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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS.

TECHNICAL MEMORANDUM 55

# RECENT PROGRESS IN THE THEORY OF AIR FLOW AS APPLIED TO AERONAUTICS.

By

L. Prandtl.

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# RECENT PROGRESS IN THE THEORY OF AIRFLOW AS APPLIED TO AERONAUTICS.*

By

L. Prandtl.

The development of the theory of flow of gases, and especially of the theory of air resistance, affords an illustration of how progress may be retarded by a false theory, especially when advocated by a scholar of world renown. I refer to Newton's theory of air resistance. We have no right to reproach this great man on this account. His service was very creditable for those times, even though he was less fortunate in this than in other matters.

Newton's law furnishes the right expression, that the air resistance is proportional to the square of the velocity of the surface presented by the object and to the density of the air, but it gives quite unsatisfactory results regarding the dependence of the air resistance on the shape of the object. According to Newton's view, the air consisted of small particles, which mutually repelled each other as far as possible and, in the event of equilibrium, remained at rest with reference to each other. If a solid body was moved through the air, the particles which were thought to be very small in comparison with the distances between them, were struck singly by the moving body, the resistance being the combined effect of all these collisions. It remained an open question as to whether the laws

* From "Zeitschrift des Vereines Deutscher Ingenieure," September 10, 1921.

sure variation occurs, there will be no appreciable changes in volume, which may accordingly be entirely neglected, in order to simplify the theoretical considerations. Volumetric changes in moving air exert an appreciable modifying influence on the motion, when the velocity is comparable with the velocity of sound. At 1/10 the velocity of sound, the variations are only about one-half of 1%, and are therefore entirely negligible.

That the air particles do not fly about at random among each other, but combine in an airflow, is explained according to the kinetic theory of gases, by the assumption that, though the individual molecules are perfectly free to move, they often collide and exchange momenta. The resultant motion, which has the mean value of the irregular individual motions, accordingly constitutes the flow of a fluid.

Pressure from any source is transmitted in all directions and it is therefore inadmissible to calculate the resistance of a body by simply adding the resistances of the individual parts, since these parts exert a mutual influence on each other and the resistance of the combined parts differs from that of the parts taken separately. The wind pressure on a roof therefore depends largely on the shape of the building it covers.

The theory of the flow of liquids, hydrodynamics, (beginning in the middle of the eighteenth century with L. Euler and D. Bernouilli), was developed under Helmholtz, Kirchhoff, Lord Kelvin, etc., to a high degree of perfection, although in only one direction, which seemed to offer but little of use to the practical man and greatly shook his confidence through contradictory results. Thus, for example, a body moving uniformly in a fluid originally at rest was supposed to experience no resistance in the direction of the motion, which was contrary to all observed facts. Calculations were made on the basis of the so-called ideal liquid, a constant-volume fluid without viscosity, because the allowance for the viscosity (whose influence on the individual particles was well known) made the calculation too difficult. Since the effects of viscosity in fluids were so slight in comparison with the effects of inertia, this method of calculation seemed to be justified.

The above-mentioned contradictory result of the absence of resistance was found first while investigating the flow around a ball and subsequently was shown to hold strictly true for the ideal fluid for all bodies without sharp edges. The hydrodynamics of the ideal fluid failed however in the problem of the resistance of actual fluids, but a more thorough investigation demonstrated that in case where the actual resistance was very small on account of the suitable shape of the bodies, the theoretical principles were satisfied in large measure. The shapes of bodies with small resistance are of the greatest practical importance in the construction of airships and airplanes. Systematic experiments in aerodynamic laboratories during the last decade have gradually developed the best shapes and shown that these shapes, excepting for the ever-present skin friction, enable the practical realization of the theory of no resistance. The fact that the flow both divides in front of the body and closes again behind it, in accordance with the theory, constitutes the main characteristic of the motion of the ideal fluid about a body. The resistance of the body of an airship, according to laboratory experiments, is between 1/30 and 1/25 of the resistance of a flat disk having the same diameter as the maximum section of the airship. The small circle in Fig. 1 represents the disk which would offer the same resistance as the airship. This resistance may be regarded as due entirely to skin friction.

Before going further into details, I will mention briefly the means of presentation and the most important theorems of hydrodynamics. The state of flow for any given instant is known when the pressure as well as the magnitude and direction of the velocity are given for every point. The velocity is ordinarily designated by the three components $u$, $v$ and $w$, according to the axes of a right-angled system of coordinates. The whole motion is known when $u$, $v$, $w$, and also the pressure $p$ are given as functions of the three-space coordinates and of the time. For the diagrammatic representation of the flows, use is made chiefly of "streamlines" showing the direction of all parts of the flow. The streamlines can be calculated from the above-mentioned functions.

The typical task of theoretical hydrodynamics is now to determine, for any given conditions on the boundaries of the fluid (for example, for a prescribed motion of solid bodies through it), the functions u, v, w and p, for the whole space occupied by the fluid. For this purpose use is made of two types of conditions:

1) The so-called continuity condition, that in every small portion of space just as much fluid flows in as out, which consequently preserves the constancy of volume;

2) A dynamic condition, that the resulting momentum of a portion of the fluid, which comes from the differences in pressure and from any other forces acting on the portion, equals the mass of that portion times its acceleration. The methods for carrying out the calculation cannot be given here in detail, but the following paragraphs will contain indications concerning them. On the other hand, several important laws will be stated here without demonstration. They are very simple laws which hold good for only the ideal fluid, and they furnish the real reason why the ideal fluid can be treated mathematically so much more easily than actual fluids.

