

REPORT No. 509

GENERAL EQUATIONS FOR THE STRESS ANALYSIS OF RINGS

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SUMMARY

In this report it is shown that the shear, axial force, and moment at one point in a simple ring subjected to any loading condition can be given by three independent equations involving certain integrals that must be evaluated regardless of the method of analysis used. It is also shown how symmetry of the ring alone or of the ring and the loading about 1 or 2 axes makes it possible to simplify the three equations and greatly to reduce the number of integrals that must be evaluated.

Application of the general equations presented in this report to practical problems in the stress analysis of rings makes it possible to shorten, simplify, and systematize the calculations for both simple and braced rings. Three illustrative problems are included to demonstrate the application of the general equations to a simple ring with different loadings.

INTRODUCTION

During the past three years several papers on the stress analysis of rings for monocoque fuselages have appeared in American aeronautical literature (references 1 to 4). In these references consideration has been given to the special case of either a circular or an elliptic ring of constant cross section. Using the method of least work, the authors have derived formulas and charts that give the shear, axial force, and moment at one or more points in the ring for a number of simple loading conditions into which the majority of complicated loading conditions on the main rings of a monocoque fuselage may be resolved. As the greater number of monocoque fuselages actually constructed are probably not mathematically circular or elliptic in shape with rings of constant cross section, the equations and charts for shear, axial force, and moment given in the above-mentioned references are not generally applicable.

In the present report, prepared in cooperation with the Bureau of Aeronautics, Navy Department, equations applicable to the general case are developed. In the presentation of the general solution no consideration is given to the many short cuts that can often be made by a judicious choice of a method of analysis.

Such considerations would only be digressions from the general case and would result in a discussion of applications to special cases.

Three problems are given in the appendix to demonstrate the simplicity and ease with which the general solution can be applied to the stress analysis of a particular ring.

SYMBOLS

Throughout the present report, the following symbols are used:

M_0 , bending moment at any point in the determinate structure.

M , bending moment at any point in the complete structure.

X_{mj} , bending moment at the j th cut.

X_{pj} , axial force at the j th cut.

X_{sj} , shearing force at the j th cut.

$\frac{\partial M}{\partial X_{mj}} = M_{mj}$, bending moment at any point due to a unit bending moment at the j th cut.

$\frac{\partial M}{\partial X_{pj}} = M_{pj}$, bending moment at any point due to a unit axial force at the j th cut.

$\frac{\partial M}{\partial X_{sj}} = M_{sj}$, bending moment at any point due to a unit shearing force at the j th cut.

P_0 , axial force at any point in the determinate structure.

P , axial force at any point in the complete structure.

$\frac{\partial P}{\partial X_{mj}} = P_{mj}$, axial force at any point due to a unit bending moment at the j th cut.

$\frac{\partial P}{\partial X_{pj}} = P_{pj}$, axial force at any point due to a unit axial force at the j th cut.

$\frac{\partial P}{\partial X_{sj}} = P_{sj}$, axial force at any point due to a unit shearing force at the j th cut.

V_0 , shearing force at any point in the determinate structure.

V , shearing force at any point in the complete structure.

$\frac{\partial V}{\partial X_{mj}} = V_{mj}$, shearing force at any point due to a unit bending moment at the j th cut.

$\frac{\partial V}{\partial X_{pj}} = V_{pj}$, shearing force at any point due to a unit axial force at the j th cut.

$\frac{\partial V}{\partial X_{vj}} = V_{vj}$, shearing force at any point due to a unit shearing force at the j th cut.

U , total strain energy.

ds , element of length along the neutral axis of member.

E , tension-compression modulus of elasticity.

G , shear modulus of elasticity.

k , factor depending upon the shape of the cross section.

R , radius of curvature of axis through the centroid of the cross section of a curved member.

e , distance between the centroid and the neutral axis of the cross section of any curved member.

A , cross-sectional area of member.

g , number of members.

n , number of cuts.

m , number of unknowns.

GENERAL LEAST-WORK ANALYSIS

The method of analysis.—The approach to a least-work analysis of a statically indeterminate structure consists of imagining the structure to be cut at a number of points with unknown values of shear, axial force, and moment acting at the cut sections. The number of cuts is just sufficient to make the structure statically determinate. An expression is then set up for the work of distortion or, what amounts to the same thing, for the internal strain energy, and this expression is differentiated partially with respect to each of the unknowns. As the principle of least work states that the internal forces and moments adjust themselves so that the energy stored in the structure is a minimum consistent with the conditions of equilibrium, each partial derivative is set equal to zero. Thus, if there are m unknown forces and moments m equations involving the unknowns are obtained. The values of the m unknowns are found by solving the m simultaneous equations. With the unknown forces and moments evaluated, the forces and moments at any other point in the structure may be obtained by statics.

Expression for strain energy.—The total strain energy stored in a length l of a curved member may be expressed as follows:

$$U = \int \left[\frac{M^2}{2AEeR} + \frac{MP}{AeR} + \frac{P^2}{2AE} + \frac{kV^2}{2AG} \right] ds \quad (1)$$

integrated over the length l (reference 5). When the depth of the curved member is small in comparison to the radius of curvature of the centroidal axis, the distribution of the bending stress over the cross section approaches a linear one and the expression for the internal strain energy may be taken as equal to that for a straight member. This assumption is usually made in the analysis of rings encountered in

aircraft design and the following expression is therefore used:

$$U = \int \left[\frac{M^2}{2EI} + \frac{P^2}{2AE} + \frac{kV^2}{2AG} \right] ds \quad (2)$$

When the ratio $\frac{\text{radius of curvature}}{\text{depth of cross section}}$ is greater than 4, the assumption that the member is straight gives the stresses in the ring within 10 percent for cross sections commonly used in aircraft structures. If in a small portion of the ring this ratio is less than 4, the approximation does not seriously affect the least-work analysis, hence the stresses calculated in the other parts of the ring. When the ratio is greater than 10, the calculated stresses will be accurate within 3 to 4 percent. These values should be considered as indicative of the approximate range of accuracy rather than as fixed limits of accuracy.

