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THEORY OF TWO-DIMENSIONAL POTENTIAL FLOW ABOUT ARBITRARY WING SECTIONS*

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SUMMARY

Three general theories treating the potential flow about an arbitrary wing section are discussed in this report. The first theory treats the method of conformal transformation as laid down by Theodorsen and Garrick; the second is a generalization of Birnbaum's theory for moderately thick airfoils; the third is a general investigation of the complex velocity function with particular reference to the relations first discussed by F. Weinig.

The relative merits of the different methods in question are illustrated on a worked-out example and will be published in a subsequent issue of this periodical.

INTRODUCTION

The present investigations relate to the two-dimensional potential flow of a frictionless, incompressible fluid around any simply connected region, particularly around airfoils. Such a flow, as is known, is completely described by a regular, analytical function - the complex velocity function. Since an analytical function, in turn, is completely described by its values along any closed curve remaining wholly within the region of regularity, it is sufficient to know this complex velocity function along a curve enveloping, once, the zone washed by the stream - for which purpose the limiting curve or profile contour itself may be chosen, with observation, of course, of any potential singularities on the limiting curve. In

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*"Theorie der ebenen Potentialströmung um beliebige Tragflügelprofile." Ingenieur-Archiv, vol. IX, no. 3, June 1938, pp. 214-240.

any case the problem may be considered solved, once the velocity along the profile contour is known in magnitude and direction.

The very next problem is to find the flow - that is, above all else, the velocity along the contour for any predetermined region. In this case the contour with its direction affords the direction of the velocity as well, leaving simply the quest of the function for the absolute value of the velocity along the contour. This, as is known (reference 1), requires the supplementary assumption of: 1) the velocity at infinity in magnitude and direction, i.e. $\alpha u V$ 2) either the circulation, i.e., the line integral of the velocity along a curve encircling the obstacle once, or else the point on the contour of one of the stagnation points of the flow.

This general flow problem can be reduced in known manner to the mathematical problem of conformal transformation of the contour of the washed region onto the contour of a circle - a problem whose solution is afforded by Riemann's law of transformation. It can be considered solved if it is possible to construct this conformal function not merely by visualization but by actual plotting, according to a method which must be rapidly converging, if infinitely many steps are necessary. Such a method was advanced and proved by T. Theodorsen and I. E. Garrick (reference 2) and in the treated cases, yields very quick results. It can also be proved that this method converges under certain sufficing assumptions and solves the transformation problem rigorously. The results, built up on it, are the mathematically exact solutions of the general flow problem.

But in aerodynamics, the opposite of the above problem is also of significance - i.e., to find a profile along which the flow about the contour is accompanied by a desired velocity distribution. F. Weinig has established the surprising fact that this problem is mathematically simpler and that, if the desired velocity distribution in the potential flow about a simply connected region can take place at all, the contour of this region can be exactly determined without an infinite method. Admittedly, the assumptions required for the velocity distribution along the contour itself cannot be summarily complied with, hence it is impossible to take such distributions directly as a starting point for the calculation of profiles. A suitable assumption, instead of this velocity distribution, is a freely available real function following from it by distortion in abscissa direction. The method is particularly suitable for developing profiles with prescribed characteristics.

In earlier airfoil theories, special conformal functions with a varying number of parameters which give the conformal transformations between circles and airfoil-shaped regions, formed the center of investigation, and logically the results were predictions about special, multiparameter airfoil families. General relations between the functions for the profile contour and for the velocity distribution could not be obtained in this manner, with the notable exception of infinitely thin airfoils by the so-called "Birnbaum theory" which, proceeding from the image of a nonuniformly covered flat vortex layer, approximately established a relation between the difference in velocity on the upper and lower profile surface and the slope of the mean line of the profile. Notwithstanding its restricted range of validity, this theory presented many advantages and made its generalization to include thick airfoils very desirable which, however, did not succeed satisfactorily with the vortex layer concept.

In the present report it is attempted to combine the past information on potential flow around airfoils, so far as they are of this functional type, with a view to obtaining data which tie the profile contour and the complex velocity, or velocity distribution along the contour in plainest and most amenable form for calculation.

In this connection, the findings by Theodorsen and Weinig are significant. It might be surmised that by vanishing profile thickness these theories lead back to the old theory for thin airfoils. *But such is not the case. It rather affords a new approximate theory - probably sufficient for all cases encountered in practice which, without restricting itself to infinitely thin profiles, contains the equation part of the theory of lifting vortex as limiting case and to that extent represents its generalization. It ultimately affords a survey over a system of integral equations, every one of which is involved as mathematical starting point for a general airfoil theory. It is shown how the theory of lifting vortex surface and Weinig's results aline themselves in this general arrangement.

1. THE POTENTIAL FLOW AROUND A CIRCULAR CYLINDER AND ITS RELATION TO FLOW AROUND GENERAL PROFILES

The physical conditions of two-dimensional vortex-free flow in frictionless, incompressible fluid are contained in the two equations

$\nabla^2 \varphi = 0$ a w exists   curl $\vec{v} = 0$, a Φ exists

$$\operatorname{div}\vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{1}$$

where u and v indicate the velocity in x and y direction. They represent the Cauchy-Riemann differential equations for the analytical function

$$\frac{dw}{dz} = w^*(z) = u(x,y) - i\, v(x,y) = \frac{\partial\varphi}{\partial x} - i\frac{\partial\varphi}{\partial y} \tag{2}$$

the so-called "complex velocity function" which completely describes a two-dimensional potential flow.

If it relates to the flow around a finite region, as, for instance, a circle or airfoil in unlimited fluid, several general predictions as to function $w^*(z)$ can be made. Since infinitely great velocities can occur only on the border of the fluid region, the function $w^*(z)$ is, above all, regular everywhere outside the region. Hence for great (Laurent) z, $w^*(z)$ can be represented by the series

$$w^*(z) = C_0 + \frac{C_1}{z} + \frac{C_2}{z^2} + \ldots \tag{2a}$$

The constant C_0 indicates the complex velocity at infinity. The constant C_1 is purely imaginary, if the contour around the obstacle is a streamline, as presumed here, in accord with classical theory. C_1 is associated with the circulation Γ of the body through the relation $C_1 = i\,\frac{\Gamma}{2\pi}$. While C_1 is decisive for the force exerted by the flow on the body, C_2 is decisive for the moment on the body. If $C_n = A_n + i\,B_n$ and ρ is the density of the flowing medium, Grammel's well-known equations (reference 3) give for the two components P_x and P_y of the force and for the moment M turning clockwise about $z=0$:

$$P_x = 2\pi\ \rho\ B_0 B_1, \quad P_y = 2\pi\ \rho\ A_0 B_1, \quad M = -\ 2\pi\ \rho (A_0 B_2 + B_0 A_2) \tag{3}$$

For the most general flow about the unit circle with the two stagnation points $z_1 = e^{i(\alpha-\beta)}$ and $z_2 = e^{i(\pi+\alpha+\beta)}$, the complex velocity function is:

$$w^*(z) = W\ e^{-i\alpha} \left[1 - \frac{e^{i(\alpha-\beta)}}{z}\right] \left[1 + \frac{e^{i(\alpha+\beta)}}{z}\right] \tag{4}$$

The flow has the absolute velocity W at infinity where it forms the angle α with the x axis. Transformed, equation (4) gives:

$$w^*(z) = W \left(e^{-i\alpha} + \frac{2i\ \sin\ \beta}{z} - \frac{e^{i\alpha}}{z^2}\right)$$

The series (2a) stops in this case with the term z^{-2}. The coefficient of z^{-1} is, as should be, purely imaginary.

On the circle circumference $z = e^{i\varphi}$ it is

$$w^*(\varphi) = 2i\ W\ e^{-i\varphi}\ (\sin\ (\varphi-\alpha) + \sin\ \beta)$$

hence the absolute velocity on the circle periphery is:

$$|w| = |w^*(\varphi)| = 2W\ |\sin\ (\varphi-\alpha) + \sin\ \beta| \tag{5}$$

The velocity is zero at the two stagnation points for $\varphi = \alpha - \beta$ and $\varphi = \pi + \alpha + \beta$. For the direction of flow at point $z = e^{i\varphi}$, we find $u/v = -\cot\ \varphi$, which confirms the fact that the function (4) actually represents a potential flow about the unit circle.

The potential flow about the circular cylinder is, as known, of fundamental importance for the general theory of potential flow about any simply connected region $\underline{B}$, because the conformal function $\bar{z}(z)$ which transfers the contour of the unit circle on the contour of $\underline{B}$, makes it possible to deduce the velocity function $\bar{w}^*(\bar{z})$ for a flow around region $\underline{B}$ from the complex velocity $w^*(z)$, according to equation (4). The relation is:

$$\bar{w}^*(\bar{z}) = w^*(z)\ \frac{dz}{d\bar{z}} \tag{6}$$

The derivation of equation (6) usually proceeds from the fact that in the conformal transformation by $\overline{z}(z)$ the orthogonal fields of the streamlines and the potential lines of both flows merge. i.e. streamlines about C → lines about B.
pot. lines .. about C → " " "

But this formula can also be understood without resorting to the flow potential. For $\overline{w}^*(\overline{z})$ is an analytical function, for whose functional values along the border of $\underline{B}$ the negative argument agrees with the direction of the contour. Function $w^*(z)$ meets this requirement for the unit circle. In the conformal transformation of $\overline{z}(z)$, point z moves toward $\overline{z}$ and the vicinity of z is turned through an angle arc $d\overline{z}/dz$. Hence, to insure the required direction of $\overline{w}^*(\overline{z})$ at the border of $\underline{B}$, it is necessary that:

$$- \operatorname{arc} \overline{w}^*(\overline{z}) = - \operatorname{arc} w^*(z) + \operatorname{arc} \frac{d\overline{z}}{dz}$$

But the analytical function which complies with this requirement is $w^*(z) \frac{dz}{d\overline{z}}$.

This process affords the potential flow about an infinitely thin, flat plate. The analytical function which transfers the outside of the unit circle in the z plane into the $\overline{z}$ plane rectilinear from -1 to $+1$ reads:

$$\overline{z} = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{or} \quad z = \overline{z} \left(\pm\right)\sqrt{\overline{z}^2 - 1} \tag{7}$$

which, written for z in equation (4) and multiplied by $dz/d\overline{z}$, according to equation (6), gives for the complex velocity of the most elementary flow about the flat plate extending from -1 to $+1$ the following (with abbreviation $\overline{W} = 2W$):

$$\overline{w}^*(\overline{z}) = \overline{W}\left(\cos\alpha - i\sin\alpha\,\frac{\overline{z}}{\sqrt{\overline{z}^2 - 1}} + \frac{i\sin\beta}{\sqrt{\overline{z}^2 - 1}}\right) \tag{8}$$

This equation describes a three-parameter system of potential flows. Two, that is, the velocity $\overline{W}$ at infinity and angle of flow α can be regulated at will. The value of the third parameter β is physically conditioned - i.e., the Joukowski condition of finite velocity at the trailing edge $\overline{z} = 1$. In the present case $\overline{w}^*(1)$ is finite if $\lim\limits_{\overline{z} \to 1} (\overline{z}\sin\alpha - \sin\beta) = 0$, that is, if $\beta = \alpha$. Thus, the velocity function for the flow around the flat plate finally reads:

$$\bar{w}^*(\bar{z}) = \bar{W}\left(\cos\alpha - i\sin\alpha\sqrt{\frac{\bar{z}-1}{\bar{z}+1}}\right) \qquad (9)$$

In order to describe the velocity distribution on the surface of the plate, the variable φ is employed. According to equation (7) $\bar{z} = \cos\varphi = \bar{x}$ for $\bar{z} = e^{i\varphi}$. Then it is for the surface of the plate:

$$\bar{w}\ (\cos\varphi) = \bar{W}\left(\cos\alpha + \sin\alpha\sqrt{\frac{1-\cos\varphi}{1+\cos\varphi}}\right) = \bar{W}\left(\cos\alpha + \sin\alpha\tan\frac{\varphi}{2}\right)$$

after which the absolute velocity at point $\bar{x} = \cos\varphi$ becomes

$$|w(\varphi)| = \bar{W}\left|\cos\alpha + \sin\alpha\tan\frac{\varphi}{2}\right| \qquad (10)$$

With this method any number of other potential flows can be mathematically described by different choice of conformal function. As it is easy to give conformal transformations which transform the contour of the unit circle into that of a simply connected airfoil-like region, it is equally possible to give profiles which may be controlled by mathematical theory.

The theories of general potential flow about airfoils discussed hereinafter, give the relation between the function for the profile contour and the complex velocity along the contour, and tie these functions in a fashion amenable to calculation. The start is made with the strict formulas for the flow about any predetermined airfoil to which the theory of Theodorsen and Garrick leads.

