \frac{\partial \ln k_1}{\partial x_2} = d_1 \frac{\partial \ln k_2}{\partial x_2}$$

$$\therefore \frac{\partial}{\partial x_2} \ln k_2 h_1^{-1/d_1} = 0$$

$$\therefore k_2 = h_1^{1/d_1} f_2(x_1) \tag{B6}$$

where $f_2(x_1)$ is arbitrary. From equations (B5) and (B6) is obtained

$$h_2 = h_1^{d_2/d_1 c_2} \left[\frac{f_2(x_1)}{f_1(x_2)} \right]^{d_2/c_2} \tag{B7}$$

Now, relations between the various constants appearing in equation (B7) must be found.

First of all, by differentiating h_2k_1/d_1 with respect to x_1 according to equation (B1),

$$\frac{1}{d_1}\left(k_1 \frac{\partial h_2}{\partial x_1} + h_2 \frac{\partial k_1}{\partial x_1}\right) = \frac{\partial^2 \ln k_1}{\partial x_1 \, \partial x_2} \tag{B8}$$

Also, from equation (B3),

$$h_1k_1 = d_2 \frac{\partial \ln k_1}{\partial x_1}$$

and differentiation of this expression with respect to x_2 gives

$$\frac{1}{d_2}\left(k_1 \frac{\partial h_1}{\partial x_2} + h_1 \frac{\partial k_1}{\partial x_2}\right) = \frac{\partial^2 \ln k_1}{\partial x_1 \, \partial x_2} \tag{B9}$$

From lemma 2 and the definition of k_1 and k_2, finally,

$$\frac{\partial h_1}{\partial x_2} = c_2 \frac{\partial h_2}{\partial x_1} \tag{B10}$$

In equation (B9) substitute for $\partial h_1/\partial x_2$ and $\partial k_1/\partial x_2$ according to equations (B10) and (B1):

$$\frac{1}{d_2}\left(k_1c_2 \frac{\partial h_2}{\partial x_1} + \frac{k_1^2h_2}{d_1} h_1\right) = \frac{\partial^2 \ln k_1}{\partial x_1 \, \partial x_2} \tag{B11}$$

Similarly, equation (B8) becomes, by equation (B3),

$$\frac{1}{d_1}\left(k_1 \frac{\partial h_2}{\partial x_1} + k_1^2 \frac{h_1h_2}{d_2}\right) = \frac{\partial^2 \ln k_1}{\partial x_1 \, \partial x_2} \tag{B12}$$

Finally, equations (B11) and (B12) yield

$$\frac{k_1c_2}{d_2} \frac{\partial h_2}{\partial x_1} = \frac{k_1}{d_1} \frac{\partial h_2}{\partial x_1}$$

Hence,

$$\frac{c_2}{d_2} d_1 = 1 \tag{B13}$$

It is now possible to evaluate d_1 by noting that the derivatives in equation (17) may be eliminated by the use of equations (B1) and (B4):

$$k_1^2 + k_2^2 + k_2^2 \frac{c_2}{d_2} + \frac{k_1^2}{d_1} = 0$$

which, by equation (B3), becomes

$$(k_1^2 + k_2^2)\left(1 + \frac{1}{d_1}\right) = 0$$

whence $d_1 = -1$ as k_1 and k_2 are assumed nonvanishing.

It therefore follows from equation (B13) that

$$\frac{c_2}{d_2} = -1$$

From the previous results, equations (B5) and (B6), and lemma 2 there results lemma 4.

Lemma 4. - Under the assumption of nonvanishing coordinate curvatures, k_1 and k_2 must be expressible as

$$k_2 = f_2(x_1)/h_1$$

$$k_1 = c_2 f_1(x_2)/h_2$$

Expressions for h_1 and h_2 can now be obtained as follows. From the definition of k_1 and k_2,

$$\frac{\partial \ln h_2}{\partial x_1} = f_2(x_2) \qquad \text{(B14)}$$

$$\frac{\partial \ln h_1}{\partial x_2} = c_2 f_1(x_1) \qquad \text{(B15)}$$

Hence, h_1 and h_2 must be expressible as a product of a function of x_1 and a function of x_2:

$$h_1 = q_1(x_1)q_2(x_2) \qquad \text{(B16)}$$

$$h_2 = p_1(x_1)p_2(x_2) \qquad \text{(B17)}$$

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Substituting expressions (B16) and (B17) into equation (B10) gives

$$q_1(x_1)q_2'(x_2) = c_2p_1'(x_1)p_2(x_2)$$

Neither $q_2'(x_2)$ nor $p_1'(x_1)$ can be identically zero under the assumption that neither of the coordinate curvatures vanishes identically. Therefore,

$$q_1(x_1) = c_3p_1'(x_1)$$

$$p_2(x_2) = c_4q_2'(x_2)$$

Hence, h_1 and h_2 can also be written in the form

$$h_1 = c_3p_1'(x_1)q_2(x_2) \tag{B18}$$

$$h_2 = c_4p_1(x_1)q_2'(x_2) \tag{B19}$$

From the definition of k_1 and k_2 and equations (B18) and (B19),

$$k_2 = \frac{1}{h_1h_2}\frac{\partial h_2}{\partial x_1} = \frac{1}{c_3p_1'(x_1)q_2(x_2)} \tag{B20}$$

$$k_1 = \frac{1}{h_1h_2}\frac{\partial h_1}{\partial x_2} = \frac{1}{c_4p_1(x_1)q_2'(x_2)} \tag{B21}$$

