NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL MEMORANDUM 1280

GRAPHICAL DETERMINATION OF WALL TEMPERATURES FOR HEAT TRANSFERS THROUGH WALLS OF ARBITRARY SHAPE

By Otto Lutz

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GRAPHICAL DETERMINATION OF WALL TEMPERATURES FOR HEAT TRANSFERS THROUGH WALLS OF ARBITRARY SHAPE*

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A graphical method is given which permits determining of the temperature distribution during heat transfer in arbitrarily shaped walls. Three examples show the application of the method.

The further development of heat engines depends to a great extent on the control of the thermal stresses in the walls. The thermal stresses stem from the nonuniform temperature distribution in heat transfer through walls which are, for structural reasons, of various thicknesses and sometimes complicated shape. Thus, it is important to know the temperature distribution in these structural parts. Following, a method is given which permits solution of this problem.

STATEMENT OF THE PROBLEM

According to figure 1, a two-dimensional heat flow through the wall is assumed; thus, the wall should extend in the direction normal to the flow sufficiently uniformly so that no components of the flow occur in this direction. Furthermore, we consider the steady state process only. It should be added that an approximately steady state flow is present even in reciprocating engines since the temperature oscillations at the wall surface are only small and are rapidly damped in the wall.[1]

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*"Zeichnerische Ermittlung der Wandtemperaturen beim Wärmedurchgang durch Wände von beliebiger Form." Zeitschrift des Vereines deutscher Ingenieure, Band 79, Nr. 34, August 1935, pp. 1041–1044.

[1]Compare G. Eichelberg: Temperaturverlauf und Wärmespannungen in Verbrennungsmotoren. VDI-Forschungsheft 263, Berlin 1923.

The problem is characterized by the fact that in the region limited by the two boundaries o_1 and o_2 (wall surfaces) and the two heat streamlines s_1 and s_2 (the course of which is assumed to be known) (fig. 2), Laplaces's equation for the potential distribution

$$\nabla^2 \vartheta \equiv \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial^2 \vartheta}{\partial y^2} = 0$$

is valid, with ϑ denoting the temperatures. The heat flow enters through the boundary o_1, and leaves through o_2. However, these two boundaries are not isotherms, but on the contrary, the boundary condition of the third kind

$$\left.\begin{aligned} \alpha_1 (\theta_1 - \vartheta_1) &= -\lambda \frac{\partial \vartheta_1}{\partial n} \\ \alpha_2 (\vartheta_2 - \theta_2) &= \lambda \frac{\partial \vartheta_2}{\partial n} \end{aligned}\right\} \qquad (1)$$

applies to them, wherein α denotes the heat-transfer coefficients, θ, the outside temperatures, ϑ, the wall temperatures, and λ, the conductivity of the wall; $\frac{\delta \vartheta}{\delta n}$ is the temperature gradient normal to the surface (directed inward).

So far, no method giving a general solution of this problem is known to me. For the present case of heat transfer, Geiger[2], Eichelberg, (footnote 1) and Lachmann[3] have striven for the solution by assuming the flow pattern (potential flow) and varying it until it corresponded to the boundary conditions, equation (1). This method is possible for simple forms only, and, if carried out correctly it is rather slow.

We take the inverse procedure: Assuming the flow direction through the boundaries, we first satisfy the boundary conditions (1), and check whether the flow pattern corresponds to a potential flow. The advantage of this method is that it leads to the solution comparatively quickly and directly shows the surface temperatures, which are of greatest interest.

[2]J. Geiger, Z. VDI Bd. 67, 1923, p. 905.

[3]K. Lachmann, Z. VDI Bd. 72, 1928, p. 1127.

DEVELOPMENT OF THE METHOD

If we assume the flow direction through the surfaces and require that the same heat quantity flow between any two flow lines thus started, we shall, in general, obtain the flow pattern shown in figure 3a if the flow direction was not, by chance, assumed correct initially. The flow direction is to be varied so that corresponding flow lines run into one another. Since, in most technically important cases, the thickness of the wall through which heat transfer takes place is small as compared to the dimensions along the surface, one will almost always get by without a determination of the potential field in the wall. It is sufficient if the condition shown in figure 3b is reached wherewith the problem is solved.

Assume that the flow entering $q(\mathrm{kcal/m^2h})$ forms the angle φ (measured positive in clockwise direction) with the perpendicular (fig. 4). For the flow components in the direction of the surface o, q_o, and q_n in the direction n normal to the surface the equations

$$\left.\begin{aligned} q_n &= -\lambda\frac{\partial\vartheta}{\partial n} \\ q_o &= -\lambda\frac{\partial\vartheta}{\partial o} \end{aligned}\right\} \qquad (2)$$

are generally valid. They are connected with each other by

$$-q_o = q_n \tan\varphi \qquad (3)$$

If we derive from the boundary condition (1) the temperature gradient along the surface, to which corresponds the flow q_o, the latter becomes

$$q_o = \frac{\lambda}{\alpha}\frac{\partial q_n}{\partial o}$$

or with equation (3)

$$-q_n \tan\varphi = \frac{\lambda}{\alpha}\frac{\partial q_n}{\partial o} \qquad (4)$$

