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GENERAL INSTABILITY CRITERION OF LAMINAR VELOCITY DISTRIBUTIONS

By W. Tollmien

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GENERAL INSTABILITY CRITERION OF LAMINAR VELOCITY DISTRIBUTIONS*

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SUMMARY

The present paper describes the results of a stability investigation on symmetrical velocity profiles in a channel and of boundary-layer profiles. The limitation to these two most important types of profiles was, however, not due to any limitation of our mathematical method. The effect of the friction was assumed to be vanishing and did not occur in the stability consideration so far as it had not been resorted to for preparatory asymptotic considerations. Proceeding on very general premises as regards the form of the velocity distribution, a proof was deduced of the elementary theorem that velocity profiles with inflection points are unstable. Aside from this comprehensive theorem, there were obtained formulas of general validity for the investigated types of profiles which disclosed certain information relating to wave length, wave velocity, and amplification of the dangerous disturbances.

Dynamically, the profiles with inflection points are identified, according to boundary-layer theory, by the existence of decelerating pressure gradients in such flows. Profiles, to which this particular stability investigation is inapplicable, may be encountered in slightly divergent channels or on the hydrodynamical rear surface (behind the pressure minimum) of cylindrical bodies, particularly if the body is short. They may also occur through superposition of the velocity distribution without inflection point, with a quasi-stationary vortex pattern, which can happen in the first stages of the formation of turbulence. The proved effect is, undoubtedly, of fundamental importance for the creation of turbulence.

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*"Ein allgemeines Kriterium der Instabilität laminarer Geschwindigkeitsverteilungen." Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen (Mathematik), vol. I, no. 5, 1935, pp. 79-114.

INTRODUCTION

The transition of laminar flow, with its clean, stratified layers of flow tubes to strongly intermingled, irregular turbulent flow, constitutes one of the most pressing problems of modern hydrodynamics. It is certain that this fundamental change in type of motion of the fluid is attributable to an instability in the laminar flow, for laminar flows of themselves would always constitute possible solutions of the hydrodynamic equations. The mathematical derivation of the expected instability of laminar flow is, as is known, beset with almost insurmountable difficulties. While it is true that the recent advances made on the subject are very promising, it is equally true that almost every stability investigation made heretofore, relates to special laminar velocity distributions. There seems to be a palpable lack of general theorems in this field which afford ready classification of laminar velocity profiles according to their stability.

The general instability criterion established hereinafter, discloses a frequent and powerful mechanism of formation of turbulence.

The analysis is to proceed on the basis of two-dimensional velocity profiles, the velocities in this fundamental flow themselves to be very simply distributed; that is, to be largely dependent only on the coordinate transverse to the direction of flow. On this flow there are then superposed disturbances which, according to the method of small oscillations, are considered as small waves advancing in the direction of flow. The disturbances themselves shall also be two-dimensional*, while the effect of friction on the disturbances will be disregarded to a certain degree. The problem posed is, When can disturbances with increasing amplitude be superposed on the basic flow?

This roughly outlined problem is by no means new. Back in 1880, Lord Rayleigh published an investigation

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*According to a note by H. B. Squire (Proceedings of the Royal Society of London A, 142, p. 621, 1933), the investigation of the stability of flows depending only on one coordinate may, if the disturbance is three-dimensional, be simply reduced to the case of two-dimensional disturbances.

which above everything else gave a readily obtainable necessary condition for instability. Subsequently, he again and again reverted to this same problem (reference 1), and even other investigators, appreciating the importance of his findings, applied themselves to the problem, but without much success beyond the initial advance. No sufficient condition for instability has been set forth thus far.

I. THE DISTURBANCE EQUATION

Let U represent the speed of the laminar fundamental flow parallel to coordinate x and dependent only on the coordinate y at right angles to x. Being assumed two-dimensional, the disturbance can be derived from a stream function $\psi\ (x,y,t)$, t being the time. The velocity component of the disturbance in x direction is $\partial\psi/\partial y$, in y direction, $-\frac{\partial\psi}{\partial x}$. Thus the equation for the stream function of the disturbance inclusive of the friction terms - considering that U itself complies with the hydrodynamic equations and with limitation to linear terms in ψ, is:

$$\frac{\partial\Delta\psi}{\partial t} + U\frac{\partial\Delta\psi}{\partial x} - \frac{d^2U}{dy^2}\frac{\partial\psi}{\partial x} = \nu\,\Delta\,\Delta\,\psi \qquad (1)$$

where Δ is the Laplace operator $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$, and ν is the kinematic viscosity. The equation is the vortex equation for the disturbance, which follows by elimination of the pressure from the hydrodynamic equations. $\Delta\psi$ is the negative vortex density of the disturbance, because $\Delta\psi = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$; $\frac{\partial\Delta\psi}{\partial t}$ gives the local time rate of change, $U\frac{\partial\Delta\psi}{\partial x}$ the transport of the disturbance turbulence by means of the fundamental motion, $-\frac{d^2U}{dy^2}\frac{\partial\psi}{\partial x}$ the vortex transport in the fundamental flow caused by the disturbance; $\nu\,\Delta\,\Delta\,\psi$ corresponds to the diffusion of the turbulence by friction. The convective change in vortex density through the disturbance motion itself is neglected as nonlinear in view of the assumed smallness of the disturbances.

Owing to the linearity of (1), the formation of φ from partial oscillations is permissible, which may be expressed as

$$\varphi(y)\ e^{i(\alpha x-\beta t)} \tag{2}$$

that is, as waves traveling in direction x. In this equation the complex method of writing is employed; for physical application, the real part of the above expression is to be selected. α is a real constant and equal to $2\pi/\lambda$, where λ is the wave length of the partial disturbance; β may be complex equivalent to $\beta_r + i\beta_i$. The real part β_r gives the cycle frequency of the disturbance, and the imaginary part β_i the logarithmic increment, becoming positive for amplified, and negative, for damped oscillations. The real part c_r of the quantity $c = \frac{\beta}{\alpha}$ is the wave velocity. Writing (2) in (1) gives the disturbance equation for the considered partial oscillation:

$$(U-c)\ (\varphi''-\alpha^2\ \varphi) - U''\ \varphi = -\ \frac{i\nu}{\alpha}\ (\varphi'''' - 2\alpha^2\ \varphi''+\alpha^4\ \varphi) \tag{3}$$

or, in nondimensional form (by measuring velocities in terms of a characteristic velocity of the basic profile, say, the maximum U_{max}, the length in terms of a characteristic width b of the profile), as

$$(U-c)\ (\varphi''-\alpha^2\ \varphi) - U''\ \varphi = -\ \frac{i\nu}{\alpha R}\ (\varphi'''' - 2\alpha^2\ \varphi''+\alpha^4\ \varphi) \tag{4}$$

Here we took the liberty of employing the same notation for the nondimensional quantities U, c, α, etc. as before. R is the Reynolds Number $\frac{U_{max}\ b}{\nu}$.

Now suppose the kinematic viscosity is very small or, in more general terms, that the Reynolds Number R is very large. Then the experiment is suggested of suppressing the friction terms altogether to obtain an insight into the behavior of the disturbance through a discussion of the frictionless disturbance equation

$$(U - c)\ (\varphi'' - \alpha^2\ \varphi) - U''\ \varphi = 0 \tag{5}$$

and this is what Rayleigh actually did. He stepped from (4) to (5) without further discussion which, however, is naturally necessary.

II. ANALYTICAL PROPERTIES OF THE INTEGRAL OF THE DISTURBANCE EQUATION AND EFFECT OF INTERNAL FRICTION

Starting with the analytical behavior of the solutions of (5), it is necessary to refer to a previous report by the author (reference 2). We assume c to be purely real - that is, consider neutral oscillations, neither amplified nor damped. Now it happens, as we shall see later, that at one point within the fluid the thereexisting fundamental velocity U becomes equal to the wave velocity c of the considered partial oscillation. In other words, at this point a fluid particle always oscillates in the same disturbance phase. This point, called the critical point, is designated with subscript o. Now the origin of the y coordinate is placed at this critical point, and the direction of y is so defined that in the vicinity $U - c > 0$ for $y > 0$; that is, so that U'_o which is not to vanish, becomes positive. U is to be capable of expansion in power series around the critical point to:

$$U = c + U'_o\, y + \frac{U''_o}{2}\, y^2 \tag{6}$$

The critical point $U_o = c$ represents a singular point of (5). Then two linear, independent solutions of (5) in the vicinity of the critical point can be represented in the following manner: With $P_1(y)$ and $P_2(y)$ as power series in y; A and a as constants determined from (5), we have a fundamental system of solutions through

$$\varphi_1 = y\, P_1(y) = y + \frac{U''_o}{2U'_o}\, y^2 + \ldots \tag{7}$$

$$\varphi_2 = P_2(y) + A\, \varphi_1 \log y = 1 + ay^2 + \ldots + \frac{U''_o}{U'_o}\, y \log y \ldots \tag{8}$$

Hereby it is to be established once for all that $\log y$ for positive real y is purely real. When U'' and likewise φ, does not disappear in the critical point, i.e., the transverse component of the disturbance, there results a logarithmically infinite (disturbance) velocity in the x-direction, which is contrary to the smallness of the disturbance velocity assumed in the method of small oscillations. Regardless of the magnitude of the Reynolds

Number, the friction must be taken into account at the critical point. The effect of the friction occurring in a small strip (of the order of magnitude of $(\alpha R U'_o)^{-1/3}$) consists, first, in a flattening of the x component of the disturbance at the critical point - naturally it then remains finite - and secondly, in the appearance of a phase discontinuity in the disturbances. Of the possible branches of the log in (8) which would be mathematically available for the continuation of the solution toward negative y (naturally outside of the minute friction layer), the writer has, on the basis of previous calculations (reference 2), chosen one which by means of the friction, is physically possible, namely, the analytical continuation of the logarithm by way of positive y through the lower complex semiplane to negative y. When φ is represented by

$$1 \ldots + \frac{U''_o}{U'_o} \varphi_1 \log y \tag{9a}$$

becomes

$$1 \ldots + \frac{U''_o}{U'_o} \varphi_1 (\log |y| - i\pi) \tag{9b}$$

when y is negative.