One very important conception for the motion of the ideal fluid is the "line integral of the velocity." If we imagine instead of the "velocity field," a "force field" such that each velocity (of given magnitude and direction) is replaced by a force of corresponding magnitude and the same direction, then the line integral of this force will represent the work performed by the force in moving a unit mass along the given line. The line integral of the velocity is obtained when each individual part of the path is multiplied by the components of the velocity falling in the direction of the path and the products are all added together. In the theory of the force field, the case is considered when the work done in moving a mass from a point A to a point B is the same for all paths. In this case, when the point A is stationary, the work done between A and any point B, and which accordingly depends only on the location of the point B, is called the potential at the point B and the whole force field is called a potential field. The same case also occurs in the motion of fluids. The value of the line integral, taken from a stationary starting point, is called the velocity potential of the motion and the velocity field is called the potential field. The velocity is then given, just as the force was given in the other case, both in magnitude and direction, by the fall of the potential. Much is gained by the introduction of the potential, for since the velocity components can be deduced from the potential, it is then necessary to determine only one function instead of three. Even the pressure in the case of potential motion may be obtained by a simple formula.

Concerning the line integral of the velocity there is an important law discovered by Lord Kelvin (Sir William Thomson), namely, that in an ideal fluid for any given closed line which is continuously formed out of the same fluid particles, the line integral can not change its value with the lapse of time.

We will immediately make an important application of this law. If, at any instant, the whole fluid is at rest, then every line integral in it has the value zero and must therefore, according to Lord Kelvin, retain the value zero for all time. If a closed line is laid through the above-mentioned points A and B, it may be easily demonstrated that the assertion that the line integral disappears for the closed path, is identical with the assertion that in both directions, around the right or around the left, from A to B, the line integral has the same value. From this we recognize that only potential flows can be generated by any kind of pressure on the surface or by setting in motion bodies existing in the ideal fluid.

The consideration of cases in which the line integral differs from zero, would bring us to the famous Helmholtz vortex theory. We can not however go further into this matter here. We will only mention that there are motions in which the characteristic of the potential motion does not hold along certain, mostly very restricted, regions in space. A line integral whose line embraces portions of this region, will then usually have a value differing from zero. In such cases, we speak of a vortex motion. The value of the line integral is called the "circulation," or, when applied to the embraced vortex, also the "strength of the vortex."

Until quite recently, it was thought necessary to draw the conclusion that vortex motions could only occur in an ideal fluid when they were present in it from the beginning through some sort of act of creation, but that their production from a condition of rest was impossible. It must not, however, be forgotten that the ideal fluid is for us only a simplified imaginary image of a real fluid, which always exhibits some viscosity. A special investigation,* which we can not take up here in detail, demonstrated that the viscosity of the fluid, even when ever so small, takes effect with finite strength in a region in the immediate vicinity of the bodies, by holding back a thin layer of the flowing fluid. Kelvin's law would not hold good for any line drawn through this region, on account of the effect of the viscosity. Any region of the fluid whose particles have previously, during the motion, come near the surface of the object, can therefore become the seat of vortices. All vortex formation in fluids with small viscosity is to be explained in this manner. We shall also see that, in a practically very important case, the theoretical conception of vortex formation has brought decisive progress.

Regular vortices are formed on sharp edges about which the fluid flows. Even in the case of perfectly rounded surfaces, like a sphere, for example, it happens after a pure potential

* See my lecture before the Heidelberger Internationalen Mathematiker-Kongress, 1904 (Proceedings of this Congress, p. 484, Leipzig, 1905), or the article "Flüssigketsbewegung," in "Handwörterbuch der Naturwissenschaften," p. 117.

motion is created at the beginning, that owing to reverse motions in the rear half of the surface layer, this layer being the seat of the friction phenomena, portions of it are first heaped up and are then liberated into the free fluid as vortices. Hitherto it has not been possible to assert much theoretically concerning these vortices, which are closely connected with the resistance to motion. Only concerning the preliminary condition for their creation, the reverse flow in the marginal layer, which causes the release of portions of this layer, it may be stated that it is connected with a retarded flow of the free fluid along the wall, and the details of this motion can be quantitatively explained. (See H. Blasius, Z. F. Math. u. Phys. 1908, p.1, and Hiemenz, Dingl. Polytechn. Journal, 1911, p.321.) It may be qualitatively explained in the following manner: The same differences in pressure, when great enough, turn back the surface layer already somewhat retarded by friction. That this causes an expansion of the surface layer is readily seen from Fig. 2, since there is no possibility of escape for the reverse flow inside the direct flow. Fig. 2 shows the incipient formation of a vortex. The quantitative results of these investigations are in accord with experience (Hiemenz a. a. O. and H. Rubach, Mitteilungen über Forschungsarbeiten, published by V.d.I., No. 185, 1916.)

Karman (See Karman and Rubach, Physikalische Zeitschrift 1912, p.49) has successfully investigated the completed vortices, which, in the event of a uniform flow, show more or less regular series of alternately right and left rotating vortices. He has shown that only one kind of vortex configuration (Fig. 3) is stable and that the resistance may be quite accurately calculated by means of purely visual observations, namely, by measuring the intervals between the vortices and the frequency of the vibratory motion, whereby the result of the calculation agrees well with the measured resistances. The relation between the vortex system and the dimensions of the vortex-generating body, which would have determined the practical applicability of the theory, has not yet been theoretically established.

Returning to the practically important problem of bodies with small resistance, I would like to take up next the investigations on airship bodies begun by my former colleague, Dr. Georg Fuhrmann, who unfortunately fell in the war. We must first find formulas for the airflow about the body of an airship. If the flow is to be considered from a stationary standpoint with relation to the airship, we must seek the velocity distribution for which the velocities are tangential to the surface of the airship. We obtain this kind of flow when we imagine the fluid continued into the inside of the airship and and adopt, on the front portion of this axis, points where the fluid is continually renewed and, on the rear portion, corresponding points where the same quantities of fluid again disappear. This flow is impossible in a physical fluid, but here (since it only concerns the flow at points where there is really no flow) it correctly represents the effect of the front part of the airship, which deflects the fluid outward on all sides and also the effect of the rear part, where the fluid again flows together.