For the performance of the graphical solution, we turn from the differential form to the difference form and obtain finally, at the surface through which the flow enters (subscript 1)

$$\Delta q_{n_1} = -\left(\frac{\Delta o_1}{\lambda/\alpha_1}\right) q_{n_1} \tan \varphi_1 \tag{5}$$

at the surface through which the flow leaves (subscript 2), according to figure 4

$$\Delta q_{n_2} = \left(\frac{\Delta o_2}{\lambda/\alpha_2}\right) q_{n_2} \tan \varphi_2 \tag{5a}$$

Therewith, a step-by-step determination of the flow distribution along the surface is possible. It will be useful to select the steps Δo as integral parts of the "fictitious transfer wall thicknesses" λ/α, thus for instance, as 1/5, 1/10, or 1/20. (According to equation (5), the fractions must be chosen smaller, the larger $\tan \varphi$, thus, the more oblique the angle of incidence between surface and permeating flow.)

We now develop the two surfaces (fig. 5), assume the flows q_1 and q_2 at the beginning of the developed regions at arbitrary linear lengths, and divide the developed surface into equal parts Δo. Starting from the center A of these parts, we draw the lines ABC given by the assumed flow angle φ_1 (at A) and the slope $\tan \psi_1 = \dfrac{\Delta o_1}{\lambda/\alpha_1}$ (at B). BD is a parallel to the base line at the distance q_1. The slope $\tan \psi_1$ is constant for the remainder of the procedure, since equal parts Δo_1 are chosen. According to equation (5), CE then represents the increase (here the decrease) Δq_{n_1} which, in addition, is to be shifted to the end of the increment Δo_1 considered (FG). The line DF . . . describes the course of the flow permeating the surface.

As is well known, for flow fields free from sources, the same partial flow must be transported in each "stream tube" or "flow tube." Accordingly, we divide the area DFHIK, representing the total flow, into t equal area parts (Q_1/t) and thus obtain the points of penetration of the (t-1) flow lines at the surface o_1. The direction of the flow lines at these points is given by the assumed orientation of the field.

Repeating the construction at the exit surface o_2 (lines A' B'C', etc.), we obtain a diagram similar to figure 3a. The field orientation at the surface is to be changed so that corresponding flow lines meet. We shall later on find a few criteria according to which the corrections may be suitably performed.

Once the flow pattern is plotted in this manner, there remains to be determined what actual flow passes through for a given difference of the outside temperatures. So far, we know only the flow distribution along the surfaces and the ratio $\frac{q_{n_1}}{q_{n_2}}$ of the entering and leaving flows of a flow tube resulting from the requirement that the total entering flow (area DHIK) must equal the leaving one (area D'H'I'K'). The ordinates q_{n_2} would have to be changed in the ratio of these areas if they are to be comparable to the corresponding ordinates q_{n_1}.

The wall is cut open along the center flow line s_m of an arbitrary flow tube (fig. 6). Since, in general, the flow q, according to

$$q = -\lambda\frac{\partial\vartheta}{\partial s} \tag{2a}$$

corresponds to the temperature gradient and, according to figure 4, $q = q_n/\cos\varphi$, we plot a gradient corresponding to this flow $q_n/\cos\varphi$ at the two surfaces. The appropriate procedure, according to figure 6, will be to represent the gradient as ratio of the flow $q_{n_1}/\cos\varphi_1$ found, according to figure 5, to the total flow Q_1 represented by DHIK. Therewith, the above-mentioned transformation of the flows q_{n_1} and q_{n_2} in the ratio of the areas Q_1 and Q_2 becomes necessary. The scale is to be selected so that one obtains slopes usable in the diagram (30° to 45°). Again both gradients are joined corresponding to the flow pattern, wherein intermediate slopes must be chosen inversely proportional to the flow widths in the flow pattern.

Now, one has at the surface o_1

$$\alpha_1\left(\theta_1 - \vartheta_1\right) = q_{n_1} = q\cos\varphi_1 \tag{6}$$

or

$$\frac{\theta_1 - \vartheta_1}{\frac{\lambda}{\alpha_1}\cos\varphi_1} = -\left(\frac{\partial\vartheta}{\partial s}\right)_1 \tag{6a}$$

Accordingly, the lines indicating the gradient are extended outside of the wall up to the distances $\frac{\lambda}{\alpha_1}\cos\varphi_1$ and $\frac{\lambda}{\alpha_2}\cos\varphi_2$; then the difference in height h corresponds to the prescribed temperature difference $\theta_1 - \theta_2$. As a proof, this construction will be carried out on two different flow tubes.