This transition substitution, however, signifies a phase discontinuity because, if the real part of $\varphi(y)\, e^{i(\alpha x - \beta t)}$ is selected for representing the disturbances, the x component of the disturbance becomes

$$\left(\frac{U''_o}{U'_o} \log y \ldots\right) \cos (\alpha x - \beta t) \tag{10a}$$

for positive y in the vicinity of the critical point, whereas

$$\left(\frac{U''_o}{U'_o} \log |y| \ldots\right) \cos (\alpha x - \beta t) + \frac{U''_o}{U'_o} \pi \sin (\alpha x - \beta t) \tag{10b}$$

for negative y. The result is a phase discontinuity which does not disappear even for the limiting case $\alpha R = \infty$.

In the present case we assumed c purely real, and only a very small amplification $(\beta_i > 0)$, that is, a

small imaginary part in $c = \frac{\beta}{\alpha} = c_r + i\, c_i$. The zero point of y is placed at the point $U = c_r$, in accordance with the previous definition. Attributing complex values to y and defining U from the power series (6), we again seek the singular point of (5) where the analytically continued $U = c = c_r + i\, c_i$. This critical point $U = c$ thus lies in the complex y plane slightly above the real axis (by $\frac{i\, c_i}{U'_o}$), so that the real axis then has no longer a singular point of (5). Then at a sufficiently large Reynolds Number $\left(\frac{c_i}{U'_o} \gg (\alpha\, R\, U'_o)^{-1/3}\right)$, no friction effect on the real axis needs to be considered. Now we introduce an η coordinate shifted with respect to y along the imaginary axis by placing the origin of η in the critical point $U = c_r + i\, c_i$ which, for sufficiently small c_i gives

$$y = \eta + \frac{i c_i}{U'_o} \tag{11}$$

(See fig. 1.)

Denoting the value of U' and U'' at $U = c$ with U'_c and U''_c, the solutions φ_1 and φ_2 in this case are written as before:

$$\varphi_1 = \eta + \frac{U''_c}{2U'_c}\eta^2 + \ldots \tag{12a}$$

$$\varphi_2 = 1 + \ldots + \frac{U''_c}{U'_c}\varphi_1(\eta) \log \eta \tag{12b}$$

We only use the values of the singular solution along the real axis y. For positive y, which is assumed to be large compared to the small quantity c_i/U'_c, we have:

$$\varphi_2 = 1 + \ldots + \frac{U''_c}{U'_c}\varphi_1(y) \log y \tag{13a}$$

while for negative y, it is:

$$\varphi_2 = 1 + \ldots + \frac{U''_c}{U'_c}\varphi_1(y) (\log |y| - i\pi) \tag{13b}$$

considering that the path, along which the logarithm must be analytically continued, lies below the singular point; that is, the same transitional substitution as for purely real c, which was obtained through a boundary transition from small friction.* This is of particular interest to us because the neutral oscillations are primarily considered as a limiting case of the amplified oscillations.

For small damping, on the other hand, the transition is from

$$\varphi_2 = 1 + \ldots + \frac{U''_c}{U'_c} \varphi_1(y) \log y \tag{14a}$$

for y positive to

$$\varphi_2 = 1 + \ldots + \frac{U''_c}{U'_c} \varphi_1(y) (\log |y| + i\pi) \tag{14b}$$

for y negative.

Following the exclusion of the vicinity of the critical point through special considerations, we have in φ_1 and φ_2, the solutions of the frictionless disturbance equation, particular solutions of the complete disturbance equation for very large αR.

The only remaining difficulty lies in the lowering of the order of the differential equation by 2 during the change from the complete to the frictionless disturbance equation. The result was, of course, only two particular solutions and consequently noncompliance to all boundary conditions. If the fluid passes between two walls, for example, the tangential and normal component of the disturbance velocity must disappear at both walls. Limited to the frictionless disturbance equation (inclusive of the necessary correction at the critical point), the stipulation is, of course, confined to the disappearance of the normal component only. But, according to the findings of various investigators (reference 3) the drop of the tangential component of the disturbance to zero on the wall at high Reynolds Numbers actually takes place in an ex-

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*Even in the case that c_i/U'_o, while positive, is no longer large compared to $(\alpha R U'_o)^{-1/3}$, the boundary transition to very large R affords the same result when the writer's analysis (reference 2, p. 27) is repeated for this case.

tremely thin layer with the result that the effect of the wall-friction becomes consistently smaller.

Besides this pair of asymptotic solutions deduced from the frictionless disturbance equation, there exist rapidly decreasing "boundary-layer-like solutions," the inclusion of which in the frictionless solutions is necessary to insure complete compliance with the boundary conditions. We shall give only the most elementary example of such a boundary-layer-like solution. Designating the distance from the wall toward the inside of the fluid y_w and considering the particular case of c having positive real and imaginary part and being sufficiently great, $\left(\left|\frac{c}{U'_w}\right| \gg |\alpha R U'_w|^{-1/3}, \quad U'_w = \text{value of } U' \text{ at wall } y_w = 0\right)$, the boundary-layer-like solution, designated φ_3 then becomes:

$$\varphi_3 = e^{\frac{3\pi i}{4}\sqrt{\alpha R c}\, y_w} \tag{15}$$

The addition of this inwardly rapidly decreasing solution necessary for compliance with the boundary condition for the tangential component has but a minor effect on the distribution of the amplitude of the disturbance and the parameters α and c, as is readily proved. By suppressing the explicit occurrence of the internal friction in first asymptotic approximation for very large Reynolds Numbers, the determination of a critical R is foregone in favor of the desired generalization of the results. On the other hand, our investigation affords some particular information about the type of dangerous disturbances and constitutes for this reason a useful preliminary for the disturbance problem with explicitly occurring internal friction.

III. FORMULATION OF CHARACTERISTIC VALUE PROBLEM FOR SYMMETRICAL VELOCITY PROFILES

Following these preparations, we can finally formulate our problem. For the sake of simplicity, we first assume two-dimensional flow through channels as the fundamental flow. The particular velocity profiles shall, as stated before, change very little in the direction of motion; that is, $U = U(y)$, and the walls of the channel

shall be approximately parallel. Furthermore, the velocity profiles shall be symmetrical with respect to the channel axis. At the wall, of course, $U = 0$. Otherwise, the form of the profiles is very little restricted. Profiles with an inflection point in each half, such as may occur in slightly divergent channels (fig. 2), are also included, but not profiles for which U itself changes sign. This includes profiles with return flow as well as separation profiles as a limiting case $\left(\frac{dU}{dy} = 0 \text{ at the wall}\right)$. At the inflection point of the included profiles (subscript s), let $U'_s > 0$, $U''_s < 0$, so that U'' is positive between the wall and point of inflection, and negative between the inflection point and the center.

For these fundamental velocities, the following boundary-value problem is to be solved:

$$\varphi'' - \alpha^2 \varphi - \frac{U''}{U - c} \varphi = 0 \tag{16}$$

with the boundary conditions that the normal component of the disturbance, that is, φ, disappears at both walls. The transitional substitution suffered by φ at a singular point $U = c$ in the case of neutral oscillations, has been previously established by a transition to the limit of vanishing friction or disappearing amplification. The characteristic-value problem here is rather unusual. The real parameter α is assumed predetermined; the quest is for c which occurs in nonlinear manner in the differential equation (16). Problems of this kind have been so little explored mathematically, that the solution in question cannot be based upon general existence theorems. The main task will be to establish necessary, and at the same time sufficient, conditions for $U(y)$, in order that complex characteristic values c may exist, for it is readily seen that, with a complex c as characteristic value the conjugate complex c together with the conjugate complex characteristic function also represents a solution. Our interest centers on the solutions with amplification, for which c has a positive imaginary part.

Since U is symmetrical, φ may be readily separated into a symmetrical and an antisymmetrical part which, individually, must satisfy the disturbance equation. As a result, we need consider only half of the y-zone, because then we have either $\varphi' = 0$ or $\varphi = 0$ in the channel center in addition to the boundary condition $\varphi = 0$ at one

wall, depending on whether the distribution of the disturbance amplitude φ about the central axis of the channel is symmetrical or antisymmetrical. This simplifies matters quite considerably because the problem narrows down to one singular point in a semichannel.

The particularly important problem of stability of boundary-layer profiles is treated in a subsequent section.

IV. RAYLEIGH'S EQUATIONS

His principal result, summed up briefly, is a necessary condition for the occurrence of amplified oscillations.

