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A RECURRENCE MATRIX SOLUTION FOR THE DYNAMIC RESPONSE OF AIRCRAFT IN GUSTS

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SUMMARY

A systematic procedure is developed for the calculation of the structural response of aircraft flying through a gust by use of difference equations and matrix notation. The use of difference equations in the solution of dynamic problems is first illustrated by means of a simple-damped-oscillator example. A detailed analysis is then given which leads to a recurrence matrix equation for the determination of the response of an airplane in a gust. The method takes into account wing bending and twisting deformations, fuselage deflection, vertical and pitching motion of the airplane, and some tail forces. The method is based on aerodynamic strip theory, but compressibility and three-dimensional aerodynamic effects can be taken into account approximately by means of over-all corrections. Either a sharp-edge gust or a gust of arbitrary shape in the spanwise or flight directions may be treated. In order to aid in the application of the method to any specific case, a suggested computational procedure is included.

The possibilities of applying the method to a variety of transient aircraft problems, such as landing, are brought out. A brief review of matrix algebra, covering the extent to which it is used in the analysis, is also included.

INTRODUCTION

In the problem of an airplane flying through gusts, accurate predictions of stresses are not always obtained if the interaction between aerodynamic loads and structural deformations is not considered. The present paper gives a method for determining the dynamic response of aircraft in gusts in which this interaction is considered. An approach is employed which is a departure from the usual modal type of solution. The time derivatives in the integro-differential equations of motion of the airplane are replaced by appropriate difference expressions and use is made of matrix notation to express conveniently the conditions of equilibrium at a number of points along the wing span.

The result is a systematic procedure which is complete and general in form. The airplane is assumed to be free to translate and pitch. Wing bending, wing twist, and fuselage flexibility are all included. Tail forces due to vertical motion, angle of attack, and gust penetration are also included in the analysis.

With the method, a gust with any gradient in the direction of flight or along the span may be treated without difficulty. The method is based on aerodynamic strip theory, but over-all compressibility and aspect-ratio corrections may be included without difficulty, if desired. One such over-all correction is indicated.

In the first part of the paper the method of using difference equations in the solution of dynamic problems is illustrated by an example in which the transient response of a simple oscillator is determined. The analysis for the determination of the response of an airplane in a gust is then given. In the following section a computational procedure is suggested. This section is not intended to describe or add to the understanding of the analysis, but by following the directions indicated, the response of any airplane may be found without following through the complete details of the analysis.

Supplementary definitions and derivations are presented in appendixes. Appendix A summarizes the various matrix coefficients and matrices that are used in the analysis, appendix B gives a derivation of the difference equations, appendix C gives a derivation of the flexibility matrices, appendix D gives a derivation of a recurrence equation for evaluating the form of Duhamel's integral which involves an exponential kernel, and appendix E presents a review of the fundamentals of matrix algebra. It is suggested that those not familiar with matrix algebra read appendix E before reading the analysis.

SYMBOLS

a	distance between leading edge of wing and elastic axis
a_1	coefficient used in unsteady lift function for sudden change in angle of attack
A	aspect ratio of wing
A_t	aspect ratio of horizontal tail
b	semispan of wing
c	chord of wing

c_o	chord at wing midspan
c_{o_t}	midspan chord of tail
c_t	mean aerodynamic chord of tail
e	distance between mass center of wing cross section and elastic axis of wing; positive when elastic axis lies forward of mass center
E	Young's modulus of elasticity
F	suddenly applied force
G	shear modulus of elasticity
i	integers 0, 1, 2, 3, 4, and 5 used to designate stations (for most part used as parenthetical numbers, that is, w(3) is deflection at station 3)
I	bending moment of inertia
J	torsional stiffness constant
k	radius of gyration of wing mass about elastic axis or elastic spring constant
l	length of section associated with a spanwise station
L	aerodynamic lift over interval l on wing
L_f	shear force transmitted to wing by fuselage
L_g	aerodynamic lift over interval l on wing due to gust
L_{g_t}	one-half aerodynamic lift on tail due to gust
L_t	one-half total aerodynamic lift on tail
L_1	part of aerodynamic lift over interval l on wing (see equation (16))
L_2	part of aerodynamic lift over interval l on wing (see equation (17))
m	mass of beam included in interval l or concentrated mass in spring oscillator

$\overline{m}$ mass m including apparent mass effect $\left(m + \frac{\pi\rho l c^2}{4}\right)$

m_A assumed over-all compressibility and aspect-ratio correction for wing $\left(\frac{A}{2 + A\sqrt{1 - M^2}}\right)$

m_{A_t} assumed over-all compressibility and aspect-ratio correction for horizontal tail $\left(\frac{A_t}{2 + A_t\sqrt{1 - M^2}}\right)$

$\overline{me}$ mass moment me including apparent mass effect $\left(me + \frac{\pi\rho l c^3}{4}\left(\frac{1}{2} - \frac{a}{c}\right)\right)$

m_f mass of fuselage per unit length

$\overline{mk^2}$ mass polar moment of inertia mk^2 including apparent mass effects $\left(mk^2 + \frac{\pi\rho l c^4}{4}\left(\frac{1}{2} - \frac{a}{c}\right)^2 + \frac{\pi\rho l c^4}{128}\right)$

M Mach number or aerodynamic moment over interval l about elastic axis of wing

M_f moment transmitted to wing by fuselage

n integers 0, 1, 2, 3, and so forth to designate number of time intervals passed

p normal load acting at a station

p_f fuselage inertia loading per unit length

q torsional load acting at a station

s distance traveled by wing in half-chords $\left(\frac{2U}{c_o} t\right.$, where midspan chord c_o is used as reference chord$\left.\right)$

Δs distance interval in half-chords $\left(\frac{2U}{c_o} \epsilon\right)$

S_t horizontal-tail area

t, τ time, zero at beginning of gust penetration

U	forward velocity of flight
v	vertical velocity of gust
w	deflection of elastic axis of wing, positive upward, or deflection of mass oscillator
w_f	deflection of fuselage, positive upward
W_1	fuselage modal function, zero at wing elastic axis and unity at tail one-quarter-chord location
x	distance along fuselage measured from wing elastic axis, positive in rearward direction
x_n	distance from foremost part of nose to elastic axis
x_t	distance from elastic axis to one-quarter-chord location on tail
y	distance along wing measured from center of airplane
z	ratio of dynamic deflection to static deflection
α_t	angle of attack of horizontal tail
β	forward-speed and aspect-ratio factor for wing $(m_A \pi \rho U)$ or coefficient of damping for spring oscillator
β_t	forward-speed and aspect-ratio factor for tail $\left(\frac{1}{2} m_{A_t} \pi \rho S_t U\right)$
γ	exponential coefficient in Φ function associated with time t, $\left(\gamma = \frac{2U}{c_o} \lambda\right)$
δ	coefficient of fuselage modal function
ϵ	time interval
λ	exponential coefficient in Φ function associated with variable s
λ_i	dimensionless interval between $i - 1$ and i stations ($\lambda_i b$ is actual length)
ρ	mass density of air
φ	angle of twist of wing, positive in stalling direction

ψ function which denotes growth of lift on rigid airfoil entering sharp-edge gust (used without subscript to indicate function for wing and with subscript t used to indicate function for tail)

ω_f natural frequency associated with W_1, radians per second

$\mathbf{1}(t)$ unit-step function

$1 - \Phi$ function which denotes growth of lift on airfoil following sudden change in angle of attack (used without subscript to indicate function for wing and with subscript t used to indicate function for tail)

$[\ ]$ square matrix

$\lceil\ \rfloor$ rectangular matrix

$|\ |$ column matrix

$\lfloor\ \rfloor$ row matrix

Subscripts:

t	tail
0,1,2,3, ... n	number of time intervals passed
0,1,2,3,4,5 or i	station (however, station is usually given as parenthetical number, such as $w(3)$ for deflection at station 3); i is also used as general subscript in appendix A

All the terms, coefficients, and matrices not defined in this section are defined in appendix A.

Dots are used to indicate derivatives with respect to time; for example, $\frac{\partial w}{\partial t} = \dot{w}$ or $\frac{\partial w}{\partial \tau} = \dot{w}$

ANALYSIS

Transient Response of a Simple Damped Oscillator

In order to illustrate the use of difference equations and to test the accuracy of the procedure that is to be used in the solution of the more complicated gust problems, the solution of a simple problem having a known analytical solution is first presented. The problem is to compute the response of the damped oscillator shown in figure 1 to a suddenly applied force. The differential equation of motion of this system due to the suddenly applied force is

$$m\ddot{w} + \beta\dot{w} + kw = F\mathbf{1}(t) \tag{1}$$

By use of difference equations this differential equation may be transformed into an equation which involves deflection ordinates at several successive values of time. Probably the most commonly used difference equations are the following (see appendix B for derivation):

$$\dot{w}_n = \frac{w_{n+1} - w_{n-1}}{2\epsilon} \tag{2}$$

$$\ddot{w}_n = \frac{w_{n+1} - 2w_n + w_{n-1}}{\epsilon^2} \tag{3}$$

which give the derivatives at the intermediate of three successive ordinates. Although these equations are quite adequate for the oscillator problem of the present paper, they cannot be used in the gust analysis which follows. Rather, for reasons which are brought out in a subsequent part of the analysis, equations that give the derivatives at the end ordinate of several successive ordinates must be used. If only three successive ordinates are used, the derivatives so found are not accurate enough to be useful. If a fourth ordinate is added, however, derivatives may be taken at the end ordinate with accuracies which are comparable to those given by equations (2) and (3). Such derivatives are derived also in appendix B and are given by the equations:

$$\dot{w}_n = \frac{11w_n - 18w_{n-1} + 9w_{n-2} - 2w_{n-3}}{6\epsilon} \tag{4}$$

$$\ddot{w}_n = \frac{2w_n - 5w_{n-1} + 4w_{n-2} - w_{n-3}}{\epsilon^2} \tag{5}$$

Although either equations (2) and (3) or equations (4) and (5) may be used in the solution of this oscillator problem, only equations (4) and (5) will be used, since only these equations can be used in the gust-problem solution presented in this paper.

If the derivatives in equation (1) are replaced by the difference equations (4) and (5), the following equation is obtained:

$$\left(2 + \frac{11}{6}\frac{\beta\epsilon}{m} + \frac{k}{m}\epsilon^2\right)w_n = \left(5 + \frac{3\beta\epsilon}{m}\right)w_{n-1} - \left(4 + \frac{9\beta\epsilon}{6m}\right)w_{n-2} + \left(1 + \frac{2\beta\epsilon}{6m}\right)w_{n-3} + \frac{\epsilon^2 F}{m} \tag{6}$$

This equation may be said to be the difference equation of motion. If the more general case of a variable applied force were being considered, the factor F in this equation would be replaced by F_n, the value of the force present at the time $t = n\epsilon$.

If a specific case is now considered, in which $\frac{k}{m} = 400$, $\frac{\beta}{2m} = 2$, $\epsilon = 0.01$, $F = 1$, and the notation $z = \frac{w}{F/k}$ (ratio of dynamic deflection to static deflection) is used, equation (6) becomes

$$z_n = 0.018927 + 2.42272\ z_{n-1} - 1.92114\ z_{n-2} + 0.47949\ z_{n-3} \tag{7}$$

This equation may be regarded as a recurrence formula; the value z_n may be interpreted as the deflection to come and may be found easily from the three preceding deflections z_{n-1}, z_{n-2}, and z_{n-3}. Then with the newly found value z_n and with z_{n-1} and z_{n-2}, the next deflection can be found, and so on. This process thus gives a step-by-step derivation of the time history of deflection and may be carried out as far as is desired. Of course the process must start with known initial values of z. These values can be found with the aid of the initial conditions of the problem by means of the following approach.

The difference equations for the first and second derivatives at the third ordinate of four successive ordinates are (see appendix B)

$$\dot{w}_n = \frac{1}{6\epsilon}\left(2w_{n+1} + 3w_n - 6w_{n-1} + w_{n-2}\right)$$

$$\ddot{w}_n = \frac{1}{\epsilon^2}\left(w_{n+1} - 2w_n + w_{n-1}\right)$$

If these equations are taken to apply at $t = 0$ $(n = 0)$, they become

$$\dot{w}_0 = \frac{1}{6\epsilon}\left(2w_1 + 3w_0 - 6w_{-1} + w_{-2}\right) \tag{8}$$

$$\ddot{w}_0 = \frac{1}{\epsilon^2}\left(w_1 - 2w_0 + w_{-1}\right) \tag{9}$$

For the problem under consideration the primary initial conditions are that, at $t = 0$, the displacement and velocity are zero. By use of equation (1) or by reasoning from Newton's second law, a secondary initial condition can be established; that is, the acceleration immediately following the application of the unit force must be $1/m$. In equation form these conditions are

$$w_0 = 0$$

$$\dot{w}_0 = 0$$

$$\ddot{w}_0 = \frac{1}{m}$$

By substitution of these values into equations (8) and (9) and by use of the notation $z = \frac{w}{F/k}$, the following relations can be found to exist between the ordinates:

$$\left.\begin{aligned} z_0 &= 0 \\ z_{-2} &= 0.24 - 8z_1 \\ z_{-1} &= 0.04 - z_1 \end{aligned}\right\} \tag{10}$$

Substitution of these values into equation (7), with n set equal to 1, gives an equation from which z_1, the deflection at $t = \epsilon$, may be evaluated. Three successive deflections can now be established: the deflection at $t = \epsilon$, the zero deflection at $t = 0$, and a fictitious deflection for $t = -\epsilon$ given by equation (10). In the real problem no deflection exists for t less than zero; the assumption that a deflection does exist before t is zero is simply a means for allowing the recurrence formula, equation (7), to apply at the origin as well as at later values of time. Furthermore, no violation is made of the conditions under consideration because, mathematically, the response after $t = 0$ is not influenced by the response that may exist before $t = 0$, so long as the initial conditions are satisfied. The process just described for treating the initial conditions is actually not different from the procedure commonly employed in difference-equation approaches, in which exterior points near a region under consideration are written in terms of the interior points by means of the boundary conditions.

With the initial values of deflection thus established the step-by-step evaluation of succeeding deflections proceeds in a straightforward manner; that is, equation (7) is now evaluated for $n = 2$, then for $n = 3$, and so on. The response of the oscillator for the physical constants listed previously is given in figure 2. The comparison between the difference solution shown in this figure and the exact solution of equation (1) is seen to be good. As a matter of interest, the solution is also shown in this figure that is obtained by the use of the parabolic end-ordinate derivative which involves only three successive ordinates. The agreement in this case is seen to be quite bad. If equations (2) and (3) had been used, on the other hand, the difference solution (in this case for w_{n+1}) would correspond to that given for the cubic end-ordinate derivative.

Recurrence Matrix Equation for Response of an Airplane in a Gust

In order to help the reader to obtain a perspective of what is to be covered in this section, the following basic phases of the analysis are given:

(1) The assumptions are stated.

(2) The coordinate system and displacements are defined.

(3) The aerodynamic lift and moment are defined.

(4) The normal and torsional dynamic loadings (inertia forces, aerodynamic forces, and fuselage forces) on the wing are derived.

(5) The equations of elastic deformation - wing vertical motion, wing rotation, and fuselage bending - are given.

(6) The dynamic loadings on the wing are transformed into difference equations.

(7) The equations of elastic deformation and the difference equations for loading are combined to give the recurrence matrix equation for response.

In succeeding sections the initial response is determined, the method for evaluating the gust forces is shown, and the method for computing the loads and stresses is indicated.

Assumptions.- In this analysis an attempt is made to obtain the simplest and most direct solution to the problem with a minimum of simplifying assumptions. The case treated is that of an airplane having an essentially straight wing and penetrating a gust of known structure. The tail is considered to penetrate subsequently the same gust as does the wing. The assumptions made are as follows:

Assumptions pertaining to elasticity and airplane motion:

(1) The usual assumptions of engineering beam theory are made.

(2) The fuselage is free to pitch and move vertically. The portion of the fuselage in front of the elastic axis of the wing is assumed for convenience to be rigid. The portion of the fuselage rearward of the elastic axis is assumed flexible, and the elastic deflection is assumed to be given by a single modal function.

(3) The tail is assumed rigid.

Assumptions pertaining to aerodynamic forces:

(1) Aerodynamic strip theory applies. Three-dimensional effects, however, may be taken into account approximately by means of over-all corrections. Some such corrections are indicated.

(2) The gust force and forces due to vertical and pitching motion are the only tail forces considered. Other forces of known character may be included, however, if desired.

(3) Aerodynamic forces on the fuselage are neglected.

Coordinate system and displacements.- Position on the airplane is denoted by an orthogonal system of axes. The origin is at the intersection of the wing elastic axis with the plane of symmetry of the airplane: the w-axis runs positive upward, the x-axis runs along the fuselage positive in the rearward direction, and the y-axis runs spanwise. The wing semispan is considered to be divided into six, not necessarily equal, sections, with a station point at the middle of each section. (See fig. 3.) More or fewer stations could be chosen, but it is believed that six is a fair compromise between the amount of labor involved in setting up a solution and the accuracy desired. The interval between stations is designated by the number of the station at the outboard end of the interval. Station 0 is located near the wing root and generally may be located where the fuselage intersects the wing. In this way the concentrated forces of the fuselage are allowed to act through station 0. The other five stations are then located in any convenient manner so as to fall at concentrated mass locations or at points which represent the average of distributed masses, station 5 being nearest the tip. The total mass within a section is assumed to be concentrated at the station point, and the average of the section geometry (chord, elastic axis position, and so on) is assumed to apply. In this way the wing is assumed to be a beam subject to six load concentrations, and as such will have a linear moment variation between each station. The further assumption is made that the $\frac{1}{EI}$ variation is linear between each station. With these assumptions for the EI variation and concentrated load locations, equations for deflection at each station point may be derived (appendix C) by direct analytical treatment.

The displacements of the cross section at each station of the wing are given as the deflection of and rotation about the wing elastic axis as shown in figure 4. The fuselage displacements are shown in figure 5 and are given by the equations:

$$w_f = w(0) - \varphi(0)x \tag{11}$$

for the forward section and

$$w_f = w(0) - \varphi(0)x + W_1\delta \tag{12}$$

for the rearward section. The function W_1 is taken as the fundamental mode of vibration of the fuselage and tail assembly, when the fuselage is considered to be clamped as a cantilever beam at the elastic-axis location of the wing, and is given in terms of a unit deflection at the $\frac{1}{4}$-chord position on the tail. With this function to represent the elastic deformation of the fuselage the deflection and angle of attack of the tail is found with the aid of equation (12) to be

$$w_f(x_t) = w(0) - \varphi(0)x_t + \delta \tag{13}$$

$$\left.\begin{aligned} \alpha_t &= -\left.\frac{dw_f}{dx}\right]_{x=x_t} \\ &= \varphi(0) - \delta\theta_1 \end{aligned}\right\} \tag{14}$$

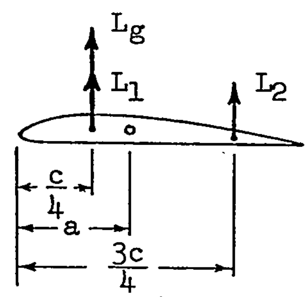
where

$$\theta_1 = \left.\frac{dW_1}{dx}\right]_{x=x_t}$$

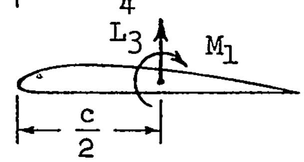
Aerodynamic lift and moment.- Before going into the details of the analysis it is felt worthwhile to define and describe the nature of the aerodynamic forces to which the wing is subjected. These forces originate from two sources: they arise directly from the gust encountered, and they arise from the ensuing airplane motion. The equations for the aerodynamic lift and moment that develops are herein set up in a convenient form on the basis of work given in references 1 to 4. In these investigations various methods for separating the lift forces have been used, but the particular method for separating these forces is not important so long as they are taken into account properly.