Therewith, the temperature $\vartheta_1{}^*$ at the point O_1 is known. Since, according to the boundary condition, equation (1) or equation (6), a linear relationship exists between the flow q_{n_1} through the surface and the surface temperature, the flow distribution found in figure 5 also reproduces the temperature distribution at the surface. The temperature scale follows from the two known points X for $q_{n_1} = 0$; thus $\vartheta_1 = \theta_1$, and Y for the surface temperature just determined $\vartheta_1{}^*$ at O_1. The corresponding method is to be applied at the surface o_2.

EXAMPLES

Wall with Protuberant and Indented Corner (Fig. 7)

We presuppose that the flows develop in the two corners, independent of each other, so that the flow lines there may be assumed as lines of symmetry at the corners. Only the region between these two flow lines will be considered.

Assumptions.- Wall thickness, 25 mm; investigated length, 125 mm; heat conductivity, $\lambda = 50$ kcal/mh°C; heat-transfer coefficients: $\alpha_1 = 500$, $\alpha_2 = 1000$ kcal/m²h°C.

Solution.- Surface o_1: assumed flow $q_1 \equiv 50$ mm

$$\Delta o_1 = 10 \text{ mm, thus } \frac{\Delta o_1}{\lambda/\alpha_2}\left(= \tan\psi_1\right) = \frac{10 \times 10^{-3}}{50/500} = \frac{1}{10}$$

Surface o_2: assumed flow $q_2 \equiv 25$ mm; in order to obtain approximately the same values Δq as for surface o_1, a value of q_2 smaller than q_1 is chosen in inverse ratio of the heat-transfer coefficients.

$$\Delta o_2 = 10 \text{ mm, thus } \frac{\Delta o_2}{\lambda/\alpha_2}\left(= \tan \psi_1\right) = \frac{10 \times 10^{-3}}{50/1000} = \frac{1}{5}.$$

The two flow areas are divided into $t = 10$ parts. Then the surface temperatures ϑ_1^* and ϑ_2^*, which establish the temperature scale, are determined: Since the surfaces are parallel, the construction given in figure 6 is simplified by determining those two points O_1 and O_2 located opposite each other where the perpendicular flow q_n entering and leaving is the same. According to the flow pattern plotted in figure 7, the temperatures at these points are $\vartheta_1^* = 42.8^\circ$, $\vartheta_2^* = 28.6^\circ$, if the entire temperature difference is assumed to be $\theta_1 - \theta_2 = 100^\circ$.

Besides, these two temperatures would occur also in case of heat transmission through a plane wall of the thickness $b = 25$ mm.

<u>Result</u>.– Thus, one obtains for the surface o_1 the excess temperature at the protuberant corner as $63^\circ - 42.8^\circ = 20.2^\circ$, or $\frac{20.2}{100 - 42.8} = 35.3$ percent of the temperature jump, from the outside to the two-dimensional wall, and the insufficient temperature at the indented corner is $42.8^\circ - 30.1^\circ = 12.7^\circ$ or 22.2 percent.

At the surface o_2, the excess temperature at the indented corner amounts to $41.2^\circ - 28.6^\circ = 12.6^\circ$ or $\frac{12.6}{28.6} = 44$ percent of the temperature jump, from the two-dimensional wall to the outside, and the insufficient temperature at the protuberant corner $28.6^\circ - 14.3^\circ = 14.3^\circ$ or 50 percent. The differences are relatively larger on the side with the larger heat-transfer coefficient. Moreover, it can be inferred that the flow lines turn their concave side more strongly toward this surface.

Protuberant Corner Rounded Inside (Fig. 8)

The same dimensions and material constants are chosen as in the first example. At a sufficient distance from the corner, the heat flow will pass through the wall at a right angle; there the entering flow q_1 then equals the leaving flow q_2. If the graphical determination is started at this point with $q_1 = q_2$, the two flow areas Q_1 and Q_2 resulting, according to figure 5, must be equal.

Surface o_1: assumed flow $q_1 \equiv 50$ mm

$$\Delta o_1 = 10 \text{ mm, thus } \frac{\Delta o_1}{\lambda/\alpha_1} \left(= \tan \psi_1\right) = \frac{10 \times 10^{-3}}{50/500} = \frac{1}{10}$$

Surface o_2: assumed flow $q_2 \equiv 50$ mm

$$\Delta o_2 = 10 \text{ mm, thus } \frac{\Delta o_2}{\lambda/\alpha_2} \left(= \tan \psi_2\right) = \frac{10 \times 10^{-3}}{50/1000} = \frac{1}{5}$$

Temperature scales: ϑ_1^* and ϑ_2^* are determined at the starting point of the investigation.

<u>Result</u>.– Surface o_1: Excess temperature at the salient corner $66.9^\circ - 42.8^\circ = 24.1^\circ$ or 42.1 percent of the temperature jump from outside to the two-dimensional wall.

Surface o_2: Excess temperature $36.8^\circ - 28.6^\circ = 8.2^\circ$ or 28.6 percent.

Due to the rounding off, the temperature differences become larger outside, and smaller inside.