2. RIGOROUS THEORY OF POTENTIAL FLOW ABOUT ANY PROFILE

The previously described classical method of obtaining the potential flow about any simply connected region yields - in conjunction with the method of Theodorsen and Garrick for obtaining the conformal function for any initial region - a rigorous solution of the basic problem of airfoil theory; that is, to find the potential flow about any given airfoil (references 2 and 4).

If the values of the complex velocity along the profile contour are known the problem is solved, because then Cauchy's integral formula gives the velocity function for any point on the outside region replete with the flowing medium. To find the velocity distribution on the profile contour, the procedure is as follows: The conformal transformation (7) which transfers the slotted plane to the outside zone of the unit circle, transforms the profile into an almost circular region. With the aid of the function which conformally transforms the contour of this region onto the contour of the unit circle, the velocity distribution on the contour is then computed for the flow around this region. From this distribution the velocity distribution along the profile contour is deduced with the aid of the first conformal transformation.

Assume the arbitrarily given profile in the $\bar{z}$ plane to be so plotted that its trailing edge coincides with point $\bar{z} = 1$ and point $\bar{z} = -1$ lies in the profile near the nose of the profile (fig. 1). If the $\bar{z}$ plane is conformally transformed by function $\zeta = \bar{z} + \sqrt{\bar{z}^2 - 1}$ on the ζ plane, then for the given position of the profile with respect to the fixed points $\bar{z} = \pm 1$, the image curve of the profile contour in the ζ plane is the contour of a simply connected region which, for the common profile forms, is in more or less satisfactory agreement with the unit circle.

The boundary curve of this region is $\zeta(\theta) = e^{\psi(\theta)+i\theta}$ (fig. 2). The function $\psi(\theta)$ is unequivocal (single-valued), steady (continuous), and periodic with the period 2π; the boundary curve, viewed from the origin $\zeta = 0$, is star-shaped.

The problem is to transform the contour of the unit circle $z = e^{i\varphi}$ on the contour of the region bounded by $\zeta(\theta)$. This conformal transformation is standardized by the condition that the infinitely remote point of the z plane and the direction of the positive real axis in it, are coordinated to infinitely remote point of the ζ plane and the direction of the positive real axis. According to Riemann, there is exactly one analytical function $\zeta(z)$ which meets this requirement and conformably transforms the two outside regions. But in view of this fact, the conformal function $\zeta(z)$ has the form

$$\zeta(z) = z\, e^{F(z)}$$

where $F(z)$ for $|z| > 1$ inclusive of $z = \infty$, is a regular analytical function which assumes a real value for $z \longrightarrow \infty$.

Owing to $z = e^{i\varphi}$ and $\zeta = e^{\psi + i\theta}$, it is

$$\boxed{\frac{\zeta}{z} = e^{\psi + i(\theta - \varphi)}}$$ transforms B $(e^{\psi} e^{i\theta})$ into the unit circle $e^{i\varphi}$ on the z plane.

on the unit circle. Thus the function $F(z)$ assumes on the circle $z = e^{i\varphi}$ the boundary values $\psi + i(\theta - \varphi)$. Given either $\theta(\varphi)$ or $\psi(\varphi)$, the function $F(z)$, and hence the conformal function $\zeta(z)$ can be directly written.[1] Since $F(z)$ takes a real value at infinity, we have here $h(\infty) = 0$; hence by steady $\psi(\varphi)$ for any z with $|z| > 1$. the equation

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[1] If $F(z) \equiv g + i\,h$ is an analytical function that is regular for $|z| > R$ including $z = \infty$ and ~~steady~~ continuous on the circle $|z| = R$, all $z = \nu\, e^{i\varphi}$ with $\nu > R$ follow the Poisson integral formulas:

$$g(\nu,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} g(R,\varphi')\,\frac{\nu^2 - R^2}{\nu^2 - 2\nu R\cos(\varphi' - \varphi) + R^2}\,d\varphi',$$

$$g(\infty) = \frac{1}{2\pi}\int_0^{2\pi} g(R,\varphi')\,d\varphi'$$

and

$$g(\nu,\varphi) = g(\infty) - \frac{1}{2\pi}\int_0^{2\pi} h(R,\varphi')\,\frac{2\nu R\sin(\varphi' - \varphi)}{\nu^2 - 2\nu R\cos(\varphi' - \varphi) + R^2}\,d\varphi'$$

which, combined, read:

$$F(z) = i\,h(\infty) + \frac{1}{2\pi}\int_0^{2\pi} g(R,\varphi')\,\frac{z + R\,e^{i\varphi'}}{z - R\,e^{i\varphi'}}\,d\varphi'$$

The second equation yields for $R = 1$, $\nu \longrightarrow R$, the important formula

$$g(1,\varphi) = g(\infty) - \frac{1}{2\pi}\int_0^{2\pi} h(1,\varphi')\cot\frac{\varphi' - \varphi}{2}\,d\varphi'$$

The ~~figurative~~ improper integral here stands for Cauchy's principal value (cf. Harry Schmidt, Aerodynamik des Fluges, p. 87).

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} \psi(\varphi) \frac{z + e^{i\varphi}}{z - e^{i\varphi}} \, d\varphi \tag{11}$$

is applicable, while on the unit circle itself, we find:

$$-\varphi + \theta(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \psi(\varphi') \cot \frac{\varphi' - \varphi}{2} \, d\varphi'$$

Introducing the phase difference $\varphi - \theta(\varphi)$ as new function $\epsilon(\varphi)$

$$\epsilon(\varphi) = \varphi - \theta(\varphi) \tag{12}$$

gives for the present

$$\epsilon(\varphi) = \frac{-1}{2\pi} \int_0^{2\pi} \psi(\varphi') \cot \frac{\varphi' - \varphi}{2} \, d\varphi'$$

If $\psi(\varphi)$ has a continuous first derivative, partial integration leads to

$$\epsilon(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{d\varphi} \ln \sin^2 \frac{\varphi' - \varphi}{2} \, d\varphi'$$

Actually this is not the function $\psi(\varphi)$ but rather $\psi(\theta)$. and the problem is to find the function $\epsilon(\theta)$ for the given $\psi(\theta)$. Then the function $\varphi(\theta)$ follows from $\epsilon(\theta)$ according to equation $\varphi(\theta) = \theta + \epsilon(\theta)$, and $\varphi(\theta)$ together with $\psi(\theta)$ gives the function $\psi(\varphi)$. With $\psi(\varphi)$ the conformal function can be explicitly written, according to equation (11).

There remains the relation between $\psi(\theta)$ and $\epsilon(\theta)$ which, however, is easily established when assuming that $d\varphi/d\theta$ is ~~steady~~ continuous and other than zero, because then

$$\int_0^{2\pi} \frac{d\psi}{d\varphi} \ln \sin^2 \frac{\varphi' - \varphi}{2} d\varphi' = \int_0^{2\pi} \frac{d\psi}{d\theta} \ln \sin^2 \left[\frac{\theta' - \theta}{2} + \frac{\epsilon(\theta') - \epsilon(\theta)}{2}\right] d\theta'$$

and consequently,

$$\epsilon(\theta') = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{d\theta} \ln \sin^2 \left[\frac{\theta' - \theta}{2} + \frac{\epsilon(\theta') - \epsilon(\theta)}{2}\right] d\theta' \tag{13}$$

This is the equation upon which Theodorsen's method is based. It is a nonlinear and singular integral equation and therefore does not aline itself in the known theory. But, according to Theodorsen, it can be successfully applied by iteration process. This method is similar to the Picard-Lindelöf method of solving differential equations. It consists in temporarily substituting the unknown $\epsilon(\theta)$ on the right-hand side for any suitable function (for instance, $\epsilon_0(\theta) \equiv 0$). Then equation

$$\epsilon_1(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{d\theta} \ln \sin^2 \frac{\theta' - \theta}{2} d\theta' \tag{13a}$$

defines $\epsilon_1(\theta)$ and generally gives $\epsilon_{k+1}(\theta)$ with the aid of $\epsilon_k(\theta)$ through the recursion formula:

$$\epsilon_{k+1}(\theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\psi}{d\theta} \ln \sin^2 \left[\frac{\theta' - \theta}{2} + \frac{\epsilon_k(\theta') - \epsilon_k(\theta)}{2}\right] d\theta'$$

~~For proof of the~~ To prove convergence it would have to be shown that the functions $\epsilon_k(\theta)$ tend toward a limiting function which satisfies the integral equation (13). This proof is not adduced in the original work of Theodorsen and Garrick, but rather in the practicability of the described method instead, by showing on several solved examples that in them the functions $\epsilon_k(\theta)$ are practically no longer distinguishable after a few iterations. The solution of the integral is effected with a quadratic formula which considers 20 points in the interval $(0, 2\pi)$.

Incidental to the present work, some searching investigations concerning the convergence of this method were carried out. It could be proved that the functions $\epsilon_k(\theta)$ converge uniformly toward a continuous ~~steady~~ limiting function $\epsilon(\theta)$, which complies with equation (13), if the initial function $\psi(\theta)$ can be represented by a Fourier series whose coefficients a_n and b_n are of the order of magnitude of $\frac{A}{(n+1)^p}$, where $p > 2$, and A is ~~are~~ smaller than an individually specifiable, positive figure A_0 different from 0. But the calculations of this proof are too extensive to be reproduced here.

It might be noted that the very first step of the method affords, in general, such a satisfactory approximation that it suffices for practical purposes. The numerical evaluation of the integral of equation (13a) may be effected with Theodorsen's calculation scheme.

Suppose that the function $\epsilon(\theta)$ is known: then, because $\varphi(\theta) = \theta + \epsilon(\theta)$ and $\frac{d\varphi}{d\theta} = 1 + \frac{d\epsilon}{d\theta}$, according to equation (11), the conformal function $\zeta(z)$ is:

$$\zeta(z) = z\, e^{\frac{1}{2\pi}\int_0^{2\pi} \psi(\theta)\, \frac{z+e^{i(\theta+\epsilon)}}{z-e^{i(\theta+\epsilon)}} \left(1 + \frac{d\epsilon}{d\theta}\right) d\theta} \tag{14}$$

from which the enlargement ratio of the conformal transformation at infinity follows immediately at

$$a = \lim_{z \to \infty} \left(\frac{d\zeta}{dz}\right) = e^{\frac{1}{2\pi}\int_0^{2\pi} \psi(\theta) \left(1 + \frac{d\epsilon}{d\theta}\right) d\theta} \tag{14a}$$

But on the boundary of the unit circle we have, because of $\zeta = e^{\psi + i\theta}$ at the point of $z = e^{i(\theta+\epsilon)} = e^{i\varphi}$

$$\left(\frac{d\zeta}{dz}\right)_{z=e^{i\varphi}} = \frac{\dfrac{d}{d\theta}\, e^{\psi+i\theta}}{\dfrac{d}{d\theta}\, e^{i(\theta+\epsilon)}} = \frac{\left(\dfrac{d\psi}{d\theta} + i\right) e^{\psi+i\theta}}{i\left(1 + \dfrac{d\epsilon}{d\theta}\right) e^{i(\theta+\epsilon)}} = \frac{1-i\,\dfrac{d\psi}{d\theta}}{1 + \dfrac{d\epsilon}{d\theta}}\, e^{\psi-i\epsilon} \tag{15}$$

By reason of this the velocity distribution in the ζ plane is possible. The values of the complex velocity function $w^{**}(\zeta)$ along the contour follow from $w^*(z)$, according to equation (4), and $d\zeta/dz$ according to

$$w^{**}(\theta) = w^{**}(\zeta)_{\zeta=e^{\psi+i\theta}} = \left(w^*(z)\,\frac{dz}{d\zeta}\right)_{z=e^{i\varphi}}$$

But on the circle periphery it is, according to equation (4)

$$w^*(z)_{z=e^{i\varphi}} = 2i\ W\ e^{-i\varphi}\ [\sin(\varphi-\alpha) + \sin\beta] =$$

$$2i\ W\ e^{-i(\theta+\epsilon)}\ [\sin(\theta+\epsilon-\alpha) + \sin\beta]$$

Hence

$$w^{**}(\theta) = 2i\ W\ e^{-\psi-i\theta}\ \frac{1 + \dfrac{d\epsilon}{d\theta}}{1-i\,\dfrac{d\psi}{d\theta}}\ [\sin(\theta+\epsilon-\alpha) + \sin\beta] \tag{16}$$

For the determination of the complex velocity $\overline{w}^*(\overline{z})$ in the $\overline{z}$ plane along the profile contour, the transfer from plane ζ to plane $\overline{z}$ must be effected. Both planes are mutually related through the conformal function $\zeta = \overline{z} + \sqrt{\overline{z}^2 - 1}$ or $\overline{z} = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)$. Using the elliptic coordinates ψ, θ the equation of the profile contour in plane $\overline{z}$ reads:

$$\overline{z}(\theta) = \frac{1}{2}\left(\zeta + \frac{1}{\zeta}\right)_{\zeta=e^{\psi+i\theta}} = \frac{1}{2}[e^{i(\theta-i\psi)} + e^{-i(\theta-i\psi)}] = \cos(\theta-i\psi)$$