Permissible Forms For U_1, $\overline{U}_2$, and g^2

It is noted once more that before a unique set of o.d.e. conditions can be prescribed for the determination of U_1 and $\overline{U}_2$, certain assumptions regarding the vanishing or nonvanishing of various terms in set (14) must be made. As the terms of interest in set (14) involve the first derivatives of U_1 and $\overline{U}_2$, the necessary assumptions can be imposed in terms of the vanishing or nonvanishing of these derivatives. By recalling from lemma 1 that $U_1 = c_1\overline{U}_2$, the following four cases cover all possibilities:

(1) Case A: $\dfrac{\partial U_1}{\partial x_1} = c_1\dfrac{\partial \overline{U}_2}{\partial x_1} \not\equiv 0$

and $\dfrac{\partial U_1}{\partial x_2} = c_1\dfrac{\partial \overline{U}_2}{\partial x_2} \not\equiv 0$

(2) Case B: $\frac{\partial U_1}{\partial x_1} \equiv 0 \quad \frac{\partial \overline{U}_2}{\partial x_1} \equiv 0$

and $\frac{\partial U_1}{\partial x_2} = c_1 \frac{\partial \overline{U}_2}{\partial x_2} \not\equiv 0$

(3) Case C: $\frac{\partial U_1}{\partial x_1} = c_1 \frac{\partial \overline{U}_2}{\partial x_1} \not\equiv 0$

and $\frac{\partial U_1}{\partial x_2} \equiv 0 \quad \frac{\partial \overline{U}_2}{\partial x_2} \equiv 0$

(4) Case D: U_1 and $\overline{U}_2$ are nonzero constants (all derivatives vanish).

First consider case A:

From the o.d.e. condition for 1 and 6 and lemma 1,

$$\frac{1}{h_1} \frac{\partial U_1}{\partial x_1} = c_5 U_1 k_2$$

$$\therefore \quad \frac{\partial \ln U_1}{\partial x_1} = c_5 k_2 h_1$$

Employing the expressions for h_1 and h_2 given in equations (B16) and (B20), respectively, gives

$$\frac{\partial \ln U_1}{\partial x_1} = c_5 \frac{d \ln p_1(x_1)}{dx_1}$$

$$\therefore \ln U_1 = \ln \left[p_1(x_1)\right]^{c_5} + f_3(x_2) \tag{B22}$$

Now, from the o.d.e. condition for 6 and 7 and lemma 1,

$$\frac{\partial \ln U_1}{\partial x_2} = c_2 h_2 k_2$$

As above, $\ln U_1$ can be evaluated to obtain

$$\ln U_1 = \ln \left[q_2(x_2)\right]^{c_6} + f_4(x_1) \tag{B23}$$

It then follows from equations (B22) and (B23) that U_1 and $\overline{U}_2$ must be expressible as

$$U_1 = c_7 p_1^n(x_1) q_2^m(x_2) = c_1 \overline{U}_2 \tag{B24}$$

where c_7, n, and m are nonzero constants.

The form of the function g^2 is obtained from the o.d.e. condition for 3 and 6 and lemma 1:

$$g^2 = c_8 U_1 k_2$$

Hence,

$$g^2 = c_9 p_1^{n-1}(x_1) q_2^{m-1}(x_2) \tag{B25}$$

If the expressions for k_1, k_2, h_1, h_2, U_1, $\overline{U}_2$, and g^2 are substituted into remaining o.d.e. conditions, it will be seen that these conditions are satisfied. Hence, for this case, necessary and sufficient conditions for obtaining ordinary differential equations from the original partial differential equations under assumption A have been found.

The analysis for cases B, C, and D follows the method outlined previously. The results are as follows:

Case B: $U_1 = (\text{constant})\ q_2^m(x_2) = c_1 \overline{U}_2$

$$g^2 = (\text{constant}) \frac{q_2^{m-1}(x_2)}{p_1(x_1)}$$

Case C: $U_1 = (\text{constant})\ p_1^n(x_1) = c_1 \overline{U}_2$

$$g^2 = (\text{constant}) \frac{p_1^{n-1}(x_1)}{q_2(x_2)}$$

Case D: $U_1 = (\text{constant}) = c_1 \overline{U}_2$

$$g^2 = \frac{\text{Constant}}{p_1(x_1) q_2(x_2)}$$

All results are summarized in Theorem 1 in the text.

APPENDIX C

NATURE OF COORDINATE SYSTEM CORRESPONDING TO $K = 0$; $k_1 \neq 0$, $k_2 \neq 0$

The following analysis will deal with the determination of the explicit form of transformation (30).

At the outset, the following observations are made. Suppose there are two coordinate systems (y^1,y^2) and (x^1,x^2) functionally related by

$$y^i = y^i(x^1,x^2)$$

If h_{ij} and g_{ij}, respectively, denote the metric tensors of the system (y_i) and the system (x_i), then

$$h_{ij} = \frac{\partial x^r}{\partial y^i}\frac{\partial x^s}{\partial y^j} g_{rs}$$

