REPORT No. 334

THE TORSION OF MEMBERS HAVING SECTIONS COMMON IN AIRCRAFT CONSTRUCTION

By GEORGE W. TRAYER and H. W. MARCH
Forest Products Laboratory

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FOREWORD

An important limitation upon the development of aircraft structures of minimum weight and maximum efficiency is the fact that the results of the basic mathematical theory of elasticity have in general not been so presented that they can be used by the average American engineer. Further, the mathematical solutions of many important specific problems are not known and our knowledge of the physical constants of the materials used is incomplete. As a contribution toward the improvement of this situation, the Bureau of Aeronautics of the Navy Department has from time to time financed work along these lines for presentation primarily from the viewpoint of the engineer. This report, submitted to the National Advisory Committee for Aeronautics for publication, covers an investigation of the torsion problem by the Forest Products Laboratory, Forest Service, Department of Agriculture, undertaken through arrangements between the Navy Department and the Department of Agriculture. The discussion and the findings, while checked largely by tests of wooden specimens, apply equally to wood and to metal, due consideration being given to the elastic properties of the materials used.

The data and the formulas presented apply strictly to the torsion phenomenon. A beam may fail either in a normal type of bending or by lateral buckling resulting from normal loading, or by twisting or wrinkling of an outstanding flange under stresses having their origin in the normal loading. Likewise, a member such as a very thin tube subjected to torsion may fail at a load less than the theoretical load calculated by the torsion formulas because of other phenomena, such as wrinkling, which have their origin in the twisting load. It is necessary either to develop criteria for freedom from such secondary failures or to apply coefficients to the calculated strength values to take care of secondary failure. Technical Note No. 189 of the National Advisory Committee for Aeronautics, which gives formulas for the variation of allowable shearing stress with change in the ratio of diameter to thickness, indicates one method of approach to this problem. The Army and Navy Standards for sizes of tubing permit a range in the ratio of diameter to thickness for seamless tubing of about 5 to 48, and tubes of higher ratio can of course be fabricated.

It should be noted that the polar moment of inertia and the polar moment of inertia divided by the distance to the extreme fiber have no significance in comparing the rigidity and the strength of sections of different form; in this respect they are not analogous to the use of the moment of inertia and the section modulus in comparing the bending of beams. It so happens that the rigorous stress formulas for circular rods and tubes as given in the report reduce to the common form $q=\frac{Tc}{J}$, *which is analogous to the stress formula* $S=\frac{MC}{I}$ *for beams. The beam formula, however, is general, while the common torsion formula is true only for circular rods and tubes. For members of other shapes an additional factor must be introduced into the formula when reduction is made to the common form.*

The results of the actual torsion tests of simple sections in Table III show large variations in observed physical properties, which may cause doubt as to the soundness of design values deduced from the results. Actually part of the material reported on, while acceptable for making tests, is outside the specified acceptable range for aircraft stock—the test material represented the entire tree. Later recommendations for design values are based on the specification range and are conservative for reasonable variations outside that range. For metals these variations (in the ratio of "G" to "E" values) are much less in amount.

Recommended design stresses as furnished by the Bureau of Aeronautics are given in Appendix C.

J. H. Towers,
Acting Chief of Bureau.

REPORT No. 334

THE TORSION OF MEMBERS HAVING SECTIONS COMMON IN AIRCRAFT CONSTRUCTION[1]

By George W. Trayer and H. W. March[2]

INTRODUCTION AND PURPOSE

Structural members designed for pure torsion are usually made with circular, elliptical, rectangular, or other regular cross sections that have already yielded to direct mathematical treatment as regards torsion. However, members designed primarily for thrust or bending and consequently as a usual thing of irregular section are subjected to torsion also, and under certain conditions they may fail by buckling and twisting through lack of sufficient torsional rigidity. In order to design against the possibility of such failure in thin deep beams or in compression members with thin, outstanding parts, we must be able to calculate their torsional rigidity. Such sections as I, T, L, and U have not been brought within the range of mathematical analysis and up to a few years ago the engineer had practically nothing on which to base an estimate of the torsional strength and rigidity of rods of irregular section. This publication presents the results of investigations of the torsion of structural members undertaken by the Forest Products Laboratory and financed by the Bureau of Aeronautics, Navy Department, under the national defense act.

Within recent years a great variety of approximate torsion formulas and drafting-room processes have been advocated. In some of these, especially where mathematical considerations are involved, the results are extremely complex and are not generally intelligible to engineers. The principal object of the investigation was to determine by experiment and theoretical investigation how accurate the more common of these formulas are and on what assumptions they are founded, and, if none of the proposed methods proved to be reasonably accurate in practice, to produce simple, practical formulas from reasonably correct assumptions, backed by experiment. A second object was to collect in readily accessible form the most useful of known results for the more common sections.

This report reviews informally the fundamental theory of torsion and shows how the more common formulas are developed from it. Formulas for all the important solid sections that have yielded to mathematical treatment are listed. Then follows a discussion of the torsion of tubular rods with formulas both rigorous and approximate.

It is shown by a series of tests of prisms of simple section that wood is a suitable material for the experimental investigation of the torsion problem. Accordingly, wood was used for the experiments on full-sized members because of the ease with which it can be worked into different shapes and because of its low torsional stiffness. The possible effect of different moduli of rigidity in radial and tangential planes was investigated mathematically, and the effects of rate of loading and of moisture content were determined experimentally. Furthermore, soap films were used in order to take advantage of a mathematical similarity that exists between the torsion problem and the problem of finding the deflection of a thin membrane under pressure. The analogy is discussed in detail in the report.

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[1] Originally submitted as "The Torsion of Cylinders and Prisms."

[2] Professor of mathematics, University of Wisconsin.

Note.—R. J. Roark, associate professor of mechanics, University of Wisconsin, collaborated with the authors in certain phases of this investigation, giving especial attention to the soap-film method.

Our experimental work with beams of irregular sections that have not yielded to mathematical treatment is described. From these experiments and certain mathematical considerations, empirical formulas are set up for irregular sections whose component parts are rectangles.

TEST MATERIAL AND PROCEDURE

DIRECT TORSION TESTS OF BEAMS

The first series of direct torsion tests was confined to rods of simple section, such as the circle, the square, the ellipse, and the equilateral triangle. The test specimens were made of carefully selected Sitka spruce and when several were to be compared directly they were cut from the same plank. The elastic properties of the material in any plank were obtained by testing small minor specimens cut from the plank and so located as to be representative. These specimens usually consisted of two bending, two compression parallel to the grain, two specific gravity, eight shear, and three torsion specimens. The three minor torsion specimens consisted of one piece approximately 1¼ inches square, one piece 1 by 3 inches quarter-sawn, and one piece 1 by 3 inches flat-sawn. Four of the shear specimens were tested radially and four tangentially. All major torsion specimens for this first series of tests were 45 inches long and the area of cross section was usually less than 2 square inches.

The second series of tests was made on beams of irregular section, such as I, T, L, and U. These beams were 96 inches long, were cut from clear Sitka spruce planks, and were matched as described for the first series.

The apparatus for the first two series of tests, which was constructed expressly for these tests, consisted essentially of an attachment for holding one end of the beam fixed against rotation and a disk for applying torque at the other end. (Fig. 1.) Rollers at the fixed end provided for longitudinal movement. The disk, which was 10 inches in radius, turned on ball bearings. It was rotated by a metal strap attached to and passing around its periphery and thence up to a yoke attached to the weighing platform of a scale, which was accurate to one one-hundredth of a pound. The entire scale was bolted to the movable head of a testing machine and load was applied by raising the head. The usual length over which distortion was read, called the "gage length" in this report, was 24 inches for the short specimens and 36 inches for the long specimens. At one end of the gage length a circular metal frame with a 20-inch radius was clamped to the specimen. On the periphery of this circular frame was attached a steel tape graduated to tenths of an inch. At the other end of the gage length a rectangular frame was also clamped to the specimen, and welded to this frame was a pointer that extended to the scale on the circular frame. As the beam was twisted the scale rotated more than the pointer. Determining the excess movement of the scale by reading the position of the pointer on it thus yielded the angle of twist over a given gage length directly, the total angle in radians being the scale reading divided by the radius, 20 inches. The angle of twist per unit length, in radians, is then this quotient divided by the gage length. Except for tests made specifically to determine the effect of rate of loading, the rate was varied with the type of specimen, in order to obtain approximately the same rate of strain in all tests. Such variation necessitated raising the movable head of the testing machine at rates of from 0.674 inch per minute to 1.40 inches per minute. For tests made to determine the effect of rate of loading, the speed of the movable head was varied from 0.023 inch per minute to 2.25 inches per minute.

SOAP-FILM TESTS

The value of soap films in determining the torsional rigidity of a twisted rod and the stresses in it depends upon an analogy between the torsion problem and that of finding the deflection of a thin membrane under the action of a uniform load. The mathematical similarity is discussed later in this report, where it is shown that if a soap film is stretched over a hole in a flat plate, the hole being the same shape as the cross section of the bar and the film being displaced from the plane of the plate by a slight difference in pressure on the two sides, the following relations hold:

1. The shear stress at any point of the cross section is proportional to the slope of the film at the corresponding point with respect to the plane of its boundary.

2. The contour lines of the film represent the direction of the resultant shear stress at every point.

3. The torsional rigidity of the section is proportional to the volume between the soap film and the plane of the plate.

In order to make use of the analogy it was necessary to design apparatus with which the slope of the film, its contour lines, and the volume of displacement could be determined. The

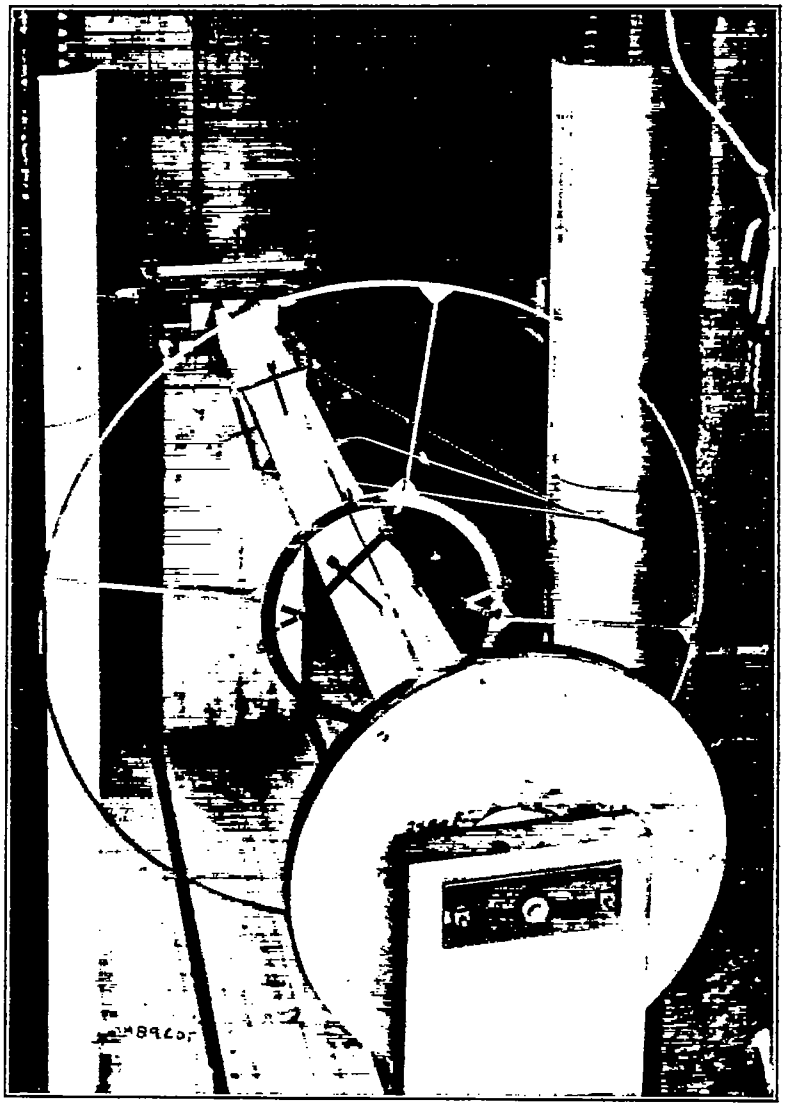

FIGURE 1.—Apparatus for applying torque to structural shapes and measuring the angle of twist

apparatus was patterned after that used by Griffith and Taylor and described by them in Advisory Committee for Aeronautics (British) Reports and Memoranda No. 333, June, 1917. As pointed out, the stresses in the bar are proportional to the inclination of the film and the stiffness of the bar is proportional to the volume generated by the film displacement. The relations hold for any number of films provided the difference in pressure on the sides of a film is the same for all. This condition is readily attainable by making more than one hole in the same test plate; it is evident that the easiest way of obtaining actual stress or rigidity values is to have a circular hole in each plate in addition to the hole that represents the section being studied. The rigidity of a circular shaft and its stresses are easily calculated, and having the two films in the same plate makes it possible to compare torsional rigidities directly by comparing volumes and to compare stresses directly by comparing slopes.