The mathematical formulation of such "sources" and "sinks" is very simple. The process of calculation would be very difficult, if it were necessary, for a given airship, to find the correct distribution of these sources and sinks. Fuhrmann proceeded in such manner, however, that he calculated the outlines of the body of an airship corresponding to suitable arbitrary distributions of sources and sinks. He also calculated the details of the flow and pressure. Concerning the connection of the pressure with the velocity of flow $v$ in a potential motion, the following may be noted. When the motion is steady, the static pressure $p$ (that is, the pressure which would be recorded by an instrument moving with the flow) plus the "velocity pressure" $\rho \frac{v^2}{2}$ forms a constant sum. The "velocity pressure" also called "dynamic pressure" or "impact pressure" is (as may be concluded from the application of the above-mentioned relationship) equal to the pressure increase in comparison with the static pressure, which appears in the opening of a tube directed up-stream against the flow. It is known that this relationship is made use of for measuring flow velocities. $\rho = \frac{\gamma}{g}$ is here the density of the medium.

The pressure distributions calculated by Fuhrmann were verified on bodies of the calculated shape by providing these bodies with perforations and measuring the pressures occurring in the perforations. The results, for three of these bodies, are given in Figs. 4 to 6, the lines denoting the calculated pressures and the small circles, the measured pressures. The observation indicates on the front end a pressure increase equal to the dynamic pressure of the artificial wind in which the experiment was performed. The calculated curve gives a like pressure increase on the rear end, which fails to appear in the experiments. This discrepancy results from the fact that in reality the flow at the rear end does not close up completely, as is assumed in the theory. This is because of the retardation of the marginal layer of air due to skin friction. In other respects the details of the pressure distribution for airship bodies of different shapes agree very well with the theory.*

A practically very important result for the theory has been obtained in the investigation of air forces on airplane wings. The reasoning processes involved may be briefly described here, though in a manner not corresponding to their historical evolution.

When an aerofoil with a shape similar to a bird's wing acquires a lift by moving swiftly through the air and is thereby in position to support the weight of the airplane, the air receives a downward pressure equal to the weight supported. Considered in detail, we find on the under side of the aerofoil, increased pressure in comparison with the pressure of the undisturbed air and a diminished pressure on the upper side. These

* The three small jogs in the lines representing the measured pressure distribution at 1/3 and 2/3 their length, are caused by defects in the construction of the three parts of the built-up model.

pressure differences, acting on the whole aerofoil taken together produce the lift.

In order to obtain information concerning the flow relations here involved, we will first seek the value of the line integral for a closed curve, which passes through the field traversed by the aerofoil. We shall find that the value of the line integral, which we will call the "circulation," generally differs from zero If we consider a closed curve which passes downward through the air strip traversed by the aerofoil and again rises in the undisturbed field, then before the passage of the aerofoil through the portion of space under consideration, the circulation was zero. During the passage of the aerofoil, we can (if we refrain from unnecessary refinements) imagine the line cut by the aerofoil. Since the pressures are different on both sides of the cut, the line integral will gradually increase (like a pipe filled with still water, at the ends of which a difference in pressure is suddenly created) in proportion to the increase in the pressure difference and to its duration. For a pipe of uniform cross-section, the line integral would be $v\,l$, in which $v$ is the velocity in the pipe and $l$ the length of the pipe. If the density is $\rho$, a pressure difference of $p_1 - p_2$ for the time $t$ produces a value of $v\,l$ of the amount

$$v\,l = \frac{p_1 - p_2}{\rho}\,t,$$

The same equation is also given by the strict theory for the circulation in the free fluid.

If the wing chord is $s$ and the flight speed $V$, we have, for the time $\tau$ while the line is cut, the equation

$$s = V\tau, \text{ hence } \tau = \frac{s}{V}$$

For the circulation, we accordingly obtain

$$\Gamma = \frac{p_1 - p_2}{\rho}\tau = \frac{(p_1 - p_2)s}{\rho V}$$

in which $(p_1 - p_2)s$ denotes the lift per unit length along the wing, which we shall designate by $a$. We then have

$$a = \Gamma \rho V \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (1)$$

a formula independently discovered by Kutta in Munich and Joukowski in Moscow by different methods. But since the line can be closed behind the wing, the circulation again becomes constant and is consequently the same for all particles of air which have touched one and the same spot on the wing.

The lift density $a$ is usually greatest in the middle of the wing and drops to zero toward the ends, since at the ends the pressure differences are equalized around the edges. What is said with regard to $a$ also applies to the distribution of the values of $\Gamma$, which correspond to the individual points of the air strip touched by the wing. This strip is accordingly the seat of vortices. The strength of the vortices in any given strip is measured by the circulation of a line encircling the individual strip. This circulation is evidently equal to the difference in $\Gamma$ on the right and left of the strip. We thus obtain, especially near the ends of the wings where $\Gamma$ drops to zero, relatively strong rotational motions in opposite directions.

If the small individual motions of the vortices are neglected, the geometric configuration of the vortex system is fully known and we can therefore calculate the latter by means of the geometrical laws which connect the vortices with the flow velocities belonging to them. The relations are the simplest when the lift density a and also the circulation $\Gamma$ are distributed according to a semi-ellipse on the span of the aerofoil (Fig. 7). In this event, the velocity $w_1$ of the descending flow behind the wing is constant and

$$w_1 = \frac{\Gamma_0}{b} = \frac{a_0}{\rho V b} = \frac{4}{\pi}\frac{A}{\rho V b^2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (2)$$

in which $A = \frac{\pi}{4} b a_0$ is the total lift.

As a more thorough investigation shows, this descending motion is first partially developed in the place where the wing is and it has, at the center of pressure of the cross-section, just half of the above-given velocity in the vortex tail.