By differentiation of the above expression coupled with other minor manipulations (ref. 20, p. 83), it is possible to show that

$$\frac{\partial^2 y^m}{\partial x^i\, \partial x^j} = {}_x\begin{Bmatrix} l \\ ij \end{Bmatrix} \frac{\partial y^m}{\partial x^l} - {}_y\begin{Bmatrix} m \\ rs \end{Bmatrix} \frac{\partial y^r}{\partial x^i}\frac{\partial y^s}{\partial x^j}$$

where ${}_x\begin{Bmatrix} l \\ ij \end{Bmatrix}$ and ${}_y\begin{Bmatrix} m \\ rs \end{Bmatrix}$ denote Christoffel symbols based on g_{rs} and h_{ij}, respectively. (See brief review at end of this appendix.) If the system (x^1,x^2) is restricted to be a Cartesian system, all Christoffel symbols vanish and the above expression becomes

$$\frac{\partial^2 y^m}{\partial x^i\, \partial x^j} - {}_x\begin{Bmatrix} l \\ ij \end{Bmatrix} \frac{\partial y^m}{\partial x^l} = 0$$

This system is then a system of second-order partial differential equations for the transformation $y^i = y^i(x^1,x^2)$. By reverting now to the notation employed in equation (30), the system can be written as the following set of first-order partial differential equations:

$$\left.\begin{aligned} \frac{\partial Y}{\partial X_i} &= \xi_i \qquad & i &= 1,2 \\ \frac{\partial \xi_i}{\partial X_j} &= \begin{Bmatrix} l \\ ij \end{Bmatrix} \xi_l \qquad & l &= 1,2 \end{aligned}\right\} \tag{C1}$$

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where Y stands for either Y^1 or Y^2. Evaluation of the Christoffel symbols gives the following results:

$$\left\{ {1 \atop 11} \right\} = \frac{1}{2} \frac{\partial \ln (h_1^*)^2}{\partial X_1} = 0$$

$$\left\{ {1 \atop 12} \right\} = \frac{1}{2} \frac{\partial \ln (h_1^*)^2}{\partial X_2} = \frac{1}{X_2}$$

$$\left\{ {2 \atop 11} \right\} = - \frac{1}{2(h_2^*)^2} \frac{\partial (h_1^*)^2}{\partial X_2} = - \frac{a^2}{b^2} \frac{X_2}{X_1^2}$$

$$\left\{ {2 \atop 12} \right\} = \frac{1}{2} \frac{\partial \ln (h_2^*)^2}{\partial X_1} = \frac{1}{X_1}$$

$$\left\{ {2 \atop 22} \right\} = \frac{1}{2} \frac{\partial \ln (h_2^*)^2}{\partial X_2} = 0$$

$$\left\{ {1 \atop 22} \right\} = - \frac{1}{2(h_1^*)^2} \frac{\partial (h_2^*)^2}{\partial X_1} = - \frac{b^2}{a^2} \frac{X_1}{X_2^2}$$

The system of equations (C1) can therefore be written as

$$\left.\begin{aligned}
&\frac{\partial Y}{\partial X_1} = \xi_1; \quad \frac{\partial Y}{\partial X_2} = \xi_2 \\
&\frac{\partial \xi_1}{\partial X_1} = - \frac{a^2}{b^2} \frac{X_2}{X_1^2} \cdot \xi_2 \\
&\frac{\partial \xi_1}{\partial X_2} = \frac{\xi_2}{X_1} + \frac{\xi_1}{X_2} \\
&\frac{\partial \xi_2}{\partial X_1} = \frac{\xi_1}{X_2} + \frac{\xi_2}{X_1} \\
&\frac{\partial \xi_2}{\partial X_2} = - \frac{b^2}{a^2} \frac{X_1}{X_2^2} \xi_1
\end{aligned}\right\} \tag{C2}$$

It is possible to give particular solutions for Y_1 and Y_2 which satisfy these equations and also satisfy the orthogonality requirement

$$\frac{\partial Y_1}{\partial X_1}\frac{\partial Y_1}{\partial X_2} + \frac{\partial Y_2}{\partial X_1}\frac{\partial Y_2}{\partial X_2} = 0$$

The functions

$$\left.\begin{aligned} Y_1 &= cX_1X_2 \cos\left(d \ln X_1 - \frac{1}{d} \ln X_2\right) \\ Y_2 &= cX_1X_2 \sin\left(d \ln X_1 - \frac{1}{d} \ln X_2\right) \end{aligned}\right\} \tag{C3}$$

where

$$c^2 = \frac{a^2b^2}{a^2 + b^2} \qquad d^2 = \frac{a^2}{b^2}$$

represent a particular solution of the system (C2). The most general solution of the system can be therefore written as

$$\overline{Y}_i = \gamma_i + \alpha_{ij}Y_j \tag{C4}$$

where the α_{ij} are the elements of an orthogonal matrix and the γ_i are constants. Equation (C4) merely represents a rotation and translation of the coordinate system defined by (Y_1,Y_2).

In order to better ascertain the nature of the (X_1,X_2) coordinate system, let

$$\left.\begin{aligned} \rho &= cX_1X_2 \\ \theta &= d \ln X_1 - \frac{\ln X_2}{d} \end{aligned}\right\} \tag{C5}$$

be a transformation of the (X_1,X_2) system to a polar coordinate system; that is, in terms of ρ and θ,

$$Y_1 = \rho \cos \theta$$

$$Y_2 = \rho \sin \theta$$

Now consider the coordinate line $X_1 = X_1^o$ = constant. The equation of this coordinate line is given parametrically by

$$\left.\begin{aligned} \rho &= cX_1^oX_2 \\ \theta &= d \ln X_1^o - \frac{1}{d} \ln X_2 \end{aligned}\right\} \tag{C6}$$

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Equation (C6) can be expressed as

$$e^{\theta} = (X_1^o)^d X_2^{-1/d} \tag{C7}$$

Hence, from equations (C6) and (C7),

$$\rho = c(X_1^o)^{d^2} e^{-\theta d} \tag{C8}$$

or

$$\rho = (\text{const.})e^{-\theta d}$$

Similarly, the equation of the coordinate line $X_2 = (\text{const.}) = X_2^o$ is expressible as

$$\rho = c(X_2^o)^{1/d^2} e^{\theta/d} \tag{C9}$$

or

$$\rho = (\text{const.})e^{\theta/d}$$

The curves defined by equations (C8) and (C9) constitute a system of mutually orthogonal logarithmic spirals relative to the (ρ,θ) system if the (ρ,θ) system is interpreted as a system of polar coordinates in the plane.