In the present paper the aerodynamic lift and moment are considered to be composed of two parts: one part, designated as the lift or moment due to circulation, which includes all lift forces or moments exclusive of aerodynamic inertia effects and the other part, which is due solely to these inertia effects. These lift forces and moments can be resolved into the force systems acting on the airfoil as shown in the following sketches:

Forces due to circulation



Inertia force and moment



The force L_g is the lift force developed by the gust. All the other forces occur as a result of motion of the airfoil. These forces, as well as the gust force, are given for an interval l of the span by the equations: For the forces due to circulation,

$$L_g = m_A \pi \rho c l U \int_0^t \frac{\partial v}{\partial \tau} \psi(t - \tau) d\tau \tag{15}$$

$$L_1 = m_A \pi \rho c l U \int_0^t \left[U\dot{\varphi} - \ddot{w} + c\left(\frac{3}{4} - \frac{a}{c}\right)\ddot{\varphi}\right]\left[1 - \Phi(t - \tau)\right] d\tau \tag{16}$$

$$L_2 = \frac{m_A \pi \rho l c^2}{4} U\dot{\varphi} \tag{17}$$

and for the inertia force and moment,

$$L_3 = \frac{\pi \rho l c^2}{4}\left[-\ddot{w} + \left(\frac{c}{2} - a\right)\ddot{\varphi}\right] \tag{18}$$

$$M_1 = -\frac{\pi \rho l c^4}{128}\ddot{\varphi} \tag{19}$$

where

m_A factor which can be used to introduce over-all compressibility and aspect-ratio corrections; in this paper the factor is assumed to be given by $\dfrac{A}{2 + A\sqrt{1 - M^2}}$

$1 - \Phi$ lift function which denotes the growth of lift on an airfoil following a sudden change in angle of attack

ψ lift function which denotes the growth of lift on a rigid airfoil entering a sharp-edge gust

The functions $1 - \Phi$ and ψ and the correction m_A are established as follows. In reference 5, approximate equations are derived which give the lift-coefficient form of the growth of lift on a finite wing following a sudden change in angle of attack or due to the penetration of a sharp-edge gust. The equations may conveniently be considered as the product of a factor, which may be regarded as a lift-curve slope, and an unsteady lift function, designated by $1 - \Phi$ for the function due to the angle-of-attack change and by ψ for the function due to the sharp-edge gust. These unsteady lift functions are shown in figures 6 and 7 and are given by the following equations: For the $1 - \Phi$ functions

$$(1 - \Phi)_{A=3} = 1 - 0.283e^{-0.540s} \tag{20a}$$

$$(1 - \Phi)_{A=6} = 1 - 0.361e^{-0.381s} \tag{20b}$$

$$(1 - \Phi)_{A=\infty} = 1 - 0.165e^{-0.045s} - 0.335e^{-0.300s} \tag{20c}$$

and for the ψ functions

$$\psi_{A=3} = 1 - 0.679e^{-0.558s} - 0.227e^{-3.20s} \tag{21a}$$

$$\psi_{A=6} = 1 - 0.448e^{-0.290s} - 0.272e^{-0.725s} - 0.193e^{-3.00s} \tag{21b}$$

$$\psi_{A=\infty} = 1 - 0.236e^{-0.058s} - 0.513e^{-0.364s} - 0.171e^{-2.42s} \tag{21c}$$

$$\psi_{A=\infty} = 1 - 0.500e^{-0.130s} - 0.500e^{-s} \tag{22}$$

Equations (21) are based on equations of reference 5; whereas equation (22) is the ψ function that is suggested for wings of infinite aspect ratio in reference 3. Inspection of equations (20) shows that the Φ function for aspect ratios 3 and 6 is given by a single exponential term. It is probable that the Φ function for higher aspect ratios, say 10 and even 20, may also be given to a sufficient approximation by a single exponential term. Therefore, the assumption is made that in general Φ may be represented by an equation of the form

$$\Phi = a_1 e^{-\lambda s} \tag{23}$$

Interpolation, for example, of the curves in figure 6 shows that the Φ function for aspect ratio 10 might be approximated by the equation:

$$\Phi_{A=10} = 0.41e^{-0.3s} \tag{24}$$

The analysis does not necessarily limit Φ to a single exponential term. Other terms could be added with some increase in labor, but the degree of refinement obtained is not expected to add much to the overall accuracy of the solution.

Although the functions given by equations (20) to (22) are known to approximate the true functions quite well over a large range in s $\left(s = \frac{2U}{c_o} t\right)$, the ψ functions given by equation (21) do not vanish, as they should, when $t = 0$. When used in the computational procedures given hereinafter, these functions, therefore, are to be taken as zero when $t = 0$. Another point to note is that the variable s

is given in terms of a reference chord c_0; thus this variable as applied to the wing is different, in general, from the variable as applied to the tail.

Examination of the values of lift-curve slope, which were stated to be present in the equations taken from reference 5, reveals that they may be approximated with good accuracy by the product of 2π and the often-used aspect-ratio correction $\frac{A}{A + 2}$ for steady incompressible flow. In the present paper it is assumed that compressibility and aspect-ratio corrections can be made by replacing this aspect-ratio correction by a compressible aspect-ratio correction defined by $\frac{A'}{A' + 2}$, where $A' = A\sqrt{1 - M^2}$, and by multiplying this correction by the Glauert-Prandtl Mach number correction $\frac{1}{\sqrt{1 - M^2}}$ to give the product m_A. The procedure then for taking into account three-dimensional and compressibility effects in the present analysis is to determine m_A from the forward speed and aspect ratio of the wing and to use the $1 - \Phi$ and ψ functions, equations (20) to (24), for the aspect ratio which is nearest that of the wing.

Some word of explanation of equation (16) might be worthwhile at this point. The $\Phi(t - \tau)$ function is associated with the lift forces which are due to the wake. Without this term the equation would yield the steady lift corresponding to the instantaneous values of angle of attack and vertical velocity. If equation (16) is integrated by parts and the conditions are stipulated that w, $\dot{w}$, φ, and $\dot{\varphi}$ are all zero at $t = 0$, the following equation may be found:

$$L_1 = \beta c l \dot{\Phi}_0 w - (1 - \Phi_0)\beta c l \dot{w} + \beta c l U\left[1 - \Phi_0 - \frac{c}{U}\left(\frac{3}{4} - \frac{a}{c}\right)\dot{\Phi}_0\right]\varphi +$$

$$(1 - \Phi_0)\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right)\dot{\varphi} + \beta c l \int_0^t w\ddot{\Phi}(t - \tau)d\tau - \beta c l U \int_0^t \varphi\dot{\Phi}(t - \tau)d\tau -$$

$$\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right)\int_0^t \varphi\ddot{\Phi}(t - \tau)d\tau \tag{25}$$

where β has been introduced as a forward-speed and aspect-ratio parameter defined by the equation

$$\beta = m_A \pi \rho U \tag{26}$$

With reference to equation (23), Φ_0 and $\dot{\Phi}_0$ in equation (25) would have the values

$$\Phi_0 = a_1$$

$$\dot{\Phi}_0 = -\frac{2U}{c_0}\lambda a_1$$

The form of L_1 given by equation (25) is presented because this form is more convenient to use in the present analysis.

For this analysis the total lift and moment acting at the elastic-axis location are desired. For the present, the total lift L and moment M of the forces due to circulation are found; the inertia force and moment are to be treated separately. Summation of all the lift forces due to circulation and summation of the moments of these forces about the elastic axis gives the desired equations for the aerodynamic lift and moment acting on the airfoil over an interval l as follows:

$$L = L_1 + L_2 + L_g \tag{27}$$

$$M = \left(a - \frac{c}{4}\right)L_1 - \left(\frac{3c}{4} - a\right)L_2 + \left(a - \frac{c}{4}\right)L_g \tag{28}$$

The loading on the wing.- The normal and torsional dynamic loads that are held in equilibrium by the elastic restoring forces of the wing may be found by considering all the aerodynamic and inertia forces that act on the wing. The mass situated at any station (see fig. 4) can be shown to have an inertia normal force equal to

$$-m\ddot{w} + me\ddot{\varphi}$$

and an inertia torsional moment about the elastic axis equal to

$$me\ddot{w} - mk^2\ddot{\varphi}$$

If the aerodynamic forces and moments (see equations (18), (19), (27), and (28)) are added to these inertia loadings, the total normal and torsional loadings on the wing at each station are found to be given, respectively, by the equations:

$$p = -m\ddot{w} + me\ddot{\varphi} + L + L_3$$

$$q = me\ddot{w} - mk^2\ddot{\varphi} + M - \left(\frac{c}{2} - a\right)L_3 + M_1$$

The terms L_3 and M_1 ordinarily would appear with the aerodynamic lift and moment values but are treated separately so that they can be combined with the structural mass terms. If use is made of equations (18) and (19), the loading equations become

$$p = -\overline{m}\ddot{w} + \overline{me}\ddot{\varphi} + L \tag{29}$$

$$q = \overline{me}\ddot{w} - \overline{mk^2}\ddot{\varphi} + M \tag{30}$$

where

$$\overline{m} = \left(m + \frac{\pi\rho l c^2}{4}\right)$$

$$\overline{me} = \left[me + \frac{\pi\rho l c^3}{4}\left(\frac{1}{2} - \frac{a}{c}\right)\right]$$

$$\overline{mk^2} = \left[mk^2 + \frac{\pi\rho l c^4}{4}\left(\frac{1}{2} - \frac{a}{c}\right)^2 + \frac{\pi\rho l c^4}{128}\right]$$

The terms appearing with the structural mass quantities in the definitions of $\overline{m}$, $\overline{me}$, and $\overline{mk^2}$ are the terms which are commonly associated with apparent mass effects.

The value of the shear forces L_f and the moment M_f transmitted to the wing by the fuselage can be found in the following manner: From equations (11) and (12) the values of the inertia loading on the forward and rearward sections of the fuselage can be shown to be given, respectively, by the equations:

$$p_f = -m_f\left[\ddot{w}(0) - \ddot{\varphi}(0)x\right] \tag{31}$$

$$p_f = -m_f\left[\ddot{w}(0) - \ddot{\varphi}(0)x + \ddot{\delta}W_1\right] \tag{32}$$

Integration of these inertia loadings over the length of the fuselage and addition of the aerodynamic tail load $2L_t$ give the value of the total load transmitted to the wing; one half of this load is designated by L_f and is assumed to act at station 0, the other half being considered to act through the corresponding station on the other half of the wing. Integration of the moment of the inertia loading about the elastic-axis location and addition of the moment $-2x_tL_t$ of the tail forces give the total moment due to the fuselage; one half of the moment is designated M_f and acts at station 0. The values of L_f

and M_f thus found can be given by the equations:

$$L_f = -M_1\ddot{w}(0) + M_2\ddot{\varphi}(0) - M_3\ddot{\delta} + L_t \tag{33}$$

$$M_f = M_2\ddot{w}(0) - M_4\ddot{\varphi}(0) + M_5\ddot{\delta} - x_tL_t \tag{34}$$

where the M_i's are considered to be generalized masses defined as follows:

$$\left.\begin{aligned}
M_1 &= \frac{1}{2}\int_{x_n}^{x_t} m_f \, dx \\
M_2 &= \frac{1}{2}\int_{x_n}^{x_t} m_f x \, dx \\
M_3 &= \frac{1}{2}\int_{0}^{x_t} m_f W_1 \, dx \\
M_4 &= \frac{1}{2}\int_{x_n}^{x_t} m_f x^2 dx \\
M_5 &= \frac{1}{2}\int_{0}^{x_t} m_f x W_1 \, dx \\
M_6 &= \frac{1}{2}\int_{0}^{x_t} m_f {W_1}^2 dx
\end{aligned}\right\} \tag{35}$$

The generalized mass constant M_6, although not appearing in equations (33) or (34), is included in this group because it occurs in a subsequent part of the analysis. In the derivation of equation (34), the aerodynamic moment of the tail about the tail $\frac{1}{4}$-chord position is neglected since it is considered to be small in comparison with the value x_tL_t. The lift on the tail L_t can be found by application of equation (27) to the tail surface. In this case the Φ function appropriate to the tail should be chosen and the values of displacement w and φ should be replaced by $w_f(x_t)$ and α_t, the values of deflection and angle of attack at the tail $\frac{1}{4}$-chord position. These values are given by equations (13) and (14).

Matrix equation of equilibrium.- The problem of computing the response may be considered to be one of the determination of the deflection and rotation of a beam which is subjected to normal and torque loadings. In differential form, the bending and rotational displacements are related to the normal and torque loadings by the well-known expressions:

$$\frac{\partial^2}{\partial y^2} EI \frac{\partial^2 w}{\partial y^2} = p \tag{36}$$

$$-\frac{\partial}{\partial y} GJ \frac{\partial \varphi}{\partial y} = q \tag{37}$$

where in this instance p and q are the loadings per unit length of beam. In addition to these two equations which are considered to apply to the wing, an equation for computing the elastic deformations of the fuselage may be found; this equation may be found in the following manner. The rearward part of the fuselage is considered to be a cantilever beam subjected to the inertia loading given by equation (32) and the tail force $2L_t$. If equation (36) is applied to the fuselage and use is made of equations (12) and (32), the following equation for fuselage bending results:

$$\delta \frac{\partial^2}{\partial x^2} EI_f \frac{\partial^2 W_1}{\partial x^2} = -m_f\left[\ddot{w}(0) - \ddot{\varphi}(0)x + \ddot{\delta} W_1\right] + 2L_t \tag{38}$$

in which L_t must be treated properly as a concentrated load and I_f is the bending moment of inertia of the fuselage. Since W_1 represents a vibration modal function, the following relation exists:

$$\frac{\partial^2}{\partial x^2} EI_f \frac{\partial^2 W_1}{\partial x^2} = m_f {\omega_f}^2 W_1$$

where ω_f is the frequency of vibration associated with W_1. Equation (38) may therefore be written

$$\delta m_f {\omega_f}^2 W_1 = -m_f\left[\ddot{w}(0) - \ddot{\varphi}(0)x + \ddot{\delta} W_1\right] + 2L_t$$

Multiplication of this equation through by W_1 and integration between 0 and x_t results in the following equation for fuselage bending

$$\omega_f^2 M_6 \delta = -M_3 \ddot{w}(0) + M_5 \ddot{\varphi}(0) - M_6 \ddot{\delta} + L_t \tag{39}$$

where M_3, M_5, and M_6 are defined by equations (35).

Equations (36), (37), and (39), when the loadings given by equations (29) and (30) are considered, are seen to be rather involved integro-differential equations but describe completely the motion of the airplane. The problem is to find functions w, φ, and δ which satisfy these equations and which satisfy both the boundary conditions and the initial conditions.

The problem of finding the w and φ functions may be simplified considerably by reducing the rather complicated equations of motion to a simplified and systematic algebraic form. The first step (see appendix C) is to replace the differential equations (36) and (37) for wing deflection and wing rotation by the following simple matrix equations:

$$\left[A\right]\left|w\right| = \left|p\right| \tag{40}$$

$$\left[B\right]\left|\varphi\right| = \left|q\right| \tag{41}$$

The matrices in these equations are defined in appendix C (see equations (C22) and (C23) and equations (C29) and (C30), respectively) and have been derived on the basis that the displacements along the semispan are given at six stations.

Equations (40) and (41) and the fuselage deflection coefficient δ are now combined in a single matrix equation of the form indicated as follows:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \left[A\right] & 0 \\ 0 & 0 & \left[B\right] \end{bmatrix} \begin{vmatrix} \delta \\ \left|w\right| \\ \left|\varphi\right| \end{vmatrix} = \begin{vmatrix} 0 \\ \left|p\right| \\ \left|q\right| \end{vmatrix} \tag{42}$$

This form is chosen because it will be useful subsequently. With the notation given in appendix A, equation (42) may be abbreviated to the form:

$$\begin{bmatrix} C \end{bmatrix} \begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix} = \begin{vmatrix} P \end{vmatrix} \tag{43}$$

This equation may be regarded as the loading matrix equation of equilibrium; it relates the loadings to the displacements by linear simultaneous equations. The boundary conditions are automatically satisfied when this equation is used because they had to be taken into account when the submatrices $\begin{bmatrix} A \end{bmatrix}$ and $\begin{bmatrix} B \end{bmatrix}$ were derived. Only the initial conditions remain to be satisfied and these are treated separately in a subsequent section.

Transformation of the loading equations into difference form.- The loading equations are now simplified by replacing the time derivatives by difference equations: If equation (5) is used to replace the derivative in equations (29) and (30) the values of the loading at the nth time interval are found to be

$$p_n = -\frac{\overline{m}}{\epsilon^2}(2w_n - 5w_{n-1} + 4w_{n-2} - w_{n-3}) + \frac{\overline{me}}{\epsilon^2}(2\varphi_n - 5\varphi_{n-1} + 4\varphi_{n-2} - \varphi_{n-3}) + L_n \tag{44}$$

$$q_n = \frac{\overline{me}}{\epsilon^2}(2w_n - 5w_{n-1} + 4w_{n-2} - w_{n-3}) - \frac{\overline{mk}^2}{\epsilon^2}(2\varphi_n - 5\varphi_{n-1} + 4\varphi_{n-2} - \varphi_{n-3}) + M_n \tag{45}$$

The values L_n and M_n are found by determining the expressions for L_1, L_2, and L_g at $t = n\epsilon$ (see equations (27) and (28)). Of these L_1 is the most complicated, since it (see equation (25)) involves three unsteady lift integrals of the Duhamel type. Fortunately, however, a rather simple recurrence relation can be developed which allows the calculation of the value of these integrals at a given time interval directly from the value at the previous time interval. This derivation is presented in appendix D and is made possible because the Φ function is of an exponential form. (Where the Φ function is given by more than one exponential term, a recurrence relation for each term may be written.) From the derivation in appendix D, therefore, the value of the three integrals at the nth time interval may be given as follows:

$$F_n + \frac{\beta c l \epsilon}{2}\ddot{\Phi}_0 w_n - \frac{\beta c l \epsilon}{2}\left[U\dot{\Phi}_0 + c\left(\frac{3}{4} - \frac{a}{c}\right)\ddot{\Phi}_0\right]\varphi_n \tag{46}$$

where

$$F_n = e^{-\gamma\epsilon} F_{n-1} + gw_{n-1} + g'\varphi_{n-1}$$

in which g and g' are defined by equations (A5) in appendix A. With this expression to replace the value of the integrals in equation (25), the value of L_{1_n} may be written

$$L_{1_n} = \beta c l\left(\dot{\Phi}_0 + \frac{\epsilon}{2}\ddot{\Phi}_0\right)w_n - (1 - \Phi_0)\beta c l\dot{w}_n + \beta c l\left[U(1 - \Phi_0) - \frac{\epsilon U}{2}\dot{\Phi}_0 - \left(\dot{\Phi}_0 + \frac{\epsilon}{2}\ddot{\Phi}_0\right)c\left(\frac{3}{4} - \frac{a}{c}\right)\right]\varphi_n + (1 - \Phi_0)\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right)\dot{\varphi}_n + F_n \tag{47}$$

With the use of difference equation (4), this equation may be transformed finally into the form:

$$L_{1_n} = d_0w_n + d_1w_{n-1} + d_2w_{n-2} + d_3w_{n-3} + d_0'\varphi_n + d_1'\varphi_{n-1} + d_2'\varphi_{n-2} + d_3'\varphi_{n-3} + F_n \tag{48}$$

where the d's are defined in appendix A. Likewise, from equations (4), (17), and (26), L_{2_n} may be written

$$L_{2_n} = \frac{\beta l c^2}{24\epsilon}\left(11\varphi_n - 18\varphi_{n-1} + 9\varphi_{n-2} - 2\varphi_{n-3}\right)$$

If L_{1_n}, L_{2_n}, and the value L_{g_n} of the gust force at $t = n\epsilon$ are introduced into equation (44), the value of p at the nth time interval can be shown to be given by the equation:

$$p_n = \eta_0w_n + \eta_1w_{n-1} + \eta_2w_{n-2} + \eta_3w_{n-3} + \eta_0'\varphi_n + \eta_1'\varphi_{n-1} + \eta_2'\varphi_{n-2} + \eta_3'\varphi_{n-3} + F_n + L_{g_n} \tag{49}$$

where the η's are coefficients which are given by equations (A3) in appendix A. In a similar manner, the equation for q (equation (45)) can be reduced to the form:

$$q_n = \nu_0w_n + \nu_1w_{n-1} + \nu_2w_{n-2} + \nu_3w_{n-3} + \nu_0'\varphi_n + \nu_1'\varphi_{n-1} + \nu_2'\varphi_{n-2} + \nu_3'\varphi_{n-3} + \left(a - \frac{c}{4}\right)F_n + \left(a - \frac{c}{4}\right)L_{g_n} \tag{50}$$

where the ν's are given by equations (A4) in appendix A.

