# REPORT No. 801

## A METHOD FOR STUDYING THE HUNTING OSCILLATIONS OF AN AIRPLANE WITH A SIMPLE TYPE OF AUTOMATIC CONTROL

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### SUMMARY

*A method is presented for predicting the amplitude and frequency, under certain simplifying conditions, of the hunting oscillations of an automatically controlled aircraft with lag in the control system or in the response of the aircraft to the controls. If the steering device is actuated by a simple right-left type of signal, the series of alternating fixed-amplitude signals occurring during the hunting may ordinarily be represented by a "square wave." Formulas are given expressing the response to such a variation of signal in terms of the response to a unit signal. A more complex type of hunting, which may involve cyclic repetition of signals of varying duration, has not been treated and requires further analysis. Several examples of application of the method are included and the results discussed.*

### INTRODUCTION

When an airplane or other aircraft is directed by a simple right-left signal from an automatic steering device, the result is usually a maintained hunting oscillation about the desired path. The amplitude of this oscillation is influenced by the amount of backlash or "dead spot" in the control system and by the damping of the motion of the airplane. In the following analysis the amplitude and frequency of these oscillations is investigated in terms of the response characteristics of the airplane.

### ANALYSIS

The analysis is based on consideration of the response of the airplane (in terms of angle of yaw or pitch) to a continued (unit) signal (fig. 1). This response may be calculated by the ordinary theory of dynamical stability and is conveniently represented in operational form (references 1 and 2) as follows:

$$R_1(t)=\overline{R}_1(D)1(t) \tag{1}$$

The unit response ordinarily occurs in the form

$$\overline{R}_1(D)=\frac{f(D)}{F(D)}$$

from which is obtained

$$R_1(t)=C(t)+(C_1e^{\lambda_1 t}+C_2e^{\lambda_2 t}+\ .\ .\ .) \tag{2}$$

where $C(t)$ is the steady-state motion, $C_1$ and $C_2$ are the constant coefficients of the Heaviside expansion, and $\lambda_1$, $\lambda_2$, and so forth, are the nonzero roots of the characteristic equation defining the natural periods of oscillation and the damping of the aircraft without signal. The function $f(D)$ and the particular solution $C(t)$ depend on the time variation of control displacement produced by a signal and on the stability characteristics of the airplane in the degrees of freedom in which the control operates. (See reference 3.) In the case of a continued signal, the usual form of the function $C(t)$ is

$$C(t)=C_{-1}+C_0t$$

where $C_0$ is the steady rate of turn called for by the signal and $C_{-1}$ is a constant. (See fig. 1.)

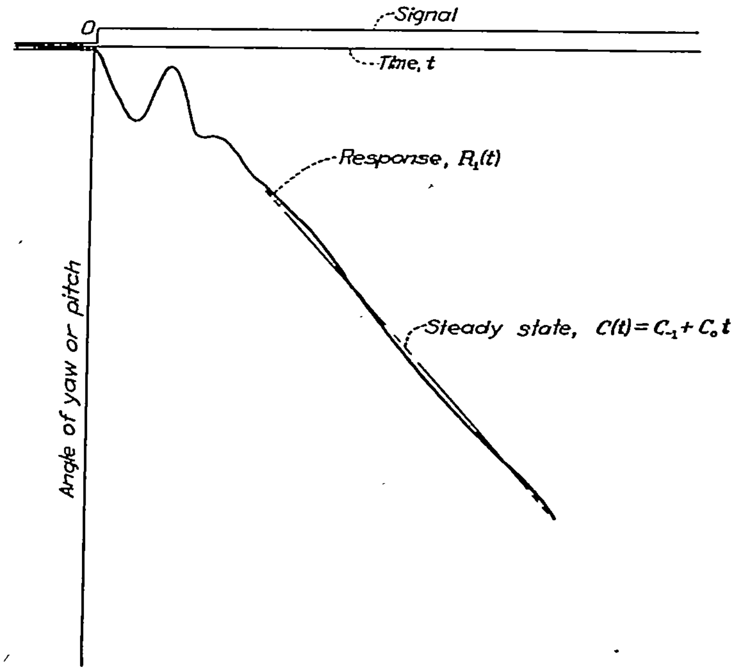


FIGURE 1.—Typical response to continued signal.