CHRISTOFFEL SYMBOLS

A brief review of the definition of Christoffel symbols is presented here as an aid in following the analyses.

Consider a surface in which an orthogonal coordinate system (x^1,x^2) is embedded. In such a system the square of the differential of arc length is given by

$$(ds)^2 = h_1^2(dx^1)^2 + h_2^2(dx^2)^2 \tag{C10}$$

when

$$h_1 = h_1(x^1,x^2)$$

$$h_2 = h_2(x^1,x^2)$$

The metric tensor g_{ij} $(i = 1,2)$ associated with (C10) is defined by

$$g_{11} = h_1^2;\ g_{22} = h_2^2;\ g_{12} = g_{21} = 0$$

The tensor g^{ij} may be defined by

$$g^{11} = \frac{1}{g_{11}};\ g^{22} = \frac{1}{g_{22}};\ g^{12} = g^{21} = 0$$

The Christoffel symbol $\left\{ {l \atop ij} \right\}$ (sometimes written Γ^{i}_{jk} is then defined by

$$\left\{ {l \atop ij} \right\} = \frac{g^{lk}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Repeated indices imply summation.

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APPENDIX D

ANALYSIS: $K = 0;\ k_1 \equiv 0,\ k_2 \not\equiv 0$. PROOF OF THEOREM 2

By following the same procedure that was used in establishing lemma 1, it can be shown that equation (24) becomes

$$k_2^2\left(1 + 2n + a_1^{-1}a_9\,\frac{\overline{U}_2^2}{U_1^2}\right) = 0$$

As $k_2 \not\equiv 0$ and a_1 and a_9 are nonzero constants, it follows that

$$U_1 = c_1\overline{U}_2 \tag{D1}$$

as in the previous case.

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From the definition of k_1,

$$k_1 = \frac{1}{h_1h_2}\,\frac{\partial h_1}{\partial x_2} \equiv 0$$

Hence,

$$h_1 = f_5(x_1) \tag{D2}$$

Recalling the definition of the Gaussian curvature of the surface K and the assumption that $K \equiv 0$ gives

$$K = -\frac{1}{h_1h_2}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{h_1}\,\frac{\partial h_2}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{h_2}\,\frac{\partial h_1}{\partial x_2}\right)\right] = 0 \tag{D3}$$

From equations (D2) and (D3),

$$\frac{\partial}{\partial x_1}\left(\frac{1}{h_1}\,\frac{\partial h_2}{\partial x_1}\right) = 0$$

$$\therefore\ \frac{1}{h_1}\,\frac{\partial h_2}{\partial x_1} = h_2k_2 = f_6(x_2)$$

or

$$k_2 = \frac{f_6(x_2)}{h_2} \tag{D4}$$

At this point, it should be noted that k_2 cannot be constant for, if it were, $h_2 \equiv \frac{k_2}{f_6(x_2)}$ and $\frac{\partial h_2}{\partial x_1} = 0$, which would imply that $k_2 \equiv 0$. It is also follows that $\frac{\partial k_2}{\partial x_1} \equiv 0$, for if it is assumed that $\frac{\partial k_2}{\partial x_1}$ is zero, then, from equation (D4),

$$\frac{\partial k_2}{\partial x_1} = - \frac{f_6(x_2)}{h_2^2} \frac{\partial h_2}{\partial x_1} = 0$$

Now, either $f_6(x_2) = 0$ or $\frac{\partial h_2}{\partial x_1} = 0$ implies $k_2 \equiv 0$, which violates the hypothesis.

Assume that $\frac{\partial k_2}{\partial x_2} \not\equiv 0$. Then, from conditions 3 and 6 and the proportionality of U_1 and U_2,

$$g^2 = b_4 U_1 k_2 \tag{D5}$$

$$\therefore \frac{\partial \ln g^2}{\partial x_2} = \frac{\partial \ln U_1}{\partial x_2} + \frac{\partial \ln k_2}{\partial x_2}$$

Hence,

$$\frac{\partial \ln k_2}{\partial x_2} = \frac{\partial \ln g^2}{\partial x_2} - \frac{\partial \ln U_1}{\partial x_2} \tag{D6}$$

Under the assumption, the right side of equation (D6) cannot be zero. Hence, the o.d.e. conditions arising from the combinations 5 and 6 and 4 and 6 along with equation (D1) yield

$$\frac{\partial \ln k_2}{\partial x_2} = b_5 h_2 k_2 \tag{D7}$$

Similarly, it can be shown that

$$\frac{\partial \ln k_2}{\partial x_1} = b_6 h_1 k_2 \tag{D8}$$

Differentiating equation (D7) with respect to x_1 and employing equation (D4) yield

$$\frac{\partial^2 \ln k_2}{\partial x_1 \, \partial x_2} = b_5 \frac{\partial}{\partial x_1} h_2 k_2 = 0 \tag{D9}$$