The value of aerodynamic lift acting at the tail $\frac{1}{4}$-chord L_t is found most conveniently by applying equation (49) to one-half of the tail surface. This application is made by modifying the η coefficients as follows: The mass value m is set equal to zero, $\frac{a}{c}$ is taken as $\frac{1}{4}$, c is replaced by c_t, and βcl is replaced by β_t, defined as the forward-speed and aspect-ratio parameter of the tail by the equation:

$$\beta_t = \frac{1}{2} m_{A_t} \pi \rho U S_t \tag{51}$$

In addition, w and φ are replaced by the deflection and rotation of the tail given by equations (13) and (14). With these substitutions the value of L_{t_n} is found to be

$$L_{t_n} = f_0 w(0)_n + f_1 w(0)_{n-1} + f_2 w(0)_{n-2} + f_3 w(0)_{n-3} + f_0{}' \varphi(0)_n + f_1{}' \varphi(0)_{n-1} + f_2{}' \varphi(0)_{n-2} + f_3{}' \varphi(0)_{n-3} + \bar{f}_0 \delta_n + \bar{f}_1 \delta_{n-1} + \bar{f}_2 \delta_{n-2} + \bar{f}_3 \delta_{n-3} + F_{t_n} + L_{g_{t_n}} \tag{52}$$

where

$$F_{t_n} = e^{-\gamma_t \epsilon} F_{t_{n-1}} + j w(0)_{n-1} + j' \varphi(0)_{n-1} + \bar{j} \delta_{n-1} \tag{53}$$

$L_{g_{t_n}}$ is one-half the tail gust force at $t = n\epsilon$ and the f's and j's are defined by equation (A7) and (A11), respectively, in appendix A.

With equation (52) and difference equations (5), equations (33) and (34) for L_f and M_f and equation (39) for fuselage bending may be reduced readily to the following form:

$$L_{f_n} = \gamma_0 w(0)_n + \gamma_1 w(0)_{n-1} + \gamma_2 w(0)_{n-2} + \gamma_3 w(0)_{n-3} + \gamma_0{}' \varphi(0)_n + \gamma_1{}' \varphi(0)_{n-1} + \gamma_2{}' \varphi(0)_{n-2} + \gamma_3{}' \varphi(0)_{n-3} + \bar{\gamma}_0 \delta_n + \bar{\gamma}_1 \delta_{n-1} + \bar{\gamma}_2 \delta_{n-2} + \bar{\gamma}_3 \delta_{n-3} + F_{t_n} + L_{g_{t_n}} \tag{54}$$

$$M_{f_n} = \mu_0 w(0)_n + \mu_1 w(0)_{n-1} + \mu_2 w(0)_{n-2} + \mu_3 w(0)_{n-3} + \mu_0{}'\varphi(0)_n + \mu_1{}'\varphi(0)_{n-1} + \mu_2{}'\varphi(0)_{n-2} + \mu_3{}'\varphi(0)_{n-3} + \bar{\mu}_0\delta_n + \bar{\mu}_1\delta_{n-1} + \bar{\mu}_2\delta_{n-2} + \bar{\mu}_3\delta_{n-3} - x_t F_{t_n} - x_t L_{g_{t_n}} \tag{55}$$

$$0 = r_0 w(0)_n + r_1 w(0)_{n-1} + r_2 w(0)_{n-2} + r_3 w(0)_{n-3} + r_0{}'\varphi(0)_n + r_1{}'\varphi(0)_{n-1} + r_2{}'\varphi(0)_{n-2} + r_3{}'\varphi(0)_{n-3} + \bar{r}_0\delta_n + \bar{r}_1\delta_{n-1} + \bar{r}_2\delta_{n-2} + \bar{r}_3\delta_{n-3} + F_{t_n} + L_{g_{t_n}} \tag{56}$$

where the γ's, μ's, and r's are given by equations (A8) to (A10) in appendix A.

The complete set of loading equations as well as the fuselage bending equation are now available in difference form. Equations (49) and (50) apply at each spanwise station and in addition the value of L_f and M_f must be introduced at station 0. The coefficients η, ν, γ, and so forth are seen to involve only the physical properties of the airplane structure, the forward-speed and aspect-ratio parameters given by equations (26) and (51), certain constants derived from the unsteady lift function, and the time interval. The time interval ϵ that is chosen should be fairly small in comparison with the natural period of the fundamental mode in bending of the wing. To serve as a guide an interval chosen near 1/30 of the estimated period of vibration of the fundamental mode appears to be quite satisfactory. Of course, some caution should be observed in the choice of this interval if the airplane is near a critical condition where some mode other than the fundamental may predominate. For example, if the airplane is flying near the flutter speed, the characteristic frequency of the response may be near the natural torsional frequency of the wing. The time interval should be modified accordingly.

Recurrence matrix equation for response.- Equations (49), (50), (54), and (55) for loading, equation (56) for fuselage bending, and the equilibrium equation (43) may now be combined to give the recurrence matrix equation for response. In order to simplify the process of combining these equations, only the abbreviated or symbolic form of the matrices which occur are used. The definitions of these matrices are given, unless otherwise stated, in a complete group in appendix A.

Application of equations (49) and (50) to each of the spanwise stations and of equations (54) and (55) to station 0 leads to a set of loading equations which may be put in the matrix form given by the following equations:

$$\left|p\right|_n = \left|\bar{\gamma}_0\right|\delta_n + \left|\bar{\gamma}_1\right|\delta_{n-1} + \left|\bar{\gamma}_2\right|\delta_{n-2} + \left|\bar{\gamma}_3\right|\delta_{n-3} + \left[\eta_0\right]\left|w\right|_n + \left[\eta_1\right]\left|w\right|_{n-1} +$$
$$\left[\eta_2\right]\left|w\right|_{n-2} + \left[\eta_3\right]\left|w\right|_{n-3} + \left[\eta_0'\right]\left|\varphi\right|_n + \left[\eta_1'\right]\left|\varphi\right|_{n-1} + \left[\eta_2'\right]\left|\varphi\right|_{n-2} +$$
$$\left[\eta_3'\right]\left|\varphi\right|_{n-3} + \Big|\left|F\right| + \left|L_g\right|\Big|_n + \left|1\right|\left(F_t + L_{g_t}\right)_n \tag{57}$$

$$\left|q\right|_n = \left|\bar{\mu}_0\right|\delta_n + \left|\bar{\mu}_1\right|\delta_{n-1} + \left|\bar{\mu}_2\right|\delta_{n-2} + \left|\bar{\mu}_3\right|\delta_{n-3} + \left[\nu_0\right]\left|w\right|_n + \left[\nu_1\right]\left|w\right|_{n-1} +$$
$$\left[\nu_2\right]\left|w\right|_{n-2} + \left[\nu_3\right]\left|w\right|_{n-3} + \left[\nu_0'\right]\left|\varphi\right|_n + \left[\nu_1'\right]\left|\varphi\right|_{n-1} + \left[\nu_2'\right]\left|\varphi\right|_{n-2} +$$
$$\left[\nu_3'\right]\left|\varphi\right|_{n-3} + \left[\left(a - \frac{c}{4}\right)\right]\Big|\left|F\right| + \left|L_g\right|\Big|_n + \left|x_t\right|\left(F_t + L_{g_t}\right)_n \tag{58}$$

where

$$\left|F\right|_n = e^{-\gamma\epsilon}\left|F\right|_{n-1} + \left[g\right]\left|w\right|_{n-1} + \left[g'\right]\left|\varphi\right|_{n-1} \tag{59}$$

$$(F_t)_n = e^{-\gamma_t\epsilon}(F_t)_{n-1} + jw(0)_{n-1} + j'\varphi(0)_{n-1} + \bar{j}\delta_{n-1} \tag{60}$$

Equations (57) and (58) and equation (56) may now be combined to form the following matrix equation:

$$\left|\begin{matrix}0\\ \left|p\right|\\ \left|q\right|\end{matrix}\right|_n = \begin{bmatrix}\bar{r}_0 & \lfloor r_0\rfloor & \lfloor r_0{}'\rfloor\\ \left|\bar{\gamma}_0\right| & \left[\eta_0\right] & \left[\eta_0{}'\right]\\ \left|\bar{\mu}_0\right| & \left[\nu_0\right] & \left[\nu_0{}'\right]\end{bmatrix}\left|\begin{matrix}\delta\\ \left|w\right|\\ \left|\varphi\right|\end{matrix}\right|_n + \begin{bmatrix}\bar{r}_1 & \lfloor r_1\rfloor & \lfloor r_1{}'\rfloor\\ \left|\bar{\gamma}_1\right| & \left[\eta_1\right] & \left[\eta_1{}'\right]\\ \left|\bar{\mu}_1\right| & \left[\nu_1\right] & \left[\nu_1{}'\right]\end{bmatrix}\left|\begin{matrix}\delta\\ \left|w\right|\\ \left|\varphi\right|\end{matrix}\right|_{n-1} +$$

$$\begin{bmatrix}\bar{r}_2 & \lfloor r_2\rfloor & \lfloor r_2{}'\rfloor\\ \left|\bar{\gamma}_2\right| & \left[\eta_2\right] & \left[\eta_2{}'\right]\\ \left|\bar{\mu}_2\right| & \left[\nu_2\right] & \left[\nu_2{}'\right]\end{bmatrix}\left|\begin{matrix}\delta\\ \left|w\right|\\ \left|\varphi\right|\end{matrix}\right|_{n-2} + \begin{bmatrix}\bar{r}_3 & \lfloor r_3\rfloor & \lfloor r_3{}'\rfloor\\ \left|\bar{\gamma}_3\right| & \left[\eta_3\right] & \left[\eta_3{}'\right]\\ \left|\bar{\mu}_3\right| & \left[\nu_3\right] & \left[\nu_3{}'\right]\end{bmatrix}\left|\begin{matrix}\delta\\ \left|w\right|\\ \left|\varphi\right|\end{matrix}\right|_{n-3} +$$

$$\begin{bmatrix}1 & 0\\ \left|1\right| & \left[I\right]\\ \left|x_t\right| & \left[\left(a - \frac{c}{4}\right)\right]\end{bmatrix}\left|\begin{matrix}F_t + L_{g_t}\\ \left|F\right| + \left|L_g\right|\end{matrix}\right|_n \tag{61}$$

For simplicity, this equation may be abbreviated to the form:

$$\left|P\right|_n = \left[S_0\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_n + \left[S_1\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-1} + \left[S_2\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-2} + \left[S_3\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-3} +$$

$$\left[R\right]\left|\left|\overline{F}\right| + \left|\overline{L}_g\right|\right|_n \qquad (62)$$

where

$$\left|\overline{F}\right|_n = \left[e\right]\left|\overline{F}\right|_{n-1} + \left[W\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-1} \qquad (63)$$

and the matrix $\left|\overline{L}_g\right|_n$ is defined in the section entitled "Derivation of Gust Forces."

Substitution of equation (62) in equation (43) gives

$$\left[C\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_n = \left[S_0\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_n + \left[S_1\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-1} + \left[S_2\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-2} + \left[S_3\right]\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-3} +$$

$$\left[R\right]\left|\left|\overline{F}\right| + \left|\overline{L}_g\right|\right|_n \qquad (64)$$

With the use of the notation

$$\left[D\right] = \left[\left[C\right] - \left[S_0\right]\right] \qquad (65)$$

and

$$\left|Q\right|_n = \left|\begin{bmatrix}S_1\end{bmatrix}\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-1} + \begin{bmatrix}S_2\end{bmatrix}\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-2} + \begin{bmatrix}S_3\end{bmatrix}\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_{n-3} + \begin{bmatrix}R\end{bmatrix}\left|\left|\overline{F}\right| + \left|\overline{L}_g\right|\right|\right|_n \tag{66}$$

equation (64) may be written simply

$$\begin{bmatrix}D\end{bmatrix}\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_n = \left|Q\right|_n \tag{67}$$

Multiplying through by the reciprocal of $\begin{bmatrix}D\end{bmatrix}$ gives finally the equation

$$\begin{vmatrix}\delta\\ w\\ \varphi\end{vmatrix}_n = \begin{bmatrix}D\end{bmatrix}^{-1}\left|Q\right|_n \tag{68}$$

This equation gives the displacements that apply at time n in terms of the displacements that occurred at several preceding values of time (see equations (63) and (66) for the definitions of $\left|\overline{F}\right|_n$ and $\left|Q\right|_n$.

From equation (68) the complete response of the airplane can be computed once the character of the gust is known. The matrix of gust-force values $\left|\overline{L}_g\right|_n$ can be determined by the procedure given in the section entitled "Derivation of Gust Forces." The initial conditions (treated in the following section) are used to obtain three successive initial sets of the displacements. With these sets of displacements the next set may be obtained by application of equation (68). With the newly found set and the preceding sets of displacements, the next set may then be found, and so forth, until a sufficient time history of the displacements is found from which maximum loading conditions may be determined.

The reason for using the difference form of the derivatives as given by equations (4) and (5) might now be given. Equation (64) may be considered a differential equation, since the matrix $\lceil C \rceil$ contains the spanwise derivative matrices $\lceil A \rceil$ and $\lceil B \rceil$ and may be likened to the differential equation which relates the load to the deflection for a beam. The unknowns are the deflections at time n. The right-hand terms correspond to the loading, the first term being the only one which is not known since it contains the unknown deflection. The subsequent inversion of the matrix $\lceil D \rceil$ leads to, in effect, the solution to this differential equation and, in the beam analogy, corresponds to the integration of the loading to obtain the deflection. When numerical methods are used, the deflection may be computed with good accuracy by integration of the loading. On the other hand, if the difference equations which apply at an interior ordinate had been used, the matrix $\lceil C \rceil$ would have appeared as an operator on one of the known deflections on the right-hand side of the equation. Effectively, its operation would be to differentiate a known deflection, corresponding in the beam analogy to the attempt to obtain the load which caused a given deflection. This process, however, cannot be done with accuracy when numerical methods are used because of the difficulty encountered in the form of small differences of large numbers. The difference equations which apply at an outer ordinate and, consequently, lead to an integration process, therefore, have to be used.

Derivation of the Initial Response

As has been mentioned, some initial values of deflection are needed before equation (68) can be used. This section shows how these values are obtained. The airplane, just before gust penetration, is considered to be in level flight, and all displacements which follow are given relative to this level-flight condition. The initial conditions are that the vertical displacements, vertical velocity, wing rotation, and angular velocity are all zero. The gust force can be shown to start from zero and, therefore, by Newton's second law the additional initial condition can be established that the acceleration must be zero at the start of the response. These conditions can be satisfied, and the beginning of the response can be found by means of the analysis which follows.

The initial response is assumed to be given in terms of four successive ordinates, denoted by w_{-2}, w_{-1}, w_0, and w_1; the w_0 ordinate is considered, as in the case of the damped oscillator, to locate the origin of time. The first and second derivatives at the w_0 ordinate are given by equations (8) and (9). By virtue of the initial conditions (the vanishing of the deflection, velocity, and accelerations at $t = 0$), the ordinate w_0 and the derivatives given by equations (8) and (9) must be zero; therefore, the ordinates w_{-2} and w_{-1} are found to be related to the ordinate w_1 by the following relations:

$$w_{-2} = -8w_1 \tag{69}$$

$$w_{-1} = -w_1 \tag{70}$$

These relations are general and must apply for deflection and rotation at each of the spanwise stations and for the fuselage deflection as well; that is, the displacements at $t = -2\epsilon$ must be minus eight times the displacements at $t = \epsilon$, and the displacements at $t = -\epsilon$ must be the negative of those at $t = \epsilon$. Substituting these conditions in equation (64), taking n as equal to 1, and using the condition that the displacements are zero at $t = 0$ give the following matrix equation in terms of the displacement at $t = \epsilon$ alone:

$$\left[\left[D\right] + \left[S_2\right] + 8\left[S_3\right]\right]\begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix}_1 = \left[R\right]\left|\bar{L}_g\right|_1 \tag{71}$$

The term $\left|\bar{F}\right|_1$ is zero and therefore does not appear in this equation. Solution of this equation gives the values of the displacements that occur at $t = \epsilon$ $(n = 1)$.

The three sets of initial displacements required to proceed with equation (68) are thus known: the set of deflections found at $t = \epsilon$, the zero set at $t = 0$, and the set at $t = -\epsilon$ given by equation (70), or simply the negative of the displacements which were found at $t = \epsilon$. In the actual case no displacements are present at $t = -\epsilon$, but these displacements may be regarded as being of a fictitious nature the only purpose of which is to allow the step-by-step evaluation of the displacements to be started easily.

Derivation of Gust Forces

The matrix $\left|\bar{L}_g\right|_n$ which appears in the response equation (68) is derived as follows. From equation (15) and the notation of equation (26), the total gust force acting over a station section at the nth time inverval may be given by the equation

$$L_{g_n} = \beta c l \int_0^{n\epsilon} \frac{\partial v}{\partial \tau} \psi(n\epsilon - \tau)d\tau \tag{72}$$

The integral in this expression is also of the Duhamel type and since the ψ function is expressed by exponential terms (see equations (21)), the integral may be evaluated quickly by a method similar to that developed in appendix D. The procedure of computing the gust force by this equation and then the response is not recommended, however, since a complete response evaluation would have to be made for each gust considered. Instead the procedure recommended is to compute the response due to a sharp-edge gust; then with this response the response to any gust may be found directly by superposition.

Thus for the case of a sharp-edge gust, equation (72) reduces simply to

$$L_{g_n} = \beta c l v \psi_n \tag{73}$$

where ψ_n is the value of the ψ function at $t = n\epsilon$. This equation when applied to each of the spanwise stations leads directly to the matrix equation for gust force:

$$\left|L_g\right|_n = \beta \begin{vmatrix} c_0 l_0 v_0 \\ c_1 l_1 v_1 \\ c_2 l_2 v_2 \\ c_3 l_3 v_3 \\ c_4 l_4 v_4 \\ c_5 l_5 v_5 \end{vmatrix} \psi_n \tag{74}$$

If the gust is uniform in the spanwise direction, the v's in this equation will all be equal.