In assembling the apparatus, a plate with the experimental hole and a circular hole cut in it (fig. 2) was clamped between the bottom and the sides of a cast-iron box (fig. 3). The box bottom, which was 11¾ inches square outside and 2 inches thick, was supported on leveling

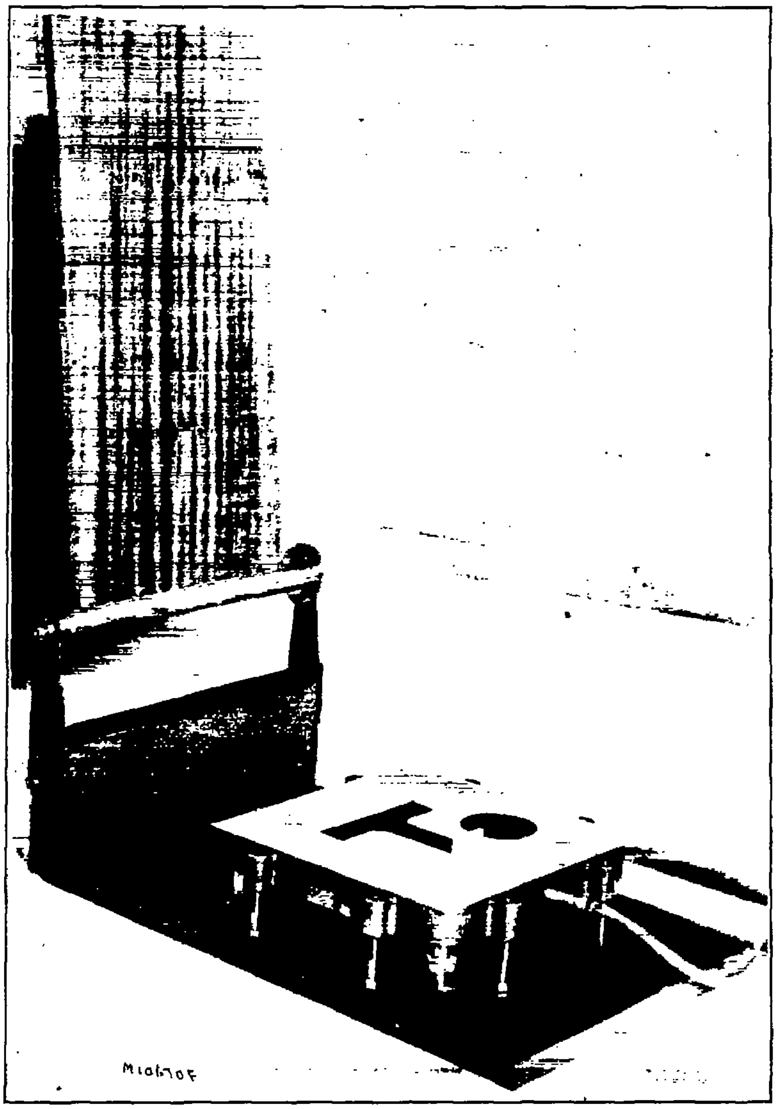

FIGURE 2.—The soap-film apparatus with the upper part of the box removed to show the perforated plate

screws. It was recessed ¼ inch inside of the ¾-inch bearing surface on which the plate rested. A square frame ¾ inch thick and 2 inches deep formed the sides of the box; both bottom and frame were provided with lugs for clamping screws. Over the frame was placed a piece of plate glass through the center of which a hole had been cut for a micrometer height gage, reading to one one-thousandth of an inch, that carried at its lower end a hardened steel needle point. Fixed axially above the needle point, extending upward from the frame supporting the gage, was a steel recording point. The position of the gage, at each reading, was recorded by pressing against it a sheet of paper attached to a board that could be swung down to the horizontal for this purpose; the board was hinged to the heavy cast-iron base on which the bubble box was leveled. Provision was made for increasing the air pressure below the soap films or decreasing it above them.

With this apparatus contour lines and hence displacement volumes could be determined. Stress could also be determined, since it is inversely proportional to the distance between con-

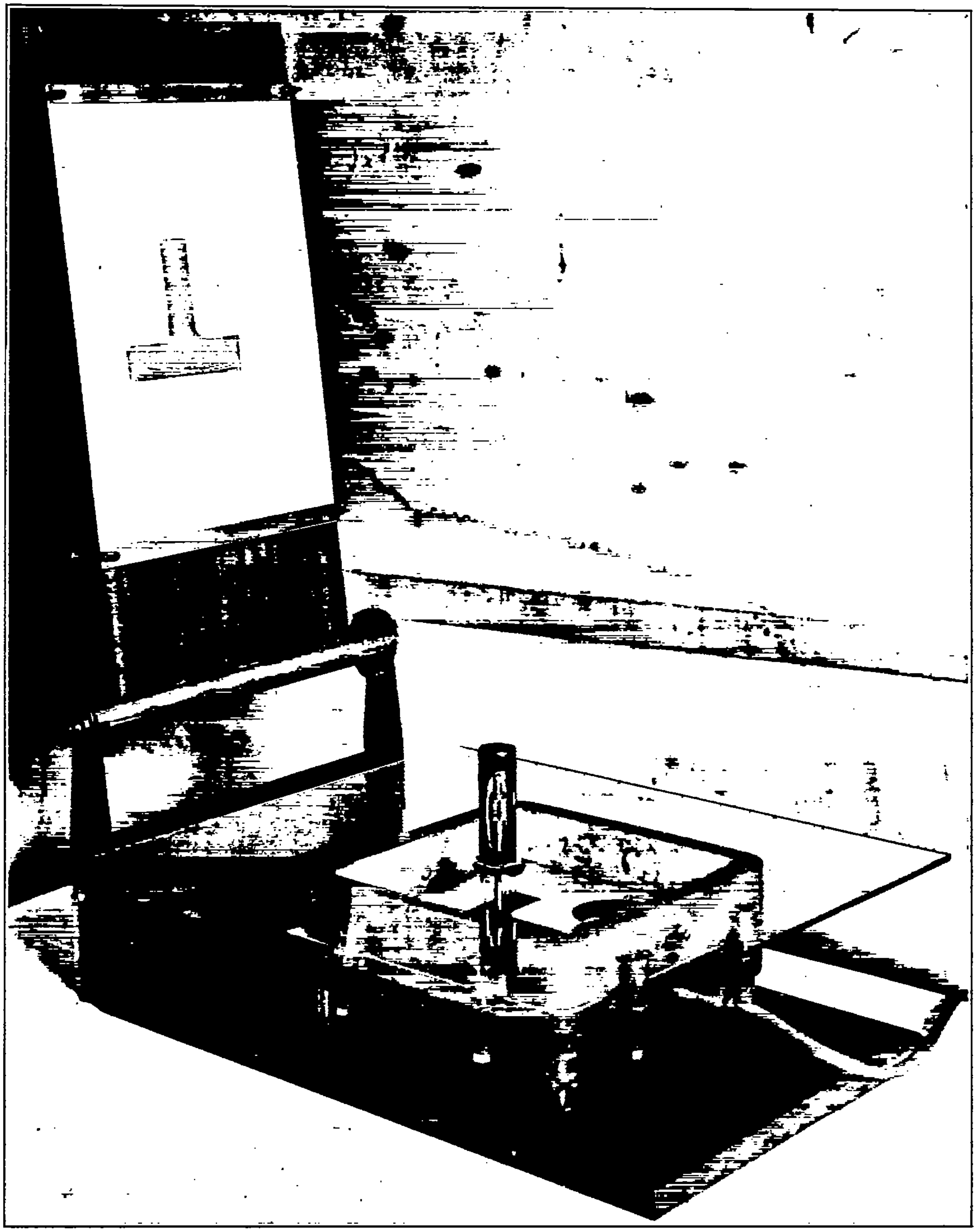

FIGURE 3.—The complete assembly of the soap-film apparatus

secutive contours. A collimator for measuring slope directly was made but time and funds allotted to the study were exhausted before it was put in use.

The test plates were cut from sheet aluminum approximately 0.05 inch thick. The edges around the test holes were beveled; the sharp edge was placed upward in the apparatus and great care was taken to keep the plates perfectly flat. When divergences from a plane were found they were corrected by propping up the plate from below or by putting small weights on top of it.

With symmetrical sections a complete boundary sometimes was not used. Griffith and Taylor found that the shape of a symmetrical film was unaltered if the film was divided by a vertical septum passing through its axis of symmetry. In our work a septum was carried down about one-eighth inch below the under side of the plate. Figure 2 shows a septum in place.

Best results were obtained with a circular hole about 3 inches in diameter and with the dimensions of the experimental hole such that the ratios of the heights of the bubbles over the two holes were between 2 and 1.

In carrying out the experimental work, a film was drawn across the holes with a strip of celluloid wet with soap solution. The blowing up was done through a burette, the bottom of which was connected to the lower end of a column of water, through a stopcock, and the top to the chamber below the test plate. As water was passed into the bottom of the burette, air was forced out of the top into the apparatus. This method was employed instead of blowing up the bubbles with air from the lungs because the carbon dioxide introduced by that method was harmful to the bubbles.

The success of the method depends largely on obtaining a soap film that will permit the taking of a great number of readings. Some difficulty was at first encountered in obtaining a suitable soap solution. All formulas investigated produced films that would last but a few minutes until a solution used by Dewar was tried. With this solution we were able to obtain films that would often last throughout a whole working day. It was made by adding a very small quantity of triethylamine oleate to a 50 per cent solution of glycerine in distilled water. The triethylamine oleate was prepared as follows, using 2 grams of triethylamine to 5 grams of oleic acid:

The amine was dissolved in warm water and the oleic acid was slowly stirred in. Excess amine in the emulsion was expelled by distillation and the water was expelled by subsequent evaporation on a steam bath. In the preparation an excess of oleic acid should be avoided, since it is not volatile.

Other oleates, such as ammonium, sodium, and potassium, were found by Dewar to be very successful, but the triethylamine solutions are by far the most resistant to atmospheric impurities.

DISCUSSION

THE TORSION PROBLEM

If a right cylinder or prism is twisted and held in equilibrium by means of couples applied at its ends, the portion of the cylinder or prism between any cross section and one end is in equilibrium under two equivalent couples, one in the plane of the cross section and the other the applied couple at the end. The couple in the plane of the cross section will be regarded as the resultant of a suitable distribution of shearing stress, which consists of tangential tractions in the plane of the section combined with equal tangential tractions along appropriate longitudinal sections. Since the cylinder or prism is in equilibrium under the action of the couples that are applied at its ends, the cylindrical surface must be free from traction.

Corresponding to the shearing stresses just referred to, there will be shearing strains of two types, one consisting of the sliding of the elements of one cross section over those of an adjoining section, the other of the relative sliding of different longitudinal elements in the direction of the length of the cylinder. The first type of strain will be expressed in terms of the angle through which the plane of the section has been rotated, the angle being assumed proportional to the distance from one end. The second type of strain, which implies that, in general, the plane cross sections are distorted into curved surfaces, will be expressed in terms of the displacement of the elements of a section in the direction of the length of the cylinder. This displacement is taken to be the same for all sections of a given cylinder or prism.

If the axis of Z be taken in the direction of the length of the prism and the components of the displacement of a point parallel to the X, Y, and Z axes be denoted by u, v, and w, respectively. the state of strain just described is a consequence of the following displacement:

$$u=-\tau yz, \qquad v=\tau zx, \qquad w=\tau\phi(x, y) \tag{1}$$

where τ is the angle of twist per unit length and ϕ is a function of x and y only, which is to be determined.

The u and v components of the displacement together express a rotation about the Z axis through an angle $z\tau$ of a section at a distance z from one end, while the w component expresses the distortion of each given section from its plane.

From the components of the displacement, the components of strain follow and from these follow the components of stress. (References 1 and 2.) It is found that the X and Y components, X_z and Y_z, respectively, of the shearing stress at a point (x, y) in any cross section, are expressed by the equations:

$$X_z=G\tau\left(\frac{\partial\phi}{\partial x}-y\right), \qquad Y_z=G\tau\left(\frac{\partial\phi}{\partial y}+x\right) \tag{2}$$

where G is the modulus of rigidity.

Associated with the stress components X_z and Y_z, which act in the plane of the cross section, are the stress components Z_x and Z_y, which are equal to X_z and Y_z, respectively, and which act in longitudinal planes parallel to the ZX and YZ planes, respectively. All other stress components are zero as a consequence of the assumed displacements (1). Thus the displacements taken in (1) lead to a system of stresses of the type described in the first paragraph of this section.

From the equations of equilibrium of the prism under the state of stress just considered, it follows that the function ϕ satisfies the differential equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \tag{3}$$

over the area of the cross section of the prism.

The requirement that the lateral surface of the cylinder or prism shall be free from traction leads to the following equation, which must be satisfied by the function ϕ on the curve bounding the cross section of the prism; namely,

$$\frac{\partial\phi}{\partial v}=y\cos(x, v)-x\cos(y, v). \tag{4}$$

In this equation, v denotes the exterior normal to the bounding curve.

The moment T of the couple in the plane of any cross section is expressed in terms of the function ϕ by the equation

$$T=C\tau, \tag{5}$$

where

$$C=G\int\int\left(x^2+y^2+x\,\frac{\partial\phi}{\partial y}-y\,\frac{\partial\phi}{\partial x}\right)dx\,dy, \tag{6}$$

the integral being extended over the area of the cross section of the prism. It is often convenient to replace C in equation (5) by GK where K is the integral by which G is multiplied in (6). Thus:

$$T=GK\tau. \tag{5'}$$

The problem of determining the torsion function ϕ subject to the differential equation (3) and the boundary condition (4) may be replaced by that of finding a function ψ conjugate to ϕ which satisfies the differential equation

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \tag{7}$$

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and the boundary condition

$$\psi-\frac{1}{2}(x^2+y^2)=\text{constant.} \tag{8}$$

The following relations connect ϕ and ψ:

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}, \qquad \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}. \tag{9}$$

If we replace ψ by the function Ψ defined in the following way:

$$\Psi=\psi-\frac{1}{2}(x^2+y^2), \tag{10}$$

we find from (7) and (8) that Ψ satisfies the differential equation

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+2=0 \tag{11}$$

subject to the condition

$$\Psi=0 \tag{12}$$

on the boundary of the section, the constant in equation (8) having been chosen to be zero.