Writing the differential term on the left-hand side of (16)

$$\varphi'' - \alpha^2 \varphi - \frac{U''}{U - c} \varphi = L(\varphi) \tag{17}$$

we form the integral term

$$\int_0^{2b} [\overline{\varphi}\, L(\varphi) - \varphi\, \overline{L(\varphi)}]\, dy_w \tag{18}$$

between the boundaries of the y-zone, the two channel walls. 2b = channel breadth. Conjugate complex quantities are overlined. Therefore,

$$\overline{L(\varphi)} = \overline{\varphi}'' - \alpha^2 \overline{\varphi} - \frac{U''}{U - \overline{c}} \overline{\varphi} \tag{17a}$$

whence φ and $\overline{\varphi}$ shall vanish, conforming to the boundary conditions at the zone boundaries. The problem is to find when a nonvanishing imaginary part of c is possible. On this premise there is no singular point on the real y axis so that the integral summarily gives:

$$- 2i\, c_i \int_0^{2b} \frac{U''}{|U - c|^2} \varphi\, \overline{\varphi}\, dy_w \tag{19}$$

This term then must vanish for solutions of the differential equation besides the boundary conditions. If U'' does not change sign, this is, however, not possible for a c_i other than zero, so that amplified oscillations are impossible except for profiles with an inflection point. The condition, however, being only necessary, Rayleigh's theo-

rem merely states that profiles without an inflection point reveal no instability in the sense used here.* Any statement as to the behavior of profiles with an inflection point is as yet impossible.

Another fact brought out by Lord Rayleigh is that, by the eventually neutral oscillations ($c_i = 0$, $c = c_r$), the wave velocity c_r of the partial oscillation must be equal to the basic velocity at one point, thus ever assuring the existence of a point with $U - c = 0$ within the fluid. We adduce a proof for this fact, which is somewhat more simple than Rayleigh's. (See reference 1, vol. VI, 1913, p. 199; also vol. I.)

Let us assume that $c > U_{max}$ (maximum value of U), so that no singular point exists within the fluid. We compare

$$\varphi'' = \left(\alpha^2 + \frac{U''}{U - c}\right)\varphi \tag{16}$$

with

$$f'' = \frac{U''}{U - c} f \tag{20}$$

that is, compare two solutions of (16) and (20) which disappear at the wall ($y_w = 0$). In addition, let $\varphi' = f'$ for $y_w = 0$. Multiplication of (16) by f and (20) by φ followed by subtraction of the results and integration starting at one wall gives:

$$f\,\varphi' - f'\,\varphi = \int_0^{y_w} \alpha^2 f\,\varphi\,dy_w \tag{21}$$

So long as φ and f are positive, it follows:

$$\frac{\varphi'}{\varphi} > \frac{f'}{f} \quad \text{and} \quad \varphi > f \tag{22}$$

after which, the solution of (20) becomes:

$$f = (c - U) \int_0^{y_w} \frac{dy_w}{(U - c)^2} \tag{23}$$

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*Many profiles without inflection points may, by applying a correction explicitly containing the internal friction - that is, in the second asymptotic approximation for very large Reynolds Numbers, become unstable, as proved by calculation on special velocity distributions. The amplification as a result of the small friction correction is, of course, of lower order of magnitude than that found here.

f is therefore always positive and greater than zero on the other wall; consequently, φ cannot disappear on the other wall, according to (22) or, in other words, it is necessary that $c < U_{max}$.*

By the same argument, c must be greater than the minimum value of the fundamental velocity (zero hereafter). The proof is the same, word for word, if we take

$$(U - c) \int_0^{y_w} \frac{dy_w}{(U - c)^2}$$

for f. It should be noted that for the proof of Rayleigh's two formulas, the assumption of symmetry of U was not utilized.

Rayleigh, being unable to advance beyond these two necessary conditions for amplified or neutral oscillations, decided to approximate the steadily curved velocity profile by a profile consisting of straight pieces (polygonal profile), in order to side-step the mathematical difficulties. The transitional conditions to be fulfilled by φ at the bends of the profile were easily established. They correspond to the condition of equal normal component of the velocities and equal pressure on both sides of the bend. So when Rayleigh** divided the profile into three strips of constant vortex density (dU/dy = constant), for example, he arrived at an ordinary quadratic equation for c and was able to show that in case of re-entrant corners, amplified oscillations are always present. But the rough approximation of the velocity profile through a polygon suppresses essential parts of the process. The profiles treated hereinafter are always curved.

V. EXISTENCE OF NEUTRAL CHARACTERISTIC OSCILLATIONS

We begin with the neutral oscillations which are possible in the case of a symmetrical channel profile, and which, according to Rayleigh, insure a singular point $c = 0$ with-

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*That $c < U_{max}$ for the special case of $U'' < 0$ (profiles without an inflection point), is readily seen by forming the integral term

$$\int_0^{2b} [\bar{\varphi}\, L(\varphi) + \varphi\, \overline{L(\varphi)}]\, dy_w.$$

**See reference 1 (Vol. III, p. 17). O. Tietjen's discussion on boundary-layer profiles is similar.

in the zone. If φ is other than zero in this critical point - say, equal to φ_0 - the previously cited transitional substitution (9) and (13), upon change from positive to negative y, that is, on passage from channel center toward the wall, suddenly creates at φ' an additive component $-\frac{U''_0}{U'_0} i \pi \varphi_0$, whereas φ remains continuous. It will be shown that such an abrupt increase is impossible with a neutral characteristic oscillation. To this end, we consider the expression:

$$\varphi_i \frac{d\varphi_r}{dy} - \varphi_r \frac{d\varphi_i}{dy} \tag{24}$$

Incidentally we add that (24) is proportional to the time average of the product of both disturbance velocities; that is, it is intimately connected with the momentum transport of the disturbance. Since φ in the case of neutral oscillations satisfies a differential equation with real coefficients, both φ_r and φ_i must individually satisfy the differential equation. Consequently, (24) is constant so long as no singular points occur in the differential equation. As a result of the cited sudden growth of φ' on passing through the singular point $U = c$, (24) also increases suddenly upon transition from greater to smaller y through the singular point, and by an amount

$$|\varphi_0|^2 \frac{U''_0}{U'_0} \pi$$

as is readily seen from a simple calculation of the behavior of φ and φ' at the singular point. Since, according to the boundary conditions, the term (24) vanishes at both the center and the wall, there is no possibility for such a sudden increase. Consequently, either φ must be equal to 0 at the singular point for the neutral characteristic oscillations, i.e., only the regular solution φ_1 supplies the neutral characteristic oscillation, or else the singular point must drop out altogether, whence $U''_0 = 0$; that is, the phase velocity must equal the fundamental velocity U_s at the inflection point, provided, of course, that the profile has one.

Starting with the possibility of $\varphi = 0$ at the critical point, we compare

$$\varphi'' = \left(\alpha^2 + \frac{U''}{U - c}\right)\varphi \tag{16}$$

with

$$f'' = \left(\frac{U''}{U - c}\right) f \tag{20}$$

and compare those solutions φ_1 of (16) with the solutions f_1 of (20), both of which disappear at the critical point and whose power expansion starts at the critical point with y. Then,

$$f_1 = \frac{U - c}{U'_o} \tag{25}$$

Multiplying (16) by f_1, and (20) by φ_1 followed by integrating from the critical point (subscript o), gives

$$f'_1 \varphi_1 - f_1 \varphi'_1 = -\int_o^y \alpha^2 f_1 \varphi_1 \, dy \tag{26}$$

f_1 and φ_1 are, in accord with the assumption for negative y (from critical point toward the wall) negative at first, and the term on the right-hand side, positive. So long as f_1 and φ_1 are negative, therefore,

$$\frac{f'_1}{f_1} \quad \frac{\varphi'_1}{\varphi_1} \quad \text{and} \quad \varphi_1 < f_1 \tag{27}$$

Since f_1 is still negative at the wall $\left(= -\frac{c}{U'_o}\right)$ rather than passing through zero, φ_1 can so much less disappear at the wall unless $c = 0$. Then, of course, both f_1 and φ_1 disappear at the wall at which the critical point then falls.

Applying the above argument in the case $c = 0$ to the zone between wall $(y_w = 0)$ and center $(y_w = b)$, gives for this zone,

$$\varphi'_1 f_1 - \varphi_1 f'_1 = \int_o^{y_w} \alpha^2 f_1 \varphi_1 \, dy_w > 0 \tag{28}$$

$$\varphi_1 > f_1 = \frac{U}{U'_w} \tag{29}$$

so long as $f_1 = \frac{U}{U'_o} = \frac{U}{U'_w}$ and φ_1 are positive. Since

in the center $f'_1 = \frac{U'}{U'_w} = 0$, it follows from (28) $\frac{U}{U'_w} \varphi'_1 > 0$, that is, $\varphi'_1 > 0$ in the center. Accordingly, no φ'_1 corresponding to $\alpha^2 > 0$ can satisfy the boundary condition at the center. This condition thus compels $\alpha = 0$ at the center which, in conjunction with the previous condition $c = 0$ leads to the regular solution $f_1 = \frac{U}{U'_w}$ or with other notation, to $\varphi = U$, which remains as the only one for a characteristic solution and actually is such a one, according to the properties of U.

As the only characteristic oscillation existing in the same manner for profiles with or without an inflection point, we obtained the abnormal oscillation with $\alpha = 0$, $c = 0$; that is, with infinite wave length and vanishing wave velocity and with the symmetrical amplitude distribution $\varphi = U$. This solution itself has been known for a long time; we prove here its singularity.

However, fundamental velocity profiles with an inflection point may also have a neutral characteristic oscillation for which $c = U_s$ (fundamental velocity at the inflection point). Mathematically, the question is as to the proof that the Sturm-Lionville formula

$$\varphi'' + \lambda \varphi - \frac{U''}{U - U_s} \varphi = 0$$

with the conditions $\varphi = 0$ for $y_w = 0$ and $\varphi' = 0$ for $y_w = b$, aside from the infinitely many positive characteristic λ values existing according to general theorems, has also a negative characteristic value $\lambda = -\alpha^2$ (fig. 3).