In a similar manner, one-half the gust force acting on the tail due to a sharp-edge gust may be shown to be

$$L_{g_{t_n}} = \beta_t v_0 \psi_{t_n} \tag{75}$$

where the gust gradient is assumed to be the same as for station 0 and ψ_{t_n} is the value of the ψ function for the tail. This equation and equation (74) may now be combined to give the desired matrix $\left|\overline{L}_g\right|_n$ as follows:

$$\left|\bar{L}_g\right|_n = \left|\begin{matrix} L_{g_t} \\ \left|L_g\right| \end{matrix}\right|_n = \begin{bmatrix} \beta_t v_0 & 0 \\ 0 & \beta c_0 l_0 v_0 \\ 0 & \beta c_1 l_1 v_1 \\ 0 & \beta c_2 l_2 v_2 \\ 0 & \beta c_3 l_3 v_3 \\ 0 & \beta c_4 l_4 v_4 \\ 0 & \beta c_5 l_5 v_5 \end{bmatrix} \left|\begin{matrix} \psi_t \\ \psi \end{matrix}\right|_n \tag{76}$$

In the application of this equation it should be kept in mind that L_{g_t} does not begin to act until the tail starts to penetrate the gust. The time interval at which penetration starts may be taken as the interval nearest to the quantity $\frac{x_t}{\epsilon U}$.

Computation of Loads and Stresses

Once the time history of the displacements has been found from equation (68), the normal or torque loading on the wing can be found with little additional work. If the notation of equation (66) is used, equation (62) may be written

$$\left|P\right|_n = \left[S_0\right] \left|\begin{matrix} \delta \\ w \\ \varphi \end{matrix}\right|_n + \left|Q\right|_n \tag{77}$$

This equation shows that the loading matrix $|P|$ may be found by adding an easily computed matrix to the matrix $|Q|$, the value of which will have already been determined in the response calculation. The loading matrix $|P|$ is remembered to be defined in terms of the normal and torque loadings, and either of these loadings may be found independently of the other.

The loadings thus found are considered to be applied statically, and the stresses are then found by ordinary static means. Since the loadings can be computed for any value of time, the complete stress history of any point in the structure may be computed. In general, the maximum stress at different points in the structure is expected to occur at different times. Some guide as to the probable time of occurrence of the most severe stress may be had, however, if the computed wing deflection is observed. It is likely that maximum stress occurs in the range where wing bending appears to be most pronounced.

The acceleration of any point in the structure may be found, if desired, with the aid of equation (5).

COMPUTATIONAL PROCEDURE

The principal results of the analysis presented in the previous sections are summarized herein in a step-by-step form. Only those steps which actually have to be performed when a determination of structural response for any airplane is being made are listed. In order to conform with standard aircraft practice the use of inch-pound-second units throughout is recommended.

The steps are as follows:

Preliminary steps:

(1) The wing semispan is divided into six sections and a station is located at the middle of each section (see fig. 3). The sections are proportioned in any convenient manner so that certain stations will fall at concentrated mass locations, such as engines or fuel tanks. Station 0 is located near where the fuselage intersects the wing and station 5 is located near the tip. The properties EI, GJ, $\overline{m}$, $\overline{me}$, and $\overline{mk^2}$ are then computed at each station.

(2) From the EI, GJ, and λ_i values determine the $\left[A\right]$ and $\left[B\right]$ matrices by the method given in appendix C.

(3) Compute the gust-force values at the successive time intervals for both the wing and the tail. (See section entitled "Derivation of Gust Forces.") The ψ functions used are taken from equations (21) or (22) for the aspect ratios which are nearest to those of the wing and tail, respectively. A time interval that appears satisfactory is one in the neighborhood of 1/30 of the estimated natural period of the fundamental bending mode of the wing.

The recurrence equation:

(4) With the quantities determined in steps (1) and (2), determine the matrix elements given by equations (A3) to (A5) at each of the spanwise stations.

(5) Compute the fuselage and tail coefficients given by equations (A8) to (A11). (See definition of M_1, M_2, M_3, M_4, M_5, and M_6 given by equations (35).)

(6) With the use of the coefficients determined in steps (4) and (5), set up the following matrices: $\left[D\right]$, $\left[S_1\right]$, $\left[S_2\right]$, $\left[S_3\right]$, $\left\lceil R\right|$, $\left[e\right|$, and $\left\lceil W\right|$. These matrices are defined in appendix A and for the most part are found from simple diagonal matrices of the coefficients determined in steps (4) and (5). The form, for example, of the $\left[S\right]$ matrices is illustrated in table 1 with randomly chosen numbers. All elements which are not shown are zero. It may be of interest to explain briefly the significance of the various terms that appear in the matrix. In order to facilitate the explanation the matrix has been partitioned into several submatrices. The terms in the upper left-hand box are associated with wing bending and the airplane vertical motion; whereas the terms in the lower right-hand box are associated with wing torsion and airplane pitching. The terms along the two subdiagonals serve to couple together the bending and twisting action. The terms in the first row and first column are associated with fuselage bending. The omission of certain terms in the matrix will lead to the matrix which applies to the more simple type of aircraft motion. For the case, for example, in which only wing bending and vertical motion are to be considered, computation of only the terms which make up the upper left-hand box is sufficient.

(7) Determine the reciprocal of the $\left[D\right]$ matrix and set up the following matrix equation:

$$\begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix}_n = \left[D\right]^{-1} \left| Q \right|_n \tag{78}$$

where

$$\left| Q \right|_n = \left| \left[S_1\right] \begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix}_{n-1} + \left[S_2\right] \begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix}_{n-2} + \left[S_3\right] \begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix}_{n-3} + \left\lceil R \right\rfloor \left| \left|\bar{F}\right| + \left|\bar{L}_g\right| \right| \right|_n$$

in which

$$\left|\bar{F}\right|_n = \left[e\right]\left|\bar{F}\right|_{n-1} + \left[W\right]\begin{vmatrix}\delta \\ w \\ \varphi\end{vmatrix}_{n-1}$$

In these equations the matrices containing δ, w, and φ are displacement matrices and are defined in appendix A. The matrix $\left|\bar{F}\right|$ takes into account the forces which develop due to the "wake effect," and $\left|\bar{L}_g\right|$ is the gust-force matrix which is derived in step (3). Equation (78) is seen to give the displacements that occur at time n in terms of the displacements which occurred at the times $n - 1$, $n - 2$, and $n - 3$.

The initial response:

(8) By use of the matrices given in step (6) and the gust forces which apply at $n = 1$, set up the following matrix equation:

$$\left[\left[D\right] + \left[S_2\right] + 8\left[S_3\right]\right]\begin{vmatrix}\delta \\ w \\ \varphi\end{vmatrix}_1 = \left[R\right]\left|\bar{L}_g\right|_1 \qquad (79)$$

The term $\left|\bar{F}\right|_1$ does not appear in this equation because it is zero.

(9) Solve equation (79) for the displacements. Any convenient method, such as the Crout method (see reference 6), may be used. The displacements found will be the value of displacements that apply at $t = \epsilon$ or $n = 1$.

The response:

(10) The response may now be found by successive application of equation (78). The response at $n = 1$ has been found in step (9); the response at $n = 2$ is next to be determined. The values of the displacements in the $n - 2$ term of the response equation are all taken to be zero (initial condition), and the values in the $n - 3$ term are taken as the negative of those found in step (9). The gust forces to use are those which apply at $n = 2$. The deflections that apply at $n = 2$ are then found by matrix algebra. For convenience the column matrix $\left|Q\right|$ is evaluated first, and then multiplication of this column

matrix by the reciprocal of the $[D]$ matrix gives the deflections at $n = 2$. With the newly found deflections at $n = 2$ and the deflections at $n = 1$ and $n = 0$, the deflections at $n = 3$ can be found, and so forth. This process is continued until the wing bending appears to be the most pronounced.

Wing loading:

(11) With the deflections known, the value of wing loading in bending or in torsion can be computed directly from equation (77). The stresses at any point can then be computed from the wing loading by ordinary static means. Since the loading may be computed at any value of time, the complete stress history of any point on the structure may be computed.

EXAMPLE

As an illustration of the method of analysis given in the present paper, the response of a typical two-engine airplane due to a sharp-edge gust is determined. For brevity the fuselage is assumed rigid and only vertical displacement and wing bending are considered. The weight variation over the wing semispan and the equivalent-weight concentrations are shown in figure 8. In this figure are shown also the station locations and the interval covered by each station section. The solution is made for a forward velocity of flight of about 210 miles per hour and a gust velocity of 10 feet per second. In tables 2, 3, and 4 are listed, respectively, the various physical characteristics and the factors which come from the unsteady lift function, the values of the ψ function and the gust-force matrix, and the matrix elements that are required for the solution (steps (1) to (5)). The Φ function for an aspect ratio of 6 was chosen (see equation (20b)); and the ψ function for an aspect ratio of infinity (equation (22)) was used.

The $[A]$ matrix as computed from the values of λ and EI listed in table 4 is shown in table 5(a). In the computation of the η values shown in table 4 for station 0, the fuselage was treated as a concentrated wing mass. This treatment is allowable since the fuselage is assumed rigid and further saves the work of computing the γ values (see equations (A8)). The $\left[[A] - [S_0]\right]$ or $[D]$ matrix, which in this case applies only to bending and vertical displacement, is shown in table 5(b). The equation which is formed from equation (78) (step (7)) and which involves the reciprocal of $[D]$ and the $[S_i]$ and $[R]$ matrices is shown in table 6. The equation for computing the initial response (step (8)) is shown in table 7.

The solution to these equations is shown in figure 9 in which deflection in inches is plotted against spanwise station points for various intervals of time. For clarity the deflections for the odd intervals have been left off. From these curves the consequent wing bending and the manner in which the airplane is swept upward by the gust can be seen. The time histories of the loads (equation (77)) that occur at each of the spanwise stations are shown in figure 10. These curves indicate the presence of some second-mode excitation in the response. The stresses that occur at stations 0, 1, and 2 are shown in figure 11. The presence of second-mode excitation is not readily discernible from the stress curves.

DISCUSSION

A method for computing the stresses and structural action of an airplane flying through a gust has been given. The method is based on aerodynamic strip theory, but over-all corrections for compressibility and three-dimensional effects can be made as is indicated by a suggested correction procedure. Some tail forces are included in the analysis and others might equally well be included if their character is known.

The analysis as given is general enough to include the wing bending and twisting flexibilities and the fuselage flexibility. In a good many cases that may be considered, however, the last two of the flexibilities may prove to be of negligible importance. Some investigators have indicated (see reference 1) that unless the forward speed of the airplane approaches the flutter or divergence speed of the wing, the torsional deformations do not have to be included. Likewise, in cases in which the fuselage is rather stiff, the effect of fuselage flexibility on the response may be rather small. In such cases in which either or both of these flexibilities may be ignored, the analysis is, of course, simplified and shortened. The example presented in the previous section illustrates this point. In the present state of understanding of gust-response analysis, enough information is not available to indicate definitely when and when not to include the various flexibilities of the aircraft structure. The analysis in the present paper may provide a useful means to assess their importance. The extent, for example, to which coupling exists between wing bending and wing torsion in any particular case may be seen by comparing the displacements obtained from the coupling terms with the displacements obtained from the noncoupling terms.

Both the symmetrical and antisymmetrical types of gusts can be handled by the analysis given in the present paper. In general, the symmetrical gust is expected to produce the most severe stress condition, and therefore only the matrices which apply for a symmetrical case have been given. These matrices were derived by using the boundary conditions for the symmetrical deformation of a free-free beam. The case of an antisymmetrical gust can be treated by replacing these matrices by the ones which apply for the antisymmetrical deformation of a free-free beam. The case of a general unsymmetrical gust can be handled by first breaking the gust into two parts, a symmetrical part and an antisymmetrical part, and then treating each part independently.

It might be of interest at this point to compare briefly the matrix method to a modal-function solution. One of the chief disadvantages of the modal-function solution is that the modes and frequencies of natural vibration of the structure have to be computed in advance. Then, a large number of integrals which involve these modes have to be determined in order to set up the problem. In the present analysis the physical properties of the airplane are used directly in the setting-up of the problem. Further, in order to make the modal solution practical the higher modes must be dropped and only the basic or fundamental modes can be used. Hence, the success of the analysis depends to a large degree on how well single modal functions, one mode each for bending and torsion, can represent the airplane distortion. In the analysis of the present paper the distortions are found for all practical purposes as the correct values at a number of spanwise stations, at least to within the accuracy to which the aerodynamic and structural parameters are known. Also, in this analysis, probably the most difficult operation is the inversion of the matrix $[D]$, which is actually not a very involved operation, especially when done by the quick and systematic procedure afforded by the Crout method (reference 6).

The present paper indicates the methods for determining the response for both a sharp-edge gust and a gust of arbitrary shape. Probably the best approach, however, is to compute only the response for a sharp-edge gust, since the response for any arbitrary gust may thereafter be computed by means of Duhamel's integral. To follow such a procedure would also save a great amount of work in the evaluation of the gust forces.

One of the important features of the method of analysis presented is that it is not restricted to the gust problem. The approach used may be used to treat other problems of a similar nature. The landing problem can be handled by simply replacing the distributed gust force by the concentrated landing forces. In the landing problem also, the problem is set up much more easily since the aerodynamic terms do not ordinarily have to be included. However, the landing problem in which aerodynamic forces are included may be solved by this method if desired. The approach used herein may also be used to solve the problem of the release of heavy objects such as bombs. This problem could be considered the inverse of the gust problem; a load is released rather than encountered. Some maneuvering problems, such as the sudden deflection of the ailerons, and a number of other transient problems might also be treated by an approach similar to that given in the present paper.

CONCLUDING REMARKS

A method for computing the stresses and structural response of an aircraft flying through a gust has been presented. The method is based on aerodynamic strip theory, but compressibility and three-dimensional effects can be taken into account approximately by means of over-all corrections. The method takes into account wing bending and twisting deformations, fuselage deflection, vertical and pitching motion of the airplane, and some tail forces. A sharp-edge gust or a gust of arbitrary shape in the spanwise or flight directions may be treated. A suggested computational procedure is given to aid in the application of the method to any specific case.

The possibilities of applying the method to a variety of transient aircraft problems, such as landing, are brought out.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., January 19, 1950

APPENDIX A

DEFINITIONS OF MATRICES USED IN ANALYSIS

For convenience in presentation, most of the matrices and matrix elements that are used in the analysis are defined in this appendix. The matrices are presented without derivations, but their origin should become evident by a study of the analysis.

Matrices.- The various matrices that are used in the analysis are defined as follows for the case in which the wing semispan is divided into six sections: (The elements which are used in the matrices are defined in the subsequent section.)

$$|w| = \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}$$

$$|\varphi| = \begin{vmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \\ \varphi(5) \\ \varphi(6) \end{vmatrix}$$

$$\begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix} = \begin{vmatrix} \delta \\ |w| \\ |\varphi| \end{vmatrix}$$

$$\left| p \right| = \left| \begin{matrix} p(0) \\ p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{matrix} \right|$$

$$\left| q \right| = \left| \begin{matrix} q(0) \\ q(1) \\ q(2) \\ q(3) \\ q(4) \\ q(5) \end{matrix} \right|$$

$$\left| P \right| = \left| \begin{matrix} 0 \\ \left| p \right| \\ \left| q \right| \end{matrix} \right|$$

$$\left.\begin{matrix} \left[A \right] \\ \left[B \right] \end{matrix}\right\}$$ See appendix C for definitions.

$$\left[C \right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \left[A \right] & 0 \\ 0 & 0 & \left[B \right] \end{bmatrix}$$

$$\left[\eta_i\right] = \begin{bmatrix} \eta_i(0) + \gamma_i & & & & & \\ & \eta_i(1) & & & & \\ & & \eta_i(2) & & & \\ & & & \eta_i(3) & & \\ & & & & \eta_i(4) & \\ & & & & & \eta_i(5) \end{bmatrix}$$

$$\left[\eta_i'\right] = \begin{bmatrix} \eta_i'(0) + \gamma_i' & & & & & \\ & \eta_i'(1) & & & & \\ & & \eta_i'(2) & & & \\ & & & \eta_i'(3) & & \\ & & & & \eta_i'(4) & \\ & & & & & \eta_i'(5) \end{bmatrix}$$

$$\left[\nu_i\right] = \begin{bmatrix} \nu_i(0) + \mu_i & & & & & \\ & \nu_i(1) & & & & \\ & & \nu_i(2) & & & \\ & & & \nu_i(3) & & \\ & & & & \nu_i(4) & \\ & & & & & \nu_i(5) \end{bmatrix}$$

$$\left[\nu_i'\right] = \begin{bmatrix} \nu_i'(0) + \mu_i' & & & & & \\ & \nu_i'(1) & & & & \\ & & \nu_i'(2) & & & \\ & & & \nu_i'(3) & & \\ & & & & \nu_i'(4) & \\ & & & & & \nu_i'(5) \end{bmatrix}$$

$$\left|\bar{\gamma}_i\right| = \begin{vmatrix} \bar{\gamma}_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\left|\bar{\mu}_i\right| = \begin{vmatrix} \bar{\mu}_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\lfloor r_i \rfloor = \lfloor r_i \; 0 \; 0 \; 0 \; 0 \; 0 \rfloor$$

$$\lfloor r_i' \rfloor = \lfloor r_i' \; 0 \; 0 \; 0 \; 0 \; 0 \rfloor$$

$$\left[S_i\right] = \begin{bmatrix} \bar{r}_i & \lfloor r_i \rfloor & \lfloor r_i' \rfloor \\ \left|\bar{\gamma}_i\right| & \left[\eta_i\right] & \left[\eta_i'\right] \\ \left|\bar{\mu}_i\right| & \left[\nu_i\right] & \left[\nu_i'\right] \end{bmatrix}$$

$$\left|1\right| = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$\left|x_t\right| = \begin{vmatrix} -x_t \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

$$[I] = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$\left[\left(a - \frac{c}{4}\right)\right] = \begin{bmatrix} \left(a - \frac{c}{4}\right)_0 & & & & & \\ & \left(a - \frac{c}{4}\right)_1 & & & & \\ & & \left(a - \frac{c}{4}\right)_2 & & & \\ & & & \left(a - \frac{c}{4}\right)_3 & & \\ & & & & \left(a - \frac{c}{4}\right)_4 & \\ & & & & & \left(a - \frac{c}{4}\right)_5 \end{bmatrix}$$

$$\lceil R \rfloor = \begin{bmatrix} 1 & 0 \\ \left|1\right| & [I] \\ \left|x_t\right| & \left[\left(a - \frac{c}{4}\right)\right] \end{bmatrix}$$

$$[e] = \begin{bmatrix} e^{-\gamma t \epsilon} & & & & & & \\ & e^{-\gamma\epsilon} & & & & & \\ & & e^{-\gamma\epsilon} & & & & \\ & & & e^{-\gamma\epsilon} & & & \\ & & & & e^{-\gamma\epsilon} & & \\ & & & & & e^{-\gamma\epsilon} & \\ & & & & & & e^{-\gamma\epsilon} \end{bmatrix}$$

$$\lfloor j \rfloor = \lfloor j \; 0 \; 0 \; 0 \; 0 \; 0 \rfloor$$

$$\lfloor j' \rfloor = \lfloor j' \; 0 \; 0 \; 0 \; 0 \; 0 \rfloor$$

$$[g] = \begin{bmatrix} g(0) & & & & & \\ & g(1) & & & & \\ & & g(2) & & & \\ & & & g(3) & & \\ & & & & g(4) & \\ & & & & & g(5) \end{bmatrix}$$

$$[g'] = \begin{bmatrix} g'(0) & & & & & \\ & g'(1) & & & & \\ & & g'(2) & & & \\ & & & g'(3) & & \\ & & & & g'(4) & \\ & & & & & g'(5) \end{bmatrix}$$

$$\lceil W \rfloor = \begin{bmatrix} \bar{j} & \lfloor j \rfloor & \lfloor j' \rfloor \\ 0 & [g] & [g'] \end{bmatrix}$$