In the following paragraph it will be further shown that the theory is capable of determining the potential flows around the wing sections and of explaining all the details which give rise to lift. These flows are definitely connected with a circulation around the wing section, of exactly the amount obtained above in equation 1. According to our reasoning, this is correct since we can close together the line passing through the strip touched by the air behind a wing element, so that it embraces the wing section in its plane. From the above-mentioned theory of wing flow, which is connected with the ideal case of an infinitely broad wing and hence of uniform flow, the result is to be anticipated that it will offer no resistance in the direction of motion, but only develop a lift perpendicular to it. We express this result now by saying: "From the circumstance that the lift drops to zero toward the ends of the wings, we have (in addition to the former flow velocities, which were present even for infinitely wide wings) a downward motion at the place where the wing is, through whose influence the whole flow, in comparison with that of an infinitely wide wing, is inclined somewhat downward and indeed roughly about the value of the angle obtained from the equation."

$$\operatorname{tg} \beta = \frac{1/2\ w_1}{V} = \frac{2\ A}{\rho \pi\ V^2 b^2} \quad . \ . \ . \ . \ . \ . \ . \ . \ . \ (3)$$

Hence the lift of each wing element will no longer be vertical, but perpendicular to this inclined direction and the flow will therefore offer a resistance or drag component of the value

$$W_i = A \sin \beta \approx A \operatorname{tg}\beta = \frac{2\ A^2}{\pi \rho V^2 b^2} \quad . \ . \ . \ . \ . \ . \ . \ . \ (4)$$

At the same time, the necessary angle of attack for obtaining a certain lift density must be increased by $\beta$, in comparison with the infinitely broad wing. The resistance just mentioned which, according to equation 4, is proportional to the square of the mean lift density $\frac{A}{b}$, does not stand in contradiction to the assumed absence of friction in the medium, since it has its exact equivalent in the kinetic energy left in the medium in the vortex wake behind the wing.

It has now been shown that, with the aid of the calculations just indicated on infinitely long wings, we can fully explain the hitherto very enigmatical great influence of the aspect ratio of the wings on their aerodynamic behavior. If we take the results of more recent measurements on a series of aerofoils of uniform cross-section, but different aspect ratio, as obtained from modern laboratories, and subtract the above-given theoretical drag from the measured drag, we find that the remaining drag, in relation to the lift per unit of surface $\frac{A}{bs}$, is almost exactly the same for the different experiments and that consequently this remaining drag is no longer dependent on the aspect ratio. The same holds true for the angle of attack of infinitely broad wings calculated in the above-given manner. On the other hand, both the remaining drag and the angle of attack of an infinitely broad wing are dependent on the wing section. For this reason, the remaining drag has been called the sectional drag. It is readily seen that, with the aid of these formulas, we can convert the experimental results obtained with one aspect ratio, to any other aspect ratio, so that in the future it will suffice to make only one experiment with only one aspect ratio.

The results of this method are shown by Figs. 8 and 9, which have been taken from the "First Report of the Göttingen Aerodynamic Laboratory" ("Ergebnisse der aerodynamischen Versuchsanstalt zu Göttingen, I. Lieferung," published by R. Oldenburg, Munich, 1921). On one side is shown lift and drag* for a series of aerofoils with different aspect ratios, presented according to measurements; on the other side, the conversion to the aspect ratio 1 : 5. The conversion of the angles of attack yields a like good agreement. The experimental values, which correspond to a square aerofoil, do not, it is true, fall in line, but this is not strange, since the theory under consideration is only a first approximation for very long surfaces. It is, on the contrary, an unexpected result that it still holds good for an aspect ratio of 1 : 3.

The theory has also been applied to biplanes and multiplanes. (See report in "Jahrbuch, 1920, der Ges. f. Luftfahrt," p.37, where further information is given. From what is given there, may be found an intimation of a noteworthy result which bears upon that distribution of the lift, on an aerofoil of any desired shape or upon a group of such aerofoils, which gives the minimum theoretical drag for a given total value of the lift. Dr. Max Munk's solution of this problem reads: "Let us imagine the space traversed by the group of aerofoils (hence a strip, or a system of strips, running along the flight path) as a rigid

* $C_a$ and $C_w$ (in Figs. 8 and 9) denote the lift and drag coefficients (Lift or drag divided by the surface area and dynamic pressure $1/2\ \rho V^2$.)

formation and let us set this in motion in an ideal fluid with a uniform velocity at all points, in the opposite direction to the lift. The resulting flow is the desired vortex motion and the distribution of the pressure at the moment of starting gives the desired lift distribution. For a monoplane, we thus obtain the above-mentioned elliptical distribution, the velocity of the rigid formation being $w_1$ according to equation 2."

In a final paragraph, we shall state briefly a theory of aerofoil sections which has today been developed to a high degree of perfection. The problem may be simplified by assuming the flow to be uniplanar or "two-dimensional." This means that the path of each air particle describes a uniplanar curve and that, in all parallel planes, the same phenomena occur. This is the case, if we imagine an aerofoil of constant cross-section infinitely extended laterally, so that every disturbance coming from the ends of the aerofoil (tending to make the flow spatial) is eliminated.

For the uniplanar potential flow, there is an especially efficacious and suitable method, the "method of orthomorphic or conformal transformation." It shows that, if we have any uniplanar potential flow, we can derive other uniplanar flows from it, by subjecting the plane of the diagram to those transformations for which its smallest geometric parts remain similar or equiangular. Through such an equiangular or conformal transformation, which can be applied several times in a series, nearly all imaginable uniplanar flows can be mathematically represented

Figs. 10 and 11 present the most common transformation in the theory of wing sections, by which a circle is transformed to a straight line AB, the surrounding circles to confocal ellipses and the radii to hyperbolas.

We have long known the potential flow about a cylinder, which, like all such flows, generates neither lift nor drag. It has been known, however, since the time of Lord Rayleigh (Messenger of Math. VII, p.14, 1877, Sc. Papers I, p.343) that a lift is generated, if a circulatory motion is superposed on the previously known motion. The flow thus obtained gives a streamline formation like Fig. 12. The above-mentioned transformation was applied first by Kutta (Illustr. aeronaut. Mitteilungen, 1902, p.133), who let the diameter AB coincide with a chord drawn through the "rest-point" Q and thus obtained the flow around a flat or cambered plate. Later Joukowski found that very beautiful sections, similar to birds' wings, were obtained by giving the diameter AB the position indicated in Fig. 12. Fig. 13 shows such a wing section with the resultant streamlines.