Along the contour, it is:

$$\left(\frac{d\overline{z}}{d\zeta}\right)_{\zeta=e^{\psi+i\theta}} = \frac{1}{2\zeta}\left(\zeta - \frac{1}{\zeta}\right)_{\zeta=e^{\psi+i\theta}} = \frac{e^{-\psi-i\theta}}{2}[e^{i(\theta-i\psi)} - e^{-i(\theta-i\psi)}] =$$

$$= i\ e^{-\psi-i\theta}\ \sin(\theta-i\psi)$$

Besides the enlargement ratio of this transformation at infinity is $\lim_{\zeta \to \infty} \left(\frac{d\bar{z}}{d\zeta}\right) = \frac{1}{2}$; hence the velocity distribution along the profile contour:

$$\bar{w}^*(\theta) = 2W \frac{1 + \frac{d\epsilon}{d\theta}}{1 - i\frac{d\psi}{d\theta}} \frac{\sin(\theta+\epsilon-\alpha) + \sin\beta}{\sin(\theta - i\psi)} \tag{17}$$

Now the parameter β must be so defined that at the trailing edge, that is, for $\theta = 0$, the velocity remains finite. The condition for this is:

$$\beta = \alpha - \epsilon_0 \quad \text{with} \quad \epsilon_0 = \epsilon(\theta)_{\theta=\sigma}$$

To make the final result agree with the result for the flat plate, equations (9) and (10), the insertion of this value for β is followed by the following changes:

$$\sin(\theta+\epsilon-\alpha) + \sin(\alpha-\epsilon_0) = \sin(\alpha-\epsilon_0) + \sin[(\theta+\epsilon-\epsilon_0)-(\alpha-\epsilon_0)] =$$

$$= \sin(\alpha-\epsilon_0) + \sin(\theta+\epsilon-\epsilon_0)\cos(\alpha-\epsilon_0) - \cos(\theta+\epsilon-\epsilon_0)\sin(\alpha-\epsilon_0) =$$

$$= \sin(\theta+\epsilon-\epsilon_0)\left[\cos(\alpha-\epsilon_0) + \sin(\alpha-\epsilon_0)\frac{1-\cos(\theta+\epsilon-\epsilon_0)}{\sin(\theta+\epsilon-\epsilon_0)}\right] =$$

$$= \sin(\theta+\epsilon-\epsilon_0)\left[\cos(\alpha-\epsilon_0) + \sin(\alpha-\epsilon_0)\tan\frac{\theta+\epsilon-\epsilon_0}{2}\right]$$

In addition, we substitute the absolute flow velocity $\bar{W}$ in the plane $\bar{z}$ for W. It is with the constant a according to equation (14a)

$$\bar{W} = \lim_{z,\zeta \to \infty} \left| w^*(z)\frac{dz}{d\zeta}\frac{d\zeta}{d\bar{z}} \right| = W\frac{2}{a} \quad \text{and hence} \quad 2W = a\bar{W}$$

giving as final profile contour

$$\bar{w}^*(\theta) = a\bar{W}\frac{\sin(\theta+\epsilon-\epsilon_0)}{\sin(\theta-i\psi)} \frac{1+\frac{d\epsilon}{d\theta}}{1-i\frac{d\psi}{d\theta}}\left[\cos(\alpha-\epsilon_0) + \right.$$

$$\left. + \sin(\alpha-\epsilon_0)\tan\frac{\theta+\epsilon-\epsilon_0}{2}\right] \tag{18}$$

But, on account of

$$|\sin(\theta - i\psi)| = \sqrt{\sin^2\theta + \sinh^2\psi} \quad \text{and} \quad \left|1 - i\,\frac{d\psi}{d\theta}\right| = \sqrt{1 + \left(\frac{d\psi}{d\theta}\right)^2}$$

the absolute velocity takes the form

$$|\overline{w}(\theta)| = a\,\overline{W} \left| \frac{\sin(\theta + \epsilon - \epsilon_0)}{\sqrt{\sin^2\theta + \sinh^2\psi}} \; \frac{1 + \frac{d\epsilon}{d\theta}}{\sqrt{1 + \left(\frac{d\psi}{d\theta}\right)^2}} \left[\cos(\alpha - \epsilon_0) + \sin(\alpha - \epsilon_0)\tan\frac{\theta + \epsilon - \epsilon_0}{2}\right] \right| \qquad (19)$$

Equations (18) and (19) are the strict equations for the velocity of the potential flow along the profile contour. To obtain an explicit expression for the complex velocity at any other point $\overline{z}$ of the outside region, the Cauchy integral formula is applied. Since the complex velocity function $\overline{w}^*(\overline{z})$ outside of the profile inclusive of infinity is regular throughout, if $\overline{z}$ denotes any point of the outside zone and $\overline{z}'$ a point of the profile contour, the equation[2] reads:

$$\overline{w}^*(\overline{z}) = \overline{w}(\infty) + \frac{1}{2\pi i} \oint_C \frac{\overline{w}^*(\overline{z}')}{\overline{z} - \overline{z}'}\, d\overline{z}'$$

$\overline{w}^*(\infty) = \overline{W}\,e^{-i\alpha}$ at infinity, and $\overline{z}' = \cos(\theta - i\psi)$, hence $d\overline{z}' = \left(1 - i\,\frac{d\psi}{d\theta}\right)\sin(\theta - i\psi)\,d\theta$ on the profile contour. Thus equation (17) gives the complex velocity function in the form

$$\overline{w}^*(\overline{z}) = \overline{W}\,e^{-i\alpha} + \frac{ia\overline{W}}{2\pi} \int_0^{2\pi} \frac{\sin(\theta + \epsilon - \alpha) + \sin(\alpha - \epsilon_0)}{\overline{z} - \cos(\theta - i\psi)} \left(1 + \frac{d\epsilon}{d\theta}\right) d\theta \qquad (20)$$

If $|\overline{z}| > |\cos(\theta - i\psi)|$, the series development for $\overline{w}^*(\overline{z})$ is

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[2] If $F(z)$ is a regular function of a simple region G bounded at infinity and everywhere else outside of a partly smooth curve C, and $F(z)$ is steady on C, every z of the outside zone of G follows Cauchy's integral formula

$$F(z) = F(\infty) + \frac{1}{2\pi i} \oint_C \frac{F(\zeta)}{z - \zeta}\, d\zeta$$

See also reference 1, p. 68.

$$\overline{w}^*(\overline{z}) = \overline{W}\, e^{-i\alpha} + \frac{C_1}{\overline{z}} + \frac{C_2}{\overline{z}^2} + \ldots$$

with

$$\left. C_n = \frac{ia\overline{W}}{2\pi} \int_0^{2\pi} [\sin(\theta + \epsilon - \alpha) + \sin(\alpha - \epsilon_0)\,\theta] \cos^{n-1}(\theta - i\psi) \left(1 + \frac{d\epsilon}{d\theta}\right) d\theta \right\} \quad (20a)$$

This integral can be computed for the coefficient C_1 and leads to a known formula. It is with $\theta + \epsilon = \varphi$

$$C_1 = \frac{ia\overline{W}}{2\pi} \int_0^{2\pi} [\sin(\theta+\epsilon-\alpha) + \sin(\alpha - \epsilon_0)] \left(1 + \frac{d\epsilon}{d\theta}\right) d\theta =$$

$$= \frac{ia\overline{W}}{2\pi} \int_0^{2\pi} [\sin(\varphi-\alpha) + \sin(\alpha - \epsilon_0)]\, d\varphi = ia\overline{W} \sin(\alpha - \epsilon_0)$$

Regarding the result (20), it should be noted that this formula naturally is not the only possible explicit presentation of the velocity function, since Cauchy's integral formula can also be applied to every regular function in the outside zone of the profile combined with $\overline{w}^*(\overline{z})$.

3. APPROXIMATE THEORY OF POTENTIAL FLOW OF THIN WING SECTIONS

There is a theory for thin wing sections which, proceeding from the image of a nonuniformly covered flat vortex layer, establishes an approximate relation between the velocity differences on the upper and lower surfaces of the wing section and the slope of its mean line (reference 5). It constituted for a long time the only one affording a functional relation between velocity distribution along the wing section and its contour, and proved very useful despite its restricted range of validity. Hence, the desire to expand this theory to include medium wing sections in approximate validity, though no satisfactory solution has been found up to now.

Theodorsen's theory, described in the preceding section, affords the strict relationship between the wing-section shape and the velocity distribution along the contour. It, moreover, affords information about thin wing sections, and it might be conjectured that for vanishing thickness the equations of this theory, with proper omissions, would become the equations of the old theory of thin airfoils, which would mean that Theodorsen's theory represents the desired generalization of the old theory. However, that is not the case, as is readily proved: Theodorsen's theory contains a theory of thin wing sections as limiting case, the results of which are a system of equations different from those of the theory of the lifting vortex surface.

The theory developed hereinafter leads, for vanishing airfoil thickness, to the formulas of the linear vortex theory and represents, in its formulas, if not in thought, a generalization of the old theory. In distinction from Theodorsen's theory, it is fundamentally an approximate theory but it involves in return no infinite method. Its result is rigorous only in the specific case of the flat plate exposed at angle α to the stream; but the more the wing section departs from the flat plate - that is, the greater its thickness and camber - the more the results assume an approximate character.

The method, like that of Theodorsen, begins with the conformal transformation of plane $\overline{z}$ into a plane z - with the given airfoil - by means of the function

$$z = \overline{z} + \sqrt{\overline{z}^2 - 1}$$

whereby the wing-section contour becomes a curve $z(\theta) = e^{\psi(\theta)+i\theta}$, not much different from the unit circle. For this almost circular contour, that flow with the complex velocity $w^*(\infty) = W\ e^{-i\alpha}$ at infinity must be determined, for which at point $z = 1$, which corresponds to the trailing edge, the velocity assumes the value zero.

But, while in Theodorsen's method the flow about the almost circular curve is rigorously computed by means of the analytical function - obtainable, to be sure, only in infinitely many steps - which transforms the contour of this curve smoothly into the curve of the unit circle, the solution of this flow is now effected approximately on the basis of the following reasoning:

At point P on the contour of the almost circular curve the tangential velocity has a certain magnitude $u_t(\theta)$. Hence the radial velocity component in P is, for small ϑ, approximately $u_t(\theta)\ \vartheta(\theta)$, whereby the radial velocity in outward direction, and the tangential velocity in direction of increasing θ carry the positive sign (fig. 3).

About the magnitude of the tangential velocity in P, the corresponding flow about the unit circle affords, for the present, some information because, when the almost circular curve changes into the unit circle in which distance PQ and angle ϑ uniformly tend toward zero for all θ, the flow changes into that about the unit circle and converges the tangential velocity $u_t(\theta)$ in P against the tangential velocity $u_{t_0}(\theta)$ at point Q of the unit circle.

Then it is assumed that the distance PQ and the angle ϑ are so small for all θ that no difference need be made between $u_{t_0}(\theta)$ and $u_t(\theta)$ in the intended approximate calculation. Hence the radial velocity in P is approximated at $u_{t_0}\ \vartheta$ and, while the change on transition from P to Q is ignored, this quantity substitutes for the normal velocity $u_n(\theta)$ at point Q of the unit circle. Accordingly, since the flow about the unit circle (with velocity $W\ e^{-i\alpha}$ at infinity and with the rear stagnation point at point $z = 1$) is $u_{t_0} = -\ 2\ W\ [\sin(\theta-\alpha) + \sin\alpha]$, we have:

$$u_n(\theta) = u_{t_0}(\theta)\ \vartheta(\theta) = -\ 2\ W\ [\sin(\theta-\alpha) + \sin\alpha]\ \vartheta(\theta) \tag{21}$$

In this fashion the problem approximately reduces to the second limiting problem of the potential theory for the unit circle. Formulated for the complex velocity, the problem consists of giving a regular analytical function in the outside of the unit circle, which has the value $W\ e^{-i\alpha}$ at infinity, a zero point at $z = 1$, and whose normal component on the unit circle takes the value prescribed by equation (21). Desired, above all else, is the explicit expression for the tangential velocity on the unit circle.

In any flow at point $z = e^{i\theta}$ of the unit circle, the following equations:

$$u_n = u_x \cos\theta + u_y \sin\theta, \qquad u_t = -u_x \sin\theta + u_y \cos\theta$$

exist between the normal component $u_n(\theta)$, the tangential component $u_t(\theta)$, and the components $u_x(\theta)$ and $u_y(\theta)$ of the velocity in x and y direction.