Wavy Wall (Fig. 9)

The waviness is assumed as sinusoidal; $\lambda = 50$ kcal/mh°C; $\alpha_1 = 375$, $\alpha_2 = 1250$ kcal/m^{2}h°C.

Surface o_1: assumed flow $q_1 \equiv 20$ mm

$$\Delta o_1 = 4 \text{ mm, thus } \frac{\Delta o_1}{\lambda/\alpha_1} \left(= \tan \psi_1\right) = \frac{4 \times 10^{-3}}{50/375} = 0.03$$

Surface o_2: assumed flow $q_2 \equiv 20$ mm

$$\Delta o_2 = 4 \text{ mm, thus } \frac{\Delta o_2}{\lambda/\alpha_2} \left(= \tan \psi_2\right) = \frac{4 \times 10^{-3}}{50/1250} = \frac{1}{10}$$

Temperature scales: The flow pattern for determination of $\vartheta_1{}^*$ and $\vartheta_2{}^*$ according to figure 6 is plotted for the boundary stream lines s_1 and s_2.

Result.— The temperature differences are only minor at the surface with the smaller heat-transfer coefficient, but considerable at the wavy surface where, moreover, the larger heat-transfer coefficient prevails.

From the graphical construction and the examples, the following criteria for correction of the assumed flow direction may be derived:

1. The concave side of the flow lines is turned more strongly toward the surface with the larger heat-transfer coefficient, or the surface with the larger heat-transfer coefficient is crossed by the flow lines at a steeper slope providing other conditions are equal.

2. The width of the flow tubes along the entrance surface decrease in the direction toward which the flow lines slope. (Compare figs. 7 and 8.)

3. The width of the flow tubes at the exit surface increases toward the side in which the flow direction is sloping.

4. The differences in the widths along a surface are greater, the more oblique the angle at which the flow lines cross the surface.

5. Other conditions being equal, the differences in the widths along the surface are greater, the larger the heat transfer coefficient.[4]

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[4]Forschg. Ing.-Wes. Bd. 6, Nr. 5, 1935, p. 240, reports on an extension of the method to include axially symmetrical forms, locally different heat-transfer coefficients, and locally different outside temperatures.

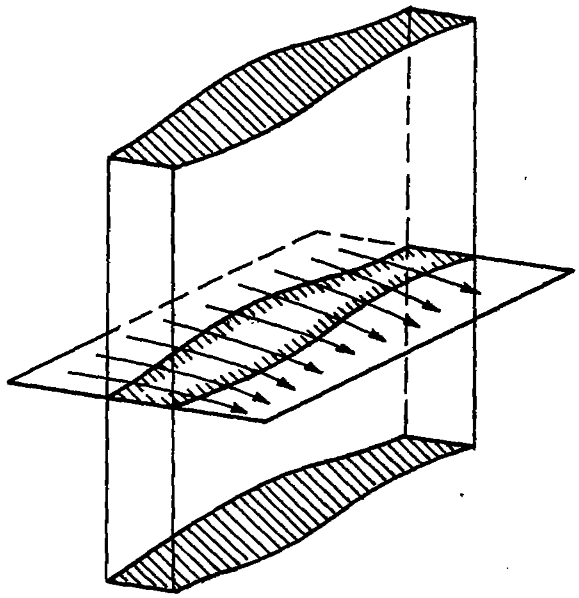

Figure 1.- Two-dimensional heat flow through a wall.

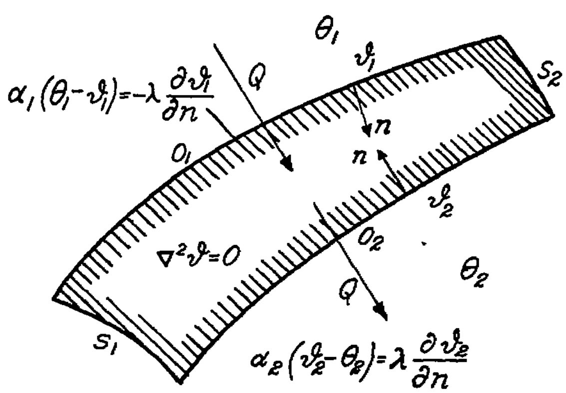

Figure 2.- Section limited by the wall boundaries o_1 and o_2 and the heat streamlines s_1 and s_2. The heat flow Q enters through the boundary o_1 and leaves through the boundary o_2.

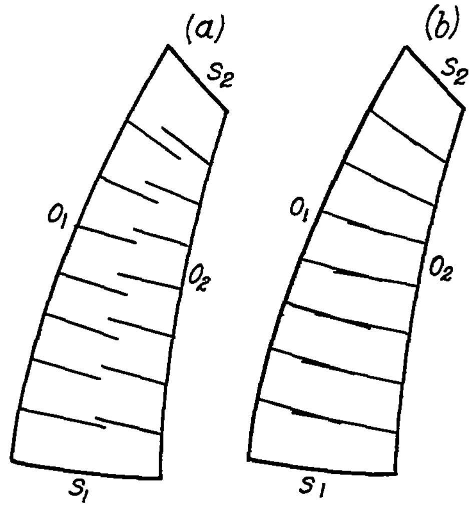

Figure 3.- Heat-flow patterns in case of incorrect (a) and correct (b) assumption of the flow direction through the surfaces o_1 and o_2.