From equations (2), (9), and (10), we find that the components of the shearing stress are simply expressed in terms of the function Ψ; namely,

$$Y_z=-G\tau\frac{\partial\Psi}{\partial x}, \qquad X_z=G\tau\frac{\partial\Psi}{\partial y}. \tag{13}$$

Hence the tangential traction at a point in any cross section of the prism has the direction of the tangent to that curve of the family

$$\Psi(x, y)=\text{constant}$$

which passes through this point. The curves, $\Psi=$constant, are therefore lines of shearing stress.

Further, the resultant shearing stress at a point in a cross section is equal to

$$\sqrt{X_z^2+Y_z^2}=G\tau\sqrt{\left(\frac{\partial\Psi}{\partial x}\right)^2+\left(\frac{\partial\Psi}{\partial y}\right)^2}=G\tau\frac{\partial\Psi}{\partial v}, \tag{14}$$

where v denotes the exterior normal to the curve $\Psi=$a constant that passes through the point in question. The resultant shearing stress at a point is therefore proportional to the gradient of the function Ψ at that point.

Further, when written in terms of the function Ψ, the expression (6) for the torsional rigidity becomes—

$$C=2G\int\int\Psi\,dx\,dy. \tag{15}$$

(Reference 3.) That is, the torsional rigidity of the prism is equal to twice the product of the modulus of rigidity and the volume inclosed between the surface

$$z=\Psi(x, y)$$

and the plane

$$z=0.$$

The solution of the torsion problem for a prism of a given section consists in determining the torsion function ϕ to satisfy the differential equation (3) and the boundary condition (4).

The torsion problem may be solved equally well by determining one of the functions ψ or Ψ from the equations (7) or (11), respectively, each subject to its appropriate boundary condition.

This is the complete theory of the torsion of thin rods. An interpretation of the displacements assumed is that all points on the Z axis remain on that axis and that every cross section of the rod except the fixed one is twisted about the Z axis. By our assumptions, cross sections do not usually remain plane but become warped. Figure 4 shows how elliptical, square, rectangular, and triangular sections become elevated in some parts and depressed in others. All originally plane sections become distorted in the same way since w, the longitudinal displacement, is not a function of z. It is clear, therefore, that the theory does not apply to sections near a fixed end nor to sections near the point where the torque is applied. That all cross sections should remain plane would require that w be constant and the only section for which this can

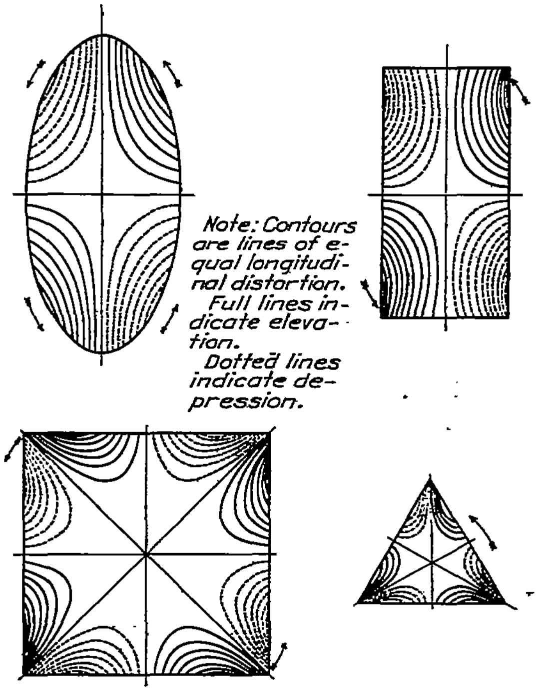

FIGURE 4.—Plane sections of noncircular rods warped in torsion

be true is the circular section. Figure 5, which is taken from Bach's "Elastizität und Festigkeit," shows the distortion in an elliptical cylinder and the lack of it in a circular cylinder.

It has been possible to solve the torsion problem rigorously for only a limited number of sections. The expressions for the torsion function ϕ or the associated function ψ for the more common sections are listed below:

(*a*) THE CIRCLE:

$$\phi=0.$$

(*b*) THE ELLIPSE:
Major and minor axes $2a$ and $2b$—

$$\phi=-\frac{a^2-b^2}{a^2+b^2}xy.$$

(*c*) THE RECTANGLE:
Sides $2a$ and $2b$—

$$\phi=-xy+4b^2\left(\frac{2}{\pi}\right)^3\sum_{n=0}^{n=\infty}\frac{(-1)^n}{(2n+1)^3}\frac{\sinh\frac{(2n+1)\pi x}{2b}}{\cosh\frac{(2n+1)\pi a}{2b}}\sin\frac{(2n+1)\pi y}{2b}.$$

(*d*) THE EQUILATERAL TRIANGLE:

Origin at centroid. Side a—

$$\psi=\frac{\sqrt{3}}{3a}\,(3xy^2-x^3).$$

Corresponding solutions are known for a sector of a circle, a curvilinear rectangle bounded by two concentric circular arcs and two radii, figures bounded by confocal ellipses and hyperbolas,

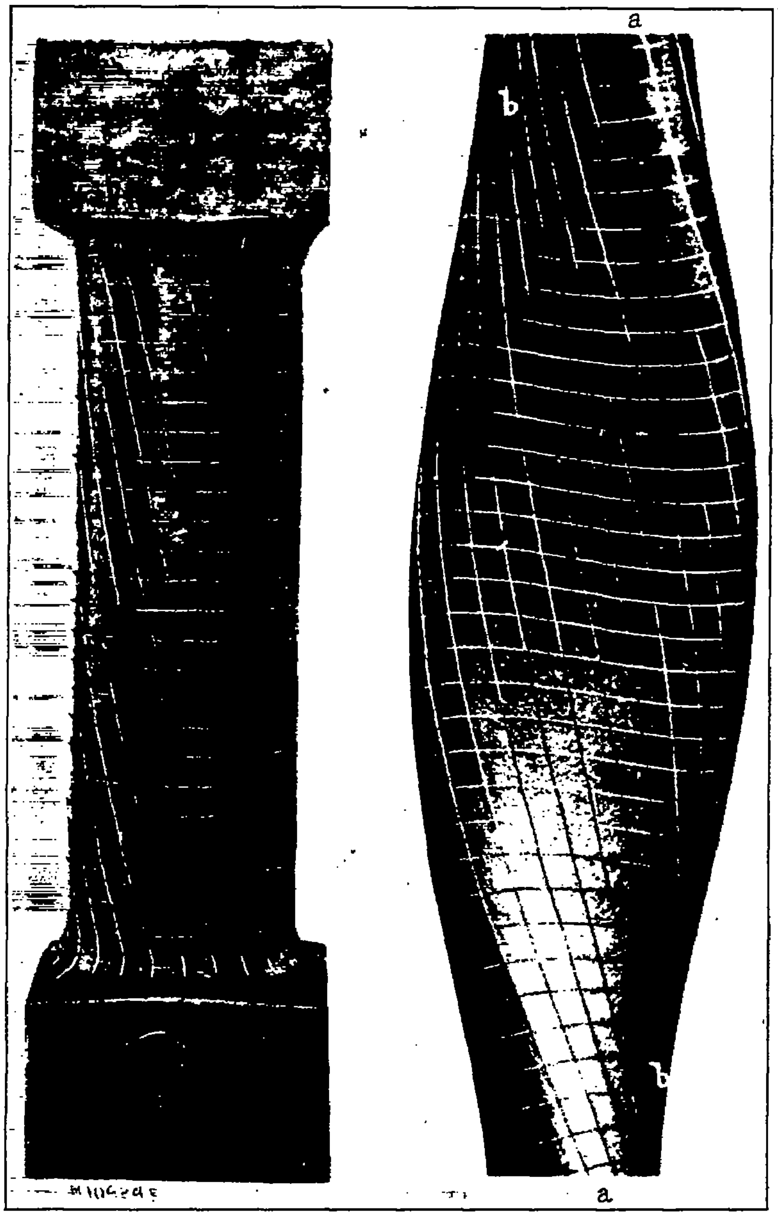

FIGURE 5.—The distortion of plane sections in an elliptical rod and the absence of such distortion in a circular rod

figures shaped like a square but with concave sides and either rounded or sharp corners, and a section somewhat resembling the section of a railway rail. (References 4, 5, and 6.)

Formulas for simple sections.

From the preceding expressions for the torsion functions, the following well-known formulas for torque and maximum stress have been derived:

Let

T = torque.

θ = total angle of twist in radians.

L = length.

G = modulus of rigidity.

q = greatest intensity of stress.

CIRCLE

$$T=\frac{\pi r^4 G\theta}{2L}, \qquad q=\frac{2T}{\pi r^3}.$$

r = radius.

ELLIPSE

$$T=\frac{\pi a^3 b^3 G\theta}{(a^2+b^2)L}, \qquad q=\frac{2T}{\pi a b^2}.$$

$2a$ = major axis.

$2b$ = minor axis.

EQUILATERAL TRIANGLE

$$T=\frac{a^4\sqrt{3}G\theta}{80L}, \qquad q=\frac{20T}{a^3}.$$

a = side of triangle.

SQUARE

$$T=\frac{s^4 G\theta}{7.11L}, \qquad q=\frac{4.808T}{s^3}.$$

s = side of square.

RECTANGLE

$$T=ab^3\mu G\frac{\theta}{L}, \qquad q=\frac{\gamma T}{\mu a b^2},$$

or

$$T=ab^3\left(\frac{16}{3}-\lambda\frac{b}{a}\right)G\frac{\theta}{L}.$$

$2a$ = long side of rectangle.

$2b$ = short side of rectangle.

The factors μ, λ, and γ are dependent upon the ratio of the sides. Their values given in Table I are from St. Venant. (Reference 5.) The maximum stress q occurs at the middle of the long side. The stress at the middle of the short side is given by

$$q_1=\frac{\gamma_1 T}{\mu b^3}$$

in which γ_1 is a factor dependent upon the ratio of the sides. Its values are also given in Table I.

TABLE I.—*Factors for calculating torsional rigidity and stress of rectangular prisms*

Ratio of sides	λ	μ	γ	γ_1	Ratio of sides	λ	μ	γ	γ_1
1.00	3.08410	2.24923	1.35063	1.35063	2.50	3.35873	3.98984	1.93614	0.59347
1.05	3.12256	2.35908	1.39651	--------	2.75	3.36023	4.11143	1.95687	--------
1.10	3.15653	2.46374	1.43956	--------	3.00	3.36079	4.21307	1.97087	--------
1.15	3.18554	2.56330	1.47990	--------	3.33	--------	--------	--------	.44545
1.20	3.21040	2.65788	1.51753	--------	3.50	3.36121	4.37299	1.98672	--------
1.25	3.23196	2.74772	1.55268	1.13782	4.00	3.36132	4.49300	1.99395	.37121
1.30	3.25035	2.83306	1.58544	--------	4.50	3.36133	4.58639	1.99724	--------
1.35	3.26632	2.91379	1.61594	--------	5.00	3.36133	4.66162	1.99874	.29700
1.40	3.28002	2.99046	1.64430	--------	6.00	3.36133	4.77311	1.99974	--------
1.45	3.29171	3.06319	1.67265	--------	6.67	3.36133	--------	--------	.22275
1.50	3.30174	3.13217	1.69512	.97075	7.00	3.36133	4.85314	1.99995	--------
1.60	3.31770	3.25977	1.73889	.91489	8.00	3.36133	4.91317	1.99999	.18564
1.70	3.32941	3.37486	1.77649	--------	9.00	3.36133	4.95985	2.00000	--------
1.75	3.33402	3.42843	1.79325	.84098	10.00	3.36133	4.99720	2.00000	.14858
1.80	3.33798	3.47890	1.80877	--------	20.00	3.36133	5.16527	2.00000	.07341
1.90	3.34426	3.57320	1.83643	--------	50.00	3.36133	5.26611	2.00000	--------
2.00	3.34885	3.65891	1.86012	.73945	100.00	3.36133	5.29972	2.00000	--------
2.25	3.35564	3.84194	1.90546	--------	∞	3.36133	5.33333	2.00000	.00000

When letters are used for the full sides and not the half sides, letting c represent the long side and d the short side, the formulas become

$$T=cd^3\beta G\frac{\theta}{L}, \qquad q=\frac{8\gamma T}{\mu cd^2},$$

or

$$T=\frac{cd^3}{3}\left(1-\frac{3\lambda}{16}\frac{d}{c}\right)G\frac{\theta}{L}$$

in which $\beta=\mu/16$, and λ has the same values as before. It can be seen that if $\frac{d}{c}$ is small, we arrive at the common approximate formula:

$$T=\frac{cd^3}{3}G\frac{\theta}{L}.$$

As the ratio $\frac{c}{d}$ varies from 1 to ∞ the expression $\frac{3\lambda}{16}$ varies from 0.578 to 0.630.

St. Venant gives the following approximate formulas for the constants, which agree with exact values within 4 per cent:

$$\gamma=\frac{3}{8}\left(1+0.6\frac{b}{a}\right)\mu,$$

$$\mu=\frac{16}{3}-3.36\frac{b}{a}\left(1-\frac{b^4}{12a^4}\right).$$

Using this value of γ

$$q=\frac{(3a+1.8b)T}{8a^2b^2}, \text{ or } q=\frac{(3c+1.8d)T}{c^2d^2}.$$

Both are common approximate expressions for the stress at the middle of the long side.