Since bending moment M , axial force P , and shearing force V usually vary throughout the entire structure, general equations must be obtained for them. By the principle of superposition,¹ these equations are

$$\left. \begin{aligned} M &= M_0 + \sum_n M_{mj} X_{mj} + \sum_n M_{pj} X_{pj} + \sum_n M_{vj} X_{vj} \\ P &= P_0 + \sum_n P_{mj} X_{mj} + \sum_n P_{pj} X_{pj} + \sum_n P_{vj} X_{vj} \\ V &= V_0 + \sum_n V_{mj} X_{mj} + \sum_n V_{pj} X_{pj} + \sum_n V_{vj} X_{vj} \end{aligned} \right\} \quad (3)$$

the significance of the summation signs being that the indicated operations should be performed for the total n cuts and summed up. The internal energy may then be written in the form

$$U = \sum_g \int \left[\frac{M^2}{2EI} + \frac{P^2}{2AE} + \frac{kV^2}{2AG} \right] ds \quad (4)$$

where the integral indicates the energy in any individual member, the summation sign indicating that the work in all the members (g in number) must be summed up.

Partial differentiation of the expression for strain energy.—Partial differentiation of U as given by equation (4) with respect to the unknowns contained in M , P , and V and setting the resulting equations equal to zero gives

$$\left. \begin{aligned} \frac{\partial U}{\partial X_{mj}} &= \sum_g \left[\int \left\{ \frac{M}{EI} M_{mj} + \frac{P}{AE} P_{mj} + \frac{kV}{AG} V_{mj} \right\} ds \right] = 0 \\ \frac{\partial U}{\partial X_{pj}} &= \sum_g \left[\int \left\{ \frac{M}{EI} M_{pj} + \frac{P}{AE} P_{pj} + \frac{kV}{AG} V_{pj} \right\} ds \right] = 0 \\ \frac{\partial U}{\partial X_{vj}} &= \sum_g \left[\int \left\{ \frac{M}{EI} M_{vj} + \frac{P}{AE} P_{vj} + \frac{kV}{AG} V_{vj} \right\} ds \right] = 0 \end{aligned} \right\} \quad (5)$$

In the general case, with n cuts, there will be n equations of each of the foregoing types. It should be remembered, however, that these equations are applicable only when the cross-sectional dimensions of the members are small in comparison to their radii of curvature.

¹ Strictly speaking, the principle of superposition does not apply when the secondary effects of deflection are considered, but these effects are usually neglected in a least-work analysis.

APPLICATION OF GENERAL LEAST-WORK ANALYSIS TO RINGS

The general equations for a least-work analysis presented in the preceding section will now be applied to the analysis of closed rings. Where a ring is heavily loaded it is sometimes desirable to brace it with a strut or stay. In the following analyses both simple and braced rings will be considered. (See fig. 1.)

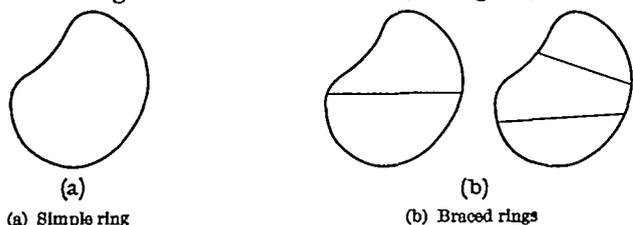


FIGURE 1.—Types of rings.

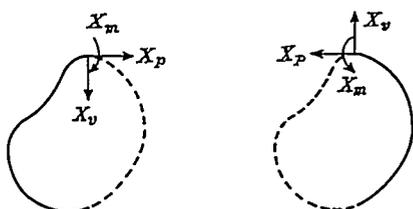


FIGURE 2.—Simple ring cut at one point with unknown values of shear, axial force, and moment acting at the section cut.

Simple ring: general case, ring of any shape.—For the simple closed ring both q and n of the general analysis become unity. Equations (5) therefore become

$$\left. \begin{aligned} \frac{\partial U}{\partial X_m} &= \int \left\{ \frac{M}{EI} M_m + \frac{P}{AE} P_m + \frac{kV}{AG} V_m \right\} ds = 0 \\ \frac{\partial U}{\partial X_p} &= \int \left\{ \frac{M}{EI} M_p + \frac{P}{AE} P_p + \frac{kV}{AG} V_p \right\} ds = 0 \\ \frac{\partial U}{\partial X_v} &= \int \left\{ \frac{M}{EI} M_v + \frac{P}{AE} P_v + \frac{kV}{AG} V_v \right\} ds = 0 \end{aligned} \right\} \quad (6)$$

For a thin ring such as considered in this report, a very close approximation may be obtained by neglecting the work due to axial forces and to shearing forces. Thus, omitting the last two terms in each of the preceding equations, equations (6) become

$$\left. \begin{aligned} \frac{\partial U}{\partial X_m} &= \int \frac{M}{EI} M_m ds = 0 \\ \frac{\partial U}{\partial X_p} &= \int \frac{M}{EI} M_p ds = 0 \\ \frac{\partial U}{\partial X_v} &= \int \frac{M}{EI} M_v ds = 0 \end{aligned} \right\} \quad (7)$$