Now, differentiating equation (D8) with respect to x_2 and recalling that $\frac{\partial h_1}{\partial x_2} \equiv 0$ (as k_1 has been assumed zero) give (with eq. (D9))

$$\frac{\partial^2 \ln k_2}{\partial x_2 \, \partial x_1} = b_6 h_1 \frac{\partial k_2}{\partial x_2} = 0$$

Thus, $\frac{\partial k_2}{\partial x_2} \neq 0$ is incompatible with $k_2 \neq 0$, $k_1 = 0$, and the only remaining possibility is that k_2 is a nonconstant function of x_1 alone. Defining $k_2 = f_7(x_1)$ gives from equation (D4)

$$h_2 = \frac{f_6(x_2)}{f_7(x_1)} \tag{D10}$$

Finally, from the definition of k_2,

$$k_2 = \frac{1}{h_1} \frac{\partial \ln h_2}{\partial x_1}$$

$$h_1 = \frac{\dfrac{\partial \ln h_2}{\partial x_1}}{k_2} = - \frac{f_7'(x_1)}{f_7^2(x_1)} \tag{D11}$$

From these results, Theorem 2 follows directly.

Because of the symmetrical nature of the equations, similar results can be obtained if it is first assumed that $k_2 \equiv 0$ and $k_1 \not\equiv 0$. In this case,

$$k_1 = f_8(x_2)$$

$$h_1 = \frac{f_9(x_1)}{f_8(x_2)}$$

and

$$h_2 = - \frac{f_8'(x_2)}{f_8^2(x_2)}$$

APPENDIX E

PROOF OF THEOREM 4

The four cases presented on page 29 will now be analyzed.

(1) Case A: $\overline{U}_2 = \text{const.}$

With $\overline{U}_2$ constant, the following o.d.e. condition from 2* and 5* (set (45)) results:

$$\frac{\partial \ln h_2^2/k_2}{\partial x_1} = \frac{\partial \ln h_2^{c_{11}}}{\partial x_1}$$

$$\therefore \frac{\partial \ln h_2^{2-c_{11}}/k_2}{\partial x_1} = 0$$

Hence,

$$h_2^{2-c_{11}} = k_2 p_1(x_2) \qquad p_1(x_2) \neq 0 \tag{E1}$$

In a similar manner from 4* and 8* there result

$$h_1^{2-c_{12}} = k_2 q_1(x_1) \qquad q_1(x_1) \neq 0 \tag{E2}$$

Substituting according to equations (43) and (44) into equation (E1) gives

$$\left[c_{10} f_{13}'(x_2) e^{\varphi(\tilde{u})}\right]^{2-c_{11}} = \frac{1}{c_{10}} \varphi' e^{-\varphi} p_1(x_2)$$

$$\therefore \frac{c_{10}^{3-c_{11}}\left[f_{13}'(x_2)\right]^{2-c_{11}}}{p_1(x_2)} = \varphi' e^{(c_{11}-3)\varphi} \tag{E3}$$

Taking the partial derivative of equation (E3) with respect to x_1 and assuming $c_{11} \neq 3$ give

$$\frac{d}{d\tilde{u}}\left[\varphi' e^{(c_{11}-3)\varphi}\right] \frac{\partial \tilde{u}}{\partial x_1} = 0$$

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Hence,

$$\frac{de^{(c_{11}-3)\varphi}}{d\tilde{u}} = \text{constant}$$

$$\therefore\ e^{(c_{11}-3)\varphi} = a\tilde{u} + b$$

or

$$\varphi = \ln(a\tilde{u} + b)^{1/(c_{11}-3)} \tag{E4}$$

A similar analysis using equations (E2), (42), and (44) yields

$$\varphi = \ln(a\tilde{u} + b)^{1/(c_{12}-3)} \tag{E5}$$

for $c_{12} \neq 3$. From equations (E4) and (E5) it follows that

$$c_{11} = c_{12}$$

If $c_{11} = 3$, there results from equation (E3) that $\varphi' = \text{constant}$ and hence

$$\varphi = a\tilde{u} + b \tag{E6}$$

Next consider the case where one of the partial derivatives of $\overline{U}_2$ does not vanish and determine the corresponding form for φ.

$$\text{(2) Case B:}\quad \frac{\partial\overline{U}_2}{\partial x_1} \not\equiv 0;\ \frac{\partial\overline{U}_2}{\partial x_2} \equiv 0$$

The o.d.e. condition for ①* and ⑤* gives

$$\frac{\partial \ln \overline{U}_2}{\partial x_1} = c_{13}\frac{\partial \ln h_2}{\partial x_1} \tag{E7}$$

As $\overline{U}_2$ is assumed to be a function of x_1 alone, from equations (E7) and (43)

$$p_2(x_1) = c_{13}\frac{\partial \ln h_2}{\partial x_1} = c_{13}\varphi'(\tilde{u})\frac{\partial u}{\partial x_1}$$

$$= c_{13}\varphi'(\tilde{u})f_{12}'(x_1) \tag{E8}$$

It follows from equation (E8) that $\varphi'(u) = \text{constant}$. Hence,

$$\varphi = a\tilde{u} + b$$

An expression for $\overline{U}_2$ can be obtained from equations (E7) and (E8) by solving

$$\frac{\partial \ln \overline{U}_2}{\partial x_1} = (\text{const.})f'_{12}(x_1)$$

There results

$$\overline{U}_2 = (\text{const.})e^{c_{14}f_{12}(x_1)} \tag{E9}$$

It can be shown that all remaining o.d.e. conditions are satisfied.