ST. VENANT'S APPROXIMATE FORMULA FOR COMPACT SECTIONS

For fairly compact sections without any reentrant angles St. Venant gives the following approximate formula for the torque:

$$T=\frac{A^4}{40J}G\frac{\theta}{L},$$

in which A is the area of the section, J the polar moment of inertia of the section, G the modulus of rigidity, and $\frac{\theta}{L}$ the angle of twist per unit of length.

Although St. Venant clearly stated that this formula applies only to fairly compact sections with no reentrant angles, it is often applied to other sections, for example, to sections made up of component rectangles. Resulting errors may amount to several hundred per cent in extreme cases. However, when restricted to sections for which it was intended the formula is fairly accurate.

Formulas for hollow prisms or tubes.

The cross section of a hollow prism or cylinder is bounded by two closed curves upon which, in accordance with equation (8), the function Ψ must take constant but, in general, different values. Denoting by Ψ_o and Ψ_i the values of Ψ on the outer and inner boundaries, respectively, and by A_o and A_i the entire areas inclosed by the respective bounding curves, the analysis that led

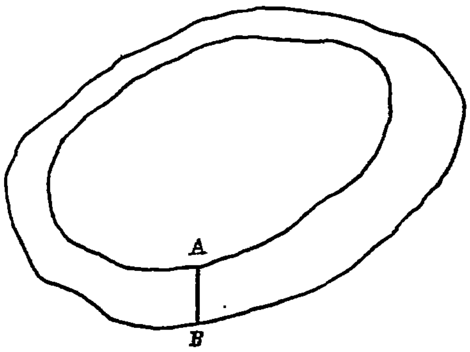

FIGURE 6

to equation (15) for a solid prism or cylinder will now lead to the following expression for the torque:

$$T=-2G\tau\Psi_o A_o+2G\tau\Psi_i A_i+2G\tau\int\int\Psi\ (x,\ y)\ dx\ dy. \tag{16}$$

(References 3 and 7.) The integration is extended over the ring-shaped section. If the ring is narrow we can replace Ψ under the integral sign by the constant

$$\Psi_m=\frac{1}{2}(\Psi_o+\Psi_i). \tag{17}$$

The last term in equation (16) then becomes

$$2G\tau\Psi_m(A_o-A_i)=G\tau(\Psi_o+\Psi_i)\ (A_o-A_i).$$

The expression for T in (16) then reduces to

$$T=2G\tau A_m(\Psi_i-\Psi_o)=2G\tau\ A_m\Delta\Psi \tag{18}$$

where

$$A_m=\frac{A_o+A_i}{2}, \qquad \Delta\Psi=\Psi_i-\Psi_o. \tag{19}$$

If we denote by t the width of the ring at any place AB (fig. 6), we obtain (equation 14) as an approximate expression for the average shearing stress at points in AB

$$q=G\tau\frac{\Delta\Psi}{t}. \tag{20}$$

Hence

$$tq=G\tau\Delta\Psi, \tag{21}$$

and (18) becomes

$$T=2tqA_m. \tag{22}$$

If we form the integral of $q\mathrm{d}s$ around the curve $\Psi=\Psi_m$ which may, if t is not too large, be taken to be the curve lying half way between the inner and outer boundaries, we find after some reduction

$$\int q\mathrm{d}s=2G\tau A_m. \tag{23}$$

(Reference 8.) Replacing q under the integral in (23) by its value from (22) we obtain

$$\frac{T}{2A_m}\int\frac{\mathrm{d}s}{t}=2A_mG\tau,$$

or

$$T=\frac{4A_m^2G\tau}{\int\frac{\mathrm{d}s}{t}} \tag{24}$$

We find from (22) as the approximate expression for the stress

$$q=\frac{T}{2tA_m}\cdot \tag{25}$$

CIRCULAR TUBES

RIGOROUS METHOD:

When the inner boundary of the tube is a line of shearing stress of a solid section that has the same outer boundary as the tube the rigidity of the tubular section may be obtained directly by subtraction.

$$T=\frac{\pi}{2}\left[\left(r+\frac{t}{2}\right)^4-\left(r-\frac{t}{2}\right)^4\right]G\frac{\theta}{L},$$

$$=2\pi tr\left(r^2+\frac{t^2}{4}\right)G\frac{\theta}{L}, \qquad q=\frac{\left(r+\frac{t}{2}\right)T}{2\pi tr\left(r^2+\frac{t^2}{4}\right)}.$$

r = mean radius.

t = thickness of wall.

APPROXIMATE METHOD:

$$\frac{T}{A}\int\frac{\mathrm{d}s}{t}=4AG\frac{\theta}{L}, \qquad q=\frac{T}{2tA},$$

$$A=\frac{\pi\left(r+\frac{t}{2}\right)^2+\pi\left(r-\frac{t}{2}\right)^2}{2}.$$

$$\int\frac{\mathrm{d}s}{t}=\frac{2\pi r}{t},$$

$$T=2\pi tr\left(r^2+\frac{t^2}{2}+\frac{t^4}{16r^2}\right)\frac{G\theta}{L}, \qquad q=\frac{T}{2\pi t\left(r^2+\frac{t^2}{4}\right)}.$$

Dropping the square and higher powers of t, we have the common approximate formulas

$$T=2\pi r^3tG\frac{\theta}{L}, \qquad q=\frac{T}{2\pi r^2t}.$$

ELLIPTICAL TUBES

RIGOROUS METHOD:

The rigorous formulas apply only when the inner and the outer ellipses are similar, that is, when the inner ellipse is a line of shearing stress for a solid shaft having the same outer boundary as the tube. The semirigorous formulas, of course, apply whether the ellipses are similar or not. Let the inner and outer boundaries be the similar ellipses:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=(1+k)^2.$$

Then the inner semimajor axis is a and the outer $a(1+k)$ and the inner semiminor axis b and the outer $b\ (1+k)$.

$$T=\frac{\pi a^3b^3}{a^2+b^2}[(1+k)^4-1]\,G\frac{\theta}{L}, \qquad q=\frac{2T}{\pi ab^2[(1+k)^4-1]}.$$

APPROXIMATE METHOD:

Neglecting the square and higher powers of k, the approximate formula (24) gives

$$T=\left[\frac{4\pi ka^3b^3}{a^2+b^2}\right]G\frac{\theta}{L}, \qquad q=\frac{T}{2\pi kab^2(1+k)}.$$

ACCURACY OF APPROXIMATE METHOD

It is apparent from a comparison of the preceding formulas for circular and elliptical tubes that the results from

$$\frac{T}{A}\int\frac{ds}{t}=4AG\frac{\theta}{L},$$

and

$$q=\frac{T}{2tA}$$

are quite accurate for small values of t. Usually a commercial tube is made with the thickness of metal constant, in which case t in $\int\frac{ds}{t}$ becomes constant. While A had best be regarded as the mean of the areas inclosed by the inner and the outer boundaries of the section, good results are obtained by drawing a curve midway between the two boundaries of the tube and taking A as the area inclosed by this curve. The quantity ds is an element of length along this curve. Further examples follow.

HOLLOW RECTANGLE

Let the outer boundaries be a and b and let a be the greater side. If t_1 is the thickness of the greater side and t the thickness of the smaller side, the sides of the mean rectangle are $(a-t)$ and $(b-t_1)$ and

$$A=(a-t)(b-t_1),$$

$$\int\frac{ds}{t}=\frac{2(a-t)}{t_1}+\frac{2(b-t_1)}{t},$$

$$T=\frac{2tt_1(a-t)^2\,(b-t_1)^2}{at+bt_1-t^2-t_1^2}G\,\frac{\theta}{L}, \qquad q=\frac{T}{2t(a-t)\,(b-t_1)}.$$

The equation for stress is true only along the sides of the rectangle where the shear lines are parallel curves. To avoid high stresses at the reentrant angles, the inner corners should be rounded.

CIRCULAR TUBE SPLIT LONGITUDINALLY

A tube of mean radius r and uniform thickness t, split longitudinally, may be regarded as a flat sheet, although this fact is not so well known as it should be. If the ratio of r to t is great. the approximate formula for a rectangle may be applied and

$$T=\frac{1}{3}(2\pi rt^3)G\frac{\theta}{L},$$

$$=\frac{2}{3}\pi rt^3G\frac{\theta}{L},$$

For the closed tube, the approximate formula is

$$T=2\pi r^3tG\frac{\theta}{L},$$

and the ratio of the torques for the same $\frac{\theta}{L}$ is

$$\frac{t^2}{3r^2}.$$

It can also be shown that for the same maximum stress the ratio of the torques is approximately equal to

$$\frac{t}{3r}.$$

The split tube, therefore, is much weaker under torsion and very much less rigid.

Solid sections of irregular shape.

We have now discussed substantially all of the sections for which practical formulas have been obtained by direct mathematical treatment. There remain such sections as the I, T, U, and L that have not yet been brought within the range of mathematical analysis. These sections normally occur in beams or in compression members and not in members designed primarily to take a torsional couple. Nevertheless, such members are all subject to torsion and the loads that they will sustain may be dependent upon their torsional rigidity. Because of its importance in this connection, our investigation has dealt largely with torsional rigidity rather than with stress.

We will first consider the calculation of the rigidity and later touch upon the matter of stress. For any section we can write

$$T=KG\frac{\theta}{L}.$$

in which K is a constant that depends solely on the shape and dimensions of the cross section and involves the fourth power of a dimension (see the preceding formulas for regular sections). This constant K is usually spoken of as the "torsion constant" of the section and will be so referred to in this report. Our problem is to determine a suitable method of calculating K for various irregular shapes.

Before embarking upon an extended series of tests, it was necessary to make some preliminary tests of wooden members of simple section in order to determine to what extent certain factors governed the torsional properties of wood. As a usual thing, the modulus of rigidity associated with a traction in a radial plane is not equal to the modulus associated with the traction in a tangential plane. In other words, the elastic constant for a shearing stress acting in a plane at right angles to the growth rings is not the same as for a shearing stress acting in a plane tangential to the growth rings. This fact introduces two moduli of rigidity into the problem. There is a third modulus of rigidity for wood which has to do with the stresses that tend to roll contiguous fibers past each other, but when a member is twisted about an axis parallel to the grain of the wood this elastic constant does not come into play. Our first problem then was to determine what difference, if any, there was between the radial modulus of rigidity and the tangential modulus. Other factors pertinent to our test procedure were the effect of moisture content and the effect of rate of strain. Each of these three factors was studied and a brief discussion of each follows.

Moduli of rigidity of spruce.

It is shown in Appendix A that a rectangular prism, two of whose axes of elastic symmetry lie in the plane of the cross section, behaves as a prism with a parallelogrammatic cross section when these axes are not parallel to the sides and as a prism with a transformed rectangular section when the axes are parallel to the sides. The modulus for the transformed section in either case is computed from the two moduli involved. It is also shown that if G_t is the modulus associated with the plane tangential to the annual rings and G_r the modulus associated with the plane perpendicular to the rings, the relations of Table II hold.

TABLE II

Sides $2a$, $2b$	Sides of transformed rectangle	Modulus
Plain-sawn board	$2a\sqrt{\frac{G_r}{G_t}}, 2b$	$G_t\sqrt{\frac{G_t}{G_r}}$
Quarter-sawn board	$2a\sqrt{\frac{G_t}{G_r}}, 2b$	$G_r\sqrt{\frac{G_r}{G_t}}$

It was possible from these relations to determine the value of G_t and G_r for any plank by testing a quarter-sawn and a plain-sawn piece cut from that plank. This was done for practically every piece used. These minor specimens were 1 by 3 inches in cross section. Occasionally slight season checks, which run radially, caused the quarter-sawn pieces to be less rigid than their corresponding plain-sawn pieces, whereas with sound material the quarter-sawn piece should be the more rigid, since G_r is greater than G_t. It was found that for Sitka spruce G_t was about 90 per cent of G_r. This means that quarter-sawn rectangular beams of Sitka spruce with a large ratio of long side to short side will average about 10 per cent more rigid in torsion than similar plain-sawn beams. Ordinarily no great error will result if the mean modulus as obtained through the test of a circular section is used in calculating the rigidity of beams. It may, of course, introduce on the average about half of the difference between the plain and the quarter-sawn values, or an error of about 5 per cent. It is much easier to make square sections than circular sections and the difference in mean modulus obtained is practically nil. Square minor specimens, therefore, were tested as a check against the values obtained from the 1 by 3 inch minor specimens.

As a check against the mathematical analysis given in Appendix A, a few series of tests were run on beams of rectangular and of elliptical sections with the annual growth rings at various angles to the axes of the sections. The results are shown in Figure 7. The curve for the ellipse was calculated by means of the relations given in Appendix A; the circles along the curve are test values. The curves for the rectangles represent the observations; they agree in form with a curve calculated for a different ratio of the two moduli.

Effect of moisture content.

In order to obviate the necessity of making moisture adjustments, the major and the minor test specimens were always kept in the same condition. They were never separated after fabrication and the time between the testing of the majors and the minors was reduced to a minimum.

A series of tests was made, however, to learn enough about the effect of moisture content on torsional properties to permit the recommendation of permissible stress values for spruce at a definite moisture content. Twelve matched square pieces were tested, 3 green, 3 at about 21½ per cent, and 3 at about 7 per cent moisture content. The results are given in Figure 8, which shows the variation with moisture content of three properties; namely, modulus of rigidity, fiber stress at elastic limit, and ultimate fiber stress. The results from this small number of tests were in agreement with relations previously established at this laboratory.

Effect of rate of loading.