Substitution of the general expression for M in the preceding equations gives

$$\left. \begin{aligned} X_m \int \frac{M_m^2}{EI} ds + X_p \int \frac{M_p M_m}{EI} ds + X_v \int \frac{M_v M_m}{EI} ds &= \\ &- \int \frac{M_0 M_m}{EI} ds \\ X_m \int \frac{M_m M_p}{EI} ds + X_p \int \frac{M_p^2}{EI} ds + X_v \int \frac{M_v M_p}{EI} ds &= \\ &- \int \frac{M_0 M_p}{EI} ds \\ X_m \int \frac{M_m M_v}{EI} ds + X_p \int \frac{M_p M_v}{EI} ds + X_v \int \frac{M_v^2}{EI} ds &= \\ &- \int \frac{M_0 M_v}{EI} ds \end{aligned} \right\} \quad (8)$$

Equations (8) are general and independent of the coordinate system used. In order that these equations may be applied to the analysis of specific problems it has been found convenient to use the following system: The origin of coordinates is assumed to be at

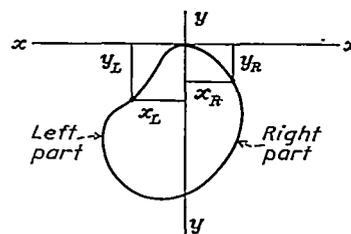


FIGURE 3.—Coordinate system used in the general solution for simple rings.

the top of the ring, where the cut is imagined to be. (See figs. 2 and 3.) The x axis is assumed to be horizontal and the y axis to be perpendicular to the x axis. By this choice of coordinates the ring is considered to be divided by the y axis into a left and a right part, each of which is treated separately in the evaluation of the integrals. The positive direction of x is to the left for the left part and to the right for the right part. The positive direction of y is downward for both the left and the right parts.

At any point in the ring the values of the shear, axial force, and moment are assumed to be positive when they have the directions indicated by the arrows in figure 2, as viewed by an observer standing on the inside of the ring, facing outward, and looking down on the part of the ring concerned. The moments M_L and M_R , in the left and right parts, respectively, may then be written as follows:

$$\left. \begin{aligned} M_L &= M_{0L} + M_{mL} X_m + M_{pL} X_p + M_{vL} X_v \\ M_R &= M_{0R} + M_{mR} X_m + M_{pR} X_p + M_{vR} X_v \end{aligned} \right\} \quad (9)$$

where

$$M_{mL} = M_{mR} = 1 \quad (10)$$

$$M_{pL} = M_{pR} = y \quad (11)$$

$$M_{vL} = x \text{ and } M_{vR} = -x \quad (12)$$

Substitution of these values in equations (8) therefore gives

$$\left. \begin{aligned} X_m \left[\int \left(\frac{1}{EI} \right)_L ds + \int \left(\frac{1}{EI} \right)_R ds \right] + X_p \left[\int \left(\frac{y}{EI} \right)_L ds + \int \left(\frac{y}{EI} \right)_R ds \right] + X_s \left[\int \left(\frac{x}{EI} \right)_L ds - \int \left(\frac{x}{EI} \right)_R ds \right] - \left[\int \left(\frac{M_0}{EI} \right)_L ds + \int \left(\frac{M_0}{EI} \right)_R ds \right] \\ X_m \left[\int \left(\frac{y}{EI} \right)_L ds + \int \left(\frac{y}{EI} \right)_R ds \right] + X_p \left[\int \left(\frac{y^2}{EI} \right)_L ds + \int \left(\frac{y^2}{EI} \right)_R ds \right] + X_s \left[\int \left(\frac{xy}{EI} \right)_L ds - \int \left(\frac{xy}{EI} \right)_R ds \right] - \left[\int \left(\frac{M_0 y}{EI} \right)_L ds + \int \left(\frac{M_0 y}{EI} \right)_R ds \right] \\ X_m \left[\int \left(\frac{x}{EI} \right)_L ds - \int \left(\frac{x}{EI} \right)_R ds \right] + X_p \left[\int \left(\frac{xy}{EI} \right)_L ds - \int \left(\frac{xy}{EI} \right)_R ds \right] + X_s \left[\int \left(\frac{x^2}{EI} \right)_L ds + \int \left(\frac{x^2}{EI} \right)_R ds \right] - \left[\int \left(\frac{M_0 x}{EI} \right)_L ds - \int \left(\frac{M_0 x}{EI} \right)_R ds \right] \end{aligned} \right\} \quad (13)$$

where the subscripts *L* and *R* indicate that the integrations are to be performed over the left and right portions, respectively.

In order to simplify the notation let

$$\left. \begin{aligned} A_1 &= \left[\int \left(\frac{1}{EI} \right)_L ds + \int \left(\frac{1}{EI} \right)_R ds \right] \\ A_2 &= \left[\int \left(\frac{y}{EI} \right)_L ds + \int \left(\frac{y}{EI} \right)_R ds \right] \\ A_3 &= \left[\int \left(\frac{x}{EI} \right)_L ds - \int \left(\frac{x}{EI} \right)_R ds \right] \\ B_1 &= \left[\int \left(\frac{y^2}{EI} \right)_L ds + \int \left(\frac{y^2}{EI} \right)_R ds \right] \\ B_2 &= \left[\int \left(\frac{xy}{EI} \right)_L ds - \int \left(\frac{xy}{EI} \right)_R ds \right] \\ C_1 &= \left[\int \left(\frac{x^2}{EI} \right)_L ds + \int \left(\frac{x^2}{EI} \right)_R ds \right] \\ D_1 &= \left[\int \left(\frac{M_0}{EI} \right)_L ds + \int \left(\frac{M_0}{EI} \right)_R ds \right] \\ D_2 &= \left[\int \left(\frac{M_0 y}{EI} \right)_L ds + \int \left(\frac{M_0 y}{EI} \right)_R ds \right] \\ D_3 &= \left[\int \left(\frac{M_0 x}{EI} \right)_L ds - \int \left(\frac{M_0 x}{EI} \right)_R ds \right] \end{aligned} \right\} \quad (14)$$

If these values are substituted in equations (13) the following system of simultaneous equations is obtained:

$$\left. \begin{aligned} A_1 X_m + A_2 X_p + A_3 X_s &= -D_1 \\ A_2 X_m + B_1 X_p + B_2 X_s &= -D_2 \\ A_3 X_m + B_2 X_p + C_1 X_s &= -D_3 \end{aligned} \right\} \quad (15)$$

The values of X_m , X_p , and X_s , having been obtained by solving the preceding equations simultaneously, the shear, axial force, and moment at all other stations around the ring may be calculated by statics, as stated previously. The stresses at each station may then be calculated by the standard beam formulas, as in the examples of reference 4.