(3) Case C: $\frac{\partial \overline{U}_2}{\partial x_1} \equiv 0; \frac{\partial \overline{U}_2}{\partial x_2} \not\equiv 0$

From symmetry considerations, this case is similar to case B, and the following results can be stated directly:

$$\varphi = a\tilde{u} + b$$

$$\overline{U}_2 = (\text{const.})e^{c_{15}f_{13}(x_2)}$$

(4) Case D: $\frac{\partial \overline{U}_2}{\partial x_1} \not\equiv 0; \frac{\partial \overline{U}_2}{\partial x_2} \not\equiv 0$

The o.d.e. condition for 1* and 5* gives

$$\frac{\partial \ln \overline{U}_2}{\partial x_1} = c_{16}\frac{\partial \ln h_2}{\partial x_1}$$

Hence,

$$\overline{U}_2 = h_2^{c_{16}}q_3(x_2) \tag{E10}$$

Similarly, from the o.d.e. condition for 6* and 8*

$$\overline{U}_2 = h_1^{c_{17}}p_3(x_1) \tag{E11}$$

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Equations (E10) and (E11) give

$$\frac{h_2^{c_{16}}}{h_1^{c_{17}}} = \frac{p_3(x_1)}{q_3(x_2)} \tag{E12}$$

Substituting according to equations (42) and (43) into (E12) and assuming $c_{16} \neq c_{17}$ yield

$$e^{(c_{16}-c_{17})\varphi} = \frac{\left[c_{10}f'_{13}(x_1)\right]^{c_{17}}}{\left[c_{10}f'_{13}(x_2)\right]^{c_{16}}} \frac{p_3(x_1)}{q_3(x_2)} = p_4(x_1)q_4(x_1) \tag{E13}$$

From equation (E13),

$$(c_{16} - c_{17})\varphi = \ln p_4(x_1) + \ln q_4(x_2) \tag{E14}$$

Since $\dfrac{\partial^2\varphi}{\partial x_1\, \partial x_2} = 0$,

$$\frac{d^2\varphi}{d\tilde{u}^2} = 0$$

$$\therefore\ \varphi = a\tilde{u} + b$$

Hence, from equation (E13),

$$p_3(x_1) = (\text{const.})e^{c_{18}f_{12}(x_1)}\left[f'_{12}(x_1)\right]^{-c_{17}} \tag{E15}$$

and

$$q_3(x_2) = (\text{const.})e^{c_{18}f_{13}(x_2)}\left[f'_{13}(x_2)\right]^{-c_{16}}$$

where

$$c_{18} = (c_{16} - c_{17})a$$

$$c_{19} = -(c_{16} - c_{17})ac_2$$

Now consider the o.d.e. condition involving ② * and ⑤ *:

$$\frac{\partial \ln h_2^2/\overline{U}_2 k_2}{\partial x_1} = c_{20} \frac{\partial \ln h_2}{\partial x_1}$$

$$\therefore \frac{\partial \ln h_2^{2-c_{20}}/\overline{U}_2 k_2}{\partial x_1} = 0 \tag{E16}$$

Substituting according to equation (E10) in (E16) gives 4975

$$\frac{\partial \ln h_2^{2-(c_{20}+c_{16})}/k_2}{\partial x_1} = 0$$

$$\therefore \left[2 - (c_{20} + c_{16})\right] \frac{\partial \ln h_2}{\partial x_1} - \frac{\partial \ln k_2}{\partial x_1} = 0$$

Therefore, from equations (43) and (44),

$$\left[2 - (c_{20} + c_{16})\right]\varphi' + \varphi' = 0$$

As $\varphi' \not\equiv 0$, it follows that

$$c_{20} + c_{16} = 3$$

From ④ * and ⑧ * a similar relation between constants results, and no new information is obtained. Remaining o.d.e. conditions are readily shown to be satisfied.

Finally, consider the possibility $c_{16} = c_{17}$. Equation (E12) then becomes

$$\left(\frac{h_2}{h_1}\right)^{c_{16}} = \frac{p_3(x_1)}{q_3(x_2)}$$

and it follows at once that

$$p_3(x_1) = c_{21}\left[f_{12}'(x_1)\right]^{-c_{16}}$$

$$q_3(x_2) = c_{21}\left[f_{13}'(x_2)\right]^{-c_{16}}$$

At this point, the analysis for φ follows the same pattern as given for case A, and the same equations for φ are determined. Theorem 4 follows at once.

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APPENDIX F

ANALYSIS: $U_1 = c_1\overline{U}_2$; $k_1 \not\equiv 0$, $k_2 \equiv 0$. PROOF OF THEOREM 6

Four possible cases are distinguished as in previous analyses.

Case A: $\overline{U}_2$ = constant

Setting $\overline{U}_2$ = constant results in three o.d.e. conditions which must be satisfied. These conditions can be obtained from the list in set (45) and, after simplification, the terms that must be proportional are the following:

$$(2)^* \quad 2k_2 - \frac{1}{h_1}\frac{\partial \ln k_2}{\partial x_1}$$

$$(4)^* \quad \frac{1}{h_2}\frac{\partial \ln k_2}{\partial x_2} \left(\text{note: } \frac{\partial h_1}{\partial x_2} \equiv 0 \quad \text{as} \quad k_1 \equiv 0\right)$$

$$(5)^* \quad k_2$$

The o.d.e. condition for $(2)^*$ and $(5)^*$ is

$$2k_2 - \frac{1}{h_1}\frac{\partial \ln k_2}{\partial x_1} = c_{22}k_2$$

Hence, either

$$\frac{\partial \ln k_2}{\partial x_1} = 0 \quad \text{and} \quad c_{22} = 2 \tag{F1}$$

or

$$(c_{22} - 2)h_1 = \frac{\partial k_2^{-1}}{\partial x_1} \qquad (c_{22} \neq 2) \tag{F2}$$