This wing-section theory - which has, in recent times, been much further developed (See Zeitschrift für Flugtechnik 1918, p.111, Karman and Trefftz, 1917, p.157, and 1920, p.68, Mises, 1921, Geckeler) and now renders it possible to calculate, for almost any given wing section, not only the value of the lift and the location of the center of pressure, but also the pressure distribution in detail - agrees also with the experiments as well as can be expected, considering that the friction is neglected. Fig. 14 shows (according to measurements by my co-worker, Dr. Betz, on a Joukowski wing section) the dependence of lift and drag on the angle of attack. In the field within which the wing section is "good" the theoretical and experimental curves run nearly parallel at a distance determined by the friction. The pressure distributions (Figs. 15 and 16) likewise agree well on the whole. The principal deviation proceeds from the fact that, due to friction, the theoretical circulation for the individual angles of attack is not fully attained in practice.

In summing up, it may be said that the hydrodynamic theories are best confirmed by experimental results for bodies with small resistance or drag and can accordingly be used in place of experimental tests.

It is evident that the theories here brought forward can also be applied to other technical phenomena. Thus their application to aircraft propellers is already settled in principle* but must be investigated further as to details. For their transfer to turbines and pumps, there is the difficulty that our calculations have in part assumed the adoption of small velocity changes and small angles of deflection. Useful results may be expected from the above, especially for machines like "Kaplan

* A. Betz, "Screw Propeller with Minimum Loss of Energy," with an appended note by L. Prandtl, Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen. Math.-Phys. Klasse 1919, p.193. Further, A. Betz, "The Principle of the Screw Propeller" (Die Vorgänge beim Schraubenpropeller), "Naturwissenschaften," 1921, No. 10.

wheels," rotary shovels, etc. The calculations are at present insufficient for other kinds of turbines, but even here advantages can be drawn from the fundamental principles.

Translated by the National Advisory Committee for Aeronautics.

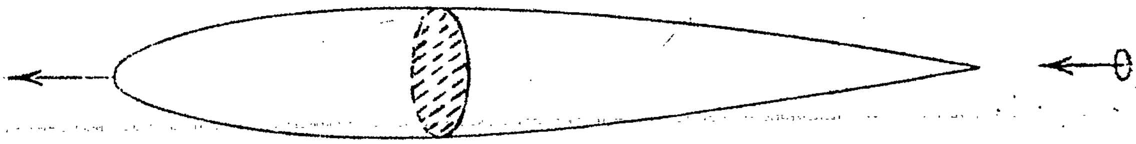


Fig. 1.

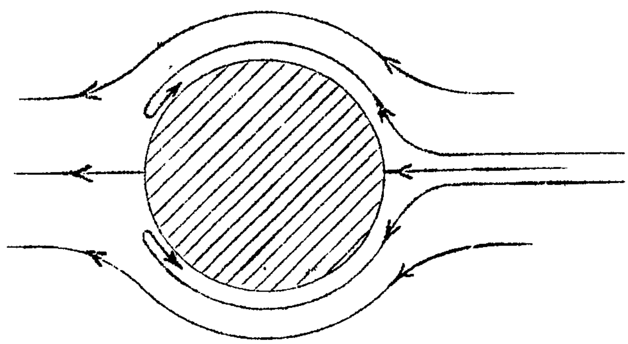


Fig. 2.