Simple ring: ring symmetrical about one axis.—In aircraft structures it is customary to build floats, hulls,

and fuselages symmetrical about their central vertical plane, even though the loading is not always symmetrical. In view of this fact, it is possible greatly to simplify equations (15) as applied to the rings or frames of these structures. Thus, if the *y* axis of the coordinate system is made to coincide with the axis of symmetry of the rings or frames,

$$\left. \begin{aligned} A_3 &= 0 \\ B_2 &= 0 \end{aligned} \right\} \quad (16)$$

and equations (15) become

$$\left. \begin{aligned} A_1 X_m + A_2 X_p &= -D_1 \\ A_2 X_m + B_1 X_p &= -D_2 \\ C_1 X_s &= -D_3 \end{aligned} \right\} \quad (17)$$

from which

$$\left. \begin{aligned} X_m &= \frac{A_2 D_2 - B_1 D_1}{A_1 B_1 - A_2^2} = -\frac{D_1 + A_2 X_p}{A_1} \\ X_p &= \frac{A_2 D_1 - A_1 D_2}{A_1 B_1 - A_2^2} = -\frac{D_1 + A_1 X_m}{A_2} \\ X_s &= -\frac{D_3}{C_1} \end{aligned} \right\} \quad (18)$$

Simple ring: ring and loading symmetrical about one axis.—In the special case where the loads as well as the ring are symmetrical about the *y* axis, D_3 is also zero. Thus, in equations (17) and (18)

$$X_s = 0 \quad (19)$$

In all other respects the equations of the preceding paragraph apply. In the evaluation of the integrals, however, it should be noted that, since both the ring and loading are symmetrical about the *y* axis of coordinates, each integral is equal to twice the value for one-half of the ring.

Simple ring: ring and loading symmetrical about two mutually perpendicular axes.—Occasionally there are aircraft structures built in which the hull or fuselage shape is symmetrical about two axes. Airship hulls formed by regular polygons with an even number of sides and fuselages of circular, elliptic, and rectangular cross sections are examples of this type.

If both the ring and the loading are symmetrical about two axes, the labor required to evaluate X_m , X_p , and X_s may be greatly reduced over that required when there is symmetry about only one axis. From the discussion of the preceding section where both the ring and loading are symmetrical about one axis, it follows that, for symmetry of the ring and loading about two axes, the shear in the ring is zero at each end of the two axes of symmetry. By a separate consideration of the equilibrium of the right and left parts of the ring, it is possible to write for this case

$$X_p = \frac{1}{2} \sum \left(\begin{array}{l} x \text{ components of all forces or loads on} \\ \text{the right or the left part of the ring} \end{array} \right) \quad (20)$$

$$\left. \begin{aligned} X_a \int \frac{M_a^2}{EI} ds + X_b \int \frac{M_b M_a}{EI} ds + \dots + X_j \int \frac{M_j M_a}{EI} ds + \dots + X_m \int \frac{M_m M_a}{EI} ds &= - \int \frac{M_o M_a}{EI} ds \\ X_a \int \frac{M_a M_b}{EI} ds + X_b \int \frac{M_b^2}{EI} ds + \dots + X_j \int \frac{M_j M_b}{EI} ds + \dots + X_m \int \frac{M_m M_b}{EI} ds &= - \int \frac{M_o M_b}{EI} ds \\ X_a \int \frac{M_a M_j}{EI} ds + X_b \int \frac{M_b M_j}{EI} ds + \dots + X_j \int \frac{M_j^2}{EI} ds + \dots + X_m \int \frac{M_m M_j}{EI} ds &= - \int \frac{M_o M_j}{EI} ds \\ X_a \int \frac{M_a M_m}{EI} ds + X_b \int \frac{M_b M_m}{EI} ds + \dots + X_j \int \frac{M_j M_m}{EI} ds + \dots + X_m \int \frac{M_m^2}{EI} ds &= - \int \frac{M_o M_m}{EI} ds \end{aligned} \right\} \quad (21)$$

In the preceding system of simultaneous equations a definite "pattern" exists. The terms in the major diagonal row running from upper left to lower right contain only integrals with M squared. All other integrals in the pattern consist of products of M ; on any diagonal row running from the lower left to the upper right the integrals symmetrically located with respect to the major diagonal are identical. A short study of the form of the pattern will enable the reader to memorize it and thus eliminate the tedious work of deriving the equations each time that a least-work analysis is made. The pattern is a result of Maxwell's law of reciprocal deflections and always has the same form regardless of how the energy is stored in the structure.

In order to illustrate the application of the preceding general equations to a ring with a strut, or stay, the problem shown in figure 4 will be considered. Since

In the evaluation of X_p by the summation process indicated by equation (20), the components of force that act in the positive direction of x (away from the y axis) are considered as positive and those that act in the negative direction (toward the y axis) as negative. With the value of X_p determined, the value of X_m may be obtained from the first equation of (18).