During a hunting oscillation, the automatic steering device reverses the signal periodically as the airplane swings through the desired heading. A typical hunting oscillation is shown in figure 2. Here it is assumed that the reversal of signal is delayed either because of a "dead spot" in the steering device or because of backlash in the control mechanism or a combination of the two. As indicated, the oscillation will have a fundamental period $2\pi/\omega$ (where $\omega$ is the angular frequency of the hunting oscillation) but may also involve components of higher frequency, depending on the natural modes of oscillation of the airplane. Ordinarily, the shorter-period components do not have sufficient amplitude to cause a reversal of the signal during a half cycle. In these cases the variation of signal with time will be represented by a simple "square wave," which may be expressed as a function of time by

$$\frac{4}{\pi}\sum_n \frac{1}{n} \sin n\omega t \qquad (n=1, 3, 5, \ldots) \quad (3)$$

or, more conveniently, by the imaginary part of the corresponding exponential series; that is,

$$I.\ P.\ \frac{4}{\pi}\sum_n \frac{1}{n}\, e^{in\omega t} \qquad (4)$$

where $t=0$ is taken to represent a time at which the signal becomes positive.

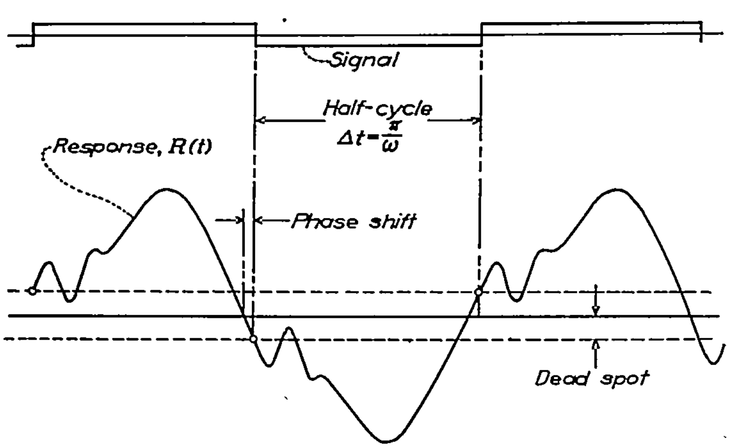


FIGURE 2.—Hunting oscillation with "square" signal.

The response to the alternating signal is obtained by substituting expression (4) for the unit function $1(t)$ in equation (1). Thus,

$$R(t)=I.\ P.\ \overline{R}_1(D)\frac{4}{\pi}\sum_n \frac{1}{n}\, e^{in\omega t} \qquad (5)$$

If the airplane is inherently stable, so that transient effects following the start of an oscillation disappear with time, the remaining steady oscillation will be represented by

$$R(t)=I.\ P.\ \frac{4}{\pi}\sum_n \frac{1}{n}\, \overline{R}_1(in\omega)e^{in\omega t} \qquad (6)$$

Equation (6) gives the forced oscillation of the airplane in response to an alternating signal in the form of a square wave of any frequency $\omega$.

By investigating the form of these forced oscillations at various frequencies it will be possible to ascertain whether such oscillations, under the conditions of automatic control, will give rise to the assumed alternating signals of equal duration, and thus to establish certain ranges of $\omega$ over which hunting of this type can occur. It will also be possible to establish, in these ranges, a correspondence between the frequency of the hunting oscillation and the magnitude of the dead spot. With the frequency determined, it is possible also to find the amplitude of the oscillation and the maximum deviation of the airplane from its path.

In the simplest cases the required information may be obtained directly from equation (6). In the case of more complex motions, further analysis will be required as follows:

As a first step, separate $\overline{R}_1(in\omega)$ into its real and imaginary parts

$$\overline{R}_1(in\omega)=A(n\omega)+iB(n\omega)$$

The functions $A$ and $B$ may be plotted against $n\omega$ as in figure 3. These functions will show peaks near values of $n\omega$