Combining u_x and u_y into the complex velocity function $u^*(z) = u_x - i\,u_y$ affords for $z = e^{i\theta}$

$$[izu^*(z)]_{z=e^{i\theta}} = e^{i\theta}(i\,u_x+u_y) = (\cos\theta + i\sin\theta)(i\,u_x+u_y) =$$

$$= -u_x \sin\theta + u_y \cos\theta + i(u_x \cos\theta + u_y \sin\theta) \quad u_t(\theta) + i\,u_n(\theta)$$

The result is a distribution of the tangential velocity over the circle periphery feasible for a potential flow with the aid of Poisson's integral as real part to the imaginary part $u_n(\theta)$, according to equation (21). This function

$$-\frac{1}{2\pi}\int_0^{2\pi} u_n(\theta')\cot\frac{\theta'-\theta}{2}\,d\theta' = \frac{W}{\pi}\int_0^{2\pi}[\sin(\theta'-\alpha) +$$

$$+ \sin\alpha]\ (\theta')\cot\frac{\theta'-\theta}{2}\,d\theta'$$

is, however, not the desired result because, since the related analytical function $i\,z\,u^*(z)$ at infinity is regular, $u^*(\infty) = 0$ is contrary to the posed problem for the related flow.

To obtain the desired flow, the most general potential flow for which the normal velocity on the circle periphery disappears and the velocity at infinity has the correct value, must be superposed. This is the flow about the unit circle with the tangential velocity

$$u_{t_0}(\theta) = -2w\,[\sin(\theta-\alpha) + \sin\beta]$$

Hence

$$\overline{u_t}(\theta) = -2W\left\{\sin(\theta-\alpha) + \sin\beta - \frac{1}{2\pi}\int_0^{2\pi} [\sin(\theta'-\alpha) + \right.$$

$$\left. + \sin\alpha]\,\vartheta(\theta')\cot\frac{\theta'-\theta}{2}\,d\theta\right\}$$

Lastly, the available parameter β must be so determined that the velocity disappears for $\theta = 0$ (Kutta-Joukowski theory). For β the equation reads:

$$\sin\beta = \sin\alpha + \frac{1}{2\pi}\int_0^{2\pi} [\sin(\theta'-\alpha) + \sin\alpha]\,\vartheta(\theta')\cot\frac{\theta'}{2}\,d\theta'$$

which finally gives the sought-for tangential velocity as

$$\left.\begin{aligned} u_t(\theta) = -2W\Big\{&\sin(\theta-\alpha) + \sin\alpha + \\ &+ \frac{1}{2\pi}\int_0^{2\pi} [\sin(\theta'-\alpha) + \sin\alpha]\,\vartheta(\theta') \\ &\qquad \left(\cot\frac{\theta'}{2} - \cot\frac{\theta'-\theta}{2}\right) d\theta'\Big\} \end{aligned}\right\} \tag{22}$$

or, transformed,

$$\sin(\theta'-\alpha) + \sin\alpha = \sin\theta'\left(\cos\alpha + \sin\alpha\tan\frac{\theta'}{2}\right)$$

$$\sin\theta'\left(\cot\frac{\theta'}{2} - \cot\frac{\theta'-\theta}{2}\right) = -\sin\theta'\,\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta'}{2}\,\sin\frac{\theta'-\theta}{2}} =$$

$$= -\sin\theta'\,\frac{\sin\theta}{2\cos\frac{\theta}{2}}\,\frac{2\cos\frac{\theta'}{2}}{\sin\theta'}\,\frac{1}{\sin\frac{\theta'-\theta}{2}} =$$

$$= -\frac{\sin\theta}{\cos\frac{\theta}{2}}\,\frac{\cos\frac{\theta'}{2}}{\sin\frac{\theta'-\theta}{2}}$$

and consequently

$$\left.\begin{aligned} u_t(\theta) = -2W \sin\theta \Bigg[& \cos\alpha + \sin\alpha \tan\frac{\theta}{2} - \\ & - \frac{1}{2\pi\cos\frac{\theta}{2}} \int_0^{2\pi} \left(\cos\alpha + \sin\alpha \tan\frac{\theta'}{2}\right) \vartheta(\theta)' \frac{\cos\frac{\theta'}{2}}{\sin\frac{\theta'-\theta}{2}}\, d\theta' \Bigg] \end{aligned}\right\} \quad (22a)$$

This equation presents the velocity distribution on the almost circular contour in plane z; from it the distribution over the airfoil contour is obtained by transformation of the z plane into the original plane with the aid of the function $\bar{z} = \frac{1}{2}\left(z + \frac{1}{z}\right)$. The enlargement factor of this transformation for $z = e^{\psi + i\theta}$ is

$$\left|\frac{d\bar{z}}{dz}\right|_{z=e^{\psi+i\theta}} = \left|\frac{1}{2z}\left(z - \frac{1}{z}\right)\right|_{z=e^{\psi+i\theta}} = \left|e^{-\psi-i\theta}\sin(\theta - i\psi)\right| =$$

$$= e^{-\psi}\sqrt{\sin^2\theta + \sinh^2\psi}$$

and the absolute velocity on the airfoil contour with $\overline{W} = 2W$ (absolute velocity at infinity) is:

$$\left.\begin{aligned} |\overline{w}(\theta)| = \overline{W} \Bigg| \frac{e^{\psi}\sin\theta}{\sqrt{\sin^2\theta + \sinh^2\psi}} \Bigg[& \cos\alpha + \sin\alpha \tan\frac{\theta}{2} - \\ & - \frac{1}{2\pi\cos\frac{\theta}{2}} \int_0^{2\pi} \left(\cos\alpha + \sin\alpha \tan\frac{\theta'}{2}\right) \vartheta(\theta') \frac{\cos\frac{\theta'}{2}}{\sin\frac{\theta'-\theta}{2}}\, d\theta \Bigg] \Bigg| \end{aligned}\right\} \quad (23)$$

In order to establish a relationship between this result and the linear vortex theory, the equation needs further conversion. To this end, the periodic function

$$f(\theta) = \left(\cos\alpha + \sin\alpha \tan\frac{\theta}{2}\right)\vartheta(\theta)$$

below the integral is divided in a component $f_1(\theta)$ symmetrical to $\theta = \pi$, whose Fourier series consists of cos terms, and an unsymmetrical component $f_2(\theta)$ whose Fourier series consists of sine terms only. If $\vartheta_1(\theta)$ and $\vartheta_2(\theta)$ are the thus-declared components of function $\vartheta(\theta)$, that is, if

$$\vartheta_1(\theta) = \frac{1}{2}\,[\vartheta(\theta)+\vartheta(2\pi-\theta)], \qquad \vartheta_2(\theta) = \frac{1}{2}\,[\vartheta(\theta)-\vartheta(2\pi-\theta)] \tag{24}$$

we have:

$$f_1(\theta) = \frac{1}{2}\,[f(\theta)+f(2\pi-\theta)] = \cos\alpha\ \vartheta_1(\theta)+\sin\alpha\ \tan\frac{\theta}{2}\ \vartheta_2(\theta)$$

$$f_2(\theta) = \frac{1}{2}\,[f(\theta)-f(2\pi-\theta)] = \cos\alpha\ \vartheta_2(\theta)+\sin\alpha\ \tan\frac{\theta}{2}\ \vartheta_1(\theta)$$

Owing to the symmetrical qualities of the functions $f_1(\theta)$ and $f_2(\theta)$, the integral in equation (23) can be substantially simplified. It is[3]

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[3]Auxiliary conversion formulas are:

$$\cos\frac{x}{2}\left(\frac{1}{\sin\frac{x-y}{2}} - \frac{1}{\sin\frac{x+y}{2}}\right) = \cos\frac{x}{2}\,\frac{\sin\frac{x+y}{2} - \sin\frac{x-y}{2}}{\sin\frac{x-y}{2}\,\sin\frac{x+y}{2}}$$

$$= \cos\frac{x}{2}\,\frac{2\cos\frac{x}{2}\sin\frac{y}{2}}{-\frac{1}{2}(\cos x - \cos y)} = -2\sin\frac{y}{2}\,\frac{2\cos^2\frac{x}{2}}{\cos x - \cos y} =$$

$$= -2\sin\frac{y}{2}\,\frac{1+\cos x}{\cos x - \cos y}$$

$$\cos\frac{x}{2}\left(\frac{1}{\sin\frac{x-y}{2}} + \frac{1}{\sin\frac{x+y}{2}}\right) = \cos\frac{x}{2}\,\frac{\sin\frac{x+y}{2} + \sin\frac{x-y}{2}}{\sin\frac{x-y}{2}\,\sin\frac{x+y}{2}} =$$

$$= \cos\frac{x}{2}\,\frac{2\sin\frac{x}{2}\cos\frac{y}{2}}{-\frac{1}{2}(\cos x - \cos y)} = -2\cos\frac{y}{2}\,\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos x - \cos y} =$$

$$= -2\cos\frac{y}{2}\,\frac{\sin x}{\cos x - \cos y}$$

$$\int_0^{2\pi} f_1(\theta') \frac{\cos\frac{\theta'}{2}}{\sin\frac{\theta'-\theta}{2}}\, d\theta' = \int_0^{\pi} f_1(\theta') \frac{\cos\frac{\theta'}{2}}{\sin\frac{\theta'-\theta}{2}}\, d\theta' +$$

$$+ \int_{\pi}^{0} f_1(2\pi-\bar{\theta}) \frac{\cos\frac{2\pi-\bar{\theta}}{2}}{\frac{2\pi-\bar{\theta}-\theta}{2}}\, d(2\pi-\bar{\theta}) =$$

$$= \int_0^{\pi} f_1(\epsilon') \left(\frac{\cos\frac{\epsilon'}{2}}{\sin\frac{\epsilon'-\epsilon}{2}} - \frac{\cos\frac{\epsilon'}{2}}{\sin\frac{\epsilon'+\epsilon}{2}} \right) d\epsilon' =$$

$$= -2 \sin\frac{\epsilon}{2} \int_0^{\pi} f_1(\epsilon') \frac{1+\cos\epsilon'}{\cos\epsilon' - \cos\epsilon}\, d\epsilon'$$

and likewise

$$\int_0^{2\pi} f_2(\epsilon') \frac{\cos\frac{\epsilon'}{2}}{\sin\frac{\epsilon'-\theta}{2}}\, d\epsilon = \int_0^{\pi} f_2(\epsilon') \frac{\cos\frac{\epsilon'}{2}}{\sin\frac{\epsilon'-\epsilon}{2}}\, d\epsilon' +$$

$$+ \int_{\pi}^{0} f_2(2\pi-\bar{\theta}) \frac{\cos\frac{2\pi-\bar{\epsilon}}{2}}{\sin\frac{2\pi-\bar{\epsilon}-\epsilon}{2}}\, d(2\pi-\bar{\epsilon}) =$$

$$= \int_0^{\pi} f_2(\epsilon') \left(\frac{\cos\frac{\epsilon'}{2}}{\sin\frac{\epsilon'-\epsilon}{2}} + \frac{\cos\frac{\epsilon'}{2}}{\sin\frac{\epsilon'+\epsilon}{2}} \right) d\epsilon' =$$

$$= -2 \cos\frac{\epsilon}{2} \int_0^{\pi} f_2(\epsilon') \frac{\sin\epsilon'}{\cos\epsilon' - \cos\epsilon}\, d\epsilon'$$

$$\left.\begin{aligned}
|\overline{w}_o(x)| &= \overline{W}\ e^{\psi_o(x)} \sqrt{\frac{1-x^2}{\cosh^2\psi_o - x^2}}\ \Bigg| \left(\cos\alpha + \sin\alpha\sqrt{\frac{1-x}{1+x}}\right) \\
&\left[1 + \frac{1}{\pi}\int_{-1}^{+1}\frac{\vartheta_2(\xi)\,d\xi}{\xi - x}\right] + \frac{1}{\pi}\cos\alpha\sqrt{\frac{1-x}{1+x}}\int_{-1}^{+1}\vartheta_1(\xi)\sqrt{\frac{1+\xi}{1-\xi}}\frac{d\xi}{\xi - x} + \\
&+ \frac{1}{\pi}\sin\alpha\int_{-1}^{+1}\vartheta_1(\xi)\sqrt{\frac{1-\xi}{1+\xi}}\frac{d\xi}{\xi - x}\ \Bigg|, \\
|\overline{w}_u(x)| &= \overline{W}\ e^{\psi_u(x)} \sqrt{\frac{1-x^2}{\cosh^2\psi_u - x^2}}\ \Bigg| \left(\cos\alpha - \sin\alpha\sqrt{\frac{1-x}{1+x}}\right) \\
&\left[1 + \frac{1}{\pi}\int_{-1}^{+1}\frac{\vartheta_2(\xi)\,d\xi}{\xi - x}\right] - \frac{1}{\pi}\cos\alpha\sqrt{\frac{1-x}{1+x}}\int_{-1}^{+1}\vartheta_1(\xi)\sqrt{\frac{1+\xi}{1-\xi}}\frac{d\xi}{\xi - x} + \\
&+ \frac{1}{\pi}\sin\alpha\int_{-1}^{+1}\vartheta_1(\xi)\sqrt{\frac{1-\xi}{1+\xi}}\frac{d\xi}{\xi - x}\ \Bigg|
\end{aligned}\right\} \quad (26)$$

This general result contains several specially noteworthy cases.