Rings with struts or stays.—Cases are encountered of rings that contain one or more struts, or stays, for strengthening and stiffening. In such rings it is necessary to imagine the ring cut at more than one point. Hence q and n in the general solution are both greater than unity. Although this problem seems complicated because more than three unknowns are involved, it may be reduced to a simple systematic solution. From inspection of equations (8) it is possible to write the form of solution for the case of n unknowns instead of repeating the analysis and deriving it as a separate case. Suppose there are m unknowns. Then, if the integral sign indicates integration over the entire structure, the m simultaneous equations are:

the strut has been assumed to be horizontal it is convenient to cut the ring at the two points shown. Two cuts having been made, there are six unknowns: X_{m1} ,

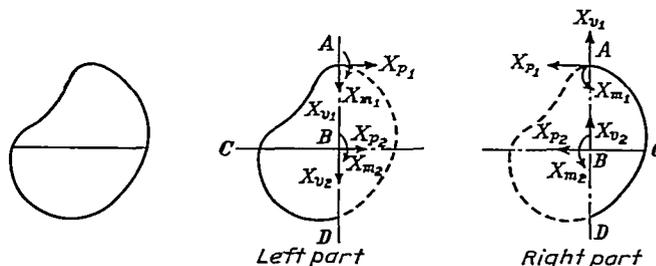


FIGURE 4.—Braced rings with one horizontal strut.

X_{p1} , X_{s1} , X_{s2} , X_{p2} , and X_{v2} . The general equations involving the unknowns are therefore six in number. From the characteristic pattern for the equations of least work, it is possible to write directly

$$\begin{aligned}
 X_{m_1} \int \frac{M_{m_1}^2}{EI} ds + X_{p_1} \int \frac{M_{p_1} M_{m_1}}{EI} ds + X_{v_1} \int \frac{M_{v_1} M_{m_1}}{EI} ds + X_{m_2} \int \frac{M_{m_2} M_{m_1}}{EI} ds + X_{p_2} \int \frac{M_{p_2} M_{m_1}}{EI} ds + X_{v_2} \int \frac{M_{v_2} M_{m_1}}{EI} ds = \\
 - \int \frac{M_0 M_{m_1}}{EI} ds \\
 X_{m_1} \int \frac{M_{m_1} M_{p_1}}{EI} ds + X_{p_1} \int \frac{M_{p_1}^2}{EI} ds + X_{v_1} \int \frac{M_{v_1} M_{p_1}}{EI} ds + X_{m_2} \int \frac{M_{m_2} M_{p_1}}{EI} ds + X_{p_2} \int \frac{M_{p_2} M_{p_1}}{EI} ds + X_{v_2} \int \frac{M_{v_2} M_{p_1}}{EI} ds = \\
 - \int \frac{M_0 M_{p_1}}{EI} ds \\
 X_{m_1} \int \frac{M_{m_1} M_{v_1}}{EI} ds + X_{p_1} \int \frac{M_{p_1} M_{v_1}}{EI} ds + X_{v_1} \int \frac{M_{v_1}^2}{EI} ds + X_{m_2} \int \frac{M_{m_2} M_{v_1}}{EI} ds + X_{p_2} \int \frac{M_{p_2} M_{v_1}}{EI} ds + X_{v_2} \int \frac{M_{v_2} M_{v_1}}{EI} ds = \\
 - \int \frac{M_0 M_{v_1}}{EI} ds \\
 X_{m_1} \int \frac{M_{m_1} M_{m_2}}{EI} ds + X_{p_1} \int \frac{M_{p_1} M_{m_2}}{EI} ds + X_{v_1} \int \frac{M_{v_1} M_{m_2}}{EI} ds + X_{m_2} \int \frac{M_{m_2}^2}{EI} ds + X_{p_2} \int \frac{M_{p_2} M_{m_2}}{EI} ds + X_{v_2} \int \frac{M_{v_2} M_{m_2}}{EI} ds = \\
 - \int \frac{M_0 M_{m_2}}{EI} ds \\
 X_{m_1} \int \frac{M_{m_1} M_{p_2}}{EI} ds + X_{p_1} \int \frac{M_{p_1} M_{p_2}}{EI} ds + X_{v_1} \int \frac{M_{v_1} M_{p_2}}{EI} ds + X_{m_2} \int \frac{M_{m_2} M_{p_2}}{EI} ds + X_{p_2} \int \frac{M_{p_2}^2}{EI} ds + X_{v_2} \int \frac{M_{v_2} M_{p_2}}{EI} ds = \\
 - \int \frac{M_0 M_{p_2}}{EI} ds \\
 X_{m_1} \int \frac{M_{m_1} M_{v_2}}{EI} ds + X_{p_1} \int \frac{M_{p_1} M_{v_2}}{EI} ds + X_{v_1} \int \frac{M_{v_1} M_{v_2}}{EI} ds + X_{m_2} \int \frac{M_{m_2} M_{v_2}}{EI} ds + X_{p_2} \int \frac{M_{p_2} M_{v_2}}{EI} ds + X_{v_2} \int \frac{M_{v_2}^2}{EI} ds = \\
 - \int \frac{M_0 M_{v_2}}{EI} ds
 \end{aligned} \tag{22}$$

Inspection of these equations shows that there is always some path, section of the ring, or stay along which the value of each integral is zero. For example, M_{p_1} is zero from B to C . Hence, any integral involving M_{p_1} is also zero from B to C . Since M_{m_2} is zero from A to C any integral involving M_{m_2} is also zero from A to C and any integral involving the product of M_{p_1} and M_{m_2} is zero over both AC and BC . The fourth integral in the second equation

$$\int \frac{M_{m_2} M_{p_1}}{EI} ds$$

need therefore be evaluated only from C to D . In a similar manner the paths for which each of the other integrals in equations (22) need to be evaluated have been determined and are listed in the following table:

EVALUATION OF EQUATIONS (22) FOR FIGURE 4

	Integrals not involving loads, left-hand side equations (22)		Integrals involving loads, right-hand side equations (22)
	First 3 terms	Second 3 terms	
First 3 equations... Second 3 equations.	AC and CD CD	CD BC and CD	AC and CD BC and CD

The integrals in equations (22) having been evaluated for the paths indicated, the values of X_{m_1} , X_{p_1} , X_{v_1} , X_{m_2} , X_{p_2} , and X_{v_2} are obtained by solving the simultaneous equations. The shear, axial force, and

moment at all stations around the ring may then be calculated by statics and the stresses by the standard beam formulas.