Matrix elements.- The matrix elements which appear in the matrices defined in the previous section are expressed for convenience in terms of the following common factors:

$$\left.\begin{aligned} \beta &= m_A \pi\rho U & \Phi_0 &= a_1 \\ \gamma &= \frac{2U}{c_0}\lambda & \dot{\Phi}_0 &= -\gamma a_1 \\ & & \ddot{\Phi}_0 &= \gamma^2 a_1 \end{aligned}\right\} \tag{A1}$$

in which the last four are associated with the Φ function for the wing. (See equation (23).) With these factors the elements that must be computed at each spanwise station are as follows:

$$\left.\begin{aligned}
d_0 &= -\frac{11}{6\epsilon}(1 - \Phi_0)\beta c l + \beta c l\left(\dot{\Phi}_0 + \frac{1}{2}\ddot{\Phi}_0\epsilon\right) \\
d_1 &= \frac{3}{\epsilon}(1 - \Phi_0)\beta c l \\
d_2 &= -\frac{3}{2\epsilon}(1 - \Phi_0)\beta c l \\
d_3 &= \frac{1}{3\epsilon}(1 - \Phi_0)\beta c l \\
d_0' &= \frac{11}{6\epsilon}(1 - \Phi_0)\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right) + \beta c l\left[U(1 - \Phi_0) - \right. \\
&\qquad \left. \frac{1}{2}\dot{\Phi}_0 U\epsilon - \left(\dot{\Phi}_0 + \frac{1}{2}\ddot{\Phi}_0\epsilon\right)c\left(\frac{3}{4} - \frac{a}{c}\right)\right] \\
d_1' &= -\frac{3}{\epsilon}(1 - \Phi_0)\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right) \\
d_2' &= \frac{3}{2\epsilon}(1 - \Phi_0)\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right) \\
d_3' &= -\frac{1}{3\epsilon}(1 - \Phi_0)\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right)
\end{aligned}\right\} \tag{A2}$$

$$\left.\begin{aligned}
\eta_0 &= -\frac{2\overline{m}}{\epsilon^2} + d_0 & \eta_0' &= \frac{2\overline{me}}{\epsilon^2} + d_0' + \frac{11}{24\epsilon}\beta c^2 l \\
\eta_1 &= \frac{5\overline{m}}{\epsilon^2} + d_1 & \eta_1' &= -\frac{5\overline{me}}{\epsilon^2} + d_1' - \frac{3}{4\epsilon}\beta c^2 l \\
\eta_2 &= -\frac{4\overline{m}}{\epsilon^2} + d_2 & \eta_2' &= \frac{4\overline{me}}{\epsilon^2} + d_2' + \frac{9}{24\epsilon}\beta c^2 l \\
\eta_3 &= \frac{\overline{m}}{\epsilon^2} + d_3 & \eta_3' &= -\frac{\overline{me}}{\epsilon^2} + d_3' - \frac{1}{12\epsilon}\beta c^2 l
\end{aligned}\right\} \tag{A3}$$

$$\left.\begin{aligned}
\nu_0 &= \frac{2\overline{me}}{\epsilon^2} + d_0\left(a - \frac{c}{4}\right) \\
\nu_1 &= -\frac{5\overline{me}}{\epsilon^2} + d_1\left(a - \frac{c}{4}\right) \\
\nu_2 &= \frac{4\overline{me}}{\epsilon^2} + d_2\left(a - \frac{c}{4}\right) \\
\nu_3 &= -\frac{\overline{me}}{\epsilon^2} + d_3\left(a - \frac{c}{4}\right) \\
\nu_0' &= -\frac{2\overline{mk}^2}{\epsilon^2} + d_0'\left(a - \frac{c}{4}\right) - \frac{11}{24\epsilon}\beta c^3 l\left(\frac{3}{4} - \frac{a}{c}\right) \\
\nu_1' &= \frac{5\overline{mk}^2}{\epsilon^2} + d_1'\left(a - \frac{c}{4}\right) + \frac{3}{4\epsilon}\beta c^3 l\left(\frac{3}{4} - \frac{a}{c}\right) \\
\nu_2' &= -\frac{4\overline{mk}^2}{\epsilon^2} + d_2'\left(a - \frac{c}{4}\right) + \frac{9}{24\epsilon}\beta c^3 l\left(\frac{3}{4} - \frac{a}{c}\right) \\
\nu_3' &= \frac{\overline{mk}^2}{\epsilon^2} + d_3'\left(a - \frac{c}{4}\right) + \frac{1}{12\epsilon}\beta c^3 l\left(\frac{3}{4} - \frac{a}{c}\right)
\end{aligned}\right\} \tag{A4}$$

$$\left.\begin{aligned}
g &= \beta c l \ddot{\Phi}_0 \epsilon e^{-\gamma\epsilon} \\
g' &= \beta c l \epsilon e^{-\gamma\epsilon}\left[-U\dot{\Phi}_0 - \ddot{\Phi}_0 c\left(\frac{3}{4} - \frac{a}{c}\right)\right]
\end{aligned}\right\} \tag{A5}$$

The coefficients which must be computed for the fuselage and tail are expressed in part in terms of the following common factors:

$$\left.\begin{aligned} \beta_t &= \frac{1}{2}m_{A_t}\pi\rho U S_t & \Phi_{t_0} &= a_{1_t} \\ \gamma_t &= \frac{2U}{c_{0_t}}\lambda_t & \dot{\Phi}_{t_0} &= -\gamma_t a_{1_t} \\ & & \ddot{\Phi}_{t_0} &= \gamma_t^2 a_{1_t} \end{aligned}\right\} \tag{A6}$$

in which the last four are associated with the Φ functions appropriate to the tail. Also used are the generalized masses given by equations (35) and the value θ_1 as given in equation (14). With these factors the coefficients for the tail and fuselage are as follows:

$$\left.\begin{aligned} f_0 &= -\frac{11}{6\epsilon}\left(1 - \Phi_{t_0}\right)\beta_t + \beta_t\left(\dot{\Phi}_{t_0} + \frac{1}{2}\ddot{\Phi}_{t_0}\epsilon\right) \\ f_1 &= \frac{3}{\epsilon}\left(1 - \Phi_{t_0}\right)\beta_t \\ f_2 &= -\frac{3}{2\epsilon}\left(1 - \Phi_{t_0}\right)\beta_t \\ f_3 &= \frac{1}{3\epsilon}\left(1 - \Phi_{t_0}\right)\beta_t \end{aligned}\right\} \tag{A7a}$$

$$\left.\begin{aligned} f_0' &= \frac{11}{6\epsilon}\left(1 - \Phi_{t_0}\right)\left(x_t + \frac{1}{2}c_t\right)\beta_t + \beta_t\Big[U\left(1 - \Phi_{t_0}\right) - \\ &\qquad \frac{1}{2}\dot{\Phi}_{t_0}U\epsilon - \left(\dot{\Phi}_{t_0} + \frac{1}{2}\ddot{\Phi}_{t_0}\epsilon\right)\left(x_t + \frac{1}{2}c_t\right)\Big] + \frac{11}{24\epsilon}\beta_t c_t \\ f_1' &= -\frac{3}{\epsilon}\left(1 - \Phi_{t_0}\right)\left(x_t + \frac{1}{2}c_t\right)\beta_t - \frac{3}{4\epsilon}\beta_t c_t \\ f_2' &= \frac{3}{2\epsilon}\left(1 - \Phi_{t_0}\right)\left(x_t + \frac{1}{2}c_t\right)\beta_t + \frac{9}{24\epsilon}\beta_t c_t \\ f_3' &= -\frac{1}{3\epsilon}\left(1 - \Phi_{t_0}\right)\left(x_t + \frac{1}{2}c_t\right)\beta_t - \frac{1}{12\epsilon}\beta_t c_t \end{aligned}\right\} \tag{A7b}$$

$$
\left.
\begin{aligned}
\bar{f}_0 &= -\frac{11}{6\epsilon}\left(1 - \Phi_{t_0}\right)\left(1 + \frac{1}{2}c_t\theta_1\right)\beta_t - \beta_t\left[U\theta_1\left(1 - \Phi_{t_0}\right) - \right. \\
&\qquad \left. \frac{1}{2}\dot{\Phi}_0 U\epsilon\theta_1 - \left(\dot{\Phi}_{t_0} + \frac{1}{2}\ddot{\Phi}_{t_0}\epsilon\right)\left(1 + \frac{1}{2}c_t\theta_1\right)\right] - \frac{11}{24\epsilon}\beta_t c_t\theta_1 \\
\bar{f}_1 &= \frac{3}{\epsilon}\left(1 - \Phi_{t_0}\right)\left(1 + \frac{1}{2}c_t\theta_1\right)\beta_t + \frac{3}{4\epsilon}\beta_t c_t\theta_1 \\
\bar{f}_2 &= -\frac{3}{2\epsilon}\left(1 - \Phi_{t_0}\right)\left(1 + \frac{1}{2}c_t\theta_1\right)\beta_t - \frac{9}{24\epsilon}\beta_t c_t\theta_1 \\
\bar{f}_3 &= \frac{1}{3\epsilon}\left(1 - \Phi_{t_0}\right)\left(1 + \frac{1}{2}c_t\theta_1\right)\beta_t + \frac{1}{12\epsilon}\beta_t c_t\theta_1
\end{aligned}
\right\} \qquad \text{(A7c)}
$$

$$
\left.
\begin{aligned}
\gamma_0 &= -\frac{2M_1}{\epsilon^2} + f_0 \\
\gamma_1 &= \frac{5M_1}{\epsilon^2} + f_1 \\
\gamma_2 &= -\frac{4M_1}{\epsilon^2} + f_2 \\
\gamma_3 &= \frac{M_1}{\epsilon^2} + f_3
\end{aligned}
\right\} \qquad \text{(A8a)}
$$

$$
\left.
\begin{aligned}
\gamma_0' &= \frac{2M_2}{\epsilon^2} + f_0' \\
\gamma_1' &= -\frac{5M_2}{\epsilon^2} + f_1' \\
\gamma_2' &= \frac{4M_2}{\epsilon^2} + f_2' \\
\gamma_3' &= -\frac{M_2}{\epsilon^2} + f_3'
\end{aligned}
\right\} \qquad \text{(A8b)}
$$

$$\left.\begin{aligned}
\bar{\gamma}_0 &= -\frac{2M_3}{\epsilon^2} + \bar{f}_0 \\
\bar{\gamma}_1 &= \frac{5M_3}{\epsilon^2} + \bar{f}_1 \\
\bar{\gamma}_2 &= -\frac{4M_3}{\epsilon^2} + \bar{f}_2 \\
\bar{\gamma}_3 &= \frac{M_3}{\epsilon^2} + \bar{f}_3
\end{aligned}\right\} \tag{A8c}$$

$$\left.\begin{aligned}
\mu_0 &= \frac{2M_2}{\epsilon^2} - x_t f_0 \\
\mu_1 &= -\frac{5M_2}{\epsilon^2} - x_t f_1 \\
\mu_2 &= \frac{4M_2}{\epsilon^2} - x_t f_2 \\
\mu_3 &= -\frac{M_2}{\epsilon^2} - x_t f_3
\end{aligned}\right\} \tag{A9a}$$

$$\left.\begin{aligned}
\mu_0' &= -\frac{2M_4}{\epsilon^2} - x_t f_0' \\
\mu_1' &= \frac{5M_4}{\epsilon^2} - x_t f_1' \\
\mu_2' &= -\frac{4M_4}{\epsilon^2} - x_t f_2' \\
\mu_3' &= \frac{M_4}{\epsilon^2} - x_t f_3'
\end{aligned}\right\} \tag{A9b}$$

$$\left.\begin{aligned}
\bar{\mu}_0 &= \frac{2M_5}{\epsilon^2} - x_t \bar{f}_0 \\
\bar{\mu}_1 &= -\frac{5M_5}{\epsilon^2} - x_t \bar{f}_1 \\
\bar{\mu}_2 &= \frac{4M_5}{\epsilon^2} - x_t \bar{f}_2 \\
\bar{\mu}_3 &= -\frac{M_5}{\epsilon^2} - x_t \bar{f}_3
\end{aligned}\right\} \tag{A9c}$$

$$\left.\begin{aligned}
r_0 &= -\frac{2M_3}{\epsilon^2} + f_0 \\
r_1 &= \frac{5M_3}{\epsilon^2} + f_1 \\
r_2 &= -\frac{4M_3}{\epsilon^2} + f_2 \\
r_3 &= \frac{M_3}{\epsilon^2} + f_3
\end{aligned}\right\} \tag{A10a}$$

$$\left.\begin{aligned}
r_0' &= \frac{2M_5}{\epsilon^2} + f_0' \\
r_1' &= -\frac{5M_5}{\epsilon^2} + f_1' \\
r_2' &= \frac{4M_5}{\epsilon^2} + f_2' \\
r_3' &= -\frac{M_5}{\epsilon^2} + f_3'
\end{aligned}\right\} \tag{A10b}$$

$$\left.\begin{aligned}
\bar{r}_0 &= -\frac{2M_6}{\epsilon^2} + \bar{f}_0 - \omega_f^2 M_6 \\
\bar{r}_1 &= \frac{5M_6}{\epsilon^2} + \bar{f}_1 \\
\bar{r}_2 &= -\frac{4M_6}{\epsilon^2} + \bar{f}_2 \\
\bar{r}_3 &= \frac{M_6}{\epsilon^2} + \bar{f}_3
\end{aligned}\right\} \tag{A10c}$$

$$\left.\begin{aligned}
j &= \beta_t \ddot{\Phi}_{t_0} \epsilon e^{-\gamma_t \epsilon} \\
j' &= \beta_t \epsilon e^{-\gamma_t \epsilon}\left[-U\dot{\Phi}_{t_0} - \ddot{\Phi}_{t_0}\left(x_t + \frac{1}{2}c_t\right)\right] \\
\bar{j} &= \beta_t \epsilon e^{-\gamma_t \epsilon}\left[U\dot{\Phi}_{t_0}\theta_1 + \ddot{\Phi}_{t_0}\left(1 + \frac{1}{2}c_t\theta_1\right)\right]
\end{aligned}\right\} \tag{A11}$$

APPENDIX B

DERIVATION OF DIFFERENCE EQUATIONS

In this appendix the parabolic and cubic difference equations for the first and second derivatives of a function are derived.

Parabolic equations.- For the parabolic difference equation, consider the function shown in figure 12(a). This function is assumed to be replaced by the arc of a parabola which passes through the three ordinates a, b, and c. It can be verified readily that such a curve can be given by the equation

$$w = \frac{a}{2}\left(\frac{y}{\epsilon} - 1\right)\left(\frac{y}{\epsilon} - 2\right) - b\frac{y}{\epsilon}\left(\frac{y}{\epsilon} - 2\right) + \frac{c}{2}\,\frac{y}{\epsilon}\left(\frac{y}{\epsilon} - 1\right) \tag{B1}$$

The first and second derivatives of this equation at $y = \epsilon$ are given by the equations

$$\left.\frac{dw}{dy}\right]_{y=\epsilon} = \frac{c - a}{2\epsilon} \tag{B2}$$

$$\left.\frac{d^2w}{dy^2}\right]_{y=\epsilon} = \frac{c - 2b + a}{\epsilon^2} \tag{B3}$$

These equations are the standard difference equations for the first and second derivatives of a function. The derivatives are purposely taken at the middle of the three ordinates because the resulting equations are suitable for use in the simplification of many problems. If the derivative had been taken at an outer ordinate, the approximation afforded would not be accurate enough to be useful.

Cubic equations.- The cubic difference equations may be derived in a manner similar to that for the parabolic equations. In this case four successive ordinates are used. (See fig. 12(b).) The function is replaced by a third-degree curve which is given by the equation

$$w = -\frac{a}{6}\left(\frac{y}{\epsilon} - 1\right)\left(\frac{y}{\epsilon} - 2\right)\left(\frac{y}{\epsilon} - 3\right) + \frac{b}{2}\frac{y}{\epsilon}\left(\frac{y}{\epsilon} - 2\right)\left(\frac{y}{\epsilon} - 3\right) - \frac{c}{2}\frac{y}{\epsilon}\left(\frac{y}{\epsilon} - 1\right)\left(\frac{y}{\epsilon} - 3\right) +$$

$$\frac{d}{6}\frac{y}{\epsilon}\left(\frac{y}{\epsilon} - 1\right)\left(\frac{y}{\epsilon} - 2\right) \tag{B4}$$

Because of the increase in accuracy that results from the use of a higher-degree curve, the first and second derivatives may be taken at an outer ordinate with an accuracy which is about equivalent to that given by equations (B2) and (B3). The derivatives at $y = 3\epsilon$ are

$$\left.\frac{dw}{dy}\right]_{y=3\epsilon} = \frac{11d - 18c + 9b - 2a}{6\epsilon} \tag{B5}$$

$$\left.\frac{d^2w}{dy^2}\right]_{y=3\epsilon} = \frac{2d - 5c + 4b - a}{\epsilon^2} \tag{B6}$$

If taken at the third of the four ordinates, the derivatives are

$$\left.\frac{dw}{dy}\right]_{y=2\epsilon} = \frac{2d + 3c - 6b + a}{6\epsilon} \tag{B7}$$

$$\left.\frac{d^2w}{dy^2}\right]_{y=2\epsilon} = \frac{d - 2c + b}{\epsilon^2} \tag{B8}$$

Equations (B5) and (B6) are used in the derivation of the response equation for an airplane in a gust. Equations (B7) and (B8) are useful in the derivation of the initial response.

APPENDIX C

DERIVATION OF MATRIX EQUATIONS OF EQUILIBRIUM

In this appendix the matrix equations

$$\begin{bmatrix}A\end{bmatrix}\begin{vmatrix}w\end{vmatrix} = \begin{vmatrix}p\end{vmatrix} \tag{C1}$$

$$\begin{bmatrix}B\end{bmatrix}\begin{vmatrix}\varphi\end{vmatrix} = \begin{vmatrix}q\end{vmatrix} \tag{C2}$$

for symmetrical bending and twisting of a free-free beam under normal and torsional loads are derived.

Bending.- In accordance with the assumptions made in this paper the wing semispan is considered to be divided into six sections with a station point at the center of each section (see fig. 3). The inertia force of the mass and the aerodynamic force that develops over each section is in turn assumed to be concentrated at the respective station points. The wing is thus effectively a beam bending under six concentrated loads and, as such, will have a linearly varying moment between each station. The following general equation for the moment between the i and $i + 1$ station may therefore be written:

$$M = a_i + b_i y \tag{C3}$$

where

$$a_i = \left[1 + \frac{y(i)}{b\lambda_{i+1}}\right]M(i) - \frac{y(i)}{b\lambda_{i+1}}M(i + 1)$$

$$b_i = \frac{1}{b\lambda_{i+1}}\left[M(i + 1) - M(i)\right]$$

in which $y(i)$ is the abscissa to the i station.