As tests were run on members of various sizes, the rate of strain was kept fairly uniform in order not to introduce this factor into the results. The ordinary test, however, took several

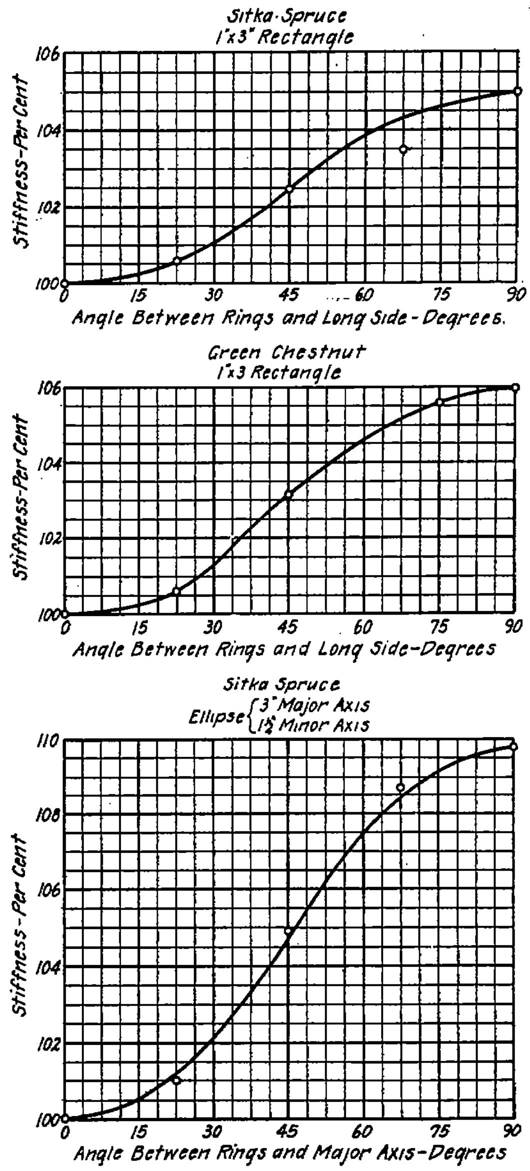

FIGURE 7.—Variation in torsional stiffness with direction of annual rings on the cross section

minutes, whereas the duration of stress assumed for aircraft stresses is three seconds. Consequently, in order to recommend torsional properties of spruce from test, it was necessary to know something about the variation in these properties with rate of strain. A matched set of cylindrical specimens was made and equal numbers of them were tested, respectively, at each of three rates; these rates were in the proportion 1 to 10 to 100. It was found that accompanying a 10 to 1 change in rate there was a 5 per cent increase in modulus of rigidity, a 10 per cent increase in ultimate fiber stress, and a 20 per cent increase in fiber stress at elastic limit. The exponential increase of stress with increased rate of fiber strain has been previously observed at this laboratory.

Torsion tests of simple sections.

As the next preliminary step before testing wooden beams of irregular section, several series of tests were made on spruce beams of simple section. Beams with the circle, the square, the ellipse, and the equilateral triangle as bounding curves of the cross sections were cut from the same plank. The dimensions and the angle of twist for a given torque as determined by test were substituted in the rigorous formula previously given for the corresponding sections and an apparent modulus of rigidity calculated thereby. Four of each type of beam were cut from each plank. The results are given in Table III. The planks were chosen so as to obtain

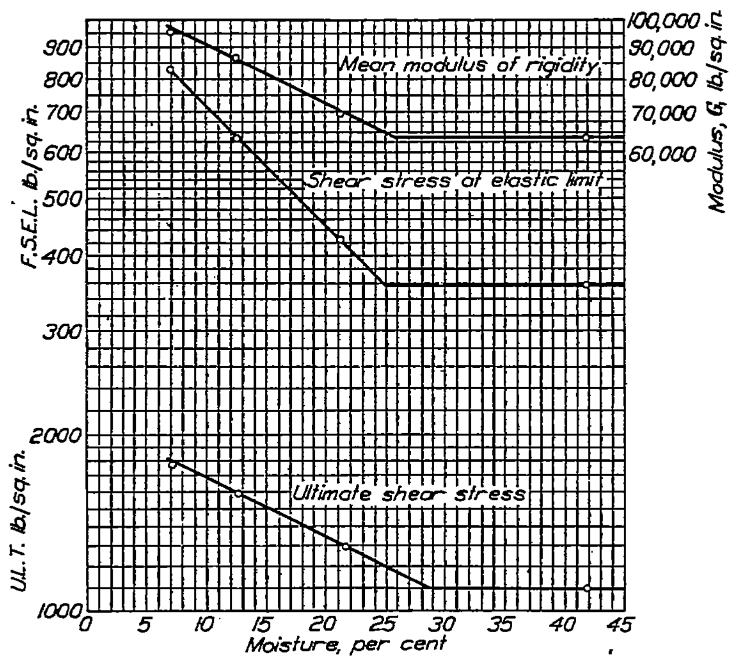

FIGURE 8.—Relation between torsional properties and moisture content of Sitka spruce

a wide range in specific-gravity values. Consequently the first two (Table III) are below the minimum specific gravity (0.36) allowable in aircraft construction.

TABLE III

Plank	Specific gravity	Moduli of rigidity				M. of E. bending
		Circle	Square	Ellipse	Equilateral triangle	
6–1–58	0.322	78,300	72,500	75,800	80,600	1,347,000
6–1–25	.344	82,500	81,600	----------	84,000	1,358,000
5–1–49	.426	116,500	119,200	122,600	121,800	1,520,000

All specimens were 45 inches long and the angle of twist was measured over a 24-inch gage length. The nominal diameter of the circular specimens and the width of the square specimens were each 1¼ inches. The major and the minor axes of the elliptical specimens were 1½ inches and 1 inch, respectively, for plank 6–1–58 and 1¾ inches and 1 inch for plank 5–1–49. The triangular specimens were 2 inches on each side. An error in grinding the shaper knives for the set with elliptical section from plank 6–1–25 necessitated the culling of that set. Averaging the results from the first and the last sets in Table III, we find that the apparent modulus of rigidity for the square is 1.6 per cent lower than for the circle, for the ellipse 1.8 per cent higher, and for the equilateral triangle 3.8 per cent higher. As pointed out in Appendix A, the direction of the annual rings with respect to the axes of the section has little effect on the rigidity of sections whose bounding curve is a square or an equilateral triangle. For the elliptical section, however, there is some difference. Consequently, the first set in the table was cut with the rings parallel to the major axis and the second set with the rings perpendicular to that axis. Had the specimens been longer, it is thought that the results for the triangular sections would have been somewhat lower. The ends of these specimens were enlarged for the application of torque. With the circular section such enlargement would make little difference as long as the points at which measurements were taken were three or four diameters away from the enlargement. This is because in the circular rod plane sections remain plane. In the triangular rod the tendency of the sections to warp is hindered by the enlargement of the ends with a consequent increase in stiffness. The same fact is true, although to a less extent, of the ellipse. The rods with square sections did not have built-up ends. Taking all these factors into consideration, the agreement as to torsional rigidity as calculated by the rigorous formulas is considered quite suitable.

Table III yields another interesting relation. If the moduli of elasticity in the last column, which were obtained from minor bending tests, are divided by the corresponding moduli of rigidity for the circular section given in the third column, the quotients will average 15.6. This relation for the average of 12 rods checks the relation obtained in 1921 for 20 rods of circular section that were tested in torsion in connection with another investigation. The mean modulus of rigidity for the 20 specimens was 100,200 p. s. i., and the average modulus of elasticity 1,569,000 p. s. i., or a ratio of 1 to 15.6. Hence the ratio for spruce is evidently between one-fifteenth and one-sixteenth, whereas for most metals it is in the neighborhood of two-fifths.

As a further check, four rods of elliptical section were made with a major axis of 2 inches and a minor axis of 1½ inches and were tested within the elastic limit, and then were cut down to a 1¾-inch major axis and 1¼-inch minor axis and retested. The apparent modulus of rigidity in the first case averaged 77,325 p. s. i., and in the second 78,162, a difference of only 1 per cent. A repetition of this series resulted in a difference of slightly over 1 per cent but with the results reversed. Sections with equilateral triangles as bounding curves were cut down in the same way, though in three steps, first with a 2-inch side, then a 1¾-inch side, and finally a 1½-inch side. The average apparent moduli for four beams were 73,350 p. s. i. for the 2-inch side, 72,250 for the 1¾-inch side, and 72,650 for the 1½-inch side. The maximum difference is about 1 per cent.

From these tests it appears that dependable results can be obtained by using wood as a test material.

Torsion tests of irregular sections.

Following these preliminary tests, additional tests were made on beams of irregular section. Such sections as **I**, **T**, **L**, **U**, and **Z** were used with and without fillets at the reentrant angles. In addition to varying the radius of fillet, the ratio of the thickness of the web to that of the flange or of one leg to that of the other was varied through a considerable range.

The beams were 8 feet long and the angle of twist was read for a gage length of 36 inches at the center. The results of these tests will be discussed later in connection with the coordination of the mathematical and the experimental work in the form of empirical formulas.

The use of soap films in solving the torsion problem for irregular sections.

The value of soap films in determining the torsional rigidity and the stress in twisted beams depends upon the mathematical analogy between the torsion problem and that of a membrane, such as a soap film, under a uniform excess of pressure on one side. Attention was first called to this analogy by Prandtl and very extensive use of it was made by Griffith and Taylor. (References 9 and 10.) The method is extremely useful in that it offers a means of determining the torsional rigidity and the stress of important irregular sections that have not yielded to mathematical treatment. An apparatus was built and, after it was found to give results for simple sections that agreed closely with calculated values, tests were made on irregular sections. The construction of the apparatus and the method of using it have already been described under the heading "Test Material and Procedure." The mathematical basis of the method and a brief discussion of the technic follow. The test results will be be dealt with later in connection with proposed formulas for irregular sections whose component parts are rectangles.

In presenting the mathematical basis of the soap-film method of test, let a very thin homogeneous membrane be stretched under uniform tension T over an opening cut in a plane sheet of rigid material and let the membrane be fixed at the edge of the opening. If a uniform excess of pressure p per unit area acts upon one face of the membrane, the small displacement z of points of the membrane will satisfy the differential equation:

$$T\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)+p=0, \tag{26}$$

and the condition that

$$z=0 \tag{27}$$

at the edge of the opening.

Let the opening and the section of the prism under consideration be identical in size and shape. If we let

$$z=\frac{p}{2T}\Psi \tag{28}$$

in equations (26) and (27), we obtain equations (11) and (12) for the function Ψ. Hence the function Ψ appropriate to the torsion problem for a section of given shape is proportional to the displacement z of a homogeneous membrane stretched over an opening of the same shape as the section. The proportionality factor in (28) is determined by means of a film stretched over a circular opening and under the same pressure as the test film. It follows from equation (15) that the torsional rigidity of a prism of the given section is proportional to the volume inclosed by the soap film and the plane of the opening. Further, the contour lines, $z=$constant, of the soap film correspond, in accordance with equations (13), to the lines of shearing stress $\Psi=$constant in the torsion problem. And the slope of the film at any point, as a consequence of equation (14), is proportional to the magnitude of the shearing stress at the corresponding point of the section.

In employing the soap-film method, an opening that represents the section of the prism to either a reduced or an enlarged scale may be used. It is necessary only to observe that the ratio of the torsional rigidities of two geometrically similar sections is equal to the fourth power of the ratio of corresponding linear dimensions.

To obtain well-defined edges coinciding with the boundary of the cross section, the edges of the openings were beveled at an angle of 45 degrees. Our experience has been that this does not entirely eliminate the errors at the edges. The film is not always attached at the upper side of the beveled edge but frequently hangs at an intermediate point. Even when great care is used to avoid a surplus of solution, there usually is a layer of solution along the edge of the film that tends to lower the level in its neighborhood and to make uncertain the actual position of the boundary. Further, at points where the stress is great and the film consequently is steep, there is a tendency for the film to run out over the plate.

Errors resulting from edge effects can be avoided by using as boundaries of the cross sections contour lines other than the actual outline of the opening in the plate. These contour lines, if taken near the edge of the opening, approximate the shape of the section with sufficient accuracy. The dimensions of the section bounded by the contour line in question can be measured. In the tables giving the results of our experiments with soap films, we have included, in general, data from one or more inner contour lines as well as from the actual outline of the opening. We have thus increased the number of sections studied. It is our feeling that the results from the inner contours are more reliable than those from the outlines of the openings. In every case an inner contour line of the spherical bubble over the circular opening was used as the boundary of the comparison cylinder.

There are various ways of finding the volume inclosed by the test bubble. A satisfactory procedure is to take contour lines at frequent intervals, planimeter the areas inclosed by these lines, and obtain the volume between the planes at different levels by the average-end-area method.

For further details in regard to the technic of the soap-film method, the reader should consult the papers by Griffith and Taylor. In our judgment, the high degree of accuracy that they attained in certain cases is not always to be expected.

Formulas for irregular solid sections.

Combining results obtained by soap-film tests with known mathematical facts, Griffith and Taylor developed an empirical method of dealing with solid rods of any section, which is explained in Appendix B of this report. The method gives results for the torsional rigidity of fairly compact sections with errors of only a few per cent. For certain sections, however, the errors are considerable. In their report on the method, they attribute a discrepancy between their results and those of published experimental work to a want of homogeneity in rolled **I** and **U** sections. Some of our soap-film experimental work on **I** and **U** beams, however, fails to check their formula by as much as 25 per cent and the discrepancy is in the same direction as that mentioned in their reports, the formula giving results that are too high.

In an extensive investigation of the torsion problem, Constantin Weber developed, on the basis of the usual mathematical theory, approximate formulas for the torsional rigidity of a large number of sections and for the maximum stress in these sections. (Reference 11.) Torsional rigidities calculated by his formulas are low in comparison with our test results.