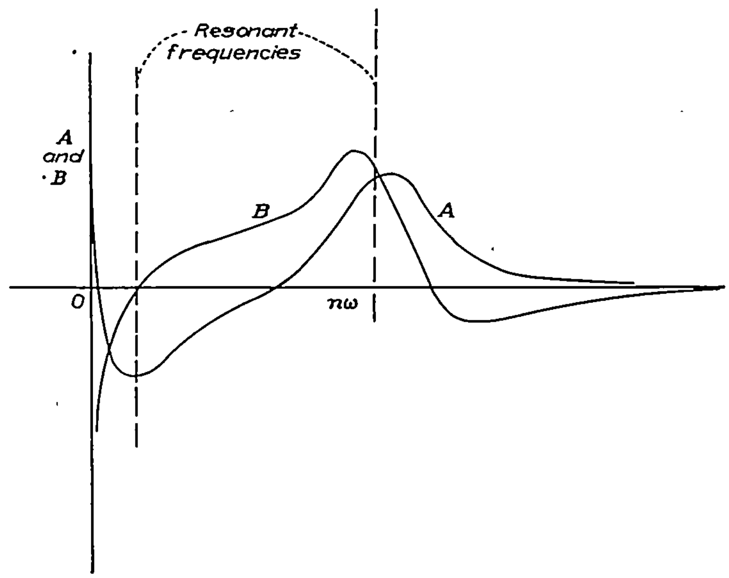


FIGURE 3.—Curves showing in-phase and out-of-phase components of response to periodic signal.

corresponding to the resonant frequencies of the airplane. Then, for any particular hunting frequency $\omega$,

$$R(t)=\frac{4}{\pi}\sum_n \frac{1}{n}\,[A(n\omega)\sin n\omega t+B(n\omega)\cos n\omega t] \qquad (7)$$

At the time of reversal of the signal $\sin n\omega t=0$ and $\cos n\omega t=\pm 1$, the sign depending on whether the signal is becoming positive or negative. The amplitude of the response at this instant is therefore

$$\pm\frac{4}{\pi}\left[B(\omega)+\frac{1}{3}B(3\omega)+\frac{1}{5}B(5\omega)+\ldots\right]$$

This amplitude will also be the amplitude of the dead spot. (See fig. 2.) A plot of

$$R_B(\omega)=\frac{4}{\pi}\sum_n \frac{1}{n}B(n\omega) \qquad (n=1, 3, 5, \ldots)$$

can readily be obtained from the curve of $B$ in figure 3 and will show the periods of the hunting oscillation corresponding to various widths of dead spot.

The slope of the response curve at this same instant is

$$R'_B=\left(\frac{dR}{dt}\right)_B=\frac{4\omega}{\pi}\,[A(\omega)+A(3\omega)+A(5\omega)+\ \ldots]$$

If the response to a positive signal is negative (as in fig. 1), in order that the motion represent a possible hunting oscillation (that is, be consistent with the assumed variation of signal), it is necessary that

(I) $$R_B \geq 0$$

for a positive dead spot, and that

(II) $$R'_B > 0$$

indicating that the airplane crosses the dead spot in the proper direction. A further condition is that no more than one complete crossing of the dead spot occurs within one-half cycle; that is,

(III) $$R(t) > -R_B$$

(See fig. 2.) The value of $R(t)$ in the middle of a half cycle is relatively simple to obtain

$$R_A = \frac{4}{\pi}\left[A(\omega) - \frac{1}{3}A(3\omega) + \frac{1}{5}A(5\omega) - \ . \ . \ .\right]$$

and may be used as a criterion, though $R_A$ is not necessarily the maximum or minimum value of $R(t)$ (see fig. 4) and condition III may not be satisfied even though $R_A > -R_B$.

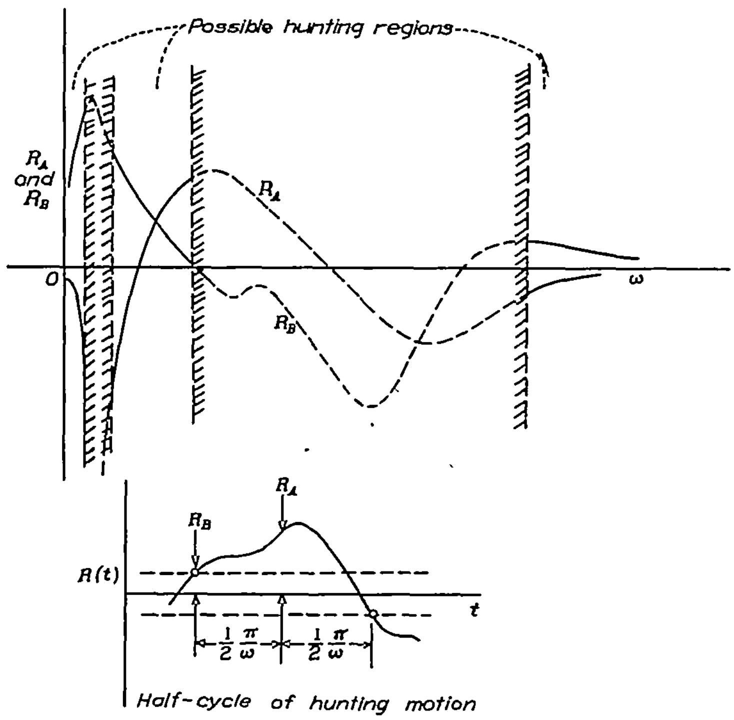


FIGURE 4.—Plot of $R_A$ and $R_B$ against frequency, showing approximate regions in which hunting oscillations are possible and width of dead spot in those regions.