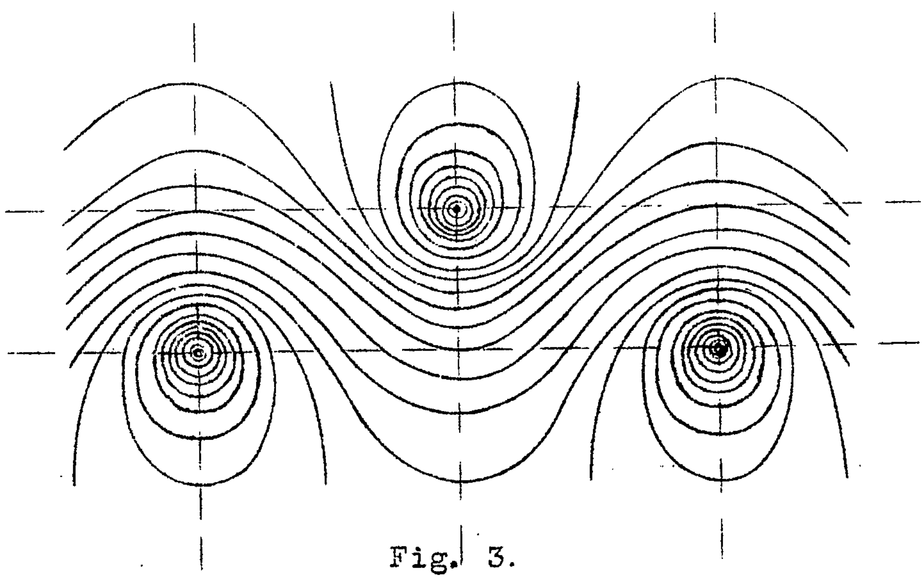
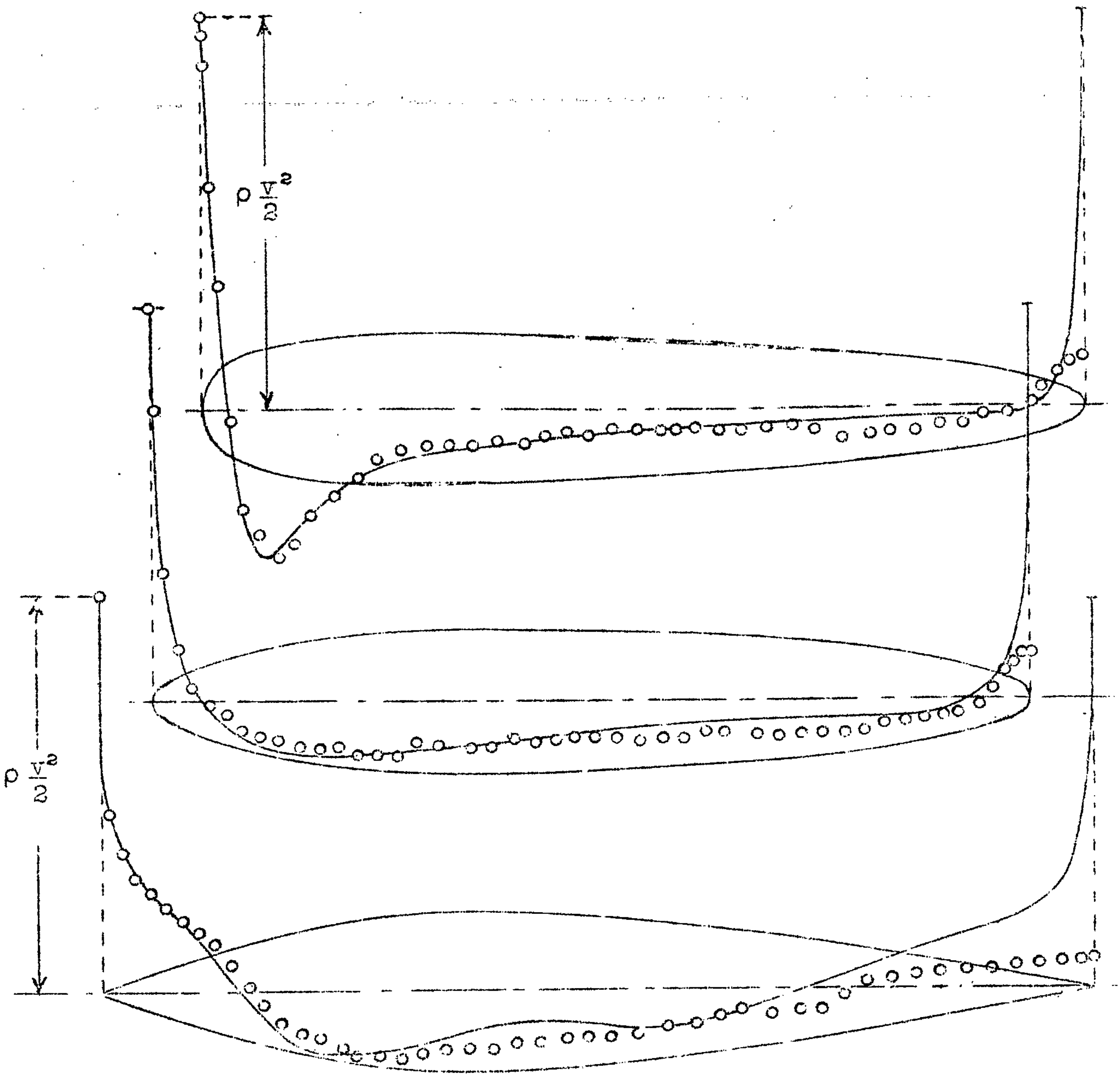


Fig. 3.




Figs. 4 to 6.