In dealing with such sections as the **L**, **U**, **Z**, **T**, and **I**, Weber replaced the given section by an equivalent rectangular section. To represent the situation at the junction of two rectangles, he chose the length of the equivalent rectangle to secure a certain desired area. Now changing the length of a rectangular section in a certain ratio does not alter its stiffness nearly so much as a corresponding change in the breadth. There is essentially an increase in breadth of section at the junction of two rectangles, for instance, at the corner of an **L**. This can not be compensated for by merely increasing the length of the equivalent rectangle in the manner chosen by Weber. Accordingly, his formulas give values of the torsional rigidity considerably below those that we have found by means of direct torsion tests and tests made by the soap-film method. It should be noted that Weber assumed that fillets were always present, their radii being equal to the width of the narrower of the component rectangles of the sections.

For sections such as **I**, **U**, and **T**, whose component parts are rectangles, the following approximate method for calculating stiffness is proposed as a result of our study; we shall first show its derivation.

The problem is to find K in

$$T = KG\frac{\theta}{L}.$$

Now K is a constant that depends solely on the shape and the dimensions of the cross section, and involves the fourth power of a dimension. Figures 9 and 10 show that at the junction of two component rectangles there occurs a hump or hill on the soap film. This hump shows that the rigidity of the complete bar is greater than the sum of the rigidities of the separate rectangular parts. The volume of the soap bubble, of course, represents the rigidity of the entire bar and the increased volume at the hump in the bubble represents the amount by which the rigidity of the bar exceeds the sum of the rigidities of its separate rectangular parts.

For an **I** beam, we write

$$K = 2K_1 + K_2 + C$$

in which K_1 is the torsion constant of one flange and K_2 that of the web, while C is the term that is to express the additional stiffness caused by the two junctions of the flanges and the web. In place of C we write $2\ \alpha D^4$ in which D is the diameter of the largest circle that can be inscribed

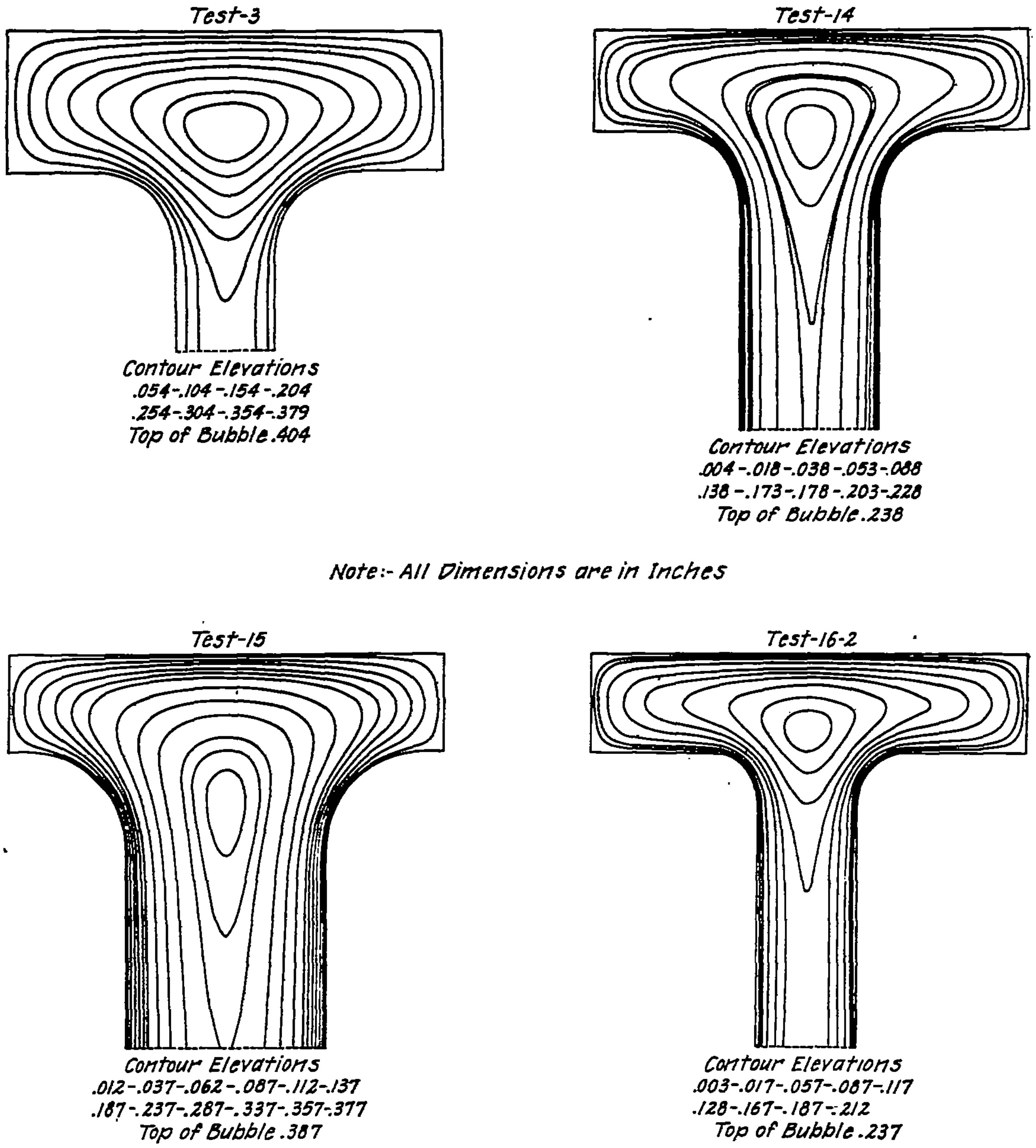

FIGURE 9.—Lines of shearing stress for I beams in torsion. (From soap-film tests on half sections)

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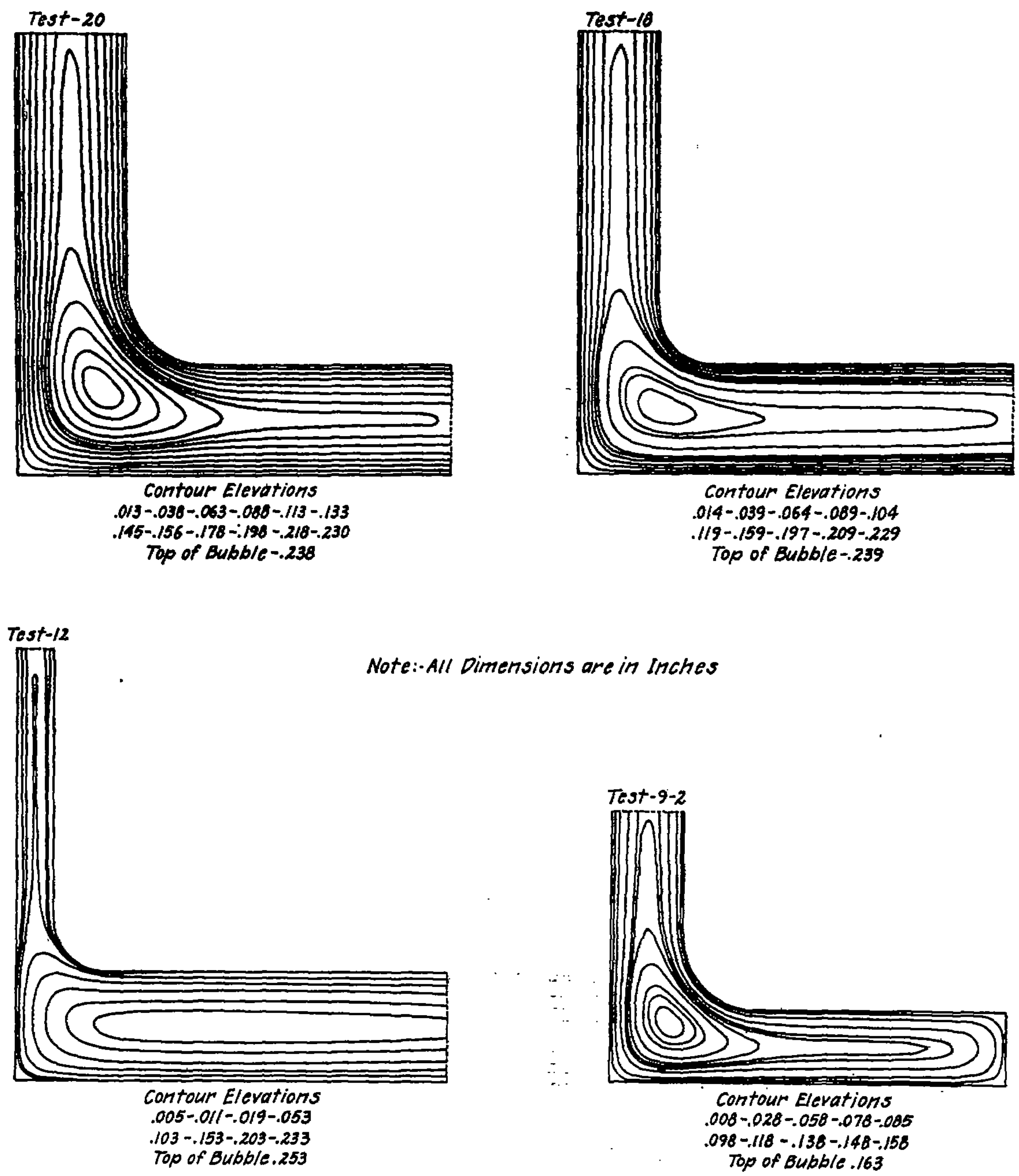

FIGURE 10.—Lines of shearing stress for U and Z beams in torsion. (From soap-film tests on half sections)

at the junction of the two component rectangles and α is a factor to be determined. We then have

$$K=2K_1+K_2+2\alpha D^4.$$

An examination of Figure 9 will show that the bubble tapers down at the ends of the flanges, as it normally would in a rectangle, while in the web it behaves more like a part of a long rectangle. For this reason, we shall calculate K_1 by the normal formula for the rectangle

$$K_1=a_1b_1^3\mu \text{ or } K_1=a_1b_1^3\left(\frac{16}{3}-\lambda\frac{b_1}{a_1}\right),$$

and K_2 by the formula

$$K_2=\frac{16}{3}a_2b_2^3.$$

The factor α for any section depends upon two things; the ratio of the radius of the fillet to the thickness of the flange and the ratio of the thickness of the narrower component rectangle to

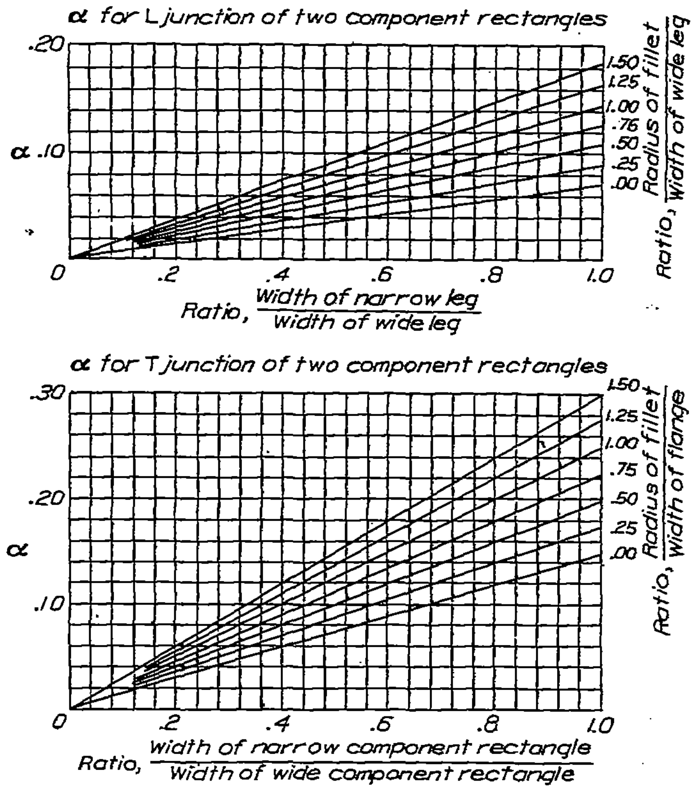

FIGURE 11.—Values of α for computing torsional rigidity of sections whose component parts are rectangles

the thickness of the wider component rectangle. The values of α for different combinations of these two factors, which were obtained through a variety of experiments with soap films and torsion tests of actual beams, are shown graphically in Figure 11. While our experiments were not extensive enough to prove conclusively that for a given ratio of radius of fillet to thickness of flange the variation in α is linear for varying ratios of the thicknesses of the two component rectangles, we feel that such a variation is close enough to the truth to warrant its use. Table IV shows how K calculated by this simple method for I sections agrees with results obtained by actual beam tests and soap-film tests.