The proof proceeds from the solution $\varphi = U - U_s$ for $\alpha = 0$, which has disappearing tangent in the center. This boundary condition in the center is maintained for the solutions studied at $\alpha \neq 0$. The addition of $\alpha^2 > 0$ slows up the drop of these solutions from the center toward the wall, with the result that the nodal point $\varphi = 0$ of these solutions, which for $\alpha = 0$ lies at the inflection point of the fundamental velocity, finally approaches the wall; that is, we have the desired characteristic oscillation (fig. 4). The conclusion of the steady dependence of nodal point position on the parameter (α^2) is, as known, made

in proof of the oscillation theorem (reference 4). Since the parameter (here α^2) may have only positive values, it summarily follows that always one, and only one, such symmetrical disturbance flow is possible, because characteristic oscillations with several nodes $\varphi = 0$ can exist only for a negative parameter. The existence of the characteristic solution could, moreover, also be proved by repeated application of our previous deduction. We call this characteristic solution φ_s and put its parameters $\alpha = \alpha_s$, $c = U_s$.

Any doubt as to the possible occurrence of an antisymmetrical disturbance flow for $c = U_s$ with $\varphi = 0$ in the center, is refuted in a subsequent section.

Lord Rayleigh himself had pointed to another possible solution for $U'' = 0$. According to the frictionless disturbance equation (5), it might be conjectured that at such a point a solution $U = c$ existed; that is, a disturbance confined only to the critical layer $U = c$ itself. Considering the high values of the differential quotients of φ in such an oscillation, the friction must in any event be taken into consideration. Since L. Hopf (reference 3) has proved that these oscillations are damped, they are merely mentioned.

VI. EXISTENCE OF AMPLIFIED CHARACTERISTIC OSCILLATIONS

Their existence under certain conditions will be proved by considering solutions adjoining the previously deduced neutral characteristic solutions. Our method is vaguely reminiscent of the well-known perturbation theory of the characteristic values (reference 5) with, however, substantial modifications imposed by the prevalent singularities. Rather than proceeding, as customary, from the known to the hypothetical solution, we attempt to double back from the hypothetical (amplified) solution to the known (neutral) characteristic solution.

Let φ_n represent the neutral characteristic solution and c_n, α_n its parameters, the parameters c (complex) and α being assumed. One solution φ_I is to satisfy the disturbance equation with these parameters without, however, being a characteristic solution, whereas φ_I is to comply with a boundary condition, say, that on the wall. In this manner φ_I is defined up to one factor; c, α, and φ_I are subsequently appropriately determined as being adjacent to

c_n, α_n, and φ_n.

Subtracting the differential equation for φ_n:

$$(U - c_n)\ (\varphi''_n - \alpha^2{}_n\ \varphi_n) - U''\ \varphi_n = 0 \tag{30}$$

from that for φ_I:

$$(U - c)\ (\varphi''_I - \alpha^2\ \varphi_I) - U''\ \varphi_I = 0 \tag{31}$$

gives

$$\begin{aligned}(U - c) \left\{(\varphi_I - \varphi_n)'' - \alpha^2\ (\varphi_I - \varphi_n)\right\} - U''\ (\varphi_I - \varphi_n) \\ = (c - c_n)\varphi''_n + \left\{(U - c)\ \alpha^2 - (U - c_n)\ \alpha^2{}_n\right\} \varphi_n\end{aligned} \tag{32}$$

or

$$(\varphi_I - \varphi_n)'' - \alpha^2\ (\varphi_I - \varphi_n) - \frac{U''}{U - c}\ (\varphi_I - \varphi_n) = g \tag{33}$$

with

$$g = \frac{c - c_n}{U - c}\ \varphi''_n - \frac{U - c_n}{U - c}\ \alpha^2{}_n\ \varphi_n + \alpha^2\ \varphi_n \tag{34}$$

g contains only φ_n and the disturbance parameters. Having assumed c as complex, we have in reality no zero point of $U - c$, that is, no singular point in the differential equation for $\varphi_I - \varphi_n$. Be it noted that the coefficients of the homogeneous part $\varphi_I - \varphi_n$ of (33) correspond to those of the differential equation for φ_I.

Then with φ_{II} as a linear solution of

$$\varphi'' - \alpha^2\ \varphi - \frac{U''}{U - c}\ \varphi = 0 \tag{31}$$

independent of φ_I, we can write $\varphi_I - \varphi_n$ in form of

$$\varphi_I - \varphi_n = \varphi_I \int_0^{y_w} g\ \varphi_{II}\ dy_w - \varphi_{II} \int_0^{y_w} g\ \varphi_I\ dy_w + C\ \varphi_I \tag{35}$$

with integration constant C in which, by normalizing

$$\varphi_I\ \varphi'_{II} - \varphi'_I\ \varphi_{II} = -1$$

The integral is to extend from the wall as subscript w

indicates. The problem then reduces to finding the necessary and sufficient condition which satisfies the boundary condition at the center through φ_I, that is, $\varphi'_I = 0$ or else $\varphi'_I - \varphi'_n = 0$. With b as the distance of the wall from the channel center, we have for $y_w = b$, according to (35):

$$\varphi'_I - \varphi'_n = \varphi'_I \int_0^b g\,\varphi_{II}\,dy_w - \varphi'_{II} \int_0^b g\,\varphi_I\,dy_w + C\,\varphi'_I \tag{36}$$

To deduce a necessary condition, it is assumed that $\varphi'_I = 0$ in the center, so that φ'_{II} becomes other than zero in view of the presumed linear independence of φ_I. Then, of course, it follows that

$$\int_0^b g\,\varphi_I\,dy_w = 0 \tag{37}$$

in accordance to (36). This condition, sufficient since for $y_w = b$, according to (36)

$$\varphi'_I\left[1 - C - \int_0^b g\,\varphi_{II}\,dy_w\right] = 0 \tag{38}$$

as well as

$$\varphi_I\left[1 - C - \int_0^b g\,\varphi_{II}\,dy_w\right] = \varphi_n \tag{39}$$

according to (35).

But φ_n is not zero in the center because there $\varphi'_n = 0$, whence the term in the brackets must likewise be other than zero. Consequently, $\varphi'_I = 0$ in (38).

Now we prove the existence of solutions with amplification for profiles with inflection points by means of (37), for it permits the calculation of c (at least near c_n) as function of α (near α_n).

We first consider the vicinity of $\varphi_n = U$, $c_n = \alpha_n = 0$. The solution φ_I adjacent to this neutral characteristic solution of the disturbance equation with small parameters c (assumed complex and with positive imaginary part) and α (real) is established through $\varphi_I = 0$ at the wall and the further condition that the derivative of φ_I at the wall equals that of φ_n, that is, $\varphi'_I = U'_w$ for $y_w = 0$.

In this case

$$g = \frac{c}{U - c} U'' + \alpha^2 U \tag{40}$$

and

$$g \varphi_I = c \varphi''_I + \alpha^2 (U - c) \varphi_I \tag{41}$$

by having recourse to the differential equation for φ_I.

Accordingly, the condition

$$\int_0^b [c \varphi''_I + \alpha^2 (U - c) \varphi_I] \, dy_w = 0 \tag{42}$$

must be met for a characteristic solution, according to (37).*

The evaluation of (42) proceeds on the derivations of several approximate representations φ_I. It does not suffice to approach φ_I through φ_n, because the operation must be effected in the dangerous proximity of a singularity. We again expand according to η from the singular point $U = c = c_r + i c_i$, so that η is approximately equal to $y - \frac{i c_i}{U'_c}$. As before, subscript c denotes the values of U', etc. at the singular point; c_r is chosen positive. The expansion is then made for a range extending on the one hand, from the singular point to the wall, where η approximates $-\frac{c}{U'_c} - \frac{U''_c c^2}{2U'^3_c}$, and on the other hand, to a small positive $y = \epsilon$; the latter (ϵ) to be chosen so as to make ϵ^2 small relative to $\frac{c_r}{|U'_c|}$, whereas ϵ is large relative to $\frac{c_i}{|U'_c|}$. Here as well as later, it is simply assumed that c_i proves of smaller order of magnitude** than c_r. Accordingly, we assume ϵ as being about equal to $\left|\frac{c_r}{U'_c}\right|^{3/4}$; φ'_I is to equal U'_w at the wall. The expan-

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*This condition (42) can also be proved direct with $\varphi'_I = 0$ in the center by substituting $(U-c)\varphi''_I - U''\varphi_I$ for $\alpha^2(U-c)\varphi_I$ according to (31).

**This relationship between c_r and c_i is already indicated by the fact that the roughest approximation for φ_I, namely, U, gives an expression for c_r according to the integral equation (42), but not for c_i.

sion, starting from the singular point, gives

$$U'_w = U'_c - \frac{U''_c c}{U'_c} - \frac{U''_c c^2}{2U'^3_c} c^2 + \frac{U'''_c}{2U'^2_c} c^2$$

The development of φ_I with two unknown coefficients A and B is:

$$\varphi_I = A\left(\eta + \frac{U''_c}{2U'_c}\eta^2 \ldots\right) + B\left(1 \ldots + \frac{U''_c}{U'_c}\eta \log \eta \ldots\right)$$

By determining A and B from the initial conditions, we obtain for φ_I in the cited zone up to terms of the approximate order of c_r^2:

$$\varphi_I = \left\{1 - c\frac{U''_c}{U'^2_c}\left[\log\left(-\frac{c}{U'_c}\right) + 1\right]\right\}\left(U'_c\,\eta + \frac{U''_c}{2}\eta^2\right)$$

$$+ \left(c - c^2\frac{U''_c}{U'^2_c}\right)\left(1 + \frac{U''_c}{U'_c}\eta \log \eta\right) \qquad (43)$$

The imaginary part of $\log\left(-\frac{c}{U'_c}\right)$ is $-i\pi$, while that of $\log \eta$ at $y = \epsilon$ is equal to $i\frac{c_i}{\epsilon U'_c}$ which, according to the definition of ϵ is very small.