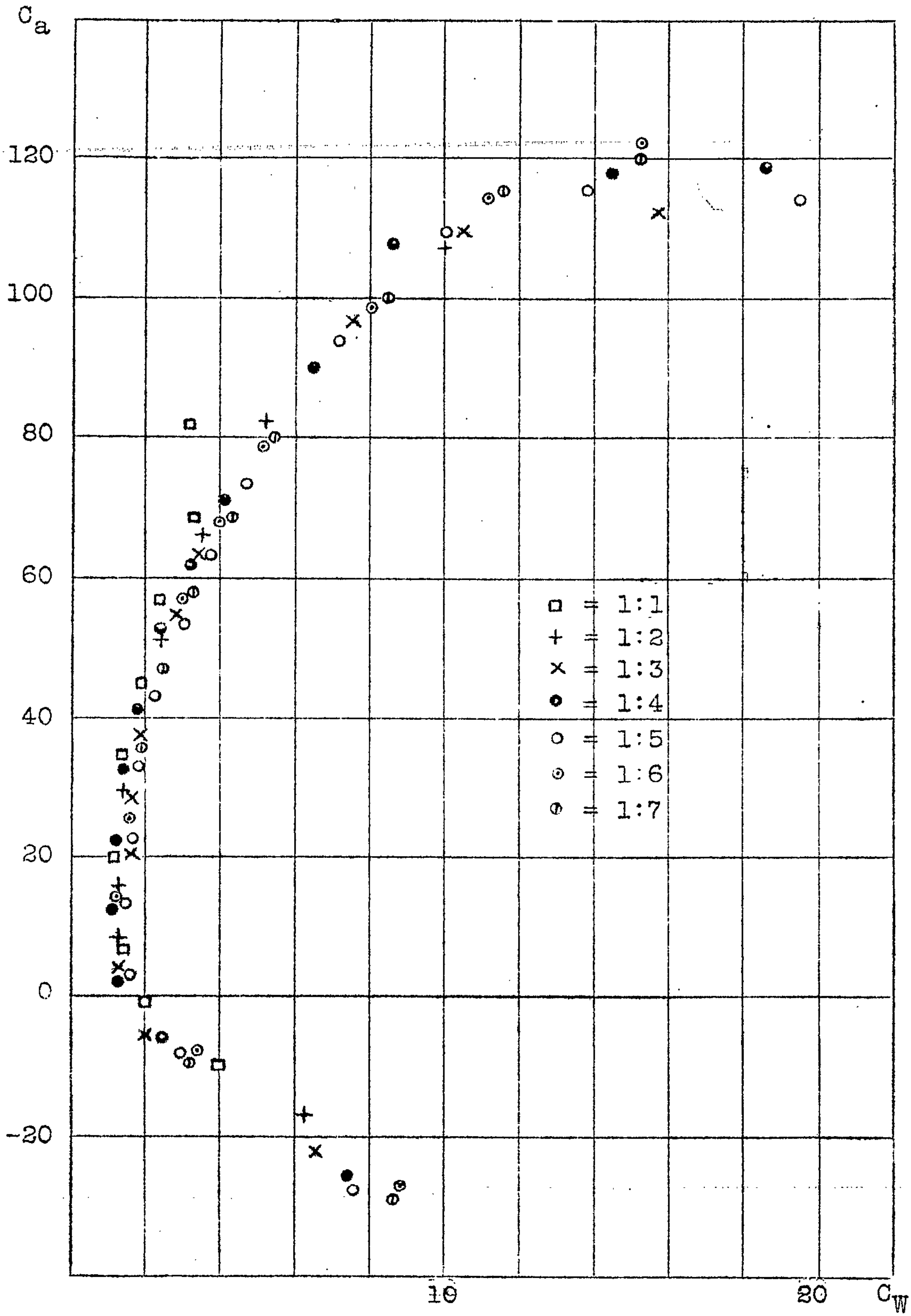



Fig. 9.

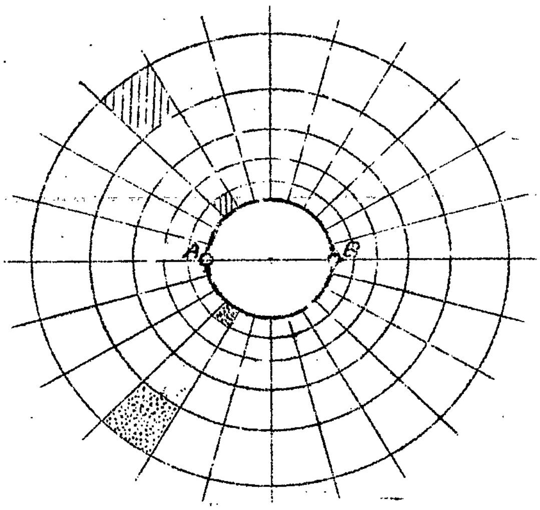



Fig..10.

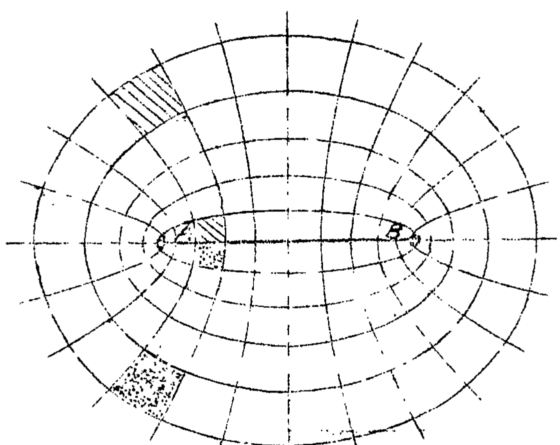



Fig.11.

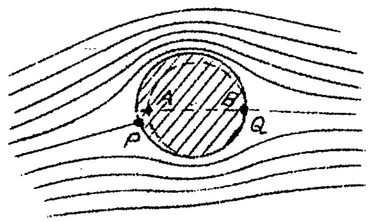



Fig. 12.

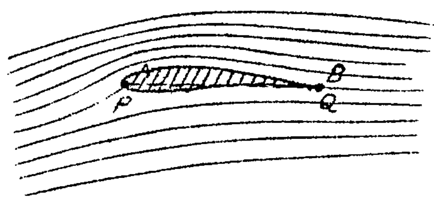



Fig. 13.

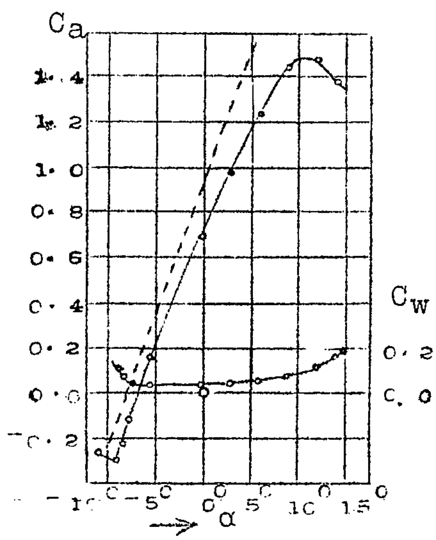



Fig. 14.