1. Flat plate.- In the case of flat plate, it is $\psi_o(x) \equiv \psi_u(x) \equiv 0$, and also $\vartheta_o(x) \equiv \vartheta_u(x) \equiv 0$, as a result of which equation (26) reduces to

$$|\overline{w}(x)| = \overline{W}\left|\ \cos\alpha \pm \sin\alpha\sqrt{\frac{1-x}{1+x}}\right|$$

in accord with the previously derived equation (10) for the absolute velocity on the plate surface. This is the only case where equations (26) give the rigorous result.

2. Symmetrical airfoil.- For this, it is $\psi_o(x) = \psi_u(x)$ and $\vartheta_o(x) = -\vartheta_u(x)$; hence $\vartheta_1(x) \equiv 0$, $\vartheta_2(x) = \vartheta_o(x)$. In this case the velocity distribution is, according to equation (26):

$$|\bar{w}(x)| = \bar{W}\, e^{\psi_o(x)} \sqrt{\frac{1 - x^2}{\cosh^2 \psi_o - x^2}}\, \left(\cos\alpha \pm \sin\alpha \sqrt{\frac{1-x}{1+x}}\right)$$

$$\left[1 + \frac{1}{\pi}\int_{-1}^{+1} \frac{\vartheta_o(\xi)\,d\xi}{\xi - x}\right] \tag{27}$$

This equation is fundamentally an approximate equation. First of all, no good agreement can be expected for the trailing-edge vicinity - that is, for x values approaching +1, if the edge angle is finite. In fact, for $\vartheta_o(1) \neq 0$, the principal value of integral

$$\int_{-1}^{+1} \frac{\vartheta_o(\xi)\,d\xi}{\xi - x}$$

increases beyond all limits if $x \longrightarrow 1$.

The explanation for this is as follows: With finite angle at the trailing edge, the tangential velocity $u_t(\theta)$ approaches zero differently for small θ than it does at the circle. Hence it is not justifiable to solve the normal velocity according to equation (21) on the basis of the velocity distribution over the circle. And on this point, equation (27) must also fail.

For practical purposes, it is recommended to figure either with a sharp trailing edge or, what amounts to the same, round off the trailing edge and locate point $z = 1$ midway between peak and center of peak curvature. In the second case, the first factor of equation (27), and with it the velocity, approaches zero for $\theta = 0$.

3. Infinitely thin airfoil.- For infinitely thin airfoils, it is $\psi_o(x) = - \psi_u(x)$ and $\vartheta_o(x) = \vartheta_u(x)$; hence $\vartheta_1(x) = \vartheta_o(x)$, $\vartheta_2(x) \equiv 0$. Since, in this case, the function $\psi_o(x)$ disappears for $x = +1$ and $x = -1$, $\psi_o(x)$ may be approximated at $\psi_o(x) \equiv 0$ if the camber is slight, while function $\vartheta_o(x) \equiv \frac{d\psi}{d\theta}(x)$ can, in this case, be expressed with $\vartheta_o(x) = -\frac{dy}{dx} = -y'$ because the airfoil follows completely in proximity of the piece of the x axis between -1 and +1. Accordingly, equation (26) gives:

$$\left.\begin{aligned} |w(x)| = \overline{W}\Bigg| & \cos\alpha - \frac{1}{\pi}\sin\alpha \int_{-1}^{+1} y'(\xi)\sqrt{\frac{1-\xi}{1+\xi}}\frac{d\xi}{\xi-x} \pm \\ & \pm\sqrt{\frac{1-x}{1+x}}\left[\sin\alpha - \frac{1}{\pi}\cos\alpha\int_{-1}^{+1} y'(\xi)\sqrt{\frac{1+\xi}{1-\xi}}\frac{d\xi}{\xi-x}\right]\Bigg| \end{aligned}\right\} \qquad (28)$$

The chief result of the linear vortex theory is a relationship between the velocity differences on the top and bottom of an infinitely thin airfoil and the slope $y'(\xi)$ of its median line. Equation (28) gives for this relation the formula

$$\overline{w}_o(x) - \overline{w}_u(x) = 2\overline{W}\sqrt{\frac{1-x}{1+x}}\left[\sin\alpha - \frac{1}{\pi}\cos\alpha\int_{-1}^{+1} y'(\xi)\sqrt{\frac{1+\xi}{1-\xi}}\frac{d\xi}{\xi-x}\right] \qquad (29)$$

which for $\alpha = 0$ reduces to

$$\overline{w}_o(x) - \overline{w}_u(x) = -\frac{2\overline{W}}{\pi}\sqrt{\frac{1-x}{1+x}}\int_{-1}^{+1} y'(\xi)\sqrt{\frac{1+\xi}{1-\xi}}\frac{d\xi}{\xi-x}$$

and represents the known result of the linear vortex theory (reference 5*). The approximate theory developed here for medium thick airfoils contains the old linear vortex theory as special case and, in that respect, represents its generalization.

5. CONTRIBUTIONS TO GENERAL POTENTIAL THEORY OF WING SECTIONS

The potential flow about an airfoil is completely described by the related complex velocity function, as exemplified in equations (4), (9), and (20), which give the velocity function flows about the unit circle, the flat plate, and about any other airfoil. The velocity function is an analytical function, regular at any point in the outside zone of the airfoil. For the vicinity of the infi-

*Fuchs-Hopf, Aerodynamik, Bd. 2, S. 87.

nitely remote point the series development (2a) is applicable. The coefficients of this series are associated with the force and moment applied by the flow on the body (equation 3).

In the following, these arguments about the complex velocity are substantially complemented, wherein relationship is established between function and velocity distribution on the airfoil contour and with the direction of the airfoil contour. This reasoning leads first to developing new integral equations for the basic problem of airfoil theory, which connect the velocity distribution along the airfoil contour with the direction of the airfoil contour at every point in exact manner; and secondly, it indicates a method for developing airfoils with required characteristics.

Assume that the airfoil is plotted in the complex ζ plane. If the flow strikes the airfoil at an angle α, then let $w^*(\alpha;\zeta)$ be the complex velocity. This function is regular in the outer region of the wing. For $\zeta \longrightarrow \infty$, it is:

$$\lim_{\zeta \longrightarrow \infty} w^*(\alpha;\zeta) = W\ e^{-i\alpha}$$

The residuum of $w^*(\alpha;\zeta)$ is purely imaginary. For the values of $w^*(\alpha;\zeta)$ along the airfoil contour, the formula

$$w^*(\alpha;\zeta(s)) = w^*(\alpha;s) = |w(\alpha;s)|\ e^{-i\vartheta(\alpha;s)} \tag{30}$$

is valid, whereby s is the arc length on the contour measured from the trailing edge in positive direction of rotation $|w(\alpha;s)|$ indicates the absolute velocity at point s, and $\vartheta(\alpha;s)$ is the angle formed by the velocity and the positive real axis (fig. 4).

Further, let $\zeta(z)$ represent the analytical function which transforms the contour of the airfoil in the ζ plane into the contour of the unit circle in the z plane. The function $\zeta(z)$ is regular for all finite z with $|z| > 1$. In the foregoing, $\zeta(z)$ had been standardized by the requirement that $\lim\limits_{z \longrightarrow \infty} \frac{d\zeta}{dz}$ should be real, the image point of the airfoil trailing edge being a wholly arbitrary point of the unit circle. In distinction from it.

a subsequent rotation with the angle β is to bring the image point of the trailing edge to point $z = 1$. Accordingly, it is at infinity

$$\lim_{z \to \infty} \frac{d\zeta}{dz} = a e^{-i\beta}$$

For great z the series development

$$\zeta(z) = a\, e^{-i\beta}\, z \left(1 + \frac{\beta_1}{z} + \frac{\beta_2}{z^2} + \ldots\right) \qquad (\beta_1, \beta_2 \ldots \text{ complex}) \tag{31}$$

Then the substitution $\zeta(z)$ in $w^*(\alpha;\zeta)$ gives an analytical function

$$\overline{w}^*(\alpha; z) = w^*(\alpha; \zeta(z)) \tag{32}$$

which is regular outside of the unit circle and assumes for $z = e^{i\varphi}$, according to equation (30), the values:

$$[\overline{w}^*(\alpha; z)]_{z=e^{i\varphi}} = |w(\alpha; s(\varphi))|\; e^{-i\vartheta(\alpha; s(\varphi))} =$$

$$= |\overline{w}(\alpha;\varphi)|\; e^{-i\overline{\vartheta}(\alpha;\varphi)} \tag{30a}$$

This function serves as basis for the subsequent investigations. The series development of $\overline{w}^*(\alpha; z)$ for great z is intended for later on. Because of equation (31) and of the development

$$w^*(\alpha;\zeta) = C_0 + \frac{C_1}{\zeta} + \frac{C_2}{\zeta^2} + \ldots, \qquad \left(C_0 = W\, e^{-i\alpha}, \quad C_1 = \frac{i\Gamma}{2\pi}\right)$$

valid for $w^*(\alpha;\zeta)$, the first terms of this series are:

$$\overline{w}^*(\alpha; z) = W\, e^{-i\alpha} + C_1 \frac{e^{i\beta}}{a z} + \left(C_2 - C_1\, \beta_1\, a\, e^{-i\beta}\right) \frac{e^{2i\beta}}{a^2 z^2} + \ldots \tag{32a}$$

The behavior of the function $\overline{w}^*(\alpha; z)$ at the point $z = +1$, resembling the trailing edge, is of special interest. If ζ_0 is the point of the trailing edge in plane ζ and γ the edge angle, we find:

$$(\zeta - \zeta_0) \sim (z - 1)^{\frac{2\pi-\gamma}{\pi}} \quad \text{or} \quad (z - 1) \sim (\zeta - \zeta_0)^{\frac{\pi}{2\pi-\gamma}}$$

for the vicinity of point $z = 1$.

Since in the flow about the circle with stagnation point at $z = 1$, the complex velocity acts like $(z - 1)$ the flow about the airfoil

$$w^*(\alpha;\zeta) \sim (z - 1)\frac{dz}{d\zeta} \sim (\zeta - \zeta_0)^{\frac{\gamma}{2\pi-\gamma}}.$$

For the function $\overline{w}^*(\alpha;z)$ resulting from substituting $\zeta(z)$, it is:

$$\overline{w}^*(\alpha;z) \sim \left((z - 1)^{\frac{2\pi-\gamma}{\pi}}\right)^{\frac{\gamma}{2\pi-\gamma}} \sim (z - 1)^{\frac{\gamma}{\pi}}$$

Hence the following result: If the airfoil has at the trailing edge the angle γ, then

$$\overline{w}^*(\alpha;z) = \text{const.}\ (z - 1)^{\frac{\gamma}{\pi}} \tag{33}$$

for the vicinity of $z = 1$.

The relation of conformal function $\zeta(z)$ with $w^*(\alpha;\zeta)$ and $\overline{w}^*(\alpha;z)$ has some surprising consequences. The function $\zeta(z)$ ties the complex velocity of flow about the arbitrary airfoil to the velocity function of the flow about the unit circle. If $W' e^{-i\alpha'}$ is the velocity at infinity for the flow about the unit circle, and $z = e^{i\varphi_1}$, the forward stagnation point, the complex velocity, since the rear stagnation point must lie in the image point of the trailing edge, i.e., at $z = + 1$ is, according to equation (4)

$$W' e^{-i\alpha'}\left(1 - \frac{1}{z}\right)\left(1 - \frac{e^{i\varphi_1}}{z}\right) \quad \text{with} \quad \varphi_1 = \pi + 2\alpha'$$

where, according to equation (6)

$$\frac{d\zeta}{dz} = W' e^{-i\alpha'}\left(1 - \frac{1}{z}\right)\left(1 - \frac{e^{i\varphi_1}}{z}\right)\frac{1}{\overline{w}^*(\alpha;z)} \tag{34}$$

From this it follows that the forward stagnation point of the flow about the unit circle is coincident with the zero point $w^*(\alpha;z)$ belonging to the forward stagnation point of the airfoil flow. Herewith φ_1 and, since $\varphi_1 = \pi + 2\alpha'$, α' itself is determined. For the infinitely remote point equation (34) gives:

$$a\ e^{-i\beta} = W'\ e^{-i\alpha'}\ W^{-1}\ e^{i\alpha}$$

hence

$$\alpha = \frac{W'}{W}, \qquad \beta = \alpha' - \alpha \tag{35}$$

The conformal function $\zeta(z)$ itself follows from equation (34) by integration. It is

$$\zeta(z)=\zeta_0+a\ W\ e^{-i(\alpha+\beta)} \int_1^z \left(1-\frac{1}{z}\right)\left[1-\frac{e^{i(\pi+2\alpha+2\beta)}}{z}\right] \frac{dz}{\overline{w}^*(\alpha;z)} \tag{36}$$

which gives for $z = e^{i\varphi}$ the profile contour $\zeta(\varphi)$ related to a predetermined function $\overline{w}^*(\alpha;z)$. From the fact that the airfoil contour is a closed curve, it follows that the residuum of the integrand in equation (36) must be equal to zero. The function $w^*(\alpha;z)$ therefore, has the important characteristic

$$\text{Residuum}\left\{\left(1-\frac{1}{z}\right)\left[1-\frac{e^{i(\pi+2\alpha+2\beta)}}{z}\right]\frac{1}{\overline{w}^*(\alpha;z)}\right\} = 0 \tag{37}$$

For the coordination of the circle periphery and the airfoil contour there exists, according to equation (34), the following relationship:

$$\frac{ds}{d\varphi} = 2W'\left|\frac{\sin(\varphi-\alpha')+\sin\alpha'}{|\overline{w}(\alpha;\varphi)|}\right| \tag{38}$$

This equation is recommended for solving $s(\varphi)$ when the distribution of the absolute values of $\overline{w}^*(\alpha;z)$, that is, the function $|\overline{w}(\alpha;\varphi)|$ on the unit circle is known. α' must be so chosen that $|\overline{w}(\alpha;\pi+2\alpha')| = 0$. Then the function $|\overline{w}(\alpha;\varphi)|$ can be replotted with $s(\varphi)$ and the velocity distribution $|w(\alpha;s)|$ on the profile contour obtained.