The region of $y = \epsilon$ up to the channel center $y_w = b$ requires a representation of the maximum imaginary part in φ_I. Since $\epsilon \gg \frac{c_i}{|U'_c|}$, we can for this range, develop

$$\varphi'' - \left(\frac{U''}{U - c} + \alpha^2\right)\varphi = 0$$

according to c_i, which affords as a first approximation:

$$\varphi'' - \left[\frac{U''}{U - c_r} + \frac{U''}{(U - c_r)^2} i\, c_i + \alpha^2\right]\varphi = 0$$

Now it is possible to set up a fundamental system of solutions, one of which satisfies the initial condition $\varphi = 1$, $\varphi' = 0$ for $y = \epsilon$, the other $\varphi = 0$, $\varphi' = 1$. In

the absence of singularities the development may be effected according to parameters $i\,c_i$ and α^2, and it is found that in this fundamental system the first imaginary components go with c_i. The indicated development of φ starts with

$$\varphi = Q_1(y) + i\,c_i\,Q_2(y) + \alpha^2\,Q_3(y)$$

where the Q has real, restricted values which are not dependent on parameters c_i and α^2. Q^1 may be explicitly expressed, according to the early statements about (20). By suitably combining these fundamental solutions, the joining to (43) may be effected for $y = \epsilon$. Here again, c_i is to be of lower order of magnitude than c_r. Remembering the remark about (43), according to which the imaginary part of $\log\left(-\frac{c}{U'_c}\right)$ equals $-\pi\,i$, while that of $\log \eta$ disappears for $y = \epsilon$, the maximum imaginary part* in φ_I is easily obtained as

$$c_r\,\frac{U''_c}{U'^2_c}\,i\,\pi\,U(y) \tag{44}$$

Now (42) can be evaluated: φ_I changes for $c = \alpha = 0$ steadily to U. With this rough approximation (42) gives the equation for c_r to the first approximation:

$$-\,c_r\,U'_w + c_r\,\varphi'_I(b) + \alpha^2 \int_0^b U^2\,dy_w = 0$$

If, however, the condition of orthogonality is complied with, then $\varphi'_I(b) = 0$, whence

$$c_r = \frac{\alpha^2}{U'_w} \int_0^b U^2\,dy_w \tag{45}$$

that is, actually positive, which proves the assumption about c_r as being in accord with the condition of orthogonality. To define c_i demands closer approximations

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*The statement about φ_I can also be confirmed insofar as an explicit expression for φ_I can be given, at least for $\alpha = 0$, that is, $\varphi_I = -\,U'_w\,c(U-c)\int_0^{y_w} \frac{dy_w}{(U-c)^2}$, as is readily seen when written in the differential equation.

(43) or (44) for φ_I. There is obtained

$$c_i = \alpha^2 c_r \frac{U''_w}{U'^3_w} \pi \int_0^b U^2 \, dy_w = c_r^2 \frac{U''_w}{U'^2_w} \pi \tag{46}$$

Equations (45) and (46) contain the necessary and at the same time sufficient conditions for the existence of amplified characteristic solutions. It remains to be established whether these conditions conflict with the assumptions relative to the disturbance parameter made incidental to φ_I. As to (45), this has been proved; as to (46), only the fact that $c_i > 0$ was utilized in the representation of φ_I should be noted. Equation (46) can be complied with only when the profiles have an inflection point, where $U''_w > 0$. For the others, $U''_w < 0$, so that we obtain a contradiction in accordance with the first formula of Rayleigh. And, as the assumptions for c_r and c_i are also confirmed, the existence of amplified characteristic oscillations in the case of profiles with inflection points is proved. It is seen that c_i increases so much more with c_r as, under otherwise identical conditions, U'' is greater at the wall.

Although this very fact proves the instability of such profiles, it should be very instructive to construct a neighboring solution with amplification for the neutral characteristic solution φ_s with the parameters $c_n = U_s$ and $\alpha_n = \alpha_s$. Again denoting this particular solution by φ_I and the parameters near U_s and α_s by c, α, we put

$$\begin{aligned} c - U_s &= \Delta c = \Delta c_r + i\, c_i \\ \alpha^2 - \alpha^2_s &= \Delta\alpha^2 \end{aligned} \tag{47}$$

c_i again being assumed positive, so that (37) suitably transformed with the aid of the differential equation for φ_s, becomes:

$$\int_0^b \frac{\Delta c\, U''}{(U - c)\,(U - U_s)} \varphi_s \varphi_I \, dy_w + \int_0^b \Delta\alpha^2 \varphi_s \varphi_I \, dy_w = 0 \tag{48}$$

Now φ_s is visualized as normalized, so that it is exactly equal to 1 at the inflection point. If y is the coordinate starting at this point, the development of φ_s starts with $1 + ky$ where k is constant, which need not

be known further. We may regard φ_I as determined by the fact that it becomes 1 at the singular point $U_c = c = U_s + \Delta c_r + i c_i$ and satisfies the boundary condition $\varphi_I = 0$. We again effect the development about the singular point in terms of $\eta = y - \frac{\Delta c}{U'_c}$, that is, at first, for an interval $-\epsilon \leqq y \leqq +\epsilon$, where $\epsilon \gg \left|\frac{\Delta c}{U'_c}\right|$, but $\epsilon^2 \ll \left|\frac{\Delta c}{U'_c}\right|$, obtainable by choosing $\epsilon = \left|\frac{\Delta c}{U'_c}\right|^{3/4}$.*

The development of φ_I, according to η starts with

$$\varphi_I = 1 + \frac{U''_c}{U'_c}\eta \log \eta + k_1 \eta \tag{49}$$

while

$$\varphi_s = 1 + k\frac{\Delta c}{U'_c} + k\eta \tag{50}$$

Owing to the steady transition from φ_I into φ_s for vanishing Δc and $\Delta\alpha^2$, the constants k and k_1, determined by the condition at the wall, differ but little. Then the integration, according to (48) for the interval of y between $-\epsilon$ and $+\epsilon$, gives for the first integral in (48) (limited to terms at least linear in Δc):

$$\frac{U''_c}{U'_c} i \pi \tag{51}$$

or, when considering that U''_c through development from the inflection point $y_w = s$ becomes equal to $\frac{U'''_s \Delta c}{U'_s}$, where U'''_s is assumed negative and U'_s positive.

$$-\frac{U'''_s \; c_i \pi}{U'^2_s} + \frac{U'''_s}{U'^2_s}\Delta c_r \pi i \tag{52}$$

Outside of this interval φ_I may again be developed,

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*ϵ is, on the one hand, chosen so small that a few terms of a development of φ_I according to η suffices for $|y| < \epsilon$, that $|U - U_s| \gg \Delta c$ for $|y| > \epsilon$ and $\frac{U''}{U - c}$ and φ_I can be developed according to the parameter Δc.

according to the parameters Δc and $\Delta\alpha^2$. The first approximation (linear in Δc) for the missing constituent of the first integral of (48) is obtained by substituting φ_s for φ_I in this zone:

$$\Delta c \left\{ \int_0^{s-\epsilon} \frac{U''}{(U - U_s)^2} \varphi_s^2 \, dy_w + \int_{s+\epsilon}^{b} \frac{U''}{(U - U_s)^2} \varphi_s^2 \, dy_w \right\} \quad (53)$$

From the determination of ϵ, it follows that the transition $\epsilon \longrightarrow 0$ can be effected (up to terms which, like ϵ, disappear) within the brackets without changing the value. We put:

$$E = \lim_{\epsilon=0} \left\{ \int_0^{s-\epsilon} \frac{U''}{(U-U_s)^2} \varphi_s^2 \, dy_w + \int_{s+\epsilon}^{b} \frac{U''}{(U-U_s)^2} \varphi_s^2 \, dy_w \right\} \quad (54)$$

E contains only known quantities and remains finite, because the constituents which become infinite, cancel out. Consequently,

$$E \, \Delta c_r - \frac{U'''_s}{U'^2_s} c_i \pi + \Delta\alpha^2 \int_0^b \varphi_s^2 \, dy_w + i \left\{ c_i E + \frac{U'''_s}{U'^2_s} \pi \, \Delta c_r \right\} = 0 \quad (55)$$

The explicit interpretation of this relation requires the knowledge of E and $\int_0^b \varphi_s^2 \, dy_w$, which would call for the calculation of the proven neutral characteristic solution φ_s. But the existence of a neighboring characteristic oscillation is readily apparent without it. To illustrate, assume $E < 0$ as can be proved (section VIII). Then the imaginary part of (55) gives:

$$c_i = - \frac{U'''_s}{U'^2_s} \frac{\Delta c_r}{E} \pi \quad (56)$$

Since c_i according to assumption is to be positive, and U'''_s negative, Δc_r should in this case, be negative. Writing c_i according to (56) in the real part of (55) gives:

$$\Delta c_r \left\{ 1 + \frac{U'''^2_s}{U'^4_s} \frac{\pi^2}{E^2} \right\} = - \frac{\Delta\alpha^2}{E} \int_0^b \varphi_s^2 \, dy_w \quad (57)$$

which proves that a satisfactory solution of the necessary and sufficient condition for amplified characteristic os-

cillations is possible as soon as $\Delta \alpha^2$ is chosen negative. It can also be proved that for $E = 0$, in which $\Delta c_r = 0$ or $E > 0$, an amplified characteristic oscillation would exist as soon as $\Delta \alpha^2$ is negative. Continuation of the characteristic solutions over α in the zone around α_s is therefore only possible in one direction, namely, for decreasing α, where E should also have the same sign. The order of magnitude of c_i, according to the above formulas is at least as great as that of Δc_r. The result is the start of the development of function $c(\alpha)$ at two points, namely, for $\alpha = 0$ and $\alpha = \alpha_s$ in general formulas.