The wing is further assumed to have a linear $1/EI$ variation between stations with the correct value of $1/EI$ at each station. This type of variation would lead to an EI curve which follows very closely the true stiffness curve of the wing and which of course has the correct values of EI at each station. A general equation for 1/EI may therefore also be written; thus,

$$\frac{1}{EI} = c_i + d_i y \tag{C4}$$

where

$$c_i = \left[1 + \frac{y(i)}{b\lambda_{i+1}}\right] \frac{1}{EI(i)} - \frac{y(i)}{b\lambda_{i+1}} \frac{1}{EI(i + 1)}$$

$$d_i = \frac{1}{b\lambda_{i+1}} \left[\frac{1}{EI(i + 1)} - \frac{1}{EI(i)}\right]$$

With equation (C3) and equation (C4) the well-known expression relating deflection to moment for a beam may be written

$$\frac{d^2w}{dy^2} = \frac{M}{EI} = (a_i + b_i y)(c_i + d_i y) \tag{C5}$$

The deflection may be found most conveniently from this equation by use of the engineering beam theorem which states that the deflection of one point on a beam relative to the tangent of the deflection curve at another point is equal to the moment about the displaced point of the M/EI diagram between the two points. In this case symmetrical loading is being considered and therefore the boundary condition at the center line is that the slope must be zero; the deflection of each station relative to this point therefore may be readily computed. Fortunately, because of the convenient analytical representation of M/EI, these deflections may be found by exact integration. The deflection, for example, at station 4 due to the M/EI variation between stations i and $i + 1$ may be given by the expression:

$$\int_{y(i)}^{y(i+1)} (a_i + b_i y)(c_i + d_i y)\left[y(4) - y\right] dy$$

Consideration of all the expressions of this sort leads to the total deflection of each station relative to the wing center line. From this deflection the more useful deflection relative to station 0 can be readily determined. The values of the deflection thus obtained are found to be expressible by the following matrix equation:

$$\begin{vmatrix} \bar{w}(1) \\ \bar{w}(2) \\ \bar{w}(3) \\ \bar{w}(4) \\ \bar{w}(5) \end{vmatrix} = \frac{b^2}{EI(0)} \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{vmatrix} M(0) \\ M(1) \\ M(2) \\ M(3) \\ M(4) \end{vmatrix} \tag{C6}$$

where the matrix elements are defined by the equations:

$$\left.\begin{aligned}
a_{11} &= \lambda_0\lambda_1 + \lambda_1{}^2A_1 \\
a_{21} &= \lambda_0(\lambda_1 + \lambda_2) + \lambda_1{}^2A_1 + \lambda_1\lambda_2B_1 \\
a_{31} &= \lambda_0(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_1{}^2A_1 + \lambda_1(\lambda_2 + \lambda_3)B_1 \\
a_{41} &= \lambda_0(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + \lambda_1{}^2A_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4)B_1 \\
a_{51} &= \lambda_0(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) + \lambda_1{}^2A_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)B_1
\end{aligned}\right\} \tag{C7a}$$

$$\left.\begin{aligned}
a_{12} &= \lambda_1{}^2C_1 \\
a_{22} &= \lambda_1{}^2C_1 + \lambda_1\lambda_2D_1 + \lambda_2{}^2A_2 \\
a_{32} &= \lambda_1{}^2C_1 + \lambda_1(\lambda_2 + \lambda_3)D_1 + \lambda_2{}^2A_2 + \lambda_2\lambda_3B_2 \\
a_{42} &= \lambda_1{}^2C_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4)D_1 + \lambda_2{}^2A_2 + \lambda_2(\lambda_3 + \lambda_4)B_2 \\
a_{52} &= \lambda_1{}^2C_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5)D_1 + \lambda_2{}^2A_2 + \lambda_2(\lambda_3 + \lambda_4 + \lambda_5)B_2
\end{aligned}\right\} \tag{C7b}$$

$$\left.\begin{aligned}
a_{23} &= \lambda_2{}^2C_2 \\
a_{33} &= \lambda_2{}^2C_2 + \lambda_2\lambda_3D_2 + \lambda_3{}^2A_3 \\
a_{43} &= \lambda_2{}^2C_2 + \lambda_2(\lambda_3 + \lambda_4)D_2 + \lambda_3{}^2A_3 + \lambda_3\lambda_4B_3 \\
a_{53} &= \lambda_2{}^2C_2 + \lambda_2(\lambda_3 + \lambda_4 + \lambda_5)D_2 + \lambda_3{}^2A_3 + \lambda_3(\lambda_4 + \lambda_5)B_3
\end{aligned}\right\} \quad \text{(C7c)}$$

$$\left.\begin{aligned}
a_{34} &= \lambda_3{}^2C_3 \\
a_{44} &= \lambda_3{}^2C_3 + \lambda_3\lambda_4D_3 + \lambda_4{}^2A_4 \\
a_{54} &= \lambda_3{}^2C_3 + \lambda_3(\lambda_4 + \lambda_5)D_3 + \lambda_4{}^2A_4 + \lambda_4\lambda_5B_4
\end{aligned}\right\} \quad \text{(C7d)}$$

$$\left.\begin{aligned}
a_{45} &= \lambda_4{}^2C_4 \\
a_{55} &= \lambda_4{}^2C_4 + \lambda_4\lambda_5D_4 + \lambda_5{}^2A_5
\end{aligned}\right\} \quad \text{(C7e)}$$

in which

$$\left.\begin{aligned}
A_i &= \frac{1}{4}\,\frac{I(0)}{I(i - 1)} + \frac{1}{12}\,\frac{I(0)}{I(i)} \\
B_i &= \frac{1}{3}\,\frac{I(0)}{I(i - 1)} + \frac{1}{6}\,\frac{I(0)}{I(i)} \\
C_i &= \frac{1}{12}\,\frac{I(0)}{I(i - 1)} + \frac{1}{12}\,\frac{I(0)}{I(i)} \\
D_i &= \frac{1}{6}\,\frac{I(0)}{I(i - 1)} + \frac{1}{3}\,\frac{I(0)}{I(i)}
\end{aligned}\right\} \quad \text{(C8)}$$

The moment $M(5)$ does not appear in equation (C6) because the boundary condition is used that the moment at station 5 is zero. For convenience equation (C6) may be given in the abbreviated form:

$$\left|\overline{w}\right| = \frac{b^2}{EI(0)} \left[H_1\right] \left|M\right| \tag{C9}$$

From the five loads $p(1)$, $p(2)$, $p(3)$, $p(4)$, and $p(5)$, the moment at each station may be found. The equations relating the moments to the loads can be shown to be given by the matrix equation:

$$\begin{vmatrix} M(0) \\ M(1) \\ M(2) \\ M(3) \\ M(4) \end{vmatrix} = b \begin{bmatrix} \lambda_1 & \lambda_1 + \lambda_2 & \lambda_1 + \lambda_2 + \lambda_3 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ 0 & \lambda_2 & \lambda_2 + \lambda_3 & \lambda_2 + \lambda_3 + \lambda_4 & \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ 0 & 0 & \lambda_3 & \lambda_3 + \lambda_4 & \lambda_3 + \lambda_4 + \lambda_5 \\ 0 & 0 & 0 & \lambda_4 & \lambda_4 + \lambda_5 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix} \begin{vmatrix} p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{vmatrix} \tag{C10}$$

which can be abbreviated simply

$$\left|M\right| = b \left[H_2\right] \left|p\right| \tag{C11}$$

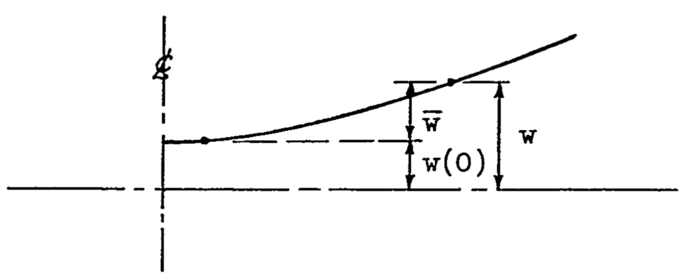
Substitution of equation (C11) into equation (C9) gives

$$\left|\bar{w}\right| = \frac{b^3}{EI(0)} \left[H_1\right]\left[H_2\right] \left|p\right| \tag{C12}$$

Multiplication through by the reciprocal of $\frac{b^3}{EI(0)} \left[H_1\right]\left[H_2\right]$ results in the equation:

$$\frac{EI(0)}{b^3} \left[\left[H_1\right]\left[H_2\right]\right]^{-1} \left|\bar{w}\right| = \left|p\right| \tag{C13}$$

This equation thus gives the loads in terms of the deflection of each station relative to station 0. In the case under consideration, however, the wing is a free body capable of motion through space and therefore to set up properly the equations of motion the deflection must be given relative to a fixed datum line. This datum line is most conveniently located as the position of the wing prior to action of the disturbing loads. Consideration of the following sketch



will show therefore that the following relation must exist:

$$\bar{w} = w - w(0) \tag{C14}$$

Substitution of this equation into equation (C13) gives

$$\frac{EI(0)}{b^3} \left[\left[H_1\right]\left[H_2\right]\right]^{-1} \left|w - w(0)\right| = \left|p\right| \tag{C15}$$

To aid in the derivation $\frac{EI(0)}{b^3}\left[\left[H_1\right]\left[H_2\right]\right]^{-1}$ is now written in the general form:

$$\frac{EI(0)}{b^3}\left[\left[H_1\right]\left[H_2\right]\right]^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{12} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{13} & b_{23} & b_{33} & b_{34} & b_{35} \\ b_{14} & b_{24} & b_{34} & b_{44} & b_{45} \\ b_{15} & b_{25} & b_{35} & b_{45} & b_{55} \end{bmatrix} \tag{C16}$$

Thus with this equation, equation (C15) may be transformed to the form:

$$\begin{bmatrix} b_{01} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{02} & b_{12} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{03} & b_{13} & b_{23} & b_{33} & b_{34} & b_{35} \\ b_{04} & b_{14} & b_{24} & b_{34} & b_{44} & b_{45} \\ b_{05} & b_{15} & b_{25} & b_{35} & b_{45} & b_{55} \end{bmatrix} \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix} = \begin{vmatrix} p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{vmatrix} \tag{C17}$$

where

$$\left.\begin{aligned} b_{01} &= -\left(b_{11} + b_{12} + b_{13} + b_{14} + b_{15}\right) \\ b_{02} &= -\left(b_{12} + b_{22} + b_{23} + b_{24} + b_{25}\right) \\ b_{03} &= -\left(b_{13} + b_{23} + b_{33} + b_{34} + b_{35}\right) \\ b_{04} &= -\left(b_{14} + b_{24} + b_{34} + b_{44} + b_{45}\right) \\ b_{05} &= -\left(b_{15} + b_{25} + b_{35} + b_{45} + b_{55}\right) \end{aligned}\right\} \tag{C18}$$

Equation (C17) is noted to express all the loads except $p(0)$ in terms of the six deflections. An additional equation in which $p(0)$ is expressed also in terms of the deflections may be established by use of the condition that all the loads acting on the wing semispan must add up to equal zero; that is,

$$p(0) + p(1) + p(2) + p(3) + p(4) + p(5) = 0 \tag{C19}$$

This condition automatically satisfies the two boundary conditions that the shear must be zero at the tip and center line of the wing. Thus if the five equations represented by equation (C17) are added, and use is made of equations (C18) and (C19), the following equation results:

$$b_{00}w(0) + b_{01}w(1) + b_{02}w(2) + b_{03}w(3) + b_{04}w(4) + b_{05}w(5) = p(0) \tag{C20}$$

where

$$b_{00} = -\left(b_{01} + b_{02} + b_{03} + b_{04} + b_{05}\right) \tag{C21}$$

This equation may now be combined with equation (C17) to give finally

$$\begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} & b_{05} \\ b_{01} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{02} & b_{12} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{03} & b_{13} & b_{23} & b_{33} & b_{34} & b_{35} \\ b_{04} & b_{14} & b_{24} & b_{34} & b_{44} & b_{45} \\ b_{05} & b_{15} & b_{25} & b_{35} & b_{45} & b_{55} \end{bmatrix} \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix} = \begin{vmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{vmatrix} \tag{C22}$$

This equation is thus the desired matrix equation which relates the normal loads to the deflection. If the square matrix is denoted by $\left[A\right]$, the equation may be abbreviated conveniently to the form

$$\left[A\right]\left|w\right| = \left|p\right| \tag{C23}$$

which is the form used in the text. (See equation (40.)

As an aid in computational work, a summary of the steps involved in the determination of $[A]$ is given to close this section:

(1) From the I values at the respective stations compute the coefficients given by equations (C8)

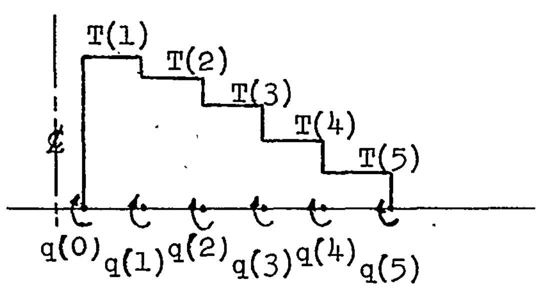
(2) With these coefficients determine the matrix elements given by equations (C7). These elements form the matrix $\left[H_1\right]$ which is defined by equations (C6) and (C9).

(3) Multiply the $\left[H_1\right]$ matrix by the $\left[H_2\right]$ matrix, which is defined by equations (C10) and (C11). The result should be a symmetrical matrix; this property serves as a very useful computational check.

(4) Invert the $\dfrac{b^3}{EI(0)}\left[\left[H_1\right]\left[H_2\right]\right]$ matrix. This matrix should also be symmetrical. (The Crout method (reference 6) serves as a rather quick and useful means for performing the inversion.)

(5) Add the columns of the inverted matrix and place the negative of these sums at the top of their respective columns such as to form a new row of matrix elements. Then add these sums and place the negative of the sum as the first matrix element of the newly formed row. A new column headed by this value is thus in the making. Fill in the remainder of the column with the respective elements of the new row; that is, the appropriate values should be inserted to make the matrix symmetrical. This final matrix is the desired $[A]$ matrix.

<u>Torsion</u>.- For the torsional case the torque loads q are assumed to be concentrated at the stations just as in the case for the normal loads p. Consideration then of the following example torque diagram

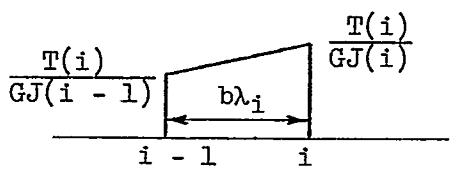


will show that the following equations must apply:

$$\left.\begin{aligned}
q(0) &= - T(1) \\
q(1) &= T(1) - T(2) \\
q(2) &= T(2) - T(3) \\
q(3) &= T(3) - T(4) \\
q(4) &= T(4) - T(5) \\
q(5) &= T(5)
\end{aligned}\right\} \qquad \text{(C24)}$$

where $T(i)$ represents the total torque present in the i interval. No torque exists between the wing center line and station 0.

To aid in the derivation, the assumption is made that $1/GJ$ varies linearly between stations. A typical T/GJ diagram between, say, the $i - 1$ and the i station would appear as follows:



From the differential relation $\frac{d\varphi}{dy} = \frac{T}{GJ}$, the fact may be observed that the change in angle of twist between two stations is equal to the area of the T/GJ diagram between the two stations; therefore,

$$\varphi(i) - \varphi(i - 1) = \frac{b\lambda_i}{2}\left[\frac{T(i)}{GJ(i - 1)} + \frac{T(i)}{GJ(i)}\right] \qquad \text{(C25)}$$

If the notation

$$j_i = \frac{2G}{b\lambda_i} \frac{1}{\frac{1}{J(i - 1)} + \frac{1}{J(i)}} \tag{C26}$$

is employed, equation (C25) may be written

$$T(i) = j_i\left[\varphi(i) - \varphi(i - 1)\right] \tag{C27}$$

Application of this equation to each of the spanwise stations gives the following equations for T:

$$\left.\begin{aligned} T(1) &= j_1\left[\varphi(1) - \varphi(0)\right] \\ T(2) &= j_2\left[\varphi(2) - \varphi(1)\right] \\ T(3) &= j_3\left[\varphi(3) - \varphi(2)\right] \\ T(4) &= j_4\left[\varphi(4) - \varphi(3)\right] \\ T(5) &= j_5\left[\varphi(5) - \varphi(4)\right] \end{aligned}\right\} \tag{C28}$$

Substitution now of these equations into equations (C24) gives the desired equations relating the torque loads to the angle of twist. The equations thus found can be given in the matrix form:

$$\begin{bmatrix} j_1 & -j_1 & 0 & 0 & 0 & 0 \\ -j_1 & (j_1 + j_2) & -j_2 & 0 & 0 & 0 \\ 0 & -j_2 & (j_2 + j_3) & -j_3 & 0 & 0 \\ 0 & 0 & -j_3 & (j_3 + j_4) & -j_4 & 0 \\ 0 & 0 & 0 & -j_4 & (j_4 + j_5) & -j_5 \\ 0 & 0 & 0 & 0 & -j_5 & j_5 \end{bmatrix} \begin{vmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \\ \varphi(5) \end{vmatrix} = \begin{vmatrix} q(0) \\ q(1) \\ q(2) \\ q(3) \\ q(4) \\ q(5) \end{vmatrix} \tag{C29}$$

which can be abbreviated to

$$\left[B\right]\left|\varphi\right| = \left|q\right| \tag{C30}$$

the form used in the test. (See equation (41).) Thus all that is involved in the computation of the matrix $[B]$ is the evaluation of the matrix elements by means of equation (C26).

APPENDIX D

RECURRENCE EQUATION FOR THE EVALUATION OF DUHAMEL'S INTEGRAL INVOLVING AN EXPONENTIAL KERNEL

The derivation of a rather simple recurrence relation for the step-by-step evaluation of the three unsteady lift integrals appearing in equation (25) is presented. This derivation is made possible because the kernels of the integrals are expressible in exponential form.