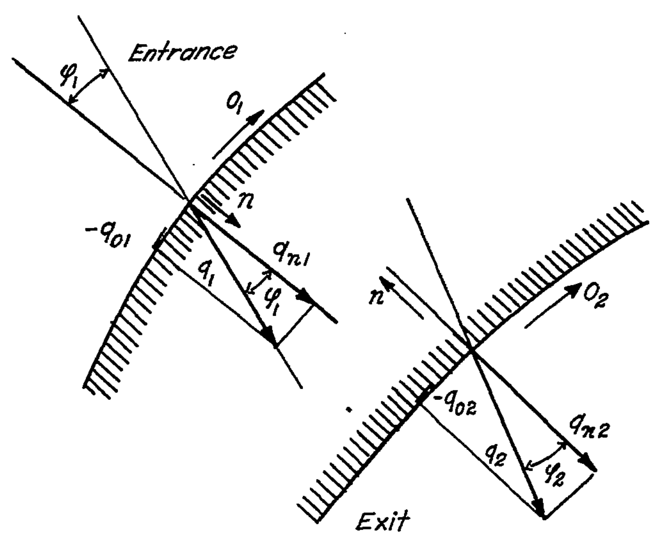

Figure 4.- Decomposition of the entering and leaving flow into its components in the direction of the surface and normal to it.

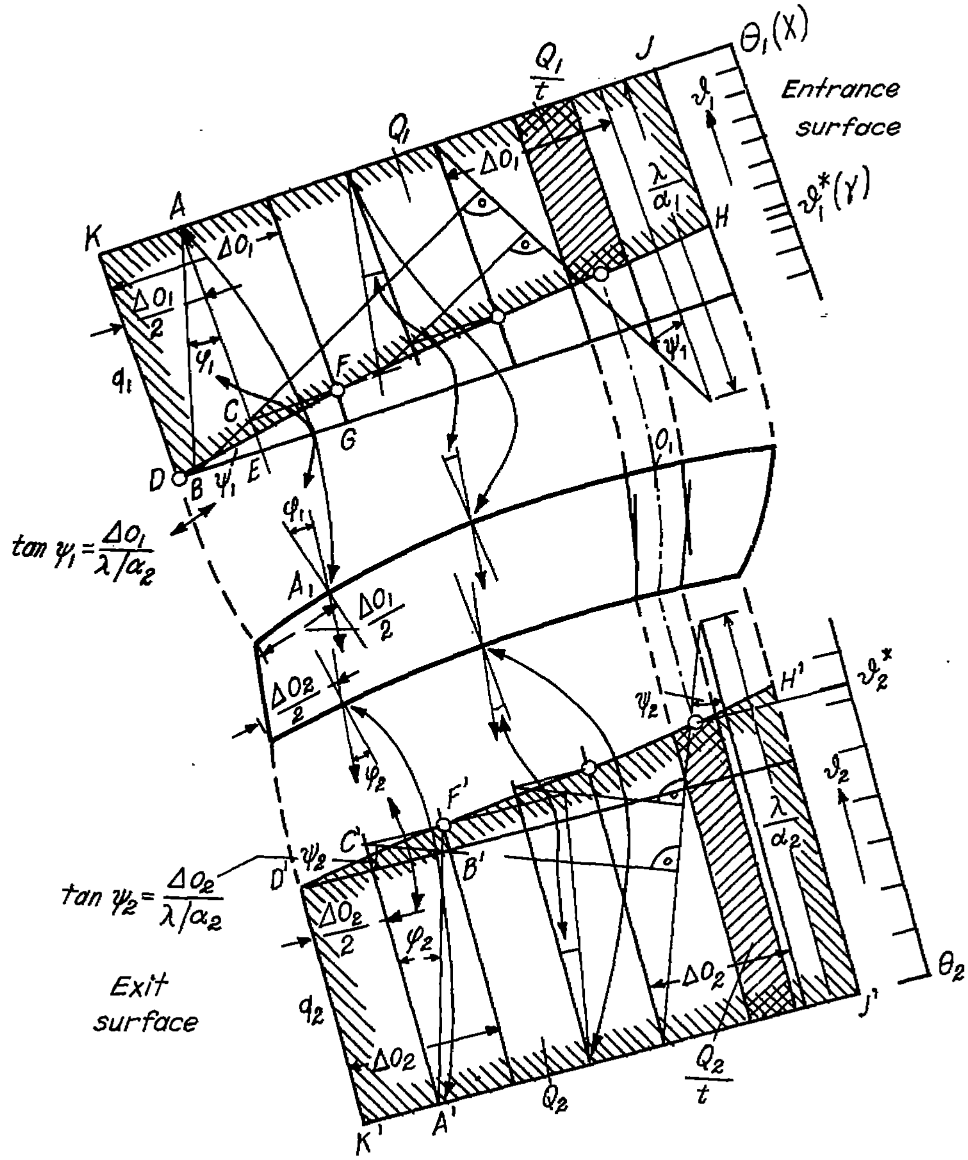

Figure 5.- Graphical determination of the course of the entering and leaving heat flow over the developed surfaces.