Conversely, for given velocity distribution $|w(\alpha;s)|$ equation (38) gives, inasmuch as point s_1 of the forward stagnation point on the profile contour corresponds with point $\varphi_1 = \pi + 2\alpha'$

$$\int_{s_1}^{s} |w(\alpha;s)|\ ds = 2W'\left|\int_{\pi+2\alpha'}^{\varphi} |\sin(\varphi-\alpha')+\sin\alpha'|\ d\varphi\right| \quad \text{or}$$

$$\int_{s_1}^{s} |w(\alpha;s)|\ ds=2W'|\cos(\varphi-\alpha')+\cos\alpha'+(\pi+2\alpha'-\varphi)\sin\alpha'| \tag{39}$$

wherein the constants α' and W' are left for determination. This is achieved by the fact that $s = 0$ relates to $\varphi = 0$ and $s = S$ to $\varphi = 2\pi$, where S is the airfoil contour. The solution for α' and W' is:

$$\left.\begin{aligned} \frac{\int_0^{s_1} |w(\alpha;s)|\,ds}{\int_{s_1}^{S} |w(\alpha;s)|\,ds} &= \frac{1 + \left(\frac{\pi}{2} + \alpha'\right)\tan\alpha'}{1 - \left(\frac{\pi}{2} - \alpha'\right)\tan\alpha'} = \mu(\alpha') \\ W' &= \frac{\int_0^{s_1} |w(\alpha;s)|\,ds}{4\left[\cos\alpha' + \left(\frac{\pi}{2} + \alpha'\right)\sin\alpha'\right]} \end{aligned}\right\} \qquad (39a)$$

The function $\mu(\alpha')$ occurring in the first equation increases between $\alpha' = -\frac{\pi}{2}$ and $\alpha' = +\frac{\pi}{2}$ monotonically from 0 to ∞. Hence there is one, and only one α' for each value of the quotient between these limits. Then the second equation gives the value W' for α'. Equation (39) gives with these values α' and W' the function $s(\varphi)$, with the aid of which the function $|\overline{w}(\alpha;\varphi)|$ can be obtained from $|\overline{w}(\alpha;s)|$.

Hence the remarkable fact that for the conformal transformation between the contours of an arbitrary airfoil and the unit circle, the coordination of the boundaries can be determined without knowing the shape of the profile if the velocity distribution along the profile contour for an arbitrary potential flow about the profile is known.

Lastly, the distribution $|w(\overline{\alpha};s)|$ for any other angle can be deduced according to equation (38), equally without knowing the profile, from the velocity distribution $|w(\alpha;s)|$ for a certain angle of attack α. Since the same function $s(\varphi)$ relates to both flows about the same profile, it is with $\varphi = \varphi(s)$

$$|w(\bar{\alpha};s)| = |w(\alpha;s)| \left| \frac{\sin(\varphi-\alpha'-\bar{\alpha}+\alpha) + \sin(\alpha'+\bar{\alpha}-\alpha)}{\sin(\varphi-\alpha') + \sin\alpha'} \right| \tag{40}$$

In particular, it gives for $\bar{\alpha} = \alpha - \alpha' = -\beta$ the velocity distribution for the circulation free flow

$$\left|w(s)\right| = \left|w(\alpha;s)\right| \left| \frac{\sin\varphi}{\sin(\varphi-\alpha') + \sin\alpha'} \right| \tag{40a}$$

Thus the velocity distribution along the profile contour related to a certain potential flow about the profile determines the velocity distribution for other angles of attack as well, and can be obtained by elementary calculation. And the fact that the related profile itself can be defined in simple manner without having recourse to an infinite method is one of the ensuing results.

Following these more general arguments about the velocity function and the velocity distributions on profile contour and circle periphery, the relationship with function $\vartheta(s)$ and $\bar{\vartheta}(\varphi)$ decisive for the shape of the contour, is discussed. Reverting back to equation (30a), we replace φ by s, so that it now reads:

$$\left[\bar{w}^*(\alpha;z)\right]_{z=e^{i\varphi}} = |\bar{w}(\alpha;\varphi)|\, e^{-i\bar{\vartheta}(\alpha;\varphi)} \tag{30a}$$

What are the connections between the functions $|\bar{w}(\alpha;\varphi)|$ and $\bar{\vartheta}(\alpha;\varphi)$?

With the complex function $\bar{w}^*(\alpha;z)$ new complex functions $f(z) = f(z,\bar{w}^*(\alpha;z))$ can be formed which are regular in the outside zone of the unit circle with inclusion of the infinitely remote point, and on the circle contour. With

$$f(z,w^*(\alpha;z)) = g(\upsilon,\varphi) + i\,h(\upsilon,\varphi)$$

the two interrelated equations exist between real and imaginary parts of $f(z)$ on the unit circle

$$g(1,\varphi) = g(\infty) - \frac{1}{2\pi}\int_0^{2\pi} h(1,\varphi')\cot\frac{\varphi' - \varphi}{2}\,d\varphi'$$

$$h(1,\varphi) = h(\infty) + \frac{1}{2\pi}\int_0^{2\pi} g(1,\varphi')\cot\frac{\varphi' - \varphi}{2}\,d\varphi'$$

at the same time. $g(1,\varphi)$ and $h(1,\varphi)$ are functions of φ, $\overline{w}(\alpha;\varphi)$ and $\overline{\vartheta}(\alpha;\varphi)$, according to equation (30a). The result therefore is a diversity of integral equations, all of which establish the relationship between the functions $|\overline{w}(\alpha;\varphi)|$ and $\overline{\vartheta}(\alpha;\varphi)$ in exact manner. A few illustrations are given:

1. The function $\overline{w}^*(\alpha;z)$ has itself the characteristics required of $f(z)$. The next example therefore is $f(z) \equiv w^*(\alpha;z)$. In this case

$$g(1,\varphi) = |\overline{w}|\cos\overline{\vartheta}, \qquad g(\infty) = W\cos\alpha$$

and

$$h(1,\varphi) = -|\overline{w}|\sin\overline{\vartheta}, \qquad h(\infty) = -W\sin\alpha$$

Hence the identities

$$|\overline{w}(\varphi)|\cos\overline{\vartheta}(\varphi) = W\cos\alpha + \\ + \frac{1}{2\pi}\int_0^{2\pi}|\overline{w}(\varphi')|\sin\overline{\vartheta}(\varphi')\cot\frac{\varphi' - \varphi}{2}\,d\varphi'$$

$$|\overline{w}(\varphi)|\sin\overline{\vartheta}(\varphi) = W\sin\alpha - \\ - \frac{1}{2\pi}\int_0^{2\pi}|\overline{w}(\varphi')|\cos\overline{\vartheta}(\varphi')\cot\frac{\varphi' - \varphi}{2}\,d\varphi'$$

Every such equation can fundamentally be used to determine the other function for given $|\overline{w}(\varphi)|$ or $\overline{\vartheta}(\varphi)$, and it should be possible in various ways to give infinite iteration methods which lead to results, and might even be advantageous for the application. This finding likewise explains the surprising fact incident to the comparison of the two theories (3 and 4) that different assumptions may lead to substantially unlike profile theories which in limiting cases, as of the infinitely thin airfoil, give outwardly entirely different equations.

2. A very interesting example is the following case:

$$f(z) = \frac{1}{2}\left(z - \frac{1}{z}\right)\left(\overline{w}^*(\alpha; z) - \overline{w}^*(\infty)\right)$$

Here $\quad [f(z)]_{z=e^{i\varphi}} = i \sin\varphi(\overline{w}\ e^{-i\overline{\vartheta}} - W\ e^{-i\alpha}) =$

$$= |\overline{w}| \sin\overline{\vartheta} \sin\varphi - W \sin\alpha \sin\varphi +$$

$$+ i(|\overline{w}| \cos\overline{\vartheta} \sin\varphi - W \cos\alpha \sin\varphi)$$

and, on account of equation 32a) with $C_1 = \frac{i\Gamma}{2\pi}$

$$f(\infty) = \lim_{z\to\infty} \frac{1}{2}\left(z - \frac{1}{z}\right)\left(C_1 \frac{e^{i\beta}}{az} + \ldots\right) =$$

$$= \frac{i\Gamma e^{i\beta}}{4\pi a} = \frac{\Gamma}{4\pi a}(-\sin\beta + i\cos\beta)$$

Hence the two identities read:

$$|\overline{w}| \cos\overline{\vartheta} \sin\varphi - W \cos\alpha \sin\varphi =$$

$$= \frac{\Gamma\cos\beta}{4\pi a} + \frac{1}{2\pi}\int_0^{2\pi}(|\overline{w}| \sin\overline{\vartheta} \sin\varphi' -$$

$$- W \sin\alpha \sin\varphi') \cot\frac{\varphi' - \varphi}{2}\, d\varphi' =$$

$$= \frac{\Gamma\cos\beta}{4\pi a} - W \sin\alpha \cos\varphi +$$

$$+ \frac{1}{2\pi}\int_0^{2\pi}|\overline{w}| \sin\overline{\vartheta} \sin\varphi' \cot\frac{\varphi' - \varphi}{2}\, d\varphi'$$

The value of the constant $\frac{\Gamma \cos \beta}{4\pi a}$ follows from the Kutta-Joukowski condition that at the trailing edge - that is, for $\varphi = 0$, the velocity must be finite. It is

$$\frac{\Gamma \cos \beta}{4\pi a} = W \sin \alpha - \frac{1}{2\pi} \int_0^{2\pi} \left|\overline{w}(\varphi')\right| \sin \overline{\vartheta}(\varphi') \sin \varphi' \cot \frac{\varphi'}{2} \, d\varphi'$$

hence the integral equation:

$$\left. \begin{aligned} \left|\overline{w}(\varphi)\right| \cos \overline{\vartheta}(\varphi) &= W \left(\cos \alpha + \sin \alpha \tan \frac{\varphi}{2}\right) + \\ &+ \frac{1}{2\pi \sin \varphi} \int_0^{2\pi} \left|\overline{w}(\varphi')\right| \sin \overline{\vartheta}(\varphi') \sin \varphi' \left(\cot \frac{\varphi' - \varphi}{2} - \cot \frac{\varphi'}{2}\right) d\varphi' \end{aligned} \right\} \quad (41)$$

This exact equation is intimately related to the theory of the lifting vortex surface. For the replacement of $|\overline{w}(\varphi)|$ below the integral by the velocity distribution of the flat plate, according to equation (10) in first approximation, leads to an equation which (substituting θ for φ, and ϑ for $\overline{\vartheta}$) agrees with equation (23) for the case $\psi = 0$. Thus the result is again among others the result of the theory of the lifting vortex surface - this time, however, not as limiting case of a somewhat more generalized approximate theory as in (4), but as first approximation of an exact equation for the most general flow about any arbitrary profile.

3. The equations obtained, according to the described principle are, as a rule, integral equations possessing, as in the two examples, combinations of functions $|\overline{w}(\varphi)|$ and $\overline{\vartheta}(\varphi)$ below and outside of the integral. However, these functions can be separated by so choosing the function $f(z)$ that its real part depends upon $|\overline{w}(\varphi)|$, and its imaginary part on $\overline{\vartheta}(\varphi)$ only.