VII. EQUILIBRIUM OF BOUNDARY-LAYER PROFILES

A particularly important case is now analyzed as to stability. Idealized to a certain extent to suit our purposes, the boundary-layer profiles are to be so defined that the velocity U rises from 0 at the wall to U_{max} at distance δ, and then remains constant to infinity (fig. 5). At $y_w = \delta$, a discontinuity in U'' may be permitted, for, integration of the differential equation for φ over a small interval around $y_w = \delta$, shows that nevertheless, in addition to φ, φ' also must remain continuous at this point.

Now a fundamental system of φ solutions can be set up for the zone $U = \text{constant} = U_{max}$, namely, $\varphi = e^{-\alpha y_w}$ and $\varphi = e^{+\alpha y_w}$, in which, when α is positive, the second may not occur because of its infinite growth in a characteristic oscillation. The characteristic function in the zone from $y_w = \delta$ to $y_w = \infty$ behaves like $e^{-\alpha y_w}$. Our analysis shall be restricted to the zone $0 \leqq y_w \leqq \delta$, so as to rule out the infinitely remote point, which is an essential singular point of the differential equation. Then the characteristic solution, aside from $\varphi = 0$ at the wall ($y_w = 0$), must satisfy the other boundary condition

$$\varphi' + \alpha \varphi = 0 \quad \text{for} \quad y_w = \delta \tag{58}$$

in order to make the joining with the cited solution form $e^{-\alpha y_w}$, possible.

The neutral characteristic solutions are established

very similarly to those for profiles in the channel, and from an analogous conclusion it follows that $0 \leqq c \leqq U_{max}$ and that as far as profiles without an inflection point are concerned, only the neutral characteristic solution $\varphi = U$, $c = \alpha = 0$ exists. Its construction proceeds from the solution of the differential equation for $c = U_s$, $\alpha = 0$, which satisfies the boundary condition for $y_w = \delta$. But this solution itself is simply $U - U_s$. Changing to $\alpha^2 > 0$ while preserving the value of φ at $y_w = \delta$ by normalizing, the φ value is increased twofold: first, as an increase of φ for $y_w = \delta$ corresponding to the boundary condition $\varphi' = -\alpha\varphi$ upon advance to smaller y_w; second, as effect of α on the coefficients of the differential equation, as known from the channel profiles. Finally, when α becomes large enough, φ ceases to be negative at the wall (fig. 6). For this $\alpha = \alpha_s$, where $\varphi = 0$ at the wall, the characteristic solution φ_s exists.

Now we construct their neighboring amplified characteristic solutions. The previous formulas for the channel profile are in any case in the neighborhood of $\varphi_n = U$, $c = \alpha = 0$ not to be transferred by simple limit process to the boundary-layer profiles, because there it gave, for instance,

$$c_r = \frac{\alpha^2}{U'_w} \int_0^b U^2 \, dy_w \tag{45}$$

so that for boundary-layer profiles, where b goes to infinity, the integral would become infinite. For this reason, a new integral condition is derived for the existence of an adjacent characteristic solution, the analysis again being restricted to the finite zone $0 \leqq y_w \leqq \delta$.

Subtracting the differential equation for the neutral characteristic solution φ_n from that for the adjacent solution φ_I, we obtain

$$(\varphi_I - \varphi_n)'' - \alpha^2(\varphi_I - \varphi_n) - \frac{U''}{U - c}(\varphi_I - \varphi_n) = g \tag{33}$$

with

$$g = \frac{c - c_n}{U - c}\varphi''_n - \frac{U - c_n}{U - c}\alpha^2{}_n \varphi_n + \alpha^2 \varphi_n \tag{34}$$

It is simplest to derive a necessary condition that φ_I be

a characteristic solution, as follows: The differential equation for φ_I

$$\varphi''_I - \alpha^2 \varphi_I - \frac{U''}{U - c} \varphi_I = 0$$

is written symbolically $L(\varphi_I) = 0$, so that the above equation for $\varphi_I - \varphi_n$ becomes:

$$L(\varphi_I - \varphi_n) = g$$

Next we formulate:

$$\int_0^\delta \left\{\varphi_I \, L(\varphi_I - \varphi_n) - (\varphi_I - \varphi_n) \, L(\varphi_I)\right\} dy_w$$

$$= \left(- \varphi_I \frac{d\varphi_n}{dy} + \varphi_n \frac{d\varphi_I}{dy}\right)_{y_w=\delta} \tag{59}$$

which with consideration to (58) gives the necessary condition:

$$\int_0^\delta \varphi_I \, g \, dy_w = (\alpha_n - \alpha) \, (\varphi_I \, \varphi_n)_{y_w=\delta} \tag{60}$$

That this condition is also sufficient, is seen in similar fashion as in section VI. We form

$$\varphi'_I - \varphi'_n + \alpha(\varphi_I - \varphi_n)$$

$$=(\varphi'_I+\alpha \, \varphi_I) \int_0^\delta g \, \varphi_{II} \, dy_w-(\varphi'_{II}+\alpha \, \varphi_{II}) \int_0^\delta g \, \varphi_I \, dy_w+C(\varphi'_I+\alpha \, \varphi_I) \tag{61}$$

for $\varphi_I - \varphi_n$ at $y_w = \delta$ according to (35).

Considering the condition (60) looked upon as fulfilled, and φ_{II} expressed as

$$\varphi_I \, \varphi'_{II} - \varphi'_I \, \varphi_{II} = - 1$$

the right-hand side becomes:

$$(\varphi'_I + \alpha \, \varphi_I) \int_0^\delta g \, \varphi_{II} \, dy_w + C(\varphi'_I + \alpha \, \varphi_I) - (\alpha - \alpha_n) \, \varphi_n$$
$$+ (\varphi'_I + \alpha \, \varphi_I) \, (\alpha - \alpha_n) \, \varphi_{II} \, \varphi_n$$

and the whole equation (61) may be written as:

$$(\varphi'_I+\alpha\varphi_I)\left\{1-C-\int_0^\delta g\,\varphi_{II}\,dy_w+(\alpha_n-\alpha)\varphi_{II}\varphi_n\right\}=\varphi'_n+\alpha_n\varphi_n=0 \qquad (62)$$

The multiplier of $\varphi'_I + \alpha\,\varphi_I$ on the left side is different from 0, because according to (35) and (60), it is:

$$\varphi_I\left\{1-C-\int_0^\delta g\,\varphi_{II}\,dy_w+(\alpha_n-\alpha)\,\varphi_{II}\,\varphi_n\right\}=\varphi_n \qquad (63)$$

for $y_w = \delta$, and $\varphi_n(\delta) \neq 0$; otherwise $\varphi'_n(\delta) = 0$ for $y_w = \delta$, that is, φ_n would have to disappear.* Consequently, the bracketed term does not disappear and $\varphi'_I + \alpha\,\varphi_I = 0$ for $y_w = \delta$, according to (62).

The determination of φ_I near $\varphi_n = U$, $c = \alpha = 0$ as before, through $\varphi_I = 0$ and $\varphi'_I = U'_w$ at the wall, is followed by the evaluation of the new integral condition (60):

$$\int_0^\delta\left\{c\,\varphi''_I+\alpha^2\,(U-c)\,\varphi_I\right\}dy_w=(\alpha_n-\alpha)\,(\varphi_I\,\varphi_n)_{y_w=\delta}$$

as

$$-\,c_r\,U'_w=-\,\alpha\,U_{max}^2$$

or

$$c_r=\frac{\alpha\,U_{max}^2}{U'_w} \qquad (64)$$

A comparison of this formula with the previous one reveals another power of α at the limiting transition from channel to boundary-layer profiles. The amplification is (deductions as before, especially with formula (44)):

$$c_i=\frac{\alpha\,c_r\,U''_w}{U'^3_w}\,\pi\,U_{max}^2=\frac{c_r^2\,U''_w}{U'^2_w} \qquad (65)$$

The relationship existing between both formulas for c_r (45) and (64) becomes so much clearer when comparing symmetrical channel profiles, which are to have a variable U from $y_w = 0$ to $y_w = \delta$, but constant $U = U_{max}$ over a greater length $2b_1$ around the center, with the boundary-layer profiles (compare third profile in fig. 2). φ_I is for the channel profile in the vicinity of constant U approximately:

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*That $\varphi_n(\delta) \neq 0$ is also readily apparent from the cited construction of the neutral characteristic solutions.

$$U_{max} \frac{e^{\alpha(b_1+\delta)} e^{-\alpha y_w} + e^{-\alpha(b_1+\delta)} e^{+\alpha y_w}}{e^{\alpha b_1} + e^{-\alpha b_1}}$$

The constituent $\int_{\delta}^{b} \alpha^2 U \varphi_I \, dy_w$ in (42) becomes equal to $\alpha U_{max}^2 \tanh \alpha b_1$ and approaches $\alpha^2 U_{max}^2 b_1$ for small αb_1 according to (45), and approaches αU_{max}^2 for large αb_1 in accordance with (64). The validity of (45) is therefore contingent upon $\alpha b_1 \ll 1$, which for boundary-layer profiles, where $b_1 \longrightarrow \infty$ is impossible, no matter how small α may be, in which case formula (64) holds for c_r.