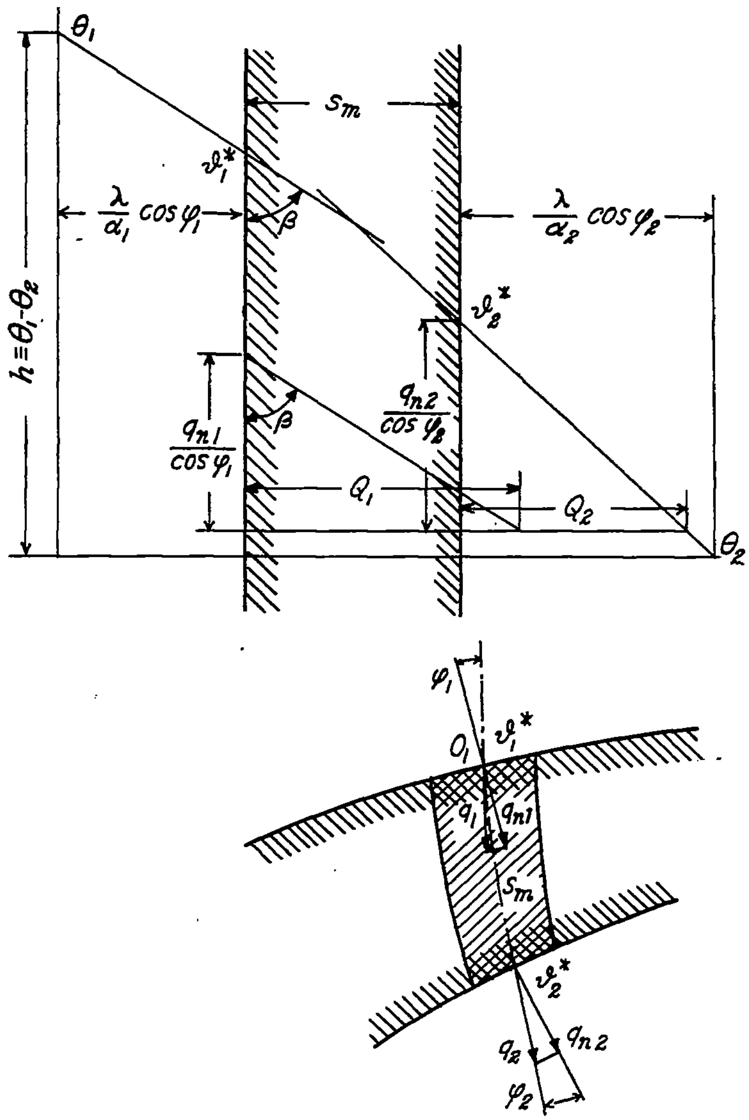

Figure 6.- Determination of the surface temperatures ϑ_1^* and ϑ_2^*. For that purpose the wall has been cut open along the center flow line s_m of an arbitrary flow tube.