If the airfoil has the trailing-edge angle γ, and the forward stagnation point of the flow lies at $e^{i\varphi_1}$, then according to equation (33),

$$f(z, \overline{w}^*(\alpha; z)) \equiv \ln \left[\left(1 - \frac{1}{z}\right)^{-\frac{\gamma}{\pi}} \left(1 - \frac{e^{i\varphi_1}}{z}\right)^{-1} \overline{w}^*(\alpha; z) \right] \quad (42)$$

is a regular function with the characteristic required for all $|z| \geqq 1$.

Now in general

$$\left[\ln\left(1-\frac{e^{i\lambda}}{z}\right)\right]_{z=e^{i\varphi}} = \ln\left(2\sin\frac{\varphi-\lambda}{2} + i\,\frac{\pi-(\varphi-\lambda)}{2}\right)$$

for $\lambda \leqq \varphi < 2\pi + \lambda$, while for $\varphi < \lambda$ and for $\varphi \geqq 2\pi + \lambda$ the two functions for real and imaginary part must be continued with the period 2π. Hence (for $\lambda = 0$ and $\lambda = \varphi_1$)

$$\left[\ln\left(1-\frac{1}{z}\right)\right]_{z=e^{i\varphi}} = \ln\left(2\sin\frac{\varphi}{2}\right) + i\,\frac{\pi-\varphi}{2} \quad \text{for} \quad 0 \leqq \varphi < 2\pi$$

and

$$\left[\ln\left(1-\frac{e^{i\varphi_1}}{z}\right)\right]_{z=e^{i\varphi}} = \ln\left(2\sin\frac{|\varphi_1-\varphi|}{2}\right) + i\left(\frac{\varphi_1-\varphi}{2} \mp \frac{\pi}{2}\right)$$

$$\text{for} \begin{cases} 0 \leqq \varphi < \varphi_1 \\ \varphi_1 < \varphi \leqq 2\pi \end{cases}$$

In this case it gives for $f(z) = g + i\,h$

$$g(1,\varphi) = \ln \frac{|\overline{w}(\varphi)|}{\left(2\sin\frac{\varphi}{2}\right)^{\frac{\gamma}{\pi}}\left(2\sin\frac{|\varphi_1-\varphi|}{2}\right)}, \qquad g(\infty) = \ln W$$

$$h(1,\varphi) = -\overline{\vartheta}(\varphi) - \frac{\gamma}{\pi}\,\frac{\pi-\varphi}{2} - \frac{\varphi_1-\varphi}{2} \pm \frac{\pi}{2} \quad \left(\text{for} \quad \varphi \lessgtr \varphi_1\right), \qquad h(\infty) = -\alpha$$

hence Poisson's integral leads to

$$|\overline{w}(\varphi)| = W\left(2\sin\frac{\varphi}{2}\right)^{\frac{\gamma}{\pi}} \left(2\sin\frac{|\varphi_1-\varphi|}{2}\right) e^{\frac{1}{2\pi}\int_0^{2\pi}\left[\overline{\vartheta}(\varphi') + \frac{\gamma}{\pi}\,\frac{\pi-\varphi'}{2} + \frac{\varphi_1-\varphi'}{2} \mp \frac{\pi}{2}\right]\cot\frac{\varphi'-\varphi}{2}\,d\varphi'} \tag{43}$$

$$\bar{\delta}(\varphi) = \alpha - \frac{\gamma}{\pi}\frac{\pi - \varphi}{2} - \frac{\varphi_1 - \varphi}{2} \pm \frac{\pi}{2} -$$

$$- \frac{1}{2\pi}\int_0^{2\pi} \ln \frac{|\bar{w}(\varphi')|}{\left(2 \sin \frac{\varphi'}{2}\right)^{\frac{\gamma}{\pi}}\left(2 \sin \frac{|\varphi_1 - \varphi'|}{2}\right)} \cot \frac{\varphi' - \varphi}{2}\, d\varphi' \qquad (44)$$

The advantage of using the function $\ln \bar{w}^*(\alpha; z)$ as basis had already been recognized by F. Weinig, who likewise cited equations (43) and (44).

The discussion of the general airfoil theories is concluded with Weinig's $\ln \bar{w}$ method, involving the question of profiles of prescribed characteristics, of profiles relating to a given velocity distribution, and the potential flow about a given profile.

5. THE WEINIG $\ln \bar{w}$ METHOD[3]

In preparation of the treatment of profiles with predetermined characteristics, the appearance of the most important profile characteristics with the function $\bar{w}^*(\alpha; z)$ is first ascertained. For this purpose equation (42) is used to formulate $w^*(\alpha; z)$ with $F(z) \equiv f(z) - f(\infty)$

$$\left.\begin{aligned} &\bar{w}^*(\alpha; z) = W\, e^{-i\alpha} \left(1 - \frac{1}{z}\right)^{\frac{\gamma}{\pi}} \left(1 - \frac{e^{i\varphi_1}}{z}\right) e^{F(z)} \\ &\text{with} \\ &F(z) = \sum_{n=1}^{\infty} \frac{c_n}{z^n} = \sum_{n=1}^{\infty} \frac{a_n + i b_n}{z^n}, \qquad \text{since} \quad F(\infty) = 0 \end{aligned}\right\} \qquad (45)$$

The parameter γ in the formula is, according to equation (33) the trailing-edge angle of the airfoil. The fact that the contour is closed affords some prediction concerning the coefficient c_1. If it relates to an airfoil with

[3] The connections between the velocity distributions at different angles of attack and the $\ln w$ method has been reported by F. Weinig in the following publications: Z.f.a.M.M., 9 (1929), p. 507; Werft Reed. Hafen 14 (1933), p. 131; Luftf.-Forschg. 12 (1935), p. 221.

fixed center of pressure, a second condition for c_2 must be added. The condition for c_1 follows from equation (37). It is

$$\text{Residuum}\left(1-\frac{1}{z}\right)\left(1-\frac{e^{i\varphi_1}}{z}\right)\frac{1}{\overline{w}^*(\alpha;z)}\Big\} =$$

$$= \frac{e^{i\alpha}}{W}\ \text{residuum}\left\{\left(1-\frac{1}{z}\right)^{1-\frac{\gamma}{\pi}} e^{-\sum\limits_{n=1}^{\infty}\frac{c_n}{z^n}}\right\} = \frac{e^{i\alpha}}{W}\left(-1+\frac{\gamma}{\pi}-c_1\right) = 0$$

hence the final condition

$$a_1 = -1+\frac{\gamma}{\pi}, \qquad b_1 = 0 \tag{46}$$

The condition of fixed center of pressure follows from the fact that an airfoil has a fixed center of pressure when the moment for zero lift is zero. In this case the aerodynamic center of v. Mises' lift parabola is zero, and the air force therefore passes through the aerodynamic center of the airfoil at every angle of attack (reference 5). This means, according to equation (3), that by series development for $\overline{w}^*(\alpha;\zeta)$ in the case of $C_1 = 0$ for $C_2 = A_2 + i\ B_2$, the relation

$$A_0B_2 + B_0A_2 = 0$$

must be fulfilled.

According to equation (32a) the disappearance of C_1 is accompanied by that of the first coefficient $\overline{C}_1$ of the series development for $\overline{w}^*(\alpha;z)$, and the second coefficient $\overline{C}_2$ is, in this case, connected to C_2 by

$$C_2 = a^2 e^{-2i\beta}\ \overline{C}_2$$

But, according to equation (45)

$$\ln\overline{w}^*(\alpha;z) = \ln W - i\ \alpha + \sum_{n=1}^{\infty}\left(c_n - \frac{\gamma}{n\pi} - \frac{e^{in\varphi_1}}{n}\right)\frac{1}{z^n}$$

hence,

$$\overline{w}^*(\alpha;z) = W\ e^{-i\alpha}\left\{1+\left(c_1-\frac{\gamma}{\pi}-e^{i\varphi_1}\right)\frac{1}{z}+\right.$$

$$\left.+\left[\left(c_2-\frac{\gamma}{2\pi}-\frac{e^{2i\varphi_1}}{2}\right)+\frac{1}{2}\left(c_1-\frac{\gamma}{\pi}-e^{i\varphi_1}\right)^2\frac{1}{z^2}+\ \ldots\right]\right\}$$

That is,

$$\overline{C}_1 = W\ e^{-i\alpha}\left(c_1 - \frac{\gamma}{\pi} - e^{i\varphi_1}\right) = -\ W\ e^{-i\alpha}\ (1 + e^{i\varphi_1})$$

therefore $\overline{C}_1 = 0$ if $\varphi_1 = \pi$; that is, if $\alpha = -\beta$. This is the condition for lift-free flow about the wing section.

For $e^{i\varphi_1} = -1$ and $\alpha = -\beta$, we have:

$$\overline{C}_2 = W\ e^{i\beta}\left(c_2 - \frac{\gamma}{2\pi} - \frac{1}{2}\right) = W\ e^{i\beta}\left(a_2 + i\ b_2 - \frac{\gamma}{2\pi} - \frac{1}{2}\right)$$

hence $$C_2 = a^2\ W\ e^{-i\beta}\left(a_2 + i\ b_2 - \frac{\gamma}{2\pi} - \frac{1}{2}\right)$$

The condition for fixed center of pressure is $A_0B_2 + B_0A_2 = 0$, which means that the imaginary part of C_0C_2 must disappear. But, since $\alpha = -\beta$, the constant is $C_0 = W\ e^{i\beta}$

$$C_0C_2 = a^2\ W^2\left(a_2 + i\ b_2 - \frac{\gamma}{2\pi} - \frac{1}{2}\right)$$

so that the condition for airfoils with fixed center of pressure reads

$$b_2 = 0 \tag{47}$$

The closing condition and the condition of fixed center of pressure are also predictions about the functions $\overline{w}(\alpha;\varphi)$ and $\overline{\vartheta}(\alpha;\varphi)$. The particular formulas can be readily obtained on the identity:

$$[\ln \overline{w}^*(\alpha;z)]_{z=e^{i\varphi}} = \ln|\overline{w}(\alpha;\varphi)| - i\ \overline{\vartheta}(\alpha;\varphi) =$$

$$= \ln W - i\ \alpha + \sum_{n=1}^{\infty}\left(c_n - \frac{\gamma}{n\pi} - \frac{e^{in\varphi_1}}{n}\right)e^{-in\varphi}$$

by division into real and imaginary parts and integration.

It is for function $\ln\ \overline{w}(\alpha;\varphi)$

$$\int_0^{2\pi} \ln|\overline{w}(\alpha;\varphi)|\ d\varphi = 2\pi \ln W \tag{48a}$$

$$\left.\begin{aligned}\int_0^{2\pi} \ln \left|\overline{w}(\alpha;\varphi)\right| \sin\varphi \, d\varphi &= -\pi \sin\varphi_1 \\ \int_0^{2\pi} \ln \left|\overline{w}(\alpha;\varphi)\right| \cos\varphi \, d\varphi &= -\pi(1+\cos\varphi_1)\end{aligned}\right\} \text{(closing condition)} \qquad (46a)$$

$$\int_0^{2\pi} \ln \left|\overline{w}(\alpha;\varphi)\right| \sin 2\varphi \, d\varphi = -\frac{\pi}{2} \sin 2\varphi_1 \qquad \text{(fixed center of pressure)} \qquad (47a)$$

and for function $\overline{\vartheta}(\alpha;\varphi)$

$$\int_0^{2\pi} \overline{\vartheta}(\alpha;\varphi) \, d\varphi = 2\pi\alpha \qquad (48b)$$

$$\left.\begin{aligned}\int_0^{2\pi} \overline{\vartheta}(\alpha;\varphi) \sin\varphi \, d\varphi &= -\pi(1+\cos\varphi_1) \\ \int_0^{2\pi} \overline{\vartheta}(\alpha;\varphi) \cos\varphi \, d\varphi &= +\pi \sin\varphi_1\end{aligned}\right\} \text{(closing condition)} \qquad (46b)$$

$$\int_0^{2\pi} \overline{\vartheta}(\alpha;\varphi) \cos 2\varphi \, d\varphi = +\frac{\pi}{2} \sin 2\varphi_1 \qquad \text{(fixed center of pressure)} \qquad (47b)$$

The general theory presented here is superior to the old method, particularly in the manner of developing mathematical profiles. The calculation needs no special conformal functions with a number of correctly chosen parameters as basis, but rather proceeds from the general function $\overline{w}^*(\alpha;z)$ which is connected with the airfoil characteristics in the described manner.

Either the coefficients c_n in equation (45) can be prescribed, noting that c_1 must always $\doteq -1 + \frac{\gamma}{\pi}$, and that $b_2 = 0$ for fixed center of pressure; or else, the function $|\overline{w}(\alpha;\varphi)|$ can be given for a certain angle of attack α, from which the velocity distribution along the profile contour (and hence the flow characteristics of the airfoil in question) can be deduced, according to equations (38) and (40). By the selection of $|\overline{w}(\alpha;\varphi)|$, it should be observed that equations (46a) are complied with and, if a fixed center of pressure is involved, equation (47a). For $\varphi = 0$, $|\overline{w}(\alpha;\varphi)|$ must, like $\varphi^{\gamma/\pi}$, become zero if the airfoil is to have the trailing-edge angle γ.