DISCUSSION

Simple rings.—The application of the general solution to problems in the stress analysis of rings and frames makes it possible to simplify and systematize the calculations for X_m , X_p , and X_v in a way that is not possible when starting from fundamental considerations each time a ring or frame is stress-analyzed. The procedure is first to locate the coordinate axes in the most judicious manner considering conditions of symmetry, and then to evaluate the integrals that appear in the general equations. If the evaluation of the integrals analytically is difficult, standard numerical or graphical methods may be used.

An examination of the integrals that must be evaluated in order to determine X_m , X_p , and X_v shows them to be of two general types. One type involves only the stiffness EI and the dimensions of the ring. The other type involves the loads in addition to the stiffness and dimensions. The following table giving the number of integrations that must be made on the two types of integrals for the various cases considered in this report is presented to show in an approximate manner the degree to which the labor is reduced by conditions of symmetry. The evaluation of any integral for one part of the ring is considered as one integration.

THE NUMBER OF INTEGRATIONS NECESSARY FOR DIFFERENT CASES

Case	Integrals involving loads	Integrals not involving loads
General case, ring with any variation in loading, material, and dimensions.....	6	12
Ring symmetrical about one axis.....	6	4
Ring and loading symmetrical about one axis.....	2	3
Ring and loading symmetrical about two axes.....	1	2

After a particular problem has been classified by the conditions of symmetry, all of the integrals involved in that case must be evaluated regardless of what method of analysis is used, unless perhaps the shape of the ring is such as to lend itself to some short-cut method. If a stress analysis of a given ring is desired for a series of loading conditions, it should be noted that only those integrals involving the loads (integrals including M_0) must again be evaluated for the different loading conditions. Thus the labor required to obtain X_m , X_p , and X_s is reduced to a minimum by application of the general equations herein developed.

In certain cases where rings of identical size or of the same relative shape may be encountered frequently, it is advantageous to resolve all complicated loading conditions into a few simple ones. (See references 1 to 4.) The shear, axial force, and moment for the simple loading conditions may be calculated either by standard methods or as outlined in this report and the appropriate values added algebraically to obtain the shear, axial force, and moment for any complicated loading condition.

It is not probable that rings and frames of identical size or of the same relative dimensions will be encountered frequently. The resolution of the complicated loading conditions into a few simple ones does therefore not always result in the same advantage. For rings of oval and other odd shapes, it is probable that the integrals involved in the general solution would have to be evaluated by standard numerical or graphical

methods. If such be the case, it is just as easy to consider the complicated loading condition as it exists. With the work properly planned for this method, it is a simple matter to obtain the moment, axial force, and shear at a number of stations around the ring after the values of X_m , X_p , and X_s have been found.

Braced rings.—The application of the general equations to the analysis of braced rings has been carried only to the extent of indicating a possible systematic solution. For each particular case the designer will be able to minimize the labor involved by a judicious choice of axes, rather than by the use of a standard set of axes such as would be necessary to carry the general solution to more detailed conclusions. In the present paper a series of braced rings with different conditions of symmetry could have been assumed and the most convenient axes chosen. The general solution could then have been completed and discussed in a manner similar to that adopted for the simple, or unbraced, ring. As the bracing used is of such a variety of forms, however, it was not thought worth while to attempt a classification of each type and present its solution.

CONCLUSIONS

1. The shear, axial force, and moment at one point in a simple ring subjected to any loading condition can be given by three independent equations involving certain integrals that must be evaluated regardless of the method of analysis used. Symmetry of the ring alone or of the ring and loading about 1 or 2 axes makes it possible to simplify the three equations and greatly to reduce the number of integrals that must be evaluated.

2. Application of the general equations presented in this report to practical problems in the stress analysis of rings makes it possible to shorten, simplify, and systematize the calculations for both simple and braced rings.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., August 10, 1934.

APPENDIX

In order to demonstrate the application of the general equation to a specific example, the values of X_m , X_p , and X_s will be calculated for a circular ring with different loading conditions. The problems were made simple so that the reader would not lose the perspective of the general solution presented in the report proper. Because the problems are simple, it may be that a shorter solution can be obtained by some method other than the general method here used. In the majority of problems, however, it will doubtless be found advantageous to use the general solution.