TABLE IV.—*Values of torsion constant K for I-beams*

[Dimensions are in inches]

ACTUAL TORSION TESTS

Test	Thickness		Total height	Total width	Fillet radius R	D	D^4	$2K_1+K_2$	2α	Test K	Proposed formula K	Difference	G. and T. formula K	Difference
	Web	Flange												
												Per cent		*Per cent*
I-7	0.500	0.500	3.46	2.240	0	0.625	0.153	0.263	0.300	0.300	0.309	+3.0	0.321	+7.0
I-8	.498	.498	3.46	2.230	0	.622	.150	.259	.300	.296	.304	+2.7	.317	+7.1
I-9	.501	.501	3.48	2.250	0	.626	.154	.266	.300	.301	.312	+3.7	.325	+8.0
I-10	.624	.876	4.00	2.740	0	.987	.947	1.164	.214	1.457	1.367	−6.1	1.416	−2.8
I-11	.624	.875	4.01	2.740	0.250	1.106	1.500	1.161	.255	1.600	1.543	−3.6	1.634	+2.1
I-12	.625	.874	4.00	2.740	0	.986	.945	1.158	.214	1.475	1.360	−7.8	1.412	−4.3
I-13	.624	.874	4.00	2.740	.250	1.104	1.485	1.157	.255	1.620	1.536	−5.2	1.631	+0.7
I-14	.623	.872	3.97	2.744	0	.990	.961	1.149	.214	1.355	1.355	0.0	1.404	+3.6
I-15	.624	.873	3.97	2.745	.250	1.108	1.481	1.158	.255	1.490	1.531	+2.7	1.625	+9.0
I-16	.622	.872	3.98	2.745	.500	1.224	2.245	1.149	.296	1.822	1.814	−0.5	1.920	+5.4
I-17	.622	.878	3.97	2.746	.750	1.340	3.305	1.152	.335	2.255	2.258	+0.1	2.356	+4.5
I-18	.507	1.045	4.49	2.750	.875	1.502	5.085	1.695	.227	2.720	2.850	+4.7	3.132	+15.2
I-19	.503	1.045	4.50	2.750	.875	1.498	5.040	1.693	.225	2.695	2.827	+4.9	3.078	+14.2
I-22	.498	1.039	4.48	2.760	.875	1.492	4.950	1.670	.224	2.734	2.779	+1.6	3.076	+12.5
I-23	.497	1.043	4.49	2.760	.875	1.492	4.950	1.689	.223	2.571	2.792	+8.6	3.096	+20.4
SOAP-FILM TESTS														
3	1.260	1.760	7.94	5.510	1.000	2.466	36.95	18.95	0.296	29.30	29.89	+2.0	31.57	+7.7
14	1.755	1.255	10.03	5.517	.875	2.276	26.84	19.78	.315	27.48	28.24	+2.8	29.17	+6.1
14-C2	1.650	1.100	9.92	5.160	.950	2.150	21.38	15.30	.315	23.23	22.04	−5.1	22.90	−1.4
15	2.507	1.260	10.03	5.512	1.250	2.929	73.55	45.71	.251	66.06	64.18	−2.8	64.45	−2.4
15-C2	2.390	1.050	9.90	5.050	1.390	2.820	63.20	38.92	.248	54.60	54.60	0.0	54.24	−0.7
16-2	1.260	1.257	10.00	5.510	.625	1.885	12.63	11.24	.400	13.74	16.29	+18.5	16.52	+20.3
16-2-C1	1.180	1.180	9.96	5.390	.640	1.793	10.45	9.26	.408	12.76	13.52	+6.0	13.69	+7.2
16-2-C2	1.105	1.105	9.98	5.260	.670	1.718	8.71	7.56	.420	10.84	11.22	+3.5	11.38	+5.0

All calculations were made with a 20-inch slide rule.
C1 and C2 indicate that first or second contour of the plate was used as the boundary of the cross section.
D=Diameter of largest inscribed circle at junction of component rectangles.
$K=2K_1+K_2+2\alpha D^4$.
K_1=torsion constant of flange.
K_2=torsion constant of web.
Results in column headed "G. and T. formula" were calculated by the Griffith and Taylor method.
Differences are expressed in per cent of test values.

An examination of the formula discloses the fact that the formula still holds at the limit where the web approaches zero thickness, since α will also approach zero. At the other limit, where the flange approaches zero thickness, α again approaches zero, butt he value $K_2=\frac{16}{3}a_2b_2^3$ is somewhat in error because the web can no longer be considered a part of a long rectangle.

From the relations holding for an I beam, we obtain results for a T beam directly. The T beam has only one junction of component rectangles and consequently only αD^4 in the formula. Also K_2 must be modified slightly. The web now closes at one end, as it would normally in a rectangle, and therefore K_2 is one-half the K of a rectangle twice as long or

$$K_2=a_2b_2^3\mu,$$

the value of μ corresponding to the ratio $2a\div b$. Our final result is

$$K=K_1+K_2+\alpha D^4.$$

For sections such as an L, we proceed in the same way. The wider leg is considered as the normal rectangle and the narrower leg as a part of a long rectangle. An examination of Figure 10 will show why this is done. Figure 11 gives the proper values for α. For sections made up of L junctions, such as U and Z sections, we proceed in the same way and add a correction for each junction.

In applying the soap-film method to U and Z sections, advantage was taken of the symmetry of these sections with respect to a line perpendicular to the bar at its middle point; L-shaped openings in the test plate, having a vertical septum at the ends of one or both legs, were used. When the legs of the L were of unequal thickness, it was desirable to have a septum at the end of each leg in order to be able to calculate two types of U or Z sections from a single test. By means of a simple calculation the effect of the septum at the end, when one is not desired, can be removed. Values of the torsion constant given in Table V for U and Z sections were obtained in this way from L-shaped openings.

The actual torsion tests on wooden beams with L, T, U, and Z sections (Table VI) yielded apparent values of the torsion constant that were considerably greater than those given by the soap-film method for the same sections. The excesses in the values thus determined are attributable to two causes:

(1) The stiffening effect of the blocks that were glued to the ends of all beams tested to make the end sections rectangular. These blocks hindered the warping of the cross sections that takes place in the twisting of all cylinders or prisms not of circular section. (Fig. 4.)

TABLE V.—*Values of torsion constant K for U or Z beams*

[All dimensions are in inches]

Test	End of legs	Thickness		Overall		Fillet radius R	D	D^4	$2K_1+K_2$	$2a$	Soap film K	Proposed formula K	Difference	G. and T. formula K	Difference
		Legs	Bar	Length leg	Width bar										
													Per cent		*Per cent*
5-1	Without septum	1.498	1.495	5.51	7.50	0	1.755	9.49	15.26	0.140	16.68	16.59	−0.5	17.71	+6.1
5-1-C1	do	1.390	1.390	5.30	7.34	0	1.630	7.06	12.08	.140	13.34	13.07	−2.0	13.85	+3.9
7-2	do	1.263	1.256	5.51	7.56	0	1.478	4.77	9.62	.140	10.12	10.29	+1.7	10.90	+5.9
7-2-C1	do	1.225	1.225	5.44	7.52	0	1.435	4.24	8.86	.140	9.39	9.45	+0.6	9.94	+5.9
7-2-C2	do	1.160	1.160	5.36	7.48	0	1.360	3.42	7.48	.140	8.06	7.96	−1.2	8.40	+4.2
9-2	do	1.006	1.008	5.51	7.54	1.00	1.515	5.27	5.20	.290	5.98	6.73	+12.5	7.49	+25.4
9-2-C1	do	.930	.940	5.41	7.48	1.04	1.451	4.43	4.12	.305	5.31	5.47	+3.0	6.14	+15.7
9-2-C2	do	.800	.810	5.26	7.38	1.14	1.339	3.21	2.65	.358	3.78	3.80	+0.5	4.18	+10.6
10	With septum	1.508	.752	6.00	11.98	.75	1.660	7.59	13.88	.107	14.94	14.69	−1.7	15.41	+3.1
10-C1	do	.710	1.470	5.97	11.96	.80	1.640	7.24	13.48	.108	14.46	14.26	−1.4	14.16	−2.1
10-C2	do	.580	1.450	5.96	11.90	.90	1.600	6.55	11.76	.093	12.78	12.37	−3.2	12.82	+0.3
11	do	1.506	1.507	6.00	12.06	.75	2.016	16.50	22.90	.215	26.18	26.45	+1.0	28.77	+9.9
11-C1	do	1.450	1.470	5.96	12.00	.79	1.980	15.38	20.80	.220	24.98	24.18	−3.2	26.39	+5.6
11-C2	do	1.353	1.373	5.96	11.98	.90	1.905	13.17	17.10	.240	20.60	20.26	−1.6	22.15	+7.5
12	do	.505	1.508	6.02	12.00	.75	1.586	6.32	13.07	.072	13.80	13.53	−1.9	13.57	−1.6
13	do	1.503	1.511	6.00	12.02	1.50	2.280	27.02	22.91	.290	30.72	30.74	0.0	34.56	+12.5
13-C1	do	1.450	1.460	5.98	11.96	1.52	2.220	24.28	20.62	.300	28.40	27.90	−1.7	31.35	+10.4
13-C2	do	1.380	1.380	5.94	11.88	1.56	2.153	21.50	17.64	.310	25.22	24.31	−3.6	27.25	+8.1
18	do	1.133	1.506	6.08	12.08	.75	1.820	10.97	17.11	.162	18.32	18.88	+3.0	20.53	+12.0
18-C1	do	1.040	1.460	6.02	12.00	.78	1.760	9.60	14.92	.157	16.16	16.42	+1.5	17.87	+10.6
18-C2	do	.900	1.370	5.98	11.86	.86	1.660	7.59	11.66	.154	12.58	12.83	+2.0	14.00	+11.3
19	do	1.506	1.507	6.01	12.00	.38	1.806	12.92	22.84	.177	23.74	25.13	+5.8	26.91	+13.4
19-C1	do	1.430	1.440	5.93	11.96	.39	1.815	10.86	19.88	.178	21.60	21.81	+1.0	23.13	+7.1
19-C2	do	1.330	1.330	5.91	11.82	.49	1.734	9.04	15.96	.193	17.16	17.70	+3.1	18.75	+9.3
20	do	1.504	1.504	6.00	12.00	1.12	2.149	21.32	22.75	.252	26.34	28.12	+6.8	31.20	+18.4
20-C1	do	1.420	1.440	5.94	11.86	1.15	2.069	18.32	19.50	.261	23.36	24.28	+4.0	26.90	+15.2
20-C2	do	1.260	1.310	5.88	11.74	1.25	1.944	14.30	14.56	.285	18.08	18.64	+3.0	20.65	+14.2

All calculations were made with a 20-inch slide rule.
Legs with septums at ends must be treated as parts of long rectangles.
C1 and C2 indicate that first or second contour of plate was used as the boundary of the cross section.
D=diameter of largest inscribed circle at the junction of component rectangles.
$K=2K_1+K_2+2aD^4$.
K_1=torsion constant of one leg.
K_2=torsion constant of bar.
Differences are expressed in per cent of soap-film values.
Results headed G. and T. formula were calculated by the Griffith and Taylor method.

TABLE VI.—*Values of torsion constant K for* L, T, U, *and* Z *beams obtained by actual test*

[All dimensions are in inches]

L-BEAMS

Test	Thickness		Overall		Fillet radius R	D	D^4	K_1+K_2	α	K		
	Long leg	Short leg	Long leg	Short leg						Test	Proposed formula	G. and T formula
L 1	0.502	0.502	3.249	2.738	0.25	0.674	0.206	0.211	0.108	0.275	0.234	0.247
L 2	.502	.502	3.245	2.750	.25	.674	.206	.212	.108	.275	.234	.247
L 3	.500	.500	3.247	2.738	0	.586	.118	.209	.070	.272	.217	.224
L 4	.500	.500	3.242	2.745	0	.586	.118	.209	.070	.265	.217	.224
L 5	.312	.496	3.250	2.740	.25	.574	.109	.126	.068	.218	.133	.140
L 6	.313	.494	3.232	2.732	.25	.572	.107	.124	.068	.190	.132	.139
L 7	.504	.498	3.243	2.727	.25	.672	.204	.207	.108	.284	.229	.242
L 8	.502	.496	3.246	2.737	.25	.668	.199	.208	.108	.307	.221	.243

T BEAMS

Test	Thickness		Total height	Total width	Fillet radius R	D	D^4	K_1+K_2	α	K			
	Flange	Web								Test	Proposed formula	G. and T. formula	Weber formula
T 1	0.504	0.504	3.240	2.755	0	0.630	0.158	0.214	0.15	0.274	0.238	0.243	—
T 2	.503	.503	3.240	2.754	0.25	.754	.323	.213	.20	.318	.278	.289	—
T 3	.503	.503	3.250	2.752	0	.629	.156	.213	.15	.275	.236	.242	—
T 4	.503	.503	3.240	2.753	.25	.754	.323	.213	.20	.312	.277	.289	—
T 5	.500	.500	3.234	2.740	0	.625	.152	.208	.15	.261	.231	.237	—
T 6	.504	.504	3.231	2.740	.25	.754	.323	.213	.20	.294	.278	.290	—
T 7	.501	.501	3.229	2.740	.50	.876	.589	.218	.25	.396	.365	.345	0.231
T 8	.502	.502	3.230	2.730	.75	.936	.768	.210	.30	.473	.460	.413	—
T 9	.504	.504	3.250	2.750	0	—	—	.208	—	.252	—	—	—
T 10	.506	.506	3.250	2.750	0	.632	.160	.217	.15	.287	.240	.247	—
T 11	.504	.504	3.250	2.750	0	—	—	.208	—	.242	—	—	—
T 12	.504	.504	3.250	2.750	0	.630	.158	.214	.15	.285	.238	.244	—

U AND Z BEAMS

Test	Thickness		Overall		Fillet radius R	D	D^4	$2K_1+K_2$	2α	K			
	Legs	Bar	Length leg	Width bar						Test	Proposed formula	G. and T. formula	Weber formula
U 1	0.502	0.502	2.750	3.750	0	0.588	0.120	0.321	0.140	0.408	0.338	0.350	—
U 2	.501	.501	2.750	3.730	0.25	.672	.204	.319	.215	.444	.363	.390	—
U 3	.502	.502	2.750	3.740	0	.588	.120	.321	.140	.436	.338	.349	—
U 4	.501	.501	2.740	3.750	.25	.672	.204	.319	.215	.448	.363	.390	—
U 5	.380	.379	2.749	3.499	.25	.530	.079	.142	.240	.251	.161	.172	—
U 6	.377	.492	2.743	3.513	.25	.600	.130	.203	.165	.324	.224	.245	—
U 7	.377	.623	2.741	3.514	.25	.692	.229	.328	.121	.452	.351	.376	—
U 8	.377	.747	2.737	3.516	.25	.790	.390	.490	.096	.634	.527	.552	—
U 9	.495	.495	2.743	3.752	0	.580	.113	.308	.140	.404	.324	.338	—
U 10	.496	.496	2.738	3.740	.25	.666	.197	.309	.215	.423	.351	.379	—
U 11	.495	.495	2.743	3.753	.50	.751	.318	.308	.290	.468	.400	.446	0.321
U 12	.496	.496	2.734	3.743	.75	.838	.493	.309	.365	.514	.489	.550	—
Z 1	.498	.498	2.735	3.715	0	.669	.200	.311	.215	.498	.354	.381	—
Z 2	.499	.499	2.736	3.718	0	.585	.117	.314	.140	.466	.330	.345	—
Z 3	.501	.501	2.734	3.708	0	.672	.204	.318	.215	.488	.360	.388	—
Z 4	.497	.497	2.736	3.714	0	.582	.115	.310	.140	.461	.326	.340	—
Z 5	.375	.375	2.750	3.750	.50	.610	.138	.141	.345	.285	.189	.208	—
Z 6	.375	.375	2.750	3.750	.50	.610	.138	.141	.345	.290	.189	.208	—
Z 7	.375	.375	2.750	3.750	.25	.525	.076	.141	.240	.248	.159	.170	—
Z 8	.376	.376	2.750	3.750	.25	.526	.077	.142	.240	.254	.161	.171	—

All calculations were made with a 20-inch slide rule.