From equation (23) the first and second derivative of the Φ function may be written

$$\dot{\Phi} = -\frac{2U}{c_o}\lambda a_1 e^{-\lambda\frac{2U}{c_o}t} = \dot{\Phi}_0 e^{-\gamma t} \tag{D1}$$

$$\ddot{\Phi} = \frac{4U^2}{c_o^{\,2}}\lambda^2 a_1 e^{-\lambda\frac{2U}{c_o}t} = \ddot{\Phi}_0 e^{-\gamma t} \tag{D2}$$

where

$$\gamma = \lambda\frac{2U}{c_o}$$

$$\dot{\Phi}_0 = -\gamma a_1$$

$$\ddot{\Phi}_0 = \gamma^2 a_1$$

With these equations the three integrals of equation (25) may be combined conveniently into the following single integral denoted by I_t:

$$I_t = \int_0^t \left\{\ddot{\Phi}_0 \beta c l w - \left[\dot{\Phi}_0 \beta c l U + \ddot{\Phi}_0 \beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right)\right]\varphi\right\}e^{-\gamma(t-\tau)}\,d\tau \tag{D3}$$

For convenience the notation

$$Y = \left\{\ddot{\Phi}_0\beta c l w - \left[\dot{\Phi}_0\beta c l U + \ddot{\Phi}_0\beta c^2 l\left(\frac{3}{4} - \frac{a}{c}\right)\right]\varphi\right\} \tag{D4}$$

is introduced and thus equation (D3) becomes

$$I_t = \int_0^t Y e^{-\gamma(t-\tau)}\, d\tau$$

or

$$I_t = e^{-\gamma t}\int_0^t Y e^{\gamma\tau}\, d\tau \tag{D5}$$

Mathematically, the integral in this equation may be interpreted to represent the area under the function given as a product of Y and $e^{\gamma\tau}$. In accordance with numerical evaluation processes, the interval 0 to t may be divided into a number of time stations of interval ϵ. The product of Y and $e^{\gamma\tau}$ may then be found at each of the time stations and from these products the area under the curve may be determined in first approximation by the trapezoidal method of determining areas. Thus, if the n time station corresponds to time t, the expression for I_t may be approximated as follows:

$$I_t \approx I_n = \epsilon e^{-\gamma n\epsilon}\left[Y_1 e^{\gamma\epsilon} + Y_2 e^{\gamma 2\epsilon} + \ldots + Y_{n-1} e^{\gamma(n-1)\epsilon} + \frac{1}{2} Y_n e^{\gamma n\epsilon}\right] \tag{D6}$$

where Y_0 does not appear since the initial conditions are used that the deflection w and rotation φ are zero at $t = 0$, and therefore Y_0 is zero. (See equation (D4).) More accurate methods, such as Simpson's method, could be used for determining the area under the curve, but because of the small interval chosen the consequent increase in accuracy is negligible. If the notation

$$F_n = \epsilon e^{-\gamma n\epsilon}\left[Y_1 e^{\gamma\epsilon} + Y_2 e^{\gamma 2\epsilon} + \ldots + Y_{n-1} e^{\gamma(n-1)\epsilon}\right] \tag{D7}$$

is introduced, equation (D6) may be written simply

$$I_n = F_n + \frac{\epsilon}{2} Y_n \tag{D8}$$

If equation (D5) is expanded similarly, only for an upper limit of $t - \epsilon$, the expanded result would be

$$I_{n-1} = \epsilon e^{-\gamma(n-1)\epsilon}\left[Y_1 e^{\gamma\epsilon} + Y_2 e^{\gamma 2\epsilon} + \ldots + Y_{n-2} e^{\gamma(n-2)\epsilon} + \frac{1}{2} Y_{n-1} e^{\gamma(n-1)\epsilon}\right] \tag{D9}$$

By analogy with equation (D7), however,

$$F_{n-1} = \epsilon e^{-\gamma(n-1)\epsilon}\left[Y_1 e^{\gamma\epsilon} + Y_2 e^{\gamma 2\epsilon} + \ldots + Y_{n-2} e^{\gamma(n-2)\epsilon}\right] \tag{D10}$$

and therefore equation (D9) becomes

$$I_{n-1} = F_{n-1} + \frac{\epsilon}{2} Y_{n-1} \tag{D11}$$

A study of equations (D7) and (D10) shows that the following relation must exist:

$$F_n = e^{-\gamma\epsilon} F_{n-1} + \epsilon e^{-\gamma\epsilon} Y_{n-1} \tag{D12}$$

Now, if equation (D4) is used to rewrite Y_n and Y_{n-1} in equations (D8) and (D12), the value of I_n may be given finally by the equation:

$$I_n = F_n + \frac{1}{2} \ddot{\Phi}_0 \beta c l \epsilon w_n - \frac{1}{2}\beta c l \epsilon\left[U\dot{\Phi}_0 + c\left(\frac{3}{4} - \frac{a}{c}\right)\ddot{\Phi}_0\right]\varphi_n \tag{D13}$$

where

$$F_n = e^{-\gamma\epsilon} F_{n-1} + \ddot{\Phi}_0 \epsilon e^{-\gamma\epsilon} \beta c l w_{n-1} - \beta c l \epsilon e^{-\gamma\epsilon}\left[U\dot{\Phi}_0 + c\left(\frac{3}{4} - \frac{a}{c}\right)\ddot{\Phi}_0\right]\varphi_{n-1} \tag{D14}$$

The value of the unsteady lift integrals is thus given by equation (D13). As regards the analysis given in the present paper, w_{n-1} and φ_{n-1} are the values of deflection and rotation which have, say, just been determined from the recurrence equation for response. The value F_{n-1} was also established and therefore F_n can be determined as a definite quantity. The value I_n is thus seen to be given in terms of the known F_n and in terms of w_n and φ_n which are the next values to be evaluated from the recurrence equation.

APPENDIX E

MATRIX ALGEBRA

This appendix is written for those not familiar with matrix notations or matrix methods. All the matrix algebra necessary for the understanding of this paper is described hereinafter by way of examples.

Matrix definition.- Some of the basic types of matrices are illustrated by the following arbitrary matrices which are of the third order:

The column matrix

$$\begin{vmatrix} 2 \\ 1 \\ -1 \end{vmatrix}$$

The row matrix

$$\lfloor 2 \quad -3 \quad 1 \rfloor$$

The square matrix

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

The diagonal matrix

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Element definition.- Each of the terms that appear in a matrix is defined as an element. Its position is usually denoted in a row by the number of terms from the left and in a column by the number of terms from the top.

Matrix addition.- The addition of two matrices produces a single matrix. Addition is performed by simply adding together corresponding elements. For example,

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ 0 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 1 \\ 1 & 5 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

Multiplication of a matrix by a scalar number.- In the multiplication of a matrix by a scalar number every element in the matrix is multiplied by the number. For example,

$$2\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 2 \\ 2 & 4 & -4 \\ -2 & -2 & 6 \end{bmatrix}$$

Multiplication of a column matrix by a row matrix.- The product of a column matrix and a row matrix is equal to the sum of the products of the corresponding elements. For example,

$$\lfloor 2 \quad -3 \quad 1 \rfloor \begin{vmatrix} 2 \\ 1 \\ -4 \end{vmatrix} = (2 \times 2) + (-3 \times 1) + \left[1 + (-4)\right] = -3$$

Multiplication of a column matrix by a square matrix.- The multiplication of a column matrix by a square matrix produces a column matrix. Consider the following set of three simultaneous equations:

$$\left.\begin{aligned} 2y_1 - 3y_2 + y_3 &= a_1 \\ y_1 + 2y_2 - 2y_3 &= a_2 \\ -y_1 - y_2 + 3y_3 &= a_3 \end{aligned}\right\} \qquad \text{(E1)}$$

The procedure adopted in matrix algebra is to write these equations in the matrix form

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{vmatrix} y_1 \\ y_2 \\ y_3 \end{vmatrix} = \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix} \tag{E2}$$

where the multiplication of the $|y|$ column matrix by each row in the square matrix produces the respective elements in the $|a|$ column matrix. (See multiplication of a column matrix by a row matrix.)

In order to simplify the presentation of an analysis, the symbolic or abbreviated matrix form is used quite often. The symbolic form of equation (E2) is simply

$$[M]|y| = |a| \tag{E3}$$

The determination of $|a|$ by the multiplication of $|y|$ by $[M]$ is illustrated with arbitrary values of y, say $y_1 = 4$, $y_2 = 5$, and $y_3 = 6$, by the following equation:

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{vmatrix} 4 \\ 5 \\ 6 \end{vmatrix} = \begin{vmatrix} (2 \times 4) + (-3 \times 5) + (1 \times 6) \\ (1 \times 4) + (2 \times 5) + (-2 \times 6) \\ (-1 \times 4) + (-1 \times 5) + (3 \times 6) \end{vmatrix} = \begin{vmatrix} -1 \\ 2 \\ 9 \end{vmatrix} = \begin{vmatrix} a_1 \\ a_2 \\ a_3 \end{vmatrix} \tag{E4}$$

<u>Multiplication of a square matrix by a square matrix</u>.- The multiplication of two square matrices produces a square matrix. Multiplication is performed by letting the multiplying matrix operate, as in the preceding section, on each of the successive columns in the matrix being multiplied to produce corresponding successive columns in the product matrix. For example,

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 1 & 3 & -2 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -12 & 14 \\ 5 & 2 & -5 \\ -5 & 2 & 5 \end{bmatrix} \tag{E5}$$

Order of multiplication.- In general the commutative multiplication law of ordinary algebra does not hold in matrix methods; that is,

$$|A|\ |B| \neq |B|\ |A|$$

Therefore, whenever the product of several matrices is indicated, these matrices must be multiplied together without changing their order.

Matrix partitioning and submatrices.- A matrix may be partitioned or divided at will into smaller matrices. For example, the left-hand side of equation (E4) may be partitioned as follows:

$$\left[\begin{array}{c|cc} 2 & -3 & 1 \\ \hline 1 & 2 & -2 \\ -1 & -1 & 3 \end{array}\right] \left|\begin{array}{c} 4 \\ \hline 5 \\ 6 \end{array}\right|$$

The matrices which are formed by the dividing lines are called submatrices. These submatrices may be treated as though they were elements when matrix operations are performed. For example, with the notation

$$a = \lfloor -3 \quad 1 \rfloor$$

$$b = \begin{vmatrix} 1 \\ -1 \end{vmatrix}$$

$$c = \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix}$$

$$d = \begin{vmatrix} 5 \\ 6 \end{vmatrix}$$

the multiplication of the foregoing partitioned matrix is as follows:

$$\left[\begin{array}{c|c} 2 & a \\ \hline b & c \end{array}\right] \begin{vmatrix} 4 \\ d \end{vmatrix} = \begin{vmatrix} 8 + ad \\ 4b + cd \end{vmatrix}$$

<u>The reciprocal of a matrix and the identity matrix</u>.- By ordinary algebraic methods the formal operation involved in the solution for x of the equation

$$mx = a$$

is the multiplication through by the reciprocal of m; thus,

$$x = m^{-1}a$$

The same formal operation may be applied to matrix equations. For example, the solution for $|y|$ in equation (E3) is simply

$$|y| = [M]^{-1} \; |a|$$

where $[M]^{-1}$ is the reciprocal, or the inverse, of $[M]$.

The reciprocal of a matrix is found as the matrix which satisfies either of the equivalent equations

$$[M]^{-1} \; [M] \quad = [I]$$

$$[M] \quad [M]^{-1} = [I]$$

where $[I]$ is the identity matrix. For equations (E2) and (E3), the reciprocal of $[M]$ is found as the matrix which satisfies the equation

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

If this equation is considered in relation to equations (E1), (E2), and (E5), the values b_1, b_2, and b_3 would simply be values of y_1, y_2, and y_3 which satisfy equation (E1) for $a_1 = 1$, $a_2 = 0$, and $a_3 = 0$; c_1, c_2, and c_3 would be the values for $a_1 = 0$, $a_2 = 1$, and $a_3 = 0$; and d_1, d_2, and d_3 would be the values for $a_1 = 0$, $a_2 = 0$, and $a_3 = 1$. For this example, the solutions are

$$\begin{array}{lll} b_1 = \frac{1}{3} & c_1 = \frac{2}{3} & d_1 = \frac{1}{3} \\ b_2 = -\frac{1}{12} & c_2 = \frac{7}{12} & d_2 = \frac{5}{12} \\ b_3 = \frac{1}{12} & c_3 = \frac{5}{12} & d_3 = \frac{7}{12} \end{array}$$

The Crout method (reference 6) provides a very quick and convenient method for determining these solutions.

The determination of y by the operation $[M]^{-1}$ on $|a|$ is illustrated as follows for $a_1 = -1$, $a_2 = 2$, and $a_3 = 9$:

$$\frac{1}{12}\begin{bmatrix} 4 & 8 & 4 \\ -1 & 7 & 5 \\ 1 & 5 & 7 \end{bmatrix} \begin{vmatrix} -1 \\ 2 \\ 9 \end{vmatrix} = \begin{vmatrix} 4 \\ 5 \\ 6 \end{vmatrix} \tag{E11}$$

The operation performed by this equation can be seen to be the inverse operation of equation (E4).

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TABLE 1

ILLUSTRATION OF THE [S] MATRICES

$$
\begin{bmatrix}
2 & 3 & & & & & & 4 & & & & & \\
1 & 10 & & & & & & 6 & & & & & \\
 & & 9 & & & & & & 5 & & & & \\
 & & & 6 & & & & & & 4 & & & \\
 & & & & 4 & & & & & & 3 & & \\
 & & & & & 2 & & & & & & 3 & \\
 & & & & & & 1 & & & & & & 1 \\
3 & 4 & & & & & & 5 & & & & & \\
 & & 3 & & & & & & 4 & & & & \\
 & & & 3 & & & & & & 3 & & & \\
 & & & & 2 & & & & & & 3 & & \\
 & & & & & 2 & & & & & & 2 & \\
 & & & & & & 1 & & & & & & 1
\end{bmatrix}
$$

TABLE 2

PHYSICAL CHARACTERISTICS AND UNSTEADY LIFT

FACTORS FOR EXAMPLE AIRPLANE

b, in.	560
c_0, in.	154
$\frac{EI_0}{b^3}$, lb/in.	165
ρ, lb/ft^3	0.0765
U, mph	210
U, in./sec	3700
v, in./sec	120
ϵ, sec	0.01
Δs, half-chords	0.48052
M	0.276
A	10
m_A	0.861
β	0.001147
λ	0.381
γ	18.3078
$e^{-\gamma\epsilon}$	0.832703
$\Phi_0 = a_1$	0.361
$\dot{\Phi}_0$	-6.60912
$\ddot{\Phi}_0$	120.9983

NACA

TABLE 3

ψ ORDINATES AND GUST-FORCE MATRIX FOR

EXAMPLE AIRPLANE

n	ψ (Equation (22))
0	0
1	.22105
2	.36744
3	.46716
4	.53741
5	.58888
6	.62830
7	.65980
8	.68594
9	.70840
10	.72820
11	.74595
12	.76215
13	.77707
14	.79086
15	.80373
16	.81574
17	.82696
18	.83748

$$\left|L_g\right| = 120 \begin{vmatrix} 17.8404 \\ 15.7552 \\ 12.1811 \\ 10.5295 \\ 8.77455 \\ 7.01964 \end{vmatrix} \psi_n$$

TABLE 4

MATRIX ELEMENTS FOR EXAMPLE AIRPLANE

Station	λ	EI	c	l	η_0	η_1	η_2	η_3	g
0	0.09	2700	154	101	-56.019712×10^4	139.84200×10^4	-111.77100×10^4	27.93800×10^4	17.9752
1	.18	1870	136	101	-31.594032	78.802027	-62.951014	15.733559	15.2697
2	.17	1100	118	90	-7.570015	18.783512	-14.956756	3.735946	12.2731
3	.16	520	102	90	-2.109675	5.151851	-4.060925	1.012428	10.6091
4	.16	225	85	90	-1.150062	2.773208	-2.168104	.539690	8.84085
5	.16	67.5	68	90	-.698450	1.664566	-1.291283	.320952	7.07268

TABLE 5

THE [A] AND [D] MATRICES FOR EXAMPLE AIRPLANE

(a) The [A] Matrix.

$$\begin{bmatrix} 82{,}192.75 & -133{,}410.07 & 61{,}949.726 & -12{,}599.08 & 2{,}094.4210 & -237.6981 \\ -133{,}410.07 & 258{,}299.66 & 172{,}806.94 & 56{,}197.447 & -9{,}339.8380 & 1{,}059.7383 \\ 61{,}959.73 & 172{,}806.94 & 194{,}219.495 & 108{,}709.511 & 28{,}579.472 & -3{,}242.2365 \\ -12{,}599.080 & 56{,}197.447 & 108{,}709.511 & 103{,}953.971 & -47{,}410.326 & 8{,}567.4988 \\ 2{,}094.4210 & -9{,}339.8380 & 28{,}579.472 & -47{,}410.326 & 36{,}607.4681 & -10{,}531.197 \\ -237.6981 & 1{,}059.7383 & -3{,}242.2365 & 8{,}567.4988 & -10{,}531.197 & 4{,}383.89451 \end{bmatrix}$$

(b) The [D] Matrix.

$$\begin{bmatrix} 642{,}389.82 & -133{,}410.07 & 61{,}959.726 & -12{,}599.080 & 2{,}094.4210 & -237.6981 \\ -133{,}410.07 & 574{,}239.980 & -172{,}806.94 & 56{,}197.447 & -9{,}339.8380 & 1{,}059.7383 \\ 61{,}959.726 & -172{,}806.94 & 269{,}919.640 & -108{,}709.511 & 28{,}579.472 & -3{,}242.2365 \\ -12{,}599.080 & 56{,}197.447 & -108{,}709.511 & 125{,}050.721 & -47{,}410.326 & 8{,}567.4988 \\ 2{,}094.4210 & -9{,}339.8380 & 28{,}579.472 & -47{,}410.326 & 48{,}108.0880 & -10{,}531.197 \\ -237.6981 & 1{,}059.7383 & -3{,}242.2365 & 8{,}567.4988 & -10{,}531.197 & 11{,}368.3945 \end{bmatrix}$$

TABLE 6

RECURRENCE EQUATION FOR RESPONSE OF EXAMPLE AIRPLANE

$$\begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}_n = \frac{1}{10,000} \begin{bmatrix} 0.016448152 & 0.003274436 & -0.002535913 & -0.002352327 & -0.0008201844 & 0.0003284241 \\ 0.003274436 & 0.022344322 & 0.01467594 & 0.002077166 & -0.002939533 & -0.002117350 \\ -0.002535913 & 0.01467594 & 0.069848456 & 0.06246892 & 0.02103792 & -0.009089948 \\ -0.002352327 & 0.002077166 & 0.062468920 & 0.19074974 & 0.15523588 & 0.01762344 \\ -0.008291844 & -0.002939533 & 0.02103792 & 0.15523588 & 0.40588417 & 0.26526111 \\ 0.0003284241 & -0.002117350 & -0.009089948 & 0.01762344 & 0.26526111 & 1.10968865 \end{bmatrix} \Bigg\| Q \Bigg\|_n$$

where

$$\Bigg\| Q \Bigg\|_n = 10,000 \left[\begin{bmatrix} 139.84200 & & & & & \\ & 78.802027 & & & & \\ & & 18.783512 & & & \\ & & & 5.151851 & & \\ & & & & 2.773208 & \\ & & & & & 1.664566 \end{bmatrix} \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}_{n-1} + \begin{bmatrix} -111.77100 & & & & & \\ & -62.951014 & & & & \\ & & -14.956756 & & & \\ & & & -4.060925 & & \\ & & & & -2.168104 & \\ & & & & & -1.291283 \end{bmatrix} \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}_{n-2} + \begin{bmatrix} 27.938000 & & & & & \\ & 15.733559 & & & & \\ & & 3.735946 & & & \\ & & & 1.012428 & & \\ & & & & 0.539690 & \\ & & & & & 0.320952 \end{bmatrix} \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}_{n-3} + \frac{1}{10,000} \Big| \big|F\big| + \big|L_g\big| \Big|_n \right]$$

in which

$$\big|F\big|_n = 0.832703 \big|F\big|_{n-1} + \begin{bmatrix} 17.9752 & & & & & \\ & 15.2697 & & & & \\ & & 12.2731 & & & \\ & & & 10.6091 & & \\ & & & & 8.84085 & \\ & & & & & 7.07268 \end{bmatrix} \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}_{n-1}$$

TABLE 7

EQUATION FOR INITIAL RESPONSE OF EXAMPLE AIRPLANE

$$10{,}000\begin{bmatrix} 175.9720 & -13.34101 & 6.195973 & -1.259908 & 0.2094421 & -0.02376981 \\ -13.34101 & 120.3415 & -17.28069 & 5.619745 & -0.9339838 & 0.1059738 \\ 6.195973 & -17.28069 & 41.92278 & -10.87095 & 2.857947 & -0.3242237 \\ -1.259908 & 5.619745 & -10.87095 & 16.54357 & -4.741033 & 0.8567499 \\ 0.2094421 & -0.9339838 & 2.857947 & -4.741033 & 6.960225 & -1.053120 \\ -0.02376981 & 0.1059738 & -0.3242237 & 0.8567499 & -1.053120 & 2.413172 \end{bmatrix} \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}_1 = 120(0.22105) \begin{vmatrix} 17.8404 \\ 15.7552 \\ 12.1811 \\ 10.5295 \\ 8.77455 \\ 7.01964 \end{vmatrix}$$

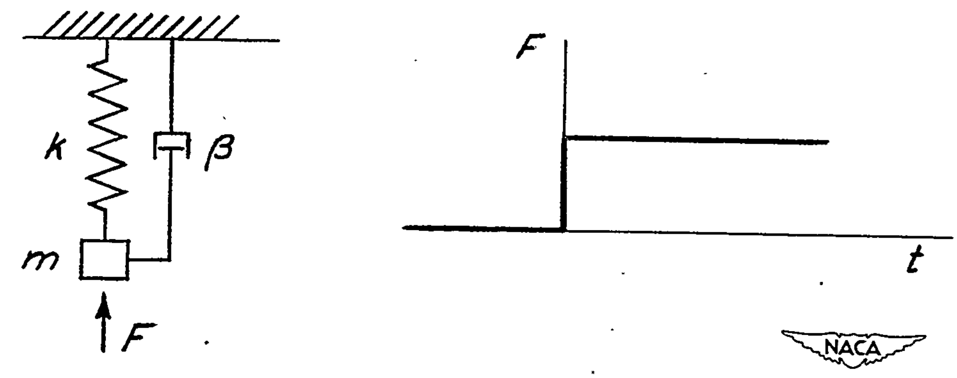


Figure 1.- Damped oscillator and suddenly applied force.