PROBLEM A

A circular ring of constant cross section is loaded as shown in figure 5. The y axis of coordinates is

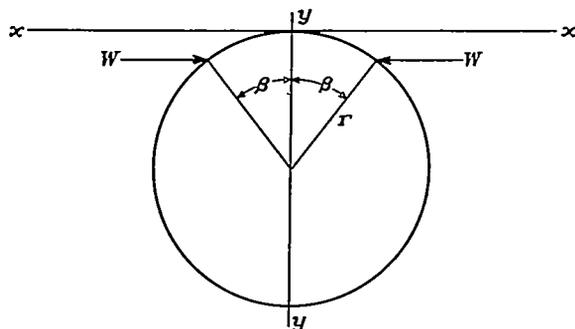


FIGURE 5.—Problem A.

made to coincide with the diameter about which the loads are symmetrical. By this choice of axes both the ring and the loading are symmetrical about the y axis. Consequently, $X_s=0$ and the equations for X_p and X_m are (see equation (18)).

$$X_p = \frac{A_2 D_1 - A_1 D_2}{A_1 B_1 - A_2^2}$$

$$X_m = -\frac{D_1 + A_2 X_p}{A_1}$$

Evaluation of the integrals.—It being assumed that the material and cross section are constant at all stations around the ring, EI may be canceled from all integrals in the numerator and denominator of the preceding equations for X_p and X_m . Thus, the evaluation of the integrals is

$$A_1 = \int ds = 2 \int_0^\pi r d\alpha = 2r \int_0^\pi d\alpha = [2r\alpha]_0^\pi = 2\pi r$$

$$A_2 = \int y ds = \int r(1 - \cos \alpha) ds = r \int ds - r \int \cos \alpha ds$$

$$= r A_1 - 2r^2 \int_0^\pi \cos \alpha d\alpha$$

$$= r A_1 - [2r^2 \sin \alpha]_0^\pi$$

$$= 2\pi r^2$$

$$B_1 = \int y^2 ds = \int r^2(1 - \cos \alpha)^2 ds$$

$$= r^2 \int ds - 2r^2 \int \cos \alpha ds + r^2 \int \cos^2 \alpha ds$$

$$= r^2 A_1 - 0 + 2r^3 \int_0^\pi \cos^2 \alpha d\alpha$$

$$= r^2 A_1 + 2r^3 \left[\frac{1}{2} \alpha + \frac{1}{4} \sin 2\alpha \right]_0^\pi$$

$$= 2\pi r^3 + \pi r^3$$

$$= 3\pi r^3$$

$$D_1 = \int M_0 ds = 2 \int_\beta^\pi W r (\cos \beta - \cos \alpha) r d\alpha$$

$$= 2W r^2 \left[\alpha \cos \beta - \sin \alpha \right]_\beta^\pi$$

$$= 2W r^2 [(\pi - \beta) \cos \beta + \sin \beta]$$

$$D_2 = \int M_0 y ds = \int M_0 r(1 - \cos \alpha) ds$$

$$= r \int M_0 ds - r \int M_0 \cos \alpha ds$$

$$= r D_1 - 2r \int_\beta^\pi W r (\cos \beta - \cos \alpha) \cos \alpha r d\alpha$$

$$= r D_1 - 2W r^3 \int_\beta^\pi (\cos \alpha \cos \beta - \cos^2 \alpha) d\alpha$$

$$= r D_1 - 2W r^3 \left[\sin \alpha \cos \beta - \frac{1}{2} \alpha - \frac{1}{4} \sin 2\alpha \right]_\beta^\pi$$

$$= 2W r^3 [(\pi - \beta) \cos \beta + \sin \beta]$$

$$- 2W r^3 \left[-\sin \beta \cos \beta - \frac{1}{2} (\pi - \beta) + \frac{1}{2} \sin \beta \cos \beta \right]$$

$$= 2W r^3 \left[(\pi - \beta) \cos \beta + \sin \beta + \frac{1}{2} \sin \beta \cos \beta + \frac{1}{2} (\pi - \beta) \right]$$

Evaluation of X_p and X_m .—Before the substitution of the integrals in the equations for X_p and X_m , it is convenient to summarize them as follows:

$$\begin{aligned} A_1 &= 2\pi r \\ A_2 &= 2\pi r^2 \\ B_1 &= 3\pi r^3 \\ D_1 &= 2Wr^2 F \\ D_2 &= 2Wr^3 (F+G) \end{aligned}$$

where $F = [(\pi - \beta) \cos \beta + \sin \beta]$

and $G = \frac{1}{2} [(\pi - \beta) + \sin \beta \cos \beta]$

Substituting the values of the integrals in the equations for X_p and X_m gives

$$\begin{aligned} X_p &= \frac{(2\pi r^2)(2Wr^2 F) - (2\pi r)[2Wr^3(F+G)]}{(2\pi r)(3\pi r^3) - (2\pi r^2)^2} \\ &= -\frac{2WG}{\pi} \\ &= -\frac{W}{\pi} [(\pi - \beta) + \sin \beta \cos \beta] \\ X_m &= -\frac{(2Wr^2 F) + (2\pi r^2) \left(-\frac{2WG}{\pi}\right)}{2\pi r} \\ &= \frac{Wr}{\pi} (2G - F) \\ &= \frac{Wr}{\pi} (1 - \cos \beta) (\pi - \beta - \sin \beta) \end{aligned}$$

PROBLEM B

In order to demonstrate further the application of the general equations to a specific example and to show how several of the integrals evaluated for problem A may be used in other problems, the values of X_p , X_p , and X_m will be calculated for a circular ring of constant cross section loaded as shown in figure 6.

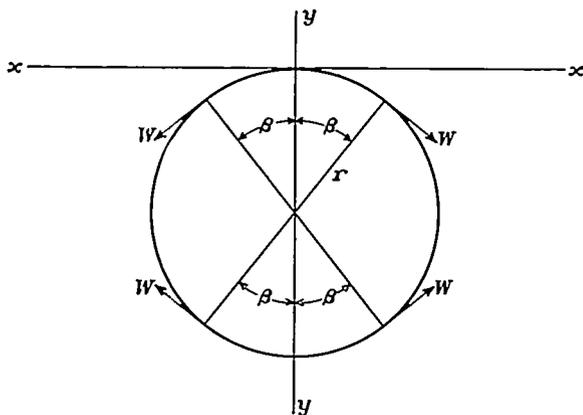


FIGURE 6.—Problem B.