As to the adjacent amplified oscillation for φ_s, $\alpha = \alpha_s$, $c = U_s$, suffice it to state that (56) and (57) are preserved; only the there-existing integral needs to be extended to $b = \infty$.

VIII. ILLUSTRATIVE EXAMPLES

These examples treat the neutral characteristic solutions φ_s, previously referred to, as well as the formulas for wave velocity and amplification in the vicinity of $\alpha = 0$ and $\alpha = \alpha_s$. First we assume sinusoidal channel profiles. Thus,

$$U = U_s + (U_{max} - U_s) \sin \left(\frac{y_w - s}{b - s} \frac{\pi}{2}\right) \tag{66}$$

The velocity U_s at the inflection point is then connected with s, the distance of the inflection point from the wall, through the relation:

$$U_s \left(1 + \sin \frac{s}{b - s} \frac{\pi}{2}\right) = U_{max} \sin \frac{s}{b - s} \frac{\pi}{2} \tag{67}$$

which follows from $U = 0$ for $y_w = 0$, whence the differential equation for a characteristic solution at $c = U_s$ runs as follows:

$$\varphi'' - \alpha^2 \varphi + \frac{1}{(b - s)^2} \frac{\pi^2}{4} \varphi = 0 \tag{68}$$

To allow at the same time for the boundary condition at the wall, we put $\varphi = \sin \frac{p y_w}{b}$; p is determined from

the boundary condition for $y_w = b$. For α, the differential equation gives:

$$\alpha^2 = \frac{1}{b^2}\left\{\frac{1}{\left(1 - \frac{s}{b}\right)^2}\frac{\pi^2}{4} - p^2\right\} \tag{69}$$

Since $0 \leqq s < \frac{b}{2}$ and α^2 is always positive, p may assume only the value $\pi/2$ in the center, according to the boundary condition $\varphi' = 0$, but not the π value, corresponding to $\varphi = 0$. The result is therefore the characteristic function

$$\varphi_s = \frac{\sin \frac{y_w}{b}\frac{\pi}{2}}{\sin \frac{s}{b}\frac{\pi}{2}} \tag{70}$$

and

$$\alpha^2 = \frac{1}{b^2}\frac{\pi^2}{4}\left\{\frac{1}{\left(1 - \frac{s}{b}\right)^2} - 1\right\} \tag{71}$$

for $s = \frac{b}{3}$, $\alpha b = 0.559\ \pi$, for example (fig. 7).

For the amplified adjacent solutions with this profile in the vicinity of $c_r = 0$, $\alpha = 0$:

$$c_r = 0.462\ \alpha^2\ b^2\ U_{max}$$

$$c_i = 7.59\ \frac{c_r^2}{U_{max}},\ \frac{\beta_i b}{U_{max}} = 11.17\left(\frac{c_r}{U_{max}}\right)^{5/2} \tag{72}$$

in the vicinity of $c_r = U_s = 0.414\ U_{max}$, $\alpha b = 0.559\ \pi$

$$c_r = 0.061\ \alpha^2\ b^2\ U_{max} + 0.226\ U_{max}$$

$$c_i = 0.194\ U_{max} - 0.468\ c_r$$

$$\frac{\beta_i b}{U_{max}} = 0.336 - 0.821\ \frac{c_r}{U_{max}} \tag{73}$$

where αb must be smaller than $0.559\ \pi$.

We proceed now to the sinusoidal boundary-layer pro-

files, for which

$$U = U_s + (U_{max} - U_s) \sin \frac{y_w - s}{\delta - s} \frac{\pi}{2} \quad \text{for} \quad y_w \leqq \delta$$
$$U = U_{max} \quad \text{for} \quad y_w \geqq \delta \tag{74}$$

The differential equation for φ in the zone $0 \leqq y_w \leqq \delta$ runs as follows:

$$\varphi'' - \alpha^2 \varphi + \frac{1}{(\delta - s)^2} \frac{\pi^2}{4} \varphi = 0 \tag{75}$$

with the boundary conditions:

$$\varphi = 0 \quad \text{for} \quad y_w = 0$$
$$\varphi' + \alpha\varphi = 0 \quad \text{for} \quad y_w = \delta$$

α was assumed positive. By making $\varphi = \sin \frac{p y_w}{\delta}$, the other boundary condition gives:

$$\alpha = - p \cot p \tag{76}$$

The differential equation furnishes the further relation:

$$\frac{1}{1 - \frac{s}{\delta}} \frac{\pi}{2} = \frac{p}{\sin p} \tag{77}$$

We consider only positive values of p, as negative p would give no new characteristic function. In view of equation (77), $\sin p > 0$. Therefore,

$$\frac{p}{\sin p} \geqq p$$

and, since

$$\frac{\pi}{2} \leqq \frac{1}{1 - \frac{s}{\delta}} \frac{\pi}{2} < \pi$$

p must be smaller than π, according to (77). As α is to be positive, $p = \pi/2$ according to (76). The solution of the transcendental equation (77) for several specific cases, follows:

For $s = 0$, $p = \frac{\pi}{2}$, $\alpha = 0$

$$\left.\begin{aligned} &\text{for } s = \frac{\delta}{3}, \quad p = 0.658\,\pi \left(= \frac{118.4}{180}\,\pi\right), \quad \alpha\delta = 0.356\,\pi \\ &\text{for } s = \frac{\delta}{2}, \quad p = 0.737\,\pi \left(= \frac{132.5}{180}\,\pi\right), \quad \alpha\delta = 0.675\,\pi \end{aligned}\right\} \quad (78)$$

The characteristic function becomes:

$$\left.\begin{aligned} \varphi_s &= \frac{\sin \frac{p y_w}{\delta}}{\sin \frac{p s}{\delta}} \qquad \text{for } 0 \leqq y_w \leqq \delta \\ \varphi_s &= \frac{\sin p}{\sin \frac{p s}{\delta}}\, e^{\alpha\delta}\, e^{-\alpha y_w} \qquad \text{for } y_w \geqq \delta \end{aligned}\right\} \quad (79)$$

For a closer description of the shape of φ_s we point to the position of the maximum at $\frac{y_w}{\delta} = \frac{\pi}{2p}$. Figure 8 shows φ_s plotted for $s = \frac{\delta}{3}$.

For the boundary-layer profile with $s = \frac{\delta}{3}$ near $c_r = 0$, $\alpha = 0$, the amplified neighboring solutions are:

$$\begin{aligned} c_r &= 1.025\ \alpha\ \delta\ U_{max} \\ c_i &= 7.59\ \frac{c_r^2}{U_{max}}, \quad \frac{\beta_i \delta}{U_{max}} = 7.41 \left(\frac{c_r}{U_{max}}\right)^3 \end{aligned} \quad (80)$$

in the vicinity of $c_r = U_s = 0.414\ U_{max}$, $\alpha\delta = 0.356\ \pi$

$$\begin{aligned} c_r &= 0.082\ \alpha^2\ \delta^2\ U_{max} + 0.312\ U_{max} \\ c_i &= 0.250\ U_{max} - 0.604\ c_r \\ \frac{\beta_i \delta}{U_{max}} &= 0.280 - 0.675\ \frac{c_r}{U_{max}} \end{aligned} \quad (81)$$

where $\alpha\delta$ must be smaller than $0.356\ \pi$.

Figures 9 and 10 show c_r versus α and β_i versus c_r according to (72), (73), (80), and (81). The start and finish of the curves are shown as heavy lines connected by dashes after interpolation, so as to give an idea of the probable amplification. No conclusions are drawn therefrom. If it is preferred to obtain the entire distribution of the cited curves the attempt at generally applicable formulas must be foregone in favor of a calculation fitting the particular case. And that itself is easy, once the difficult problem of the existence has been explained.

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REFERENCES

1. Rayleigh, Lord: Scientific Papers, vol. I, 1880, p. 474; vol. III, 1887, p. 17; vol. IV, 1895, p. 203; vol. VI, 1913, p. 197.

2. Tollmien, W.: The Production of Turbulence. T.M. No. 609, N.A.C.A., 1931.

3. Hopf, L.: Ann. d. Phys. 43, 1914, p. 1, and 59, 1919, p. 538.

   Prandtl, L.: Bemerkungen über die Entstehung der Turbulenz. Z.f.a.M.M., vol. I, 1921, p. 431.

   Tietjens, O.: Beiträge zur Entstehung der Turbulenz. Z.f.a.M.M., vol. 5, 1926, p. 200.

4. Frank, Ph. and Mises, R. v.: Die Differential- und Integralgleichungen der Mechanik und Physik, vol. I, Braunschweig, 1930, p. 357.