It should be noted that, in the regions excluded by the foregoing conditions, a more complex type of hunting oscillation involving a sequence of signals of different durations may occur. In these regions, the curves of $R_A$ and $R_B$ derived for the square-wave signal no longer apply to the condition of automatic control. These oscillations require analysis beyond that presented in this report.

## EXAMPLES

In order to demonstrate and check the procedure described, assume a simple response characteristic in which the airplane immediately starts turning at a constant rate, as directed by the signal. With this response

$$\overline{R}_1(D) = -\frac{C_0}{D}$$

$$\overline{R}_1(n\omega i) = \frac{C_0}{n\omega} i$$

and, from equation (7),

$$R(t) = \frac{4}{\pi}\sum_n \frac{C_0}{n^2\omega} \cos n\omega t$$

which is the Fourier series for a "saw-tooth" wave 90° out of phase with the signal. (See fig. 5.) In this case the response occurs without lag and the amplitude of the hunting is exactly equal to the dead spot. The frequency $\omega$ is $\pi C_0/2$ divided by the width of the dead spot.

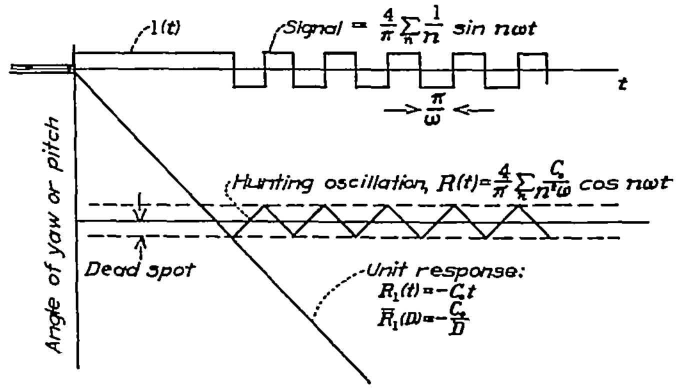


FIGURE 5.—Example in which response is instantaneous.

A simple example nearer the practical case is one in which the signal causes a force $F$ to act on a mass $m$. In this case the response to a unit signal is

$$\overline{R}_1(D) = -\frac{F}{m}\frac{1}{D^2}$$

and the hunting oscillation is seen to be

$$R(t) = \frac{4}{\pi}\frac{F}{m}\sum_n \frac{1}{n^3\omega^2} \sin n\omega t$$

The expression is recognized as the Fourier series for a succession of parabolic segments (fig. 6). It should be noted that there is no component out of phase with the signal, with the result that $R_B$ is zero for all values of $\omega$. Hence the calculation shows no possibility of hunting with a finite dead spot. In fact, it can be seen from energy considerations that if a dead spot existed the oscillation would be divergent.

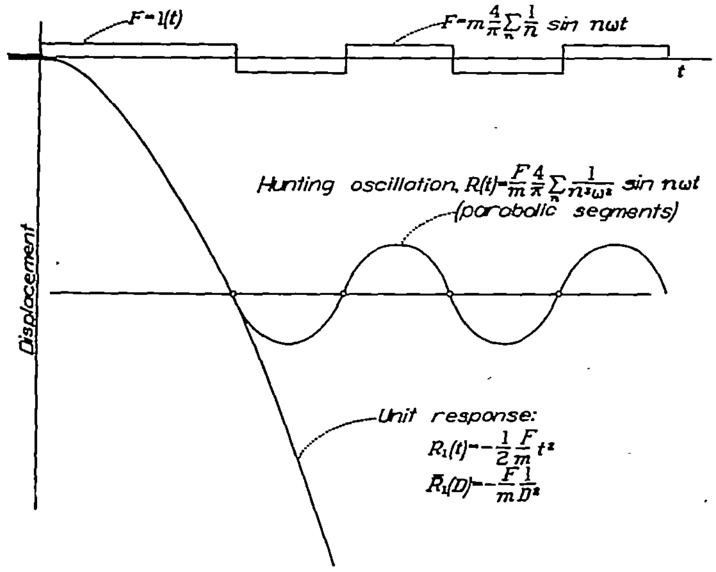


FIGURE 6.—Hunting oscillation of mass acted on by force.