Also from $(4)^*$ and $(5)^*$ there is the possibility that either

$$\frac{\partial \ln k_2}{\partial x_2} = 0 \tag{F3}$$

or

$$h_2 = c_{23}\frac{\partial k^{-1}}{\partial x_2} \tag{F4}$$

If equations (F1) and (F3) hold, then k_2 = constant and

$$h_2 = p_1^n(x_1)q_1(x_2)$$

$$h_1 = \frac{p_1'(x_1)}{p_1(x_1)} \tag{F5}$$

If it is assumed that equation (F4) is valid, there is obtained upon differentiation

$$\frac{\partial h_2}{\partial x_1} = c_{23} \frac{\partial^2 k^{-1}}{\partial x_2 \, \partial x_1} \tag{F6}$$

However, from equation (F1) or (F2) it can be shown that $\frac{\partial^2 k^{-1}}{\partial x_2 \, \partial x_1} \equiv 0$. But this would imply that $\frac{\partial h_2}{\partial x_1} = h_1 h_2 k_2 = 0$, which violates the hypothesis. Hence, the one remaining possibility is

$$\frac{\partial \ln k_1}{\partial x_2} = 0; \; \frac{\partial \ln k_2}{\partial x_1} \not\equiv 0$$

Therefore,

$$k_2 = p_1(x_1) \tag{F7}$$

From equation (F2) it then follows that

$$h_1 = \frac{1}{2 - c_{22}} \frac{p_1'(x_1)}{p_1^2(x_1)} \tag{F8}$$

Now,

$$\frac{\partial \ln h_2}{\partial x_1} = h_1 k_2 = \frac{1}{2 - c_{22}} \frac{p_1'(x_1)}{p_1(x_1)}$$

$$\therefore h_2 = \left[p_1(x_1)\right]^{1/(2-c_{22})} q_1(x_2) \tag{F9}$$

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$$\text{Case B:} \quad \frac{\partial \overline{U}_2}{\partial x_2} \not\equiv 0; \ \frac{\partial \overline{U}_2}{\partial x_1} \equiv 0$$

By an analysis similar to that given for case A, it can be shown that h_1, h_2, k_2, and $\overline{U}_2$ must have the following forms:

$$\overline{U}_2 = (\text{const.})e^{\int q_1(x_2)dx_2}$$

$$h_1 = -\frac{p_1'(x_1)}{p_1^2(x_1)}$$

$$h_2 = \frac{q_1(x_2)}{p_1(x_1)}$$

$$k_2 = p_1(x_1)$$

$$\text{Case C:} \quad \frac{\partial \overline{U}_2}{\partial x_1} \not\equiv 0; \ \frac{\partial \overline{U}_2}{\partial x_2} \equiv 0$$

The solutions for h_1, h_2, k_2, and $\overline{U}_2$ are $\overline{U}_2 = \left[p_1(x_1) + \text{const.}\right]$ with h_1 and h_2 defined by equation (F5) or $\overline{U}_2 = (\text{const.})\dfrac{h_2}{q_1(x_2)}$ with k_2, h_1, and h_2 defined by equations (F7), (F8), and (F9), respectively.

$$\text{Case D:} \quad \frac{\partial \overline{U}_2}{\partial x_1} \not\equiv 0 \ \text{ and } \ \frac{\partial \overline{U}_2}{\partial x_2} \not\equiv 0$$

The solutions for h_1, h_2, k_2, and $\overline{U}_2$ are

$$\overline{U}_2 = (\text{const.})p_1^n(x_1)e^{\int q_1(x_2)dx_2} \quad (n \neq 0)$$

$$h_1 = -\frac{p_1'(x_1)}{p_1^2(x_1)}$$

$$h_2 = \frac{q_1(x_2)}{p_1(x_1)}$$

$$k_2 = p_1(x_1)$$

As mentioned previously, the case where $k_2 \equiv 0$ gives similar results, with the roles of x_1 and x_2 interchanged (from symmetry considerations). Theorem 6 readily follows.