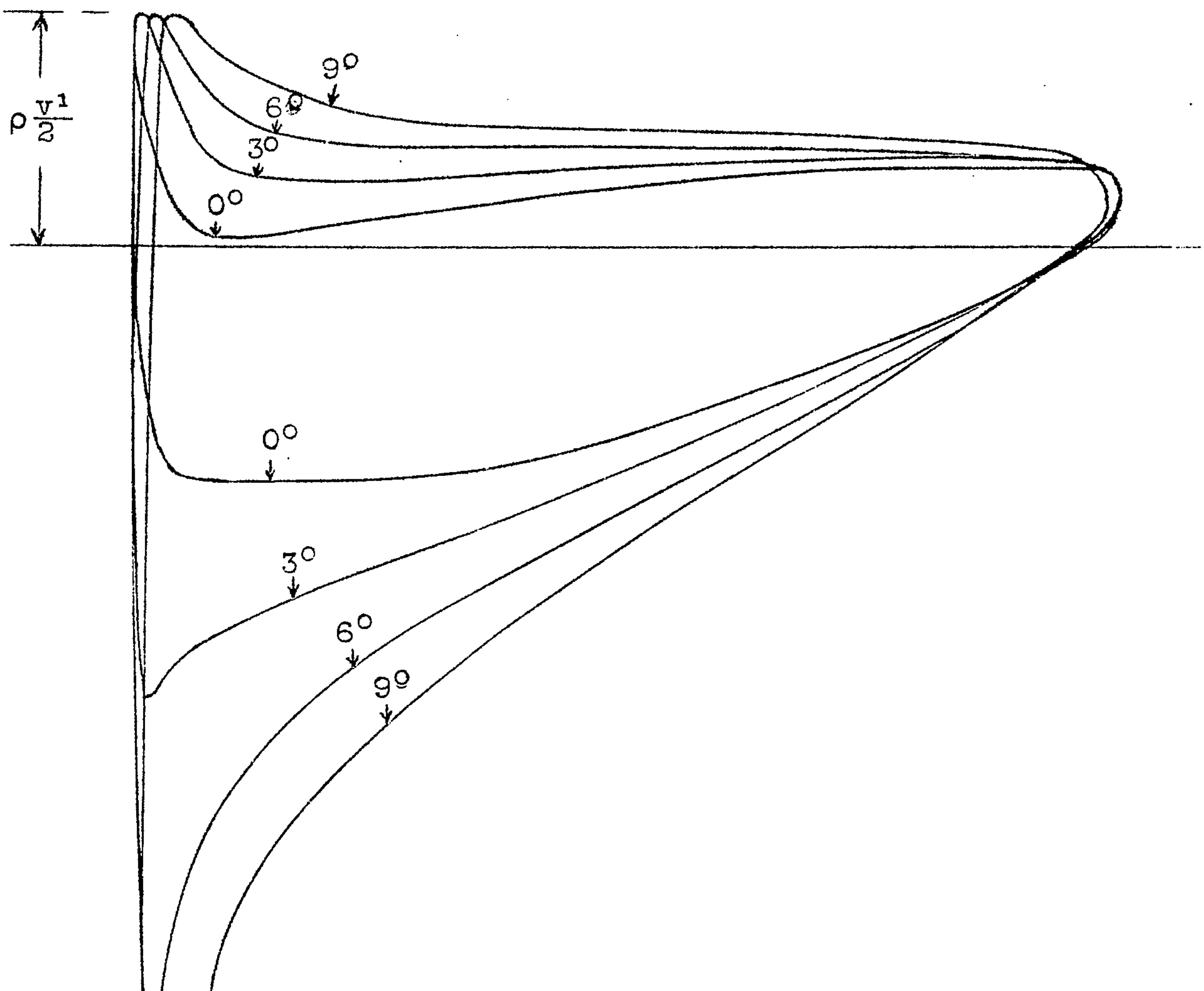



Fig. 15.

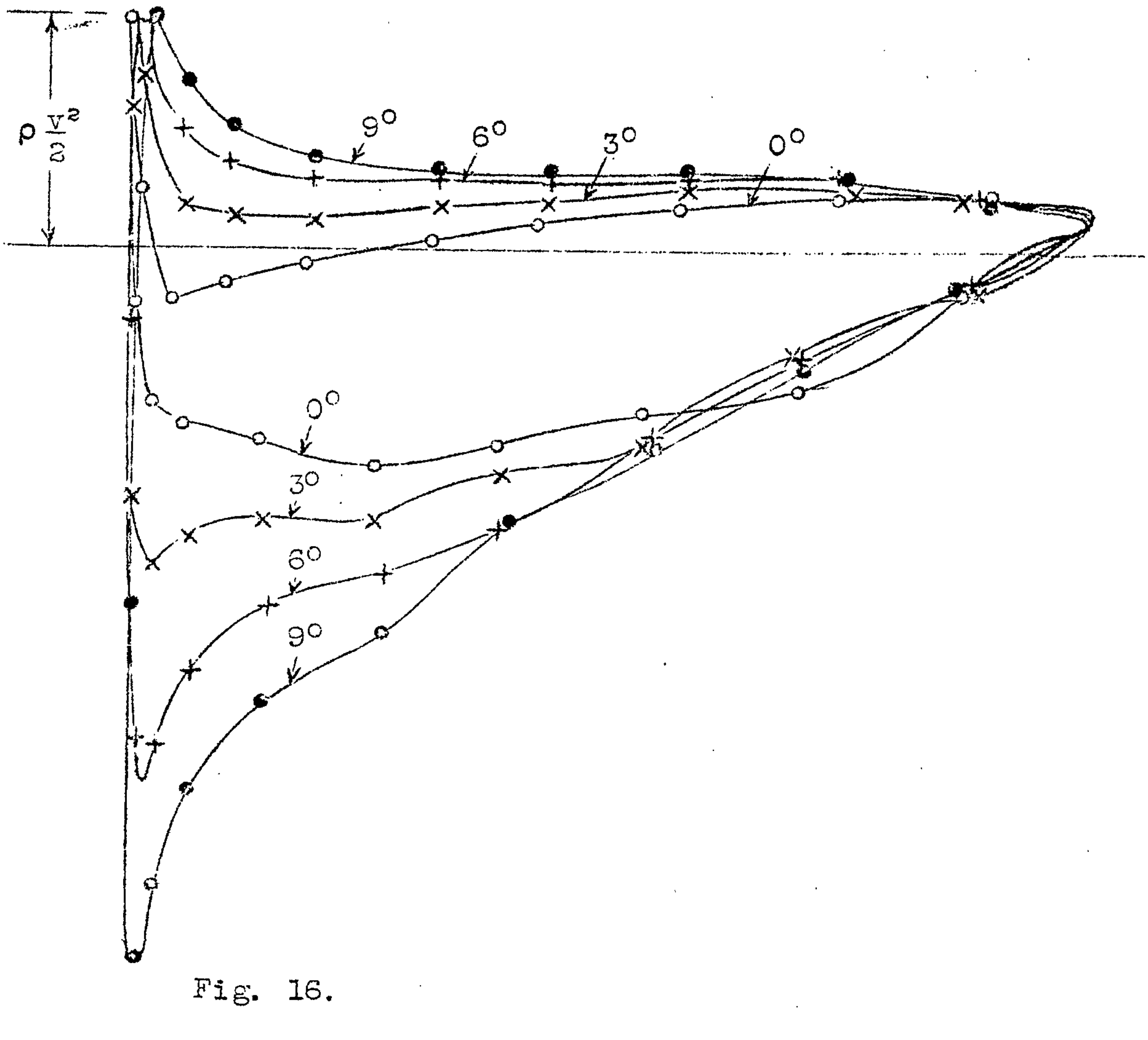



Fig. 16.