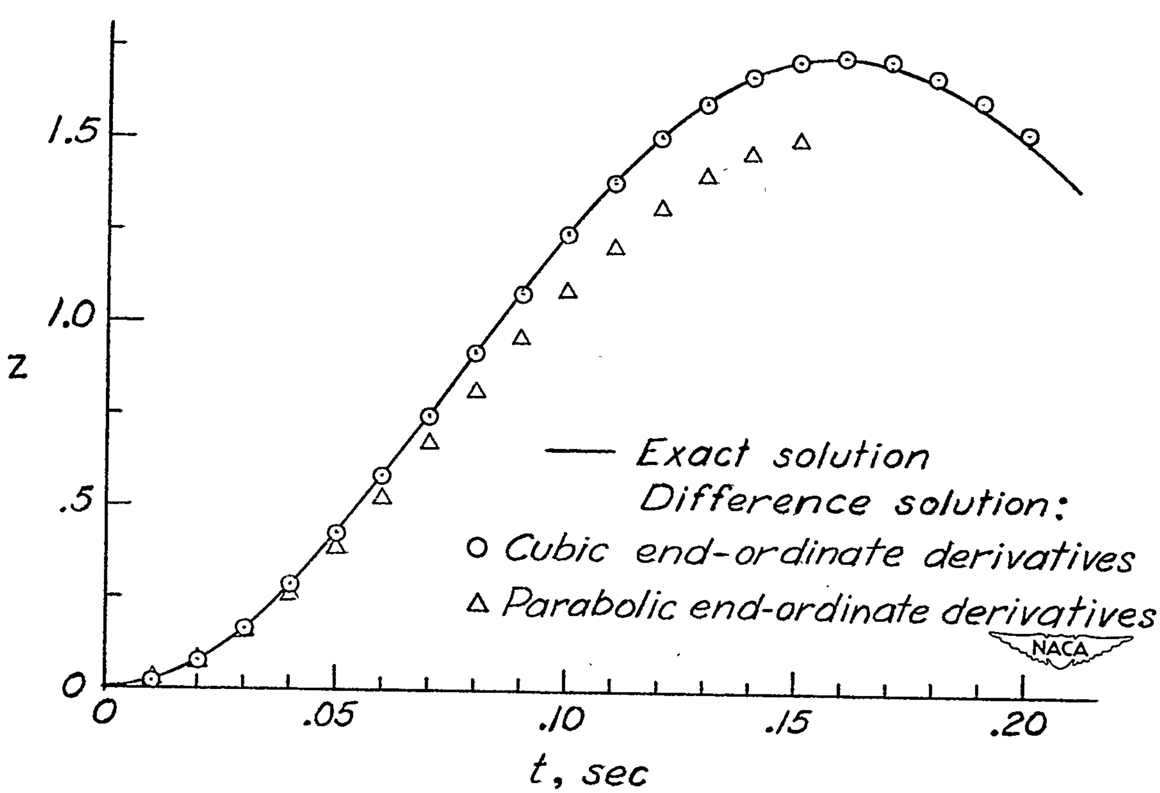


Figure 2.- Comparison of exact and difference-equation solutions for response of damped oscillator.

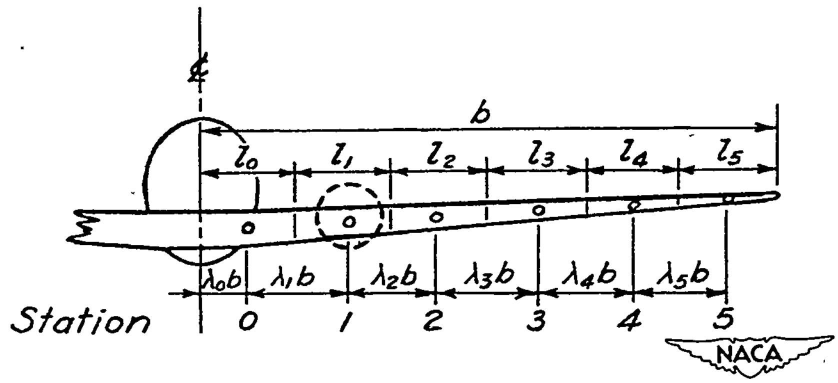


Figure 3.- Division of wing into sections.

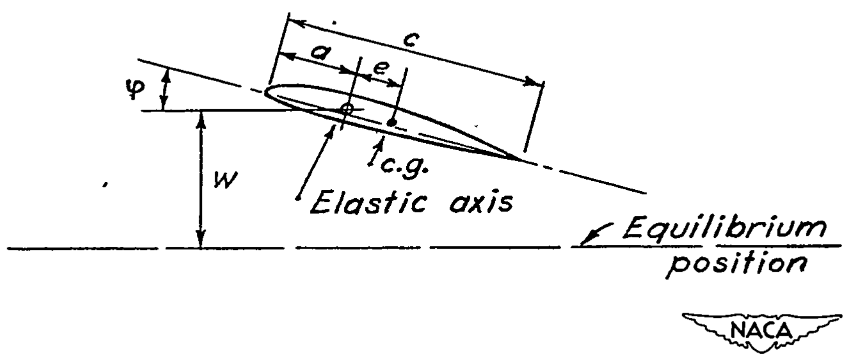


Figure 4.- Displacements of a wing cross section.

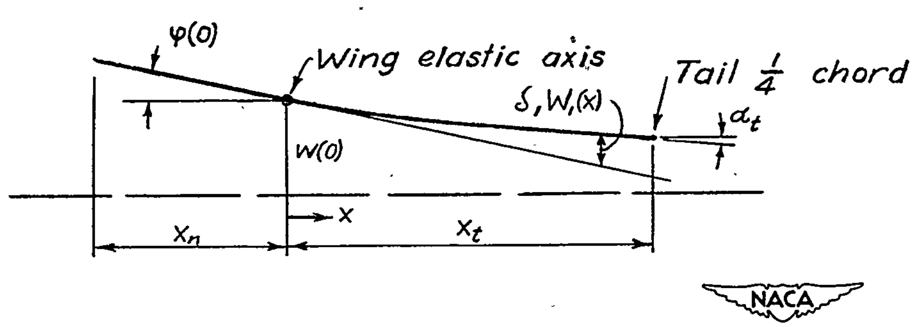


Figure 5.- Coordinate system for fuselage displacement.

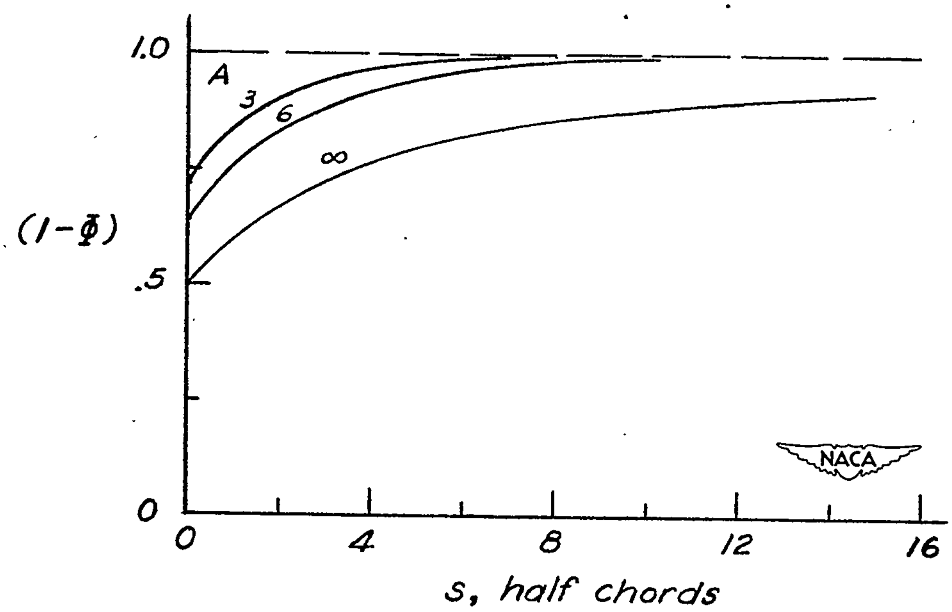


Figure 6.- Lift functions for sudden change in angle of attack. (See equation (20).)

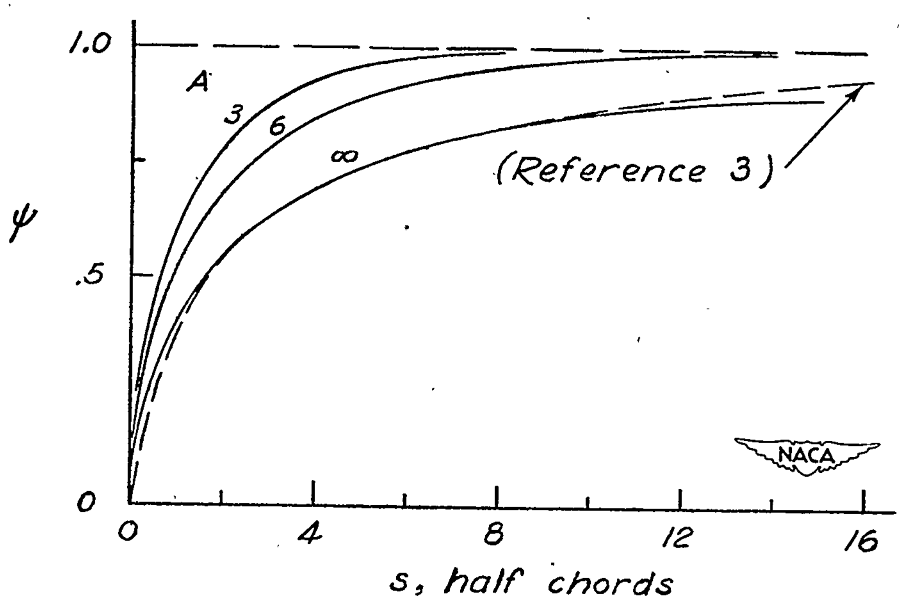


Figure 7.- Lift functions for wings entering a sharp-edge gust. (See equations (21) and (22).)

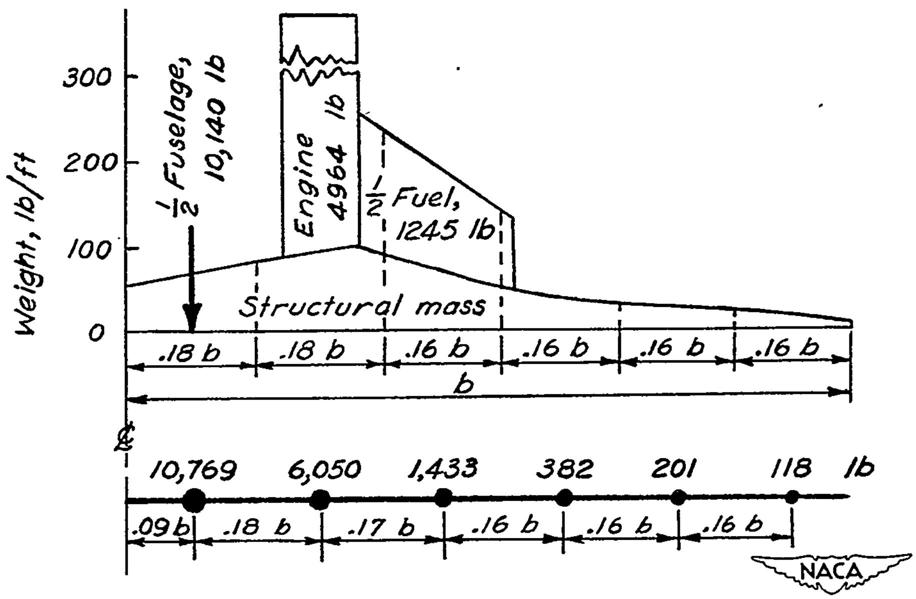


Figure 8.- Weight distribution and equivalent concentrations for example two-engine aircraft.

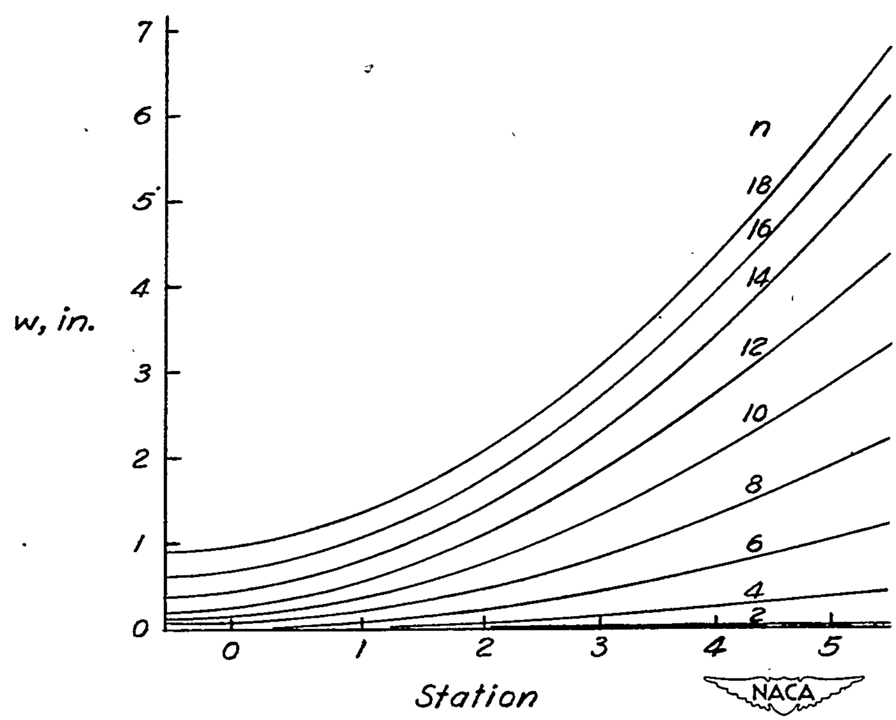


Figure 9.- Response of example airplane due to 10-foot-per-second sharp-edge gust. U = 210 miles per hour.

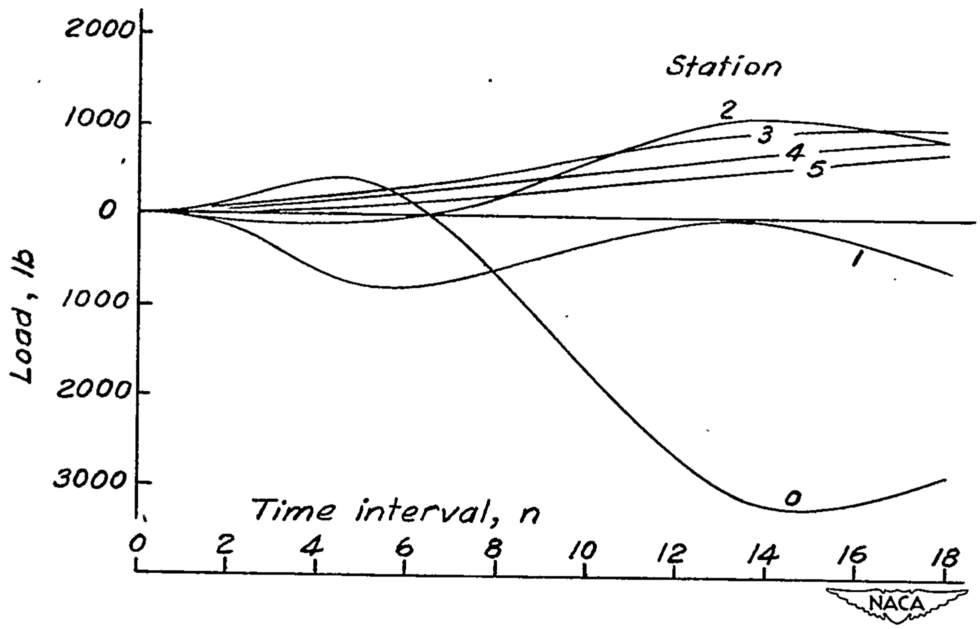


Figure 10.- Time history of station loads for example airplane.

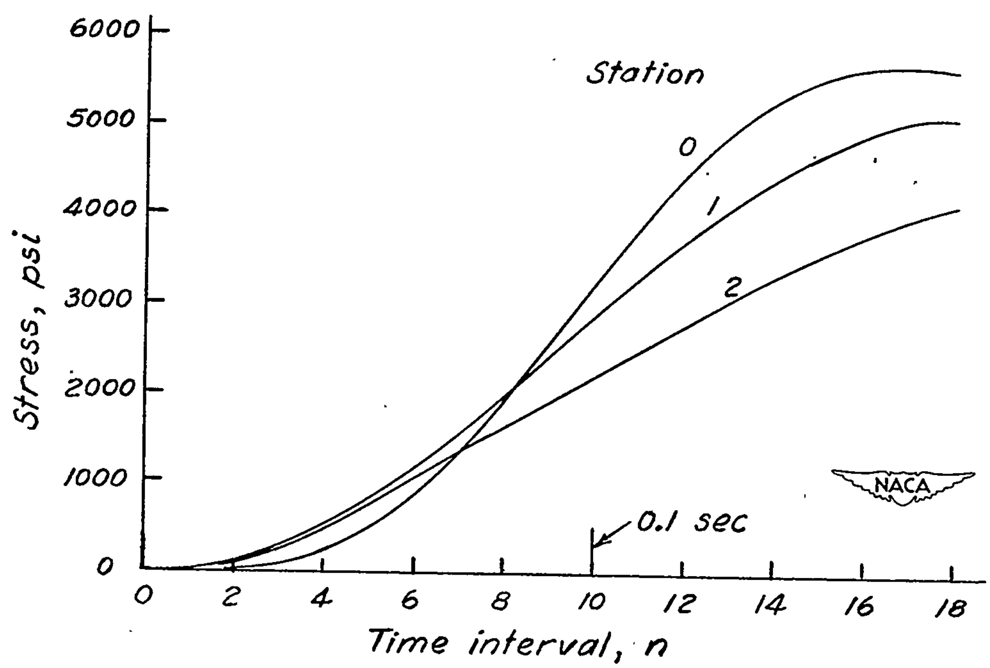
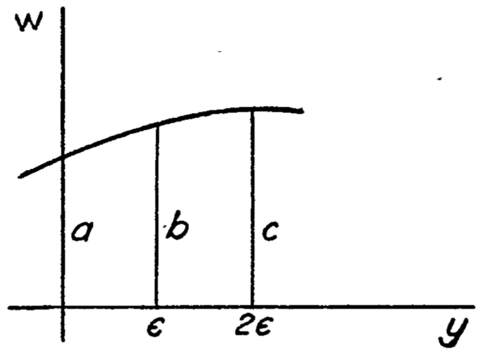
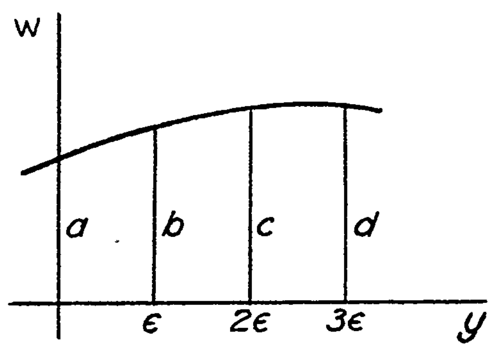


Figure 11.- Bending stress developed in example airplane due to 10-foot-per-second sharp-edge gust.



(a) Parabolic.



(b) Cubic.

Figure 12.- Functional notation used in the derivation of parabolic and cubic difference equations.

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