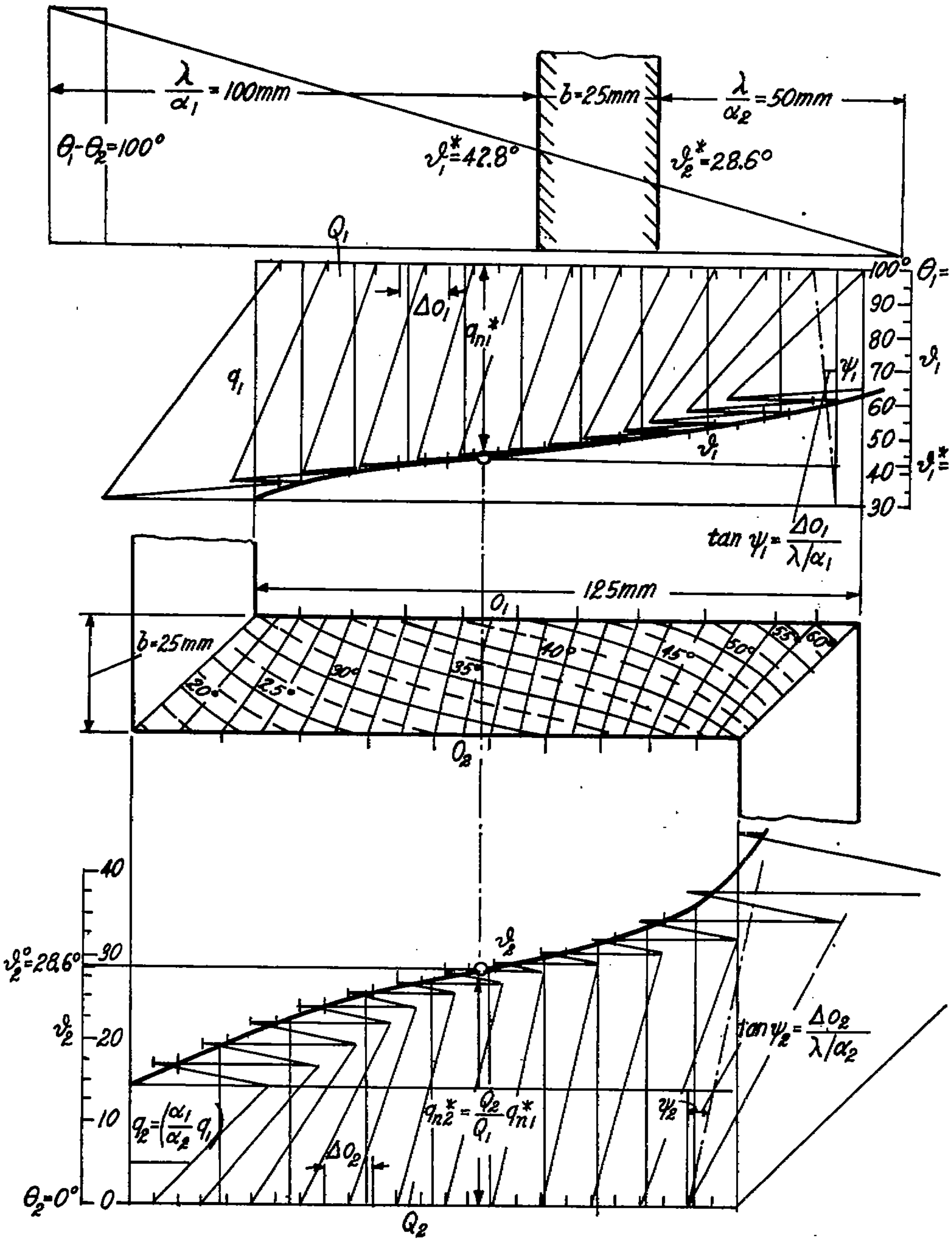

Figure 7.- Heat transmission through a wall with protuberant and indented corners. Temperature difference between the two sides $\theta_1 - \theta_2 = 100^\circ$; ϑ_1 surface temperature on the entrance side, ϑ_2 on the exit side; heat conductivity $\lambda = 50$ kcal/mh°C; heat-transfer coefficients $\alpha_1 = 500$, $\alpha_2 = 1000$ kcal/m²h°C.