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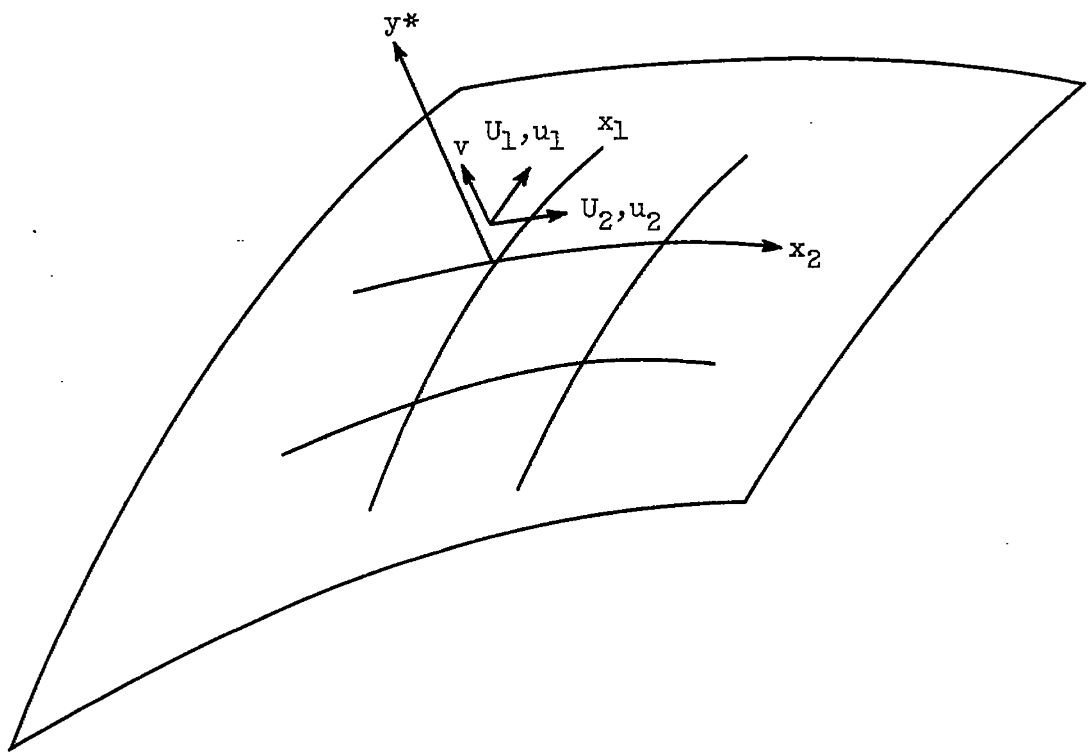

Figure 1. - Coordinate system and orientation of velocity components for flow over a surface.

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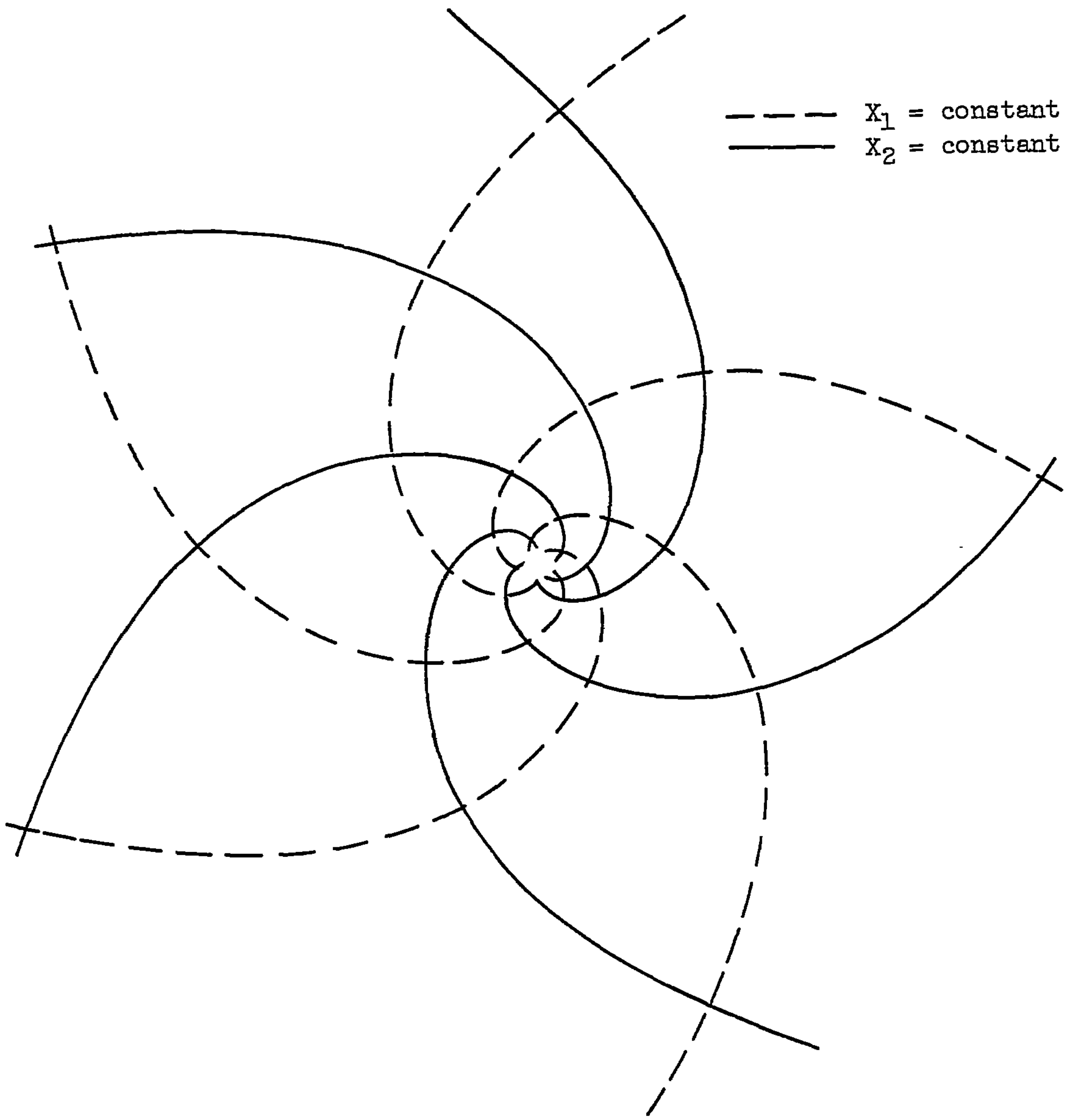

Figure 2. - System of spiral coordinates.

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