In this problem the y axis will be made to coincide with the vertical diameter, one of the two diameters about which the ring and loads are symmetrical. By this choice of axes

$$X_p = 0$$

$$X_p = W \cos \beta \text{ (See equation (20).)}$$

$$X_m = -\frac{D_1 + A_2 X_p}{A_1}$$

Since the ring has not changed from that considered in problem A, the integrals A_1 and A_2 have not changed. The integral D_1 must again be evaluated because the loading has changed.

$$\begin{aligned} D_1 &= \int M_0 ds = 2 \int_{\beta}^{\pi} Wr [1 - \cos(\alpha - \beta)] r d\alpha \\ &\quad + 2 \int_{\pi - \beta}^{\pi} Wr [1 - \cos(\alpha - \pi + \beta)] r d\alpha \\ &= -2Wr^2 \left[\alpha - \sin(\alpha - \beta) \right]_{\beta}^{\pi} \\ &\quad + 2Wr^2 \left[\alpha - \sin(\alpha - \pi + \beta) \right]_{\pi - \beta}^{\pi} \\ &= -2Wr^2 (\pi - \beta - \sin \beta) + 2Wr^2 (\beta - \sin \beta) \\ &= 2Wr^2 (2\beta - \pi) \end{aligned}$$

Substituting the values of the integrals and X_p in the equation for X_m gives

$$X_m = Wr \left(1 - \frac{2\beta}{\pi} - \cos \beta \right)$$

PROBLEM C

In this problem a circular ring of constant cross section is loaded as shown in figure 7. Both the ring

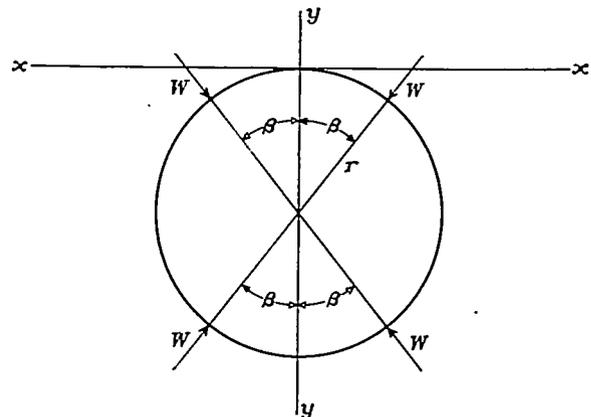


FIGURE 7.—Problem C.

and loading are symmetrical about two axes so, if the y axis is made to coincide with the vertical diameter as shown,

$$X_p = 0$$

$$X_p = -W \sin \beta$$

$$X_m = -\frac{D_1 + A_2 X_p}{A_1}$$

The integrals A_1 and A_2 are still the same as for problem A. The integral D_1 is dependent upon the

loading condition and must be evaluated for the particular problem.

$$\begin{aligned} D_1 &= \int M_0 ds = 2 \int_{\beta}^{\pi} W r \sin(\alpha - \beta) r d\alpha \\ &\quad + 2 \int_{\pi - \beta}^{\pi} W r \sin(\alpha - \pi + \beta) r d\alpha \\ &= -2 \left[W r^2 \cos(\alpha - \beta) \right]_{\beta}^{\pi} - 2 \left[W r^2 \cos(\alpha - \pi + \beta) \right]_{\pi - \beta}^{\pi} \\ &= 4 W r^2 \end{aligned}$$

Substituting the values of the integrals and X_p in the equation for X_m gives

$$X_m = W r \left[\sin \beta - \frac{2}{\pi} \right].$$

It should be noted in problems B and C that the integral D_1 was evaluated by taking the sum of two integrals, each of which considered the separate moments caused by the two forces on one part of the ring. It was found that this method of evaluating the integral involved fewer terms than would have been involved had the moment caused by the first force been integrated between the limits β and $\pi - \beta$ and then the combined moment caused by the two forces integrated between the limits $\pi - \beta$ and π . This method of evaluating the D integrals as a sum of the integrals for the separate loads is often advantageous in problems where the loadings are likely to change.

REFERENCES

1. Miller, Roy A.: A Solution of the Circular Ring. *Airway Age*, May 1931, pp. 465-468.

The author breaks down a complicated loading condition into a number of simple ones and applies the method

of least work to obtain shear, axial force, and moment in the ring. The application of the method of least work is long and laborious in that the differentiations of the expressions for the work of distortion were not made until after the integrations.

2. Sibert, H. W.: Circular Ring with Concentrated Loads. *Air Corps Information Circular No. 673*, Matériel Division, Army Air Corps, 1932.

The solutions of Miller, presented in reference 1, are simplified by differentiating the expressions for the work of distortion before their integration.

3. Burke, Walter F.: Working Charts for the Stress Analysis of Elliptic Rings. T. N. No. 444, N. A. C. A., 1933.

Charts are presented which give in coefficient form the axial force and moment at one end of the axis of symmetry for an elliptic ring of constant cross section subjected to two simple loading conditions into which the majority of symmetrical loading conditions on the main frames of a monocoque fuselage may be resolved. An illustrative problem is included to demonstrate the application of the charts.

4. Miller, Roy A., and Wood, Karl D.: Formulas for the Stress Analysis of Circular Rings in a Monocoque Fuselage. T. N. No. 462, N. A. C. A., 1933.

No consideration is given to the application of the method of least work, the object being merely to summarize the formulas derived for certain simplified loading conditions for circular rings of uniform cross section and to show how they may be used in practical stress-analysis work by calculating the stresses in one of the main rings of a monocoque fuselage of circular section.

5. Timoshenko, S.: *Strength of Materials, Part II*. D. Van Nostrand Co., Inc., 1930.