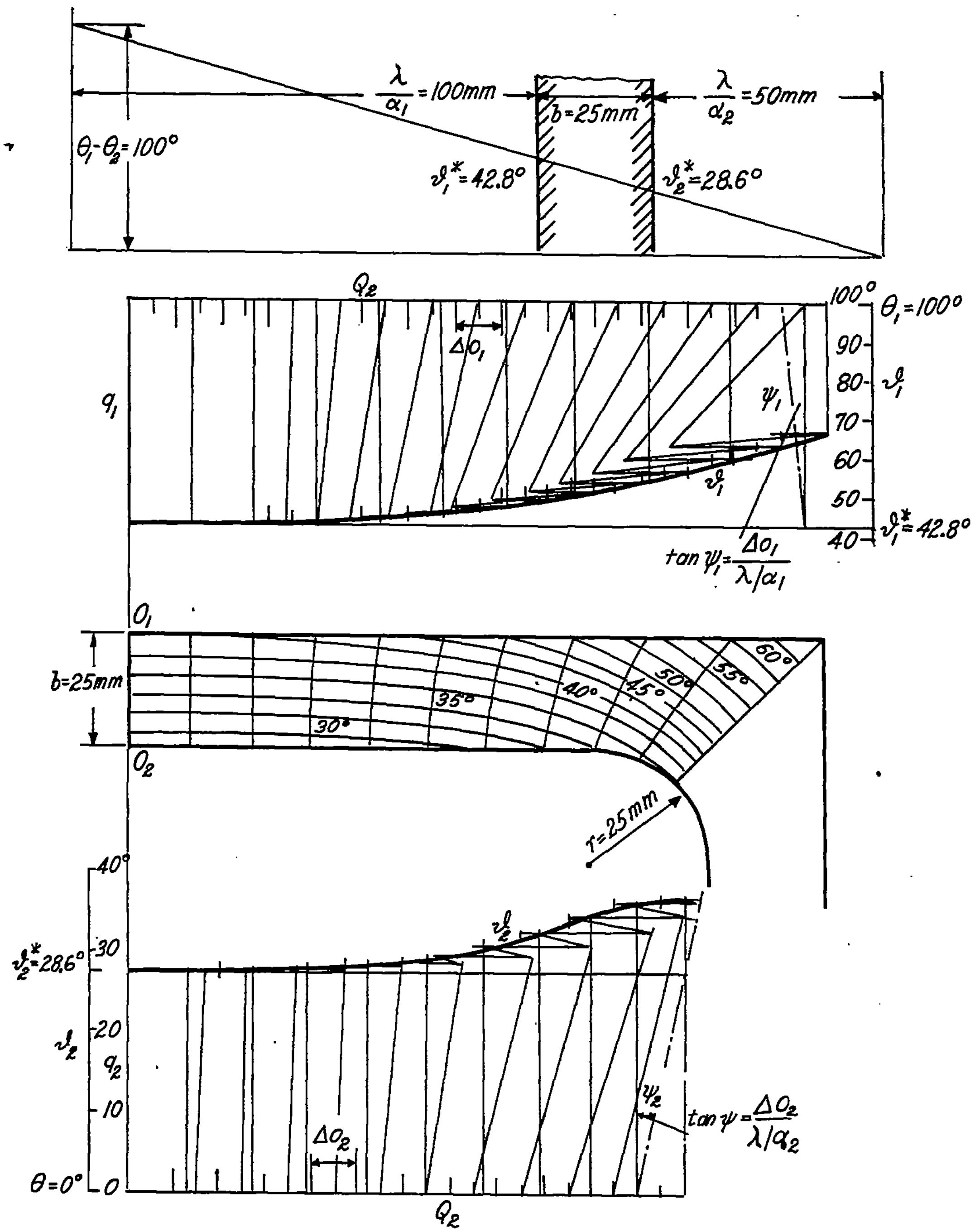

Figure 8.- Heat transmission through a protuberant corner with rounding on the inside. Outer temperatures and material constants as in figure 7.

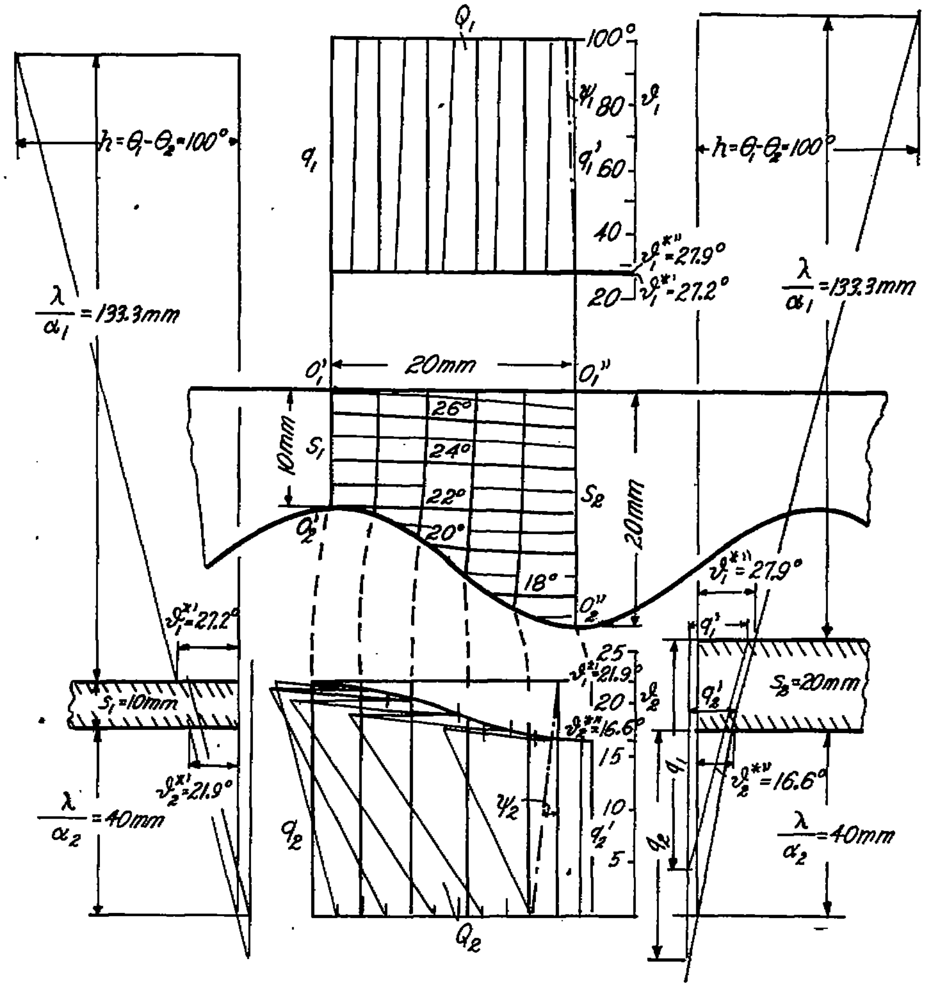

Figure 9.- Heat transmission through a wavy wall. Difference of the outside temperatures $\theta_1 - \theta_2 = 100^{\circ}$; heat conductivity $\lambda = 50$ kcal/mh°C; heat-transfer coefficients $\alpha_1 = 375$, $\alpha_2 = 1250$ kcal/m^{2}h°C.