5. Schrödinger, E.: Abhandlungen zur Wellenmechanik, Leipzig, 1928, p. 88.

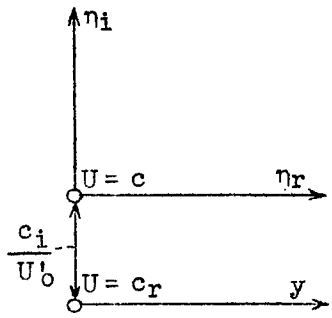
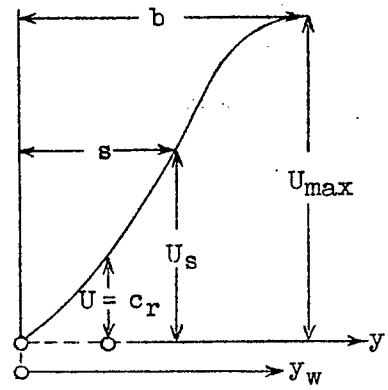


Figure 1.



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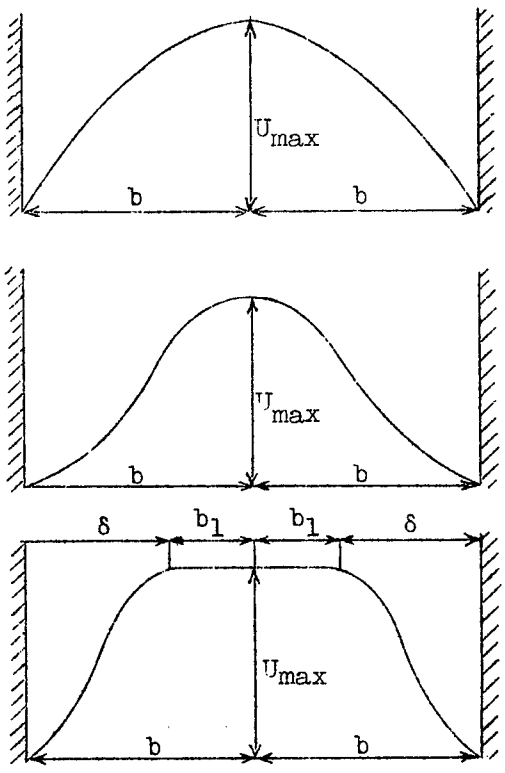

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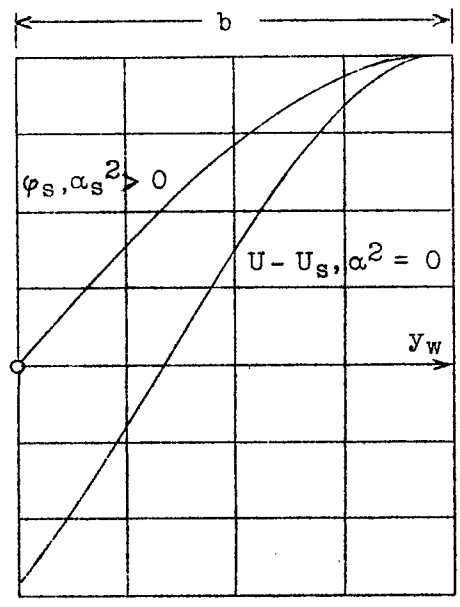

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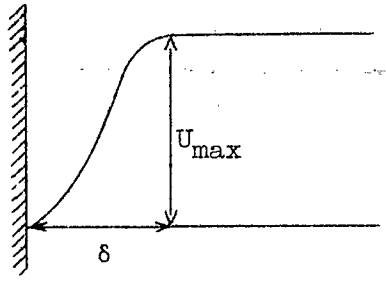

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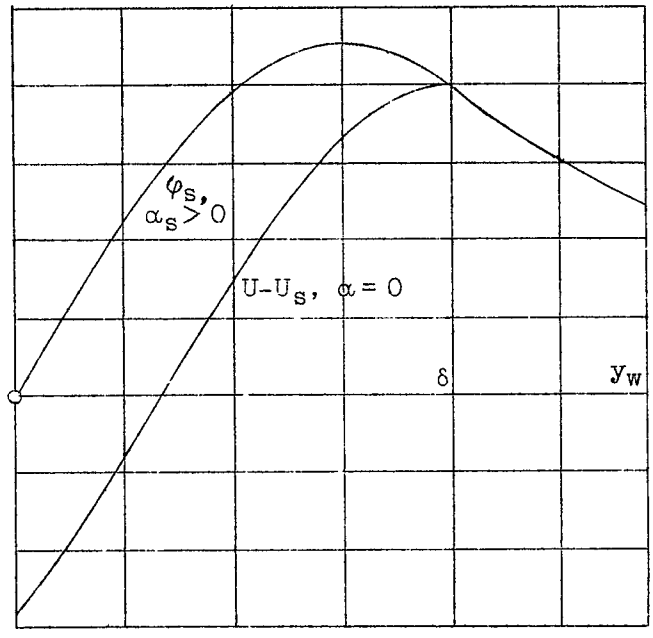

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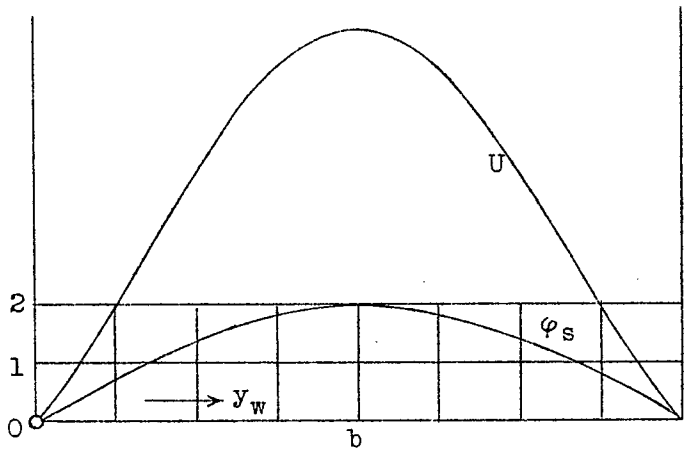

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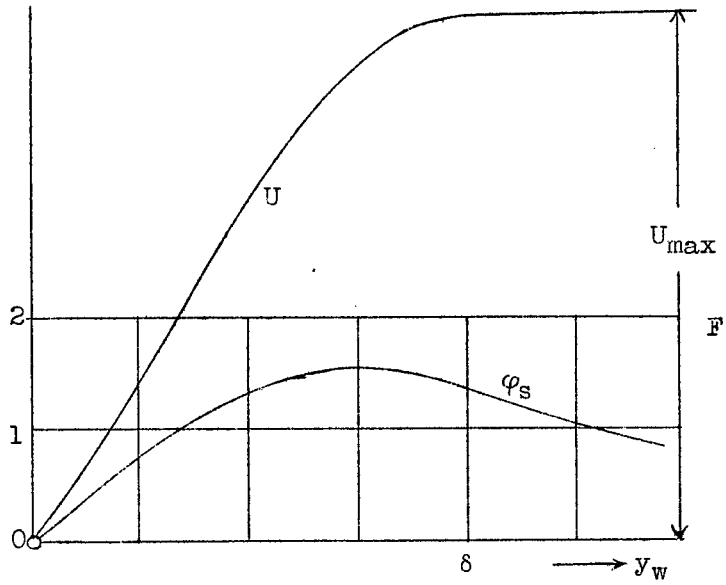
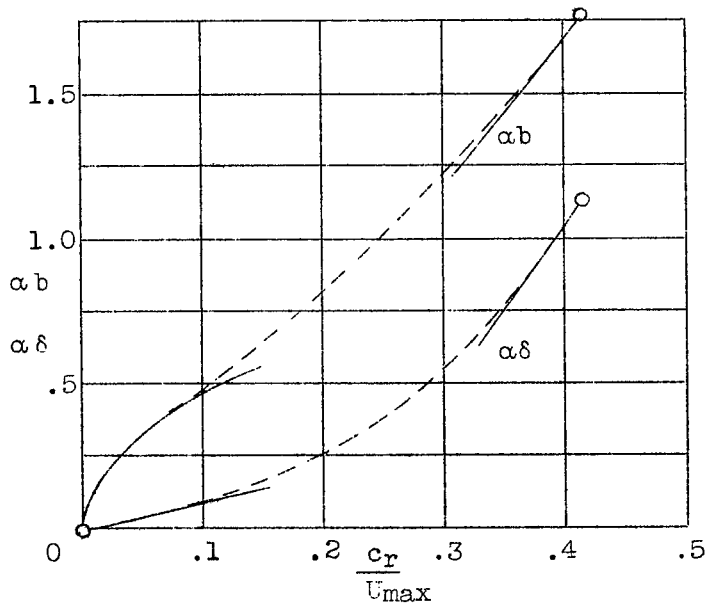


Figure 8.



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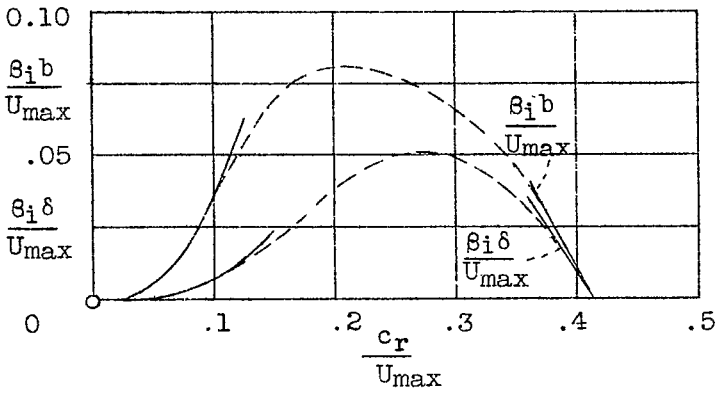

Figure 10.