With this chosen function $|\overline{w}(\alpha;\varphi)|$

$$\ln \frac{|\overline{w}(\alpha;\varphi)|}{\left(2\,\sin\frac{\varphi}{2}\right)^{\frac{\gamma}{\pi}}\left(2\,\sin\frac{|\varphi_1-\varphi|}{2}\right)}$$

must be formulated and developed in Fourier series. If this development reads

$$\ln \frac{|\overline{w}(\alpha;\varphi)|}{\left(2\,\sin\frac{\varphi}{2}\right)^{\frac{\gamma}{\pi}}\left(2\,\sin\frac{|\varphi_1-\varphi|}{2}\right)} = \ln W +$$

$$+ \sum_{n=1}^{\infty} (a_n \cos n\,\varphi + b_n \sin n\,\varphi) =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \ln w(\alpha;\varphi)\,d\varphi - \left(1 - \frac{\gamma}{\pi}\right)\cos\varphi +$$

$$+ \sum_{n=2}^{\infty} (a_n \cos n\,\varphi + b_n \sin n\,\varphi)$$

the related function $\overline{\vartheta}(\alpha;\varphi)$ is, according to equation (44):

$$\overline{\vartheta}(\alpha;\varphi) = \alpha - \frac{\gamma}{\pi}\,\frac{\pi - \varphi}{2} - \frac{\varphi_1 - \varphi}{2} \pm \frac{\pi}{2} - \left(1 - \frac{\gamma}{\pi}\right)\sin\varphi +$$

$$+ \sum_{n=2}^{\infty} (a_n \sin n\,\varphi - b_n \cos n\,\varphi)$$

and the profile contour itself, according to equation (36),

$$\zeta(z) = \zeta_o + W' e^{-i\alpha'} \int_1^z \left(1 - \frac{1}{z}\right)\left(1 - \frac{e^{i\varphi_1}}{z}\right) \frac{dz}{\overline{w}^*(\alpha;z)}$$

with $z = e^{i\varphi}$. Hereby, according to equation (4),

$$z = e^{i\varphi}, \quad \int_1^z \ldots\, dz = \int_0^\varphi \ldots\, i\, e^{i\varphi}\, d\varphi,$$

$$[\overline{w}^*(\alpha;z)]_{z=e^{i\varphi}} = |\overline{w}(\alpha;\varphi)|\; e^{-i\overline{\vartheta}(\alpha;\varphi)},$$

$$\left[W' e^{-i\alpha'}\left(1 - \frac{1}{z}\right)\left(1 - \frac{e^{i\varphi_1}}{z}\right)\right]_{z=e^{i\varphi}} =$$

$$= 2i\, W' e^{-i\varphi} \left(\sin\,(\varphi - \alpha') + \sin\,\alpha'\right)$$

and consequently,

$$\zeta(\varphi) = \zeta_o - 2a\, W \int_0^\varphi \frac{\sin(\varphi - \alpha') + \sin\,\alpha'}{|\overline{w}(\alpha;\varphi)|}\, e^{i\overline{\vartheta}(\alpha;\varphi)}\, d\varphi \qquad (36a)$$

The separation of real and imaginary parts gives for the profile contour the parameters

$$\left.\begin{aligned} \xi(\varphi) &= \xi_o - 2a\, W \int_0^\varphi \frac{\sin(\varphi-\alpha')+\sin\,\alpha'}{|\overline{w}(\alpha;\varphi)|} \cos\,\overline{\vartheta}(\alpha;\varphi)\, d\varphi \\ \eta(\varphi) &= \eta_o - 2a\, W \int_0^\varphi \frac{\sin(\varphi-\alpha')+\sin\,\alpha'}{|\overline{w}(\alpha;\varphi)|} \sin\,\overline{\vartheta}(\alpha;\varphi)\, d\varphi \end{aligned}\right\} \qquad (36b)$$

The constant a is a scale factor for the profile. These integrals can be graphically evaluated.

As regards the question of profile for a given velocity distribution $|w(\alpha;s)|$ along the profile contour, raised previously, the following can now be stated. If it has previously been proved that $|w(\alpha;s)|$ belongs to a potential flow about a closed profile, then $|\overline{w}(\alpha;\varphi)|$ can be ascertained from equation (39), and the cited method gives the profile contour. Hence the problem of determining the profile for a certain velocity distribution is, as originally discussed by Weinig, explicitly solvable and consequently essentially simpler than the reversed problem. Nevertheless, the function $|w(\alpha;s)|$ is no suitable starting point for profile calculations - for the choice of $|\overline{w}(\alpha;\varphi)|$ as the choice of $|w(\alpha;s)|$ is restricted by the final condition (46a). These equations are, as follows by substitution of $\varphi(s)$, a very complex requirement for function $|w(\alpha;s)|$ which is not summarily met, and would involve a return to $|\overline{w}(\alpha;\varphi)|$, which would bring us back to the previously described method.

Lastly, the problem of computing the velocity distribution for a prescribed airfoil can be treated by the $\ln \overline{w}$ method. The deciding condition is equation (43). But, since with given profile (and forward stagnation point) the function $\vartheta(\alpha;s)$ is exactly known, while $\overline{\vartheta}(\alpha;\varphi)$ is not known, the problem leads progressively to the executed approximation of the function $\overline{\vartheta}(\alpha;\varphi)$ which, however, means that the principal problem of the airfoil theory leads here also to an infinite method.

The first formula for $\overline{\vartheta}(\alpha;\varphi)$ can be obtained by the following process:

Select the junction line between trailing edge and airfoil nose as profile axis. Draw a straight line from point $P(s)$ on the contour to this line. If the base point P' has the coordinate x, and t is the profile chord, put

$$\varphi(s) = \operatorname{arc} \cos \frac{x(s)}{t/2}$$

and coordinate the upper surface of the profile to the interval $0 < \varphi < \pi$, and the lower surface to interval $\pi < \varphi < 2\pi$ (fig. 5).

Now, $\vartheta(\alpha;s)$ is replotted on the basis of this approximately valid coordination of s and φ; let, thereby, the function $\underline{\underline{\vartheta}}(\alpha;\varphi)$ originate. Since $s(\varphi)$ is not accurately known, this $\underline{\underline{\vartheta}}(\alpha;\varphi)$ generally does not comply with the final condition (46b), and therefore is not as yet practical for the calculation, according to equation (43). For this reason, a correction of the form $\Delta\overline{\vartheta}(\alpha;\varphi) = \Delta a \sin \varphi - \Delta b \cos \varphi$ with the free constants Δa and Δb, is added to $\underline{\underline{\vartheta}}(\alpha;\varphi)$ in such a way that

$$\bar{\vartheta}_1(\alpha;\varphi) = \bar{\bar{\vartheta}}(\alpha;\varphi) + \Delta\,\bar{\bar{\vartheta}}(\alpha;\varphi)$$

complies with equation (46b). With this $\bar{\vartheta}_1(\alpha;\varphi)$ a first approximation $|\bar{w}_1(\alpha;\varphi)|$ of the velocity distribution is then attempted by means of equation (43).

This correction in respect to (46b) results in the suppression of the cos term in the Fourier series for $\bar{\bar{\vartheta}}(\alpha;\varphi) + \frac{\gamma}{\pi}\,\frac{\pi - \gamma}{2} + \frac{\varphi_1 - \varphi}{2} \mp \frac{\pi}{2}$ and the addition of the coefficient $-1 + \frac{\gamma}{\pi}$ to the sine term. Then the series for the integrand of (43) reads:

$$\bar{\vartheta}_1(\alpha;\varphi) + \frac{\gamma}{\pi}\,\frac{\pi - \varphi}{2} + \frac{\varphi_1 - \varphi}{2} \mp \frac{\pi}{2} = -\left(1 - \frac{\gamma}{\pi}\right)\sin\varphi +$$

$$+ \sum_{n=2}^{\infty}(a_n \sin n\varphi - b_n \cos n\varphi)$$

and the first approximation $|\bar{w}_1(\alpha;\varphi)|$ is, according to equation (43):

$$|\bar{w}_1(\alpha;\varphi)| = W\left(2\sin\frac{\varphi}{2}\right)^{\frac{\gamma}{\pi}} \times$$

$$\times \left(2\sin\frac{\varphi_1 - \varphi}{2}\right)e^{-\left(1 - \frac{\gamma}{\pi}\right)\cos\varphi + \sum_{n=2}^{\infty}(a_n \cos n\varphi + b_n \sin n\varphi)}$$

With this function $|\bar{w}_1(\alpha;\varphi)|$ the connection between s and φ can then be newly computed, according to equation (38), and a new approximation $\bar{\vartheta}_2(\alpha;\varphi)$ obtained from $\vartheta(\alpha;s)$. The method can be continued at will. Its convergence is not easily followed mathematically. But there is good reason to surmise that results are quickly achieved, and also confirmed by Weinig.

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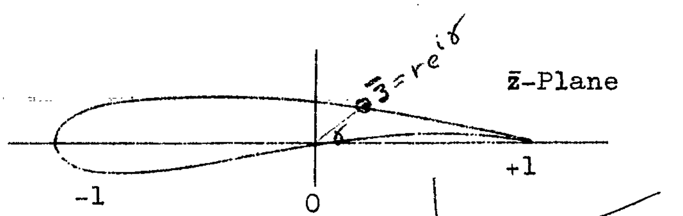

Figure 1.

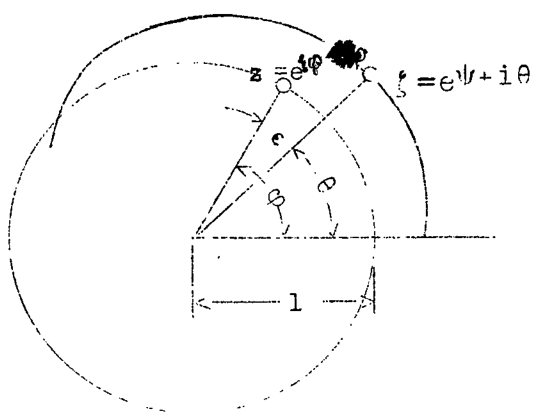

Figure 2.

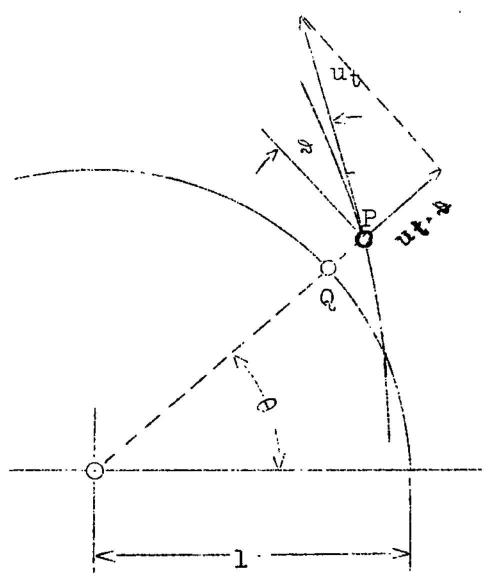

Figure 3.

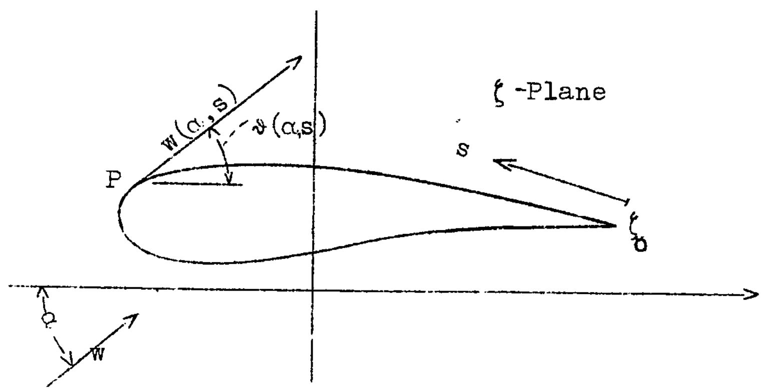

Figure 4

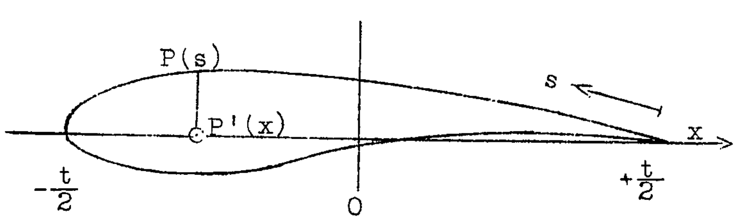

Figure 5.