Interesting applications of the method are furnished by cases in which the response to a signal shows a lag, possibly due to backlash in the control mechanism, in addition to a dead spot. A simple example of this kind is illustrated in figure 7. Here the response is similar to that in the first example (fig. 5) except for the time lag $\tau$. Use is made of the well-known lag operator $e^{-\tau D}$. Thus,

$$e^{-\tau D}f(t)=f(t-\tau)$$

Applying this operator to the response in figure 5 gives

$$\overline{R}_1(D)=-e^{-\tau D}\frac{C_0}{D}$$

$$\overline{R}_1(in\omega)=\frac{C_0}{n\omega}(\sin n\omega\tau+i\cos n\omega\tau)=A+iB$$

and, finally (equation (7)),

$$R(t)=\frac{4}{\pi}\sum_n\frac{C_0}{n^2\omega}(\sin n\omega\tau\sin n\omega t+\cos n\omega\tau\cos n\omega t)$$

$$=\frac{4}{\pi}\sum_n\frac{C_0}{n^2\omega}\cos n\omega(t-\tau)$$

With the lagging response, the hunting oscillation is not confined to the amplitude of the dead spot and, in fact, hunting will occur with no dead spot. It is easily seen by reference to figure 7 that the half period of the oscillation in this case (no dead spot) is

$$\frac{\pi}{\omega}=2\tau$$

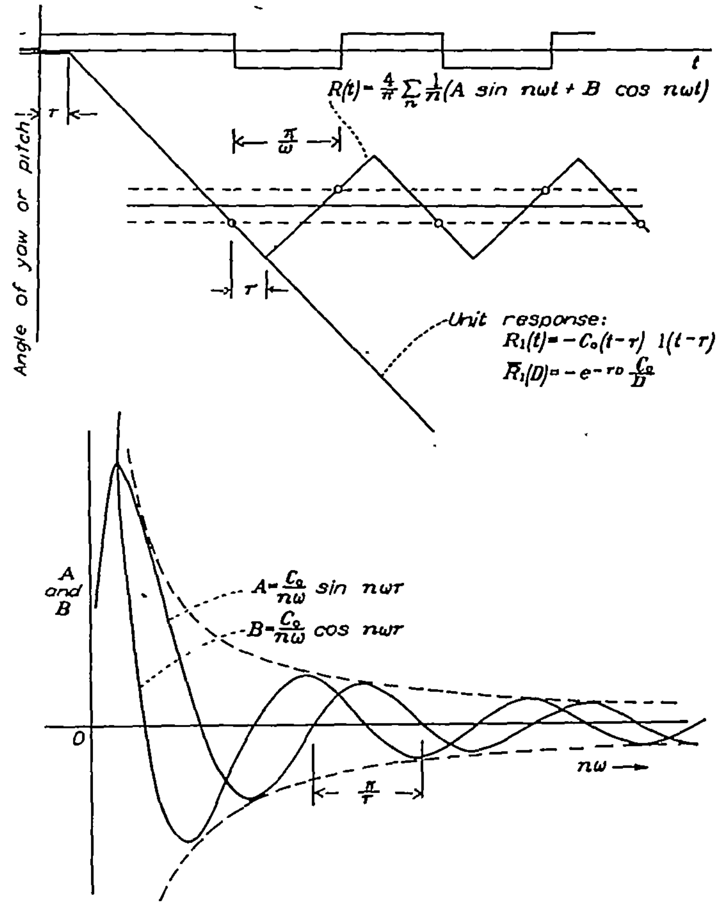


FIGURE 7.—Example in which response shows lag.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *May 5, 1944.*

## REFERENCES

1. Jeffreys, Harold: Operational Methods in Mathematical Physics. Cambridge Tracts in Mathematics and Mathematical Physics, No. 23. Second ed., Cambridge Univ. Press, 1931.
2. Jones, Robert T.: A Simplified Application of the Method of Operators to the Calculation of Disturbed Motions of an Airplane. NACA Rep. No. 560, 1936.
3. Jones, Robert T.: Calculation of the Motion of an Airplane under the Influence of Irregular Disturbances. Jour. Aero. Sci., vol. 3, no. 12, Oct. 1936, pp. 419–425.