Weber formula assumes a radius of fillet equal to the thickness of the narrower component rectangle.

T 9 and T 11 did not have web glued to flange. They act, therefore, as two separate pieces except for additional stiffness resulting from filler blocks glued at ends to make end section rectangular.

For a discussion of the discrepancy between calculated and test values of K, see concluding remarks under "Formulas for Irregular Solid Sections."

(2) The combination of bending and torsion caused by the fact that in many instances the axis of twist did not coincide with the axis of the figure.

Neither of these causes would be as effective with I beams as with the sections just mentioned.

The soap-film method furnishes the value of the torsion constant K associated with pure torsion under ideal conditions as to the application of torque at the ends. Usually an actual beam will have a margin of safety as regards torsional rigidity because of the fixity of its ends.

CONCLUSIONS

The soap-film method proved to be a valuable aid in the solution of the torsion problem for cylinders and prisms for which no rigorous mathematical solution has been found. Not only is the method capable of furnishing the torsional rigidities and the stresses with considerable accuracy but it also gives a visual representation of the actual situation as regards torsional stresses, a representation that can be readily interpreted by the observer.

From a study of the soap-film tests and the actual torsion tests, it has been possible to conclude that the torsional rigidity of prisms with sections such as **I, T, L, U,** and **Z**, which are composed of rectangles, is equal to the sum of the torsional rigidities of prisms whose sections are the component rectangles, corrected by a simple additive term to take account of the increased stiffness resulting from the junctions of the rectangles.

The formulas developed by C. Weber for such sections were found to be fairly accurate when the widths of the component rectangles are extremely small in comparison with their lengths, as with many rolled-steel sections. (Reference 11.) For sections of wooden beams for which the component rectangles are wider (say the width greater than one-fifth the length), Weber's formulas give torsional rigidities that are much too low. His formulas always assume the presence of fillets, the radii of which are equal to the width of the narrower component rectangle. With thicker sections, such as those that we have tested, the variation of the torsional rigidity with the radii of the fillets can not be neglected. In our opinion, the reasoning employed by Weber in deriving his formulas is open to objections. The errors introduced, however, are negligible for very thin sections.

Griffith and Taylor developed rules for calculating the torsional rigidities of prisms of any section. The application of these rules to sections of the kind that we are considering is rather an intricate process as compared with the simple computation required by our proposed formula. The results obtained by Griffith and Taylor's rules are good for fairly compact sections. For sections made up of component rectangles, the results calculated by their rules appear to be somewhat too high.

Our tests show that the torsional stiffness of a beam may be materially increased by the way in which it is fastened at the ends. Two other factors are important in connection with the torsional behavior of wooden beams. They are rate of fiber strain and moisture content. Corrections for their influence on torsional properties were determined. We have concluded that a third factor, which has to do with the difference between the moduli of rigidity of wood referred to planes radial and tangential to the annual rings, may, in general, be neglected in design and a mean modulus used. For Sitka spruce this mean modulus is between one-fifteenth and one-sixteenth of Young's modulus parallel to the grain.

SUMMARY

This report reviews briefly the fundamental theory of torsion and shows how the more common torsion formulas have been developed from that theory. Formulas for solid and tubular sections that have yielded to mathematical treatment are given, and empirical formulas are developed for irregular sections whose component parts are rectangles. The empirical formulas are a result both of direct torsion tests of wooden specimens and of the application of the soap-film method of investigation to the sections in question. The mathematical analogy upon which the soap-film method is based is explained.

The effect of a lack of isotropy in wood, caused by the presence of the annual growth rings, is discussed and is shown to be relatively unimportant.

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APPENDIX A

PRISMS OF NONISOTROPIC MATERIAL

In order to solve the torsion problem for a wooden beam, we shall consider a prism of nonisotropic material in which there are three mutually perpendicular planes of elastic symmetry, one of which is perpendicular to the direction of length of the prism. It will be shown that the solution of the torsion problem for such a prism can be reduced to the solution of the same problem for an isotropic prism whose section is obtained by transforming the boundaries of the original section through a linear transformation and whose modulus of rigidity is expressed in terms of the moduli of the original material.

Let the axis of Z lie along the direction of the length of the prism and the axes of X and Y be axes to which the boundary of the section is conveniently referred. (Fig. 12.) Let the planes ZX' and ZY' be the longitudinal planes of elastic symmetry and let G_1 and G_2 be the

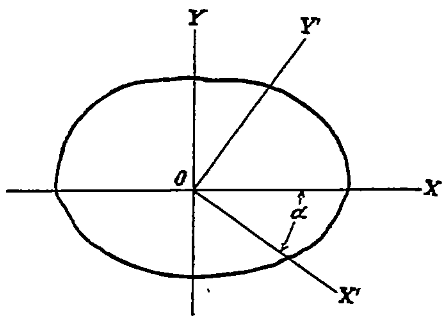

FIGURE 12

moduli of rigidity associated with shearing strains corresponding to the pairs of directions of the axes of Z and X' and of Z and Y', respectively.

We form the same general picture of the state of stress and strain as for the isotropic prism (p. 10) and accordingly we again assume that the components of the displacement parallel to the X', Y', and Z axes, respectively, are expressed as follows:

$$u=-\tau y' z, \qquad v=\tau z x', \qquad w=\tau\phi\,(x', y'), \tag{1}$$

where τ is the angle of twist per unit length and ϕ is a function of x' and y' only, which is to be determined.

As a consequence of the type of displacement given by (1), all of the components of strain vanish except

$$\begin{aligned} e_{y'z}&=\frac{\partial w}{\partial y'}+\frac{\partial v}{\partial z}=\tau\left(\frac{\partial\phi}{\partial y'}+x'\right),\\ e_{zx'}&=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x'}=\tau\left(\frac{\partial\phi}{\partial x'}-y'\right). \end{aligned} \tag{2}$$

(Reference 1.) These are shearing strains corresponding to the pair of directions zy' and zx', respectively. Then all stress components vanish except the components X'_z and Y'_z of shearing stress and these are given by

$$X'_z=G_1\,e_{zx'}, \qquad Y'_z=G_2\,e_{y'z}. \tag{3}$$

(Reference 2.) Referred to the axes X and Y, which make an angle α with the axes X' and Y', the stress components are

$$\begin{aligned} X_z &= G_2\, e_{y'z} \sin\alpha - G_1\, e_{zx'} \cos\alpha, \\ Y_z &= G_2\, e_{y'z} \cos\alpha - G_1\, e_{zx'} \sin\alpha. \end{aligned} \tag{4}$$

Entering the values of $e_{y'z}$ and $e_{zx'}$ from (2), noting that

$$x' = x\cos\alpha - y\sin\alpha, \qquad y' = x\sin\alpha + y\cos\alpha,$$

and using the abbreviations

$$\begin{aligned} \kappa &= G_2 \sin^2\alpha + G_1 \cos^2\alpha, \\ \lambda &= (G_2 - G_1)\sin\alpha\cos\alpha, \\ \mu &= G_2\cos^2\alpha + G_1\sin^2\alpha, \end{aligned} \tag{5}$$

we find that equations (4) become

$$\begin{aligned} X_z &= \tau\left(\kappa\frac{\partial\phi}{\partial x} + \lambda\frac{\partial\phi}{\partial y} + \lambda x - \kappa y\right), \\ Y_z &= \tau\left(\lambda\frac{\partial\phi}{\partial x} + \mu\frac{\partial\phi}{\partial y} + \mu x - \lambda y\right). \end{aligned} \tag{6}$$

From the equations of equilibrium and equations (6), we obtain the differential equation which the unknown function ϕ must satisfy; namely,

$$\kappa\frac{\partial^2\phi}{\partial} + 2\lambda\frac{\partial^2\phi}{\partial x\,\partial y} + \mu\frac{\partial^2\phi}{\partial y^2} = 0. \tag{7}$$

This equation for the determination of ϕ corresponds to equation (3), page 11, for an isotropic prism.

The requirement that the lateral surface of the prism shall be free from traction leads to the following condition, which ϕ must satisfy on the curve $f(x, y) = 0$, the boundary of the cross section:

$$\begin{aligned} &\left(\kappa\frac{\partial\phi}{\partial x} + \lambda\frac{\partial\phi}{\partial y}\right)\frac{\partial f}{\partial x} + \left(\lambda\frac{\partial\phi}{\partial x} + \mu\frac{\partial\phi}{\partial y}\right)\frac{\partial f}{\partial y} \\ &= (\kappa y - \lambda x)\frac{\partial f}{\partial x} + (\lambda y - \mu x)\frac{\partial f}{\partial y}. \end{aligned} \tag{8}$$

After the change of independent variables

$$\xi = \delta\, x, \qquad \eta = y - \gamma x, \tag{9}$$

where

$$\delta = \frac{\sqrt{G_1 G_2}}{\kappa}, \qquad \gamma = \frac{\lambda}{\kappa}, \tag{10}$$

the differential equation (7) becomes

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} = 0. \tag{11}$$

If the equation of the boundary $f(x, y) = 0$ is transformed into

$$F(\xi, \eta) = 0 \tag{12}$$

by the change of variables (9) the boundary condition on ϕ in equation (8) becomes

$$\delta\left(\frac{\partial\phi}{\partial\xi}\frac{\partial F}{\partial\xi} + \frac{\partial\phi}{\partial\eta}\frac{\partial F}{\partial\eta}\right) = \eta\frac{\partial F}{\partial\xi} - \xi\frac{\partial F}{\partial\eta}. \tag{13}$$

If we let

$$\phi' = \delta\phi, \tag{14}$$

this condition reduces further to

$$\frac{\partial \phi'}{\partial \xi}\frac{\partial F}{\partial \xi}+\frac{\partial \phi'}{\partial \eta}\frac{\partial F}{\partial \eta}=\eta\frac{\partial F}{\partial \xi}-\xi\frac{\partial F}{\partial \eta}, \tag{15}$$

or

$$\frac{\partial \phi'}{\partial v}=\eta \cos(\xi, v)-\xi \cos(\eta, v), \tag{16}$$

where v denotes the normal to the new boundary.

From (12) and (14)

$$\frac{\partial^2 \phi'}{\partial \xi^2}+\frac{\partial^2 \phi'}{\partial \eta^2}=0. \tag{17}$$

The solution of (17) subject to the boundary condition (16) corresponds to the solution of the torsion problem for a prism whose section has the new boundary (12) and which is composed of isotropic material.

It will now be shown that the torsional rigidity of the original prism can be expressed in terms of the torsional rigidity of the transformed prism. For the couple T we have

$$T=\int\int(Y_z x-X_z y)\mathrm{d}x\,\mathrm{d}y.$$

Entering for X_z and Y_z their expressions in terms of ϕ (equation 6) and changing the variables of integration to ξ and η by equations (9) we obtain

$$T=\frac{\tau}{\delta}\int\int\left\{\sqrt{G_1G_2}\left(\xi\frac{\partial\phi}{\partial\eta}-\eta\frac{\partial\phi}{\partial\xi}\right)+\kappa(\xi^2+\eta^2)\right\}\mathrm{d}\xi\,\mathrm{d}\eta, \tag{18}$$

where the integration is now extended over the area of the transformed cross section. It follows at once from equations (18) and (10) that the torsional rigidity C of the original prism is given by

$$C=\frac{\kappa}{\delta}\int\int\left(\xi\frac{\partial\phi'}{\partial\eta}-\eta\frac{\partial\phi'}{\partial\xi}+\xi^2+\eta^2\right)\mathrm{d}\xi\,\mathrm{d}\eta. \tag{19}$$

The right-hand member of this equation is the torsional rigidity of an isotropic prism whose cross section is obtained from that of the original prism by the transformation (9) and whose modulus of rigidity is

$$\overline{G}=\frac{\kappa}{\delta}=\frac{(G_2 \sin^2\alpha+G_1 \cos^2\alpha)^2}{\sqrt{G_1G_2}}. \tag{20}$$

LINES OF SHEARING STRESS AND INTENSITY OF SHEARING STRESS IN A NONISOTROPIC PRISM

Let Ψ' be a function associated with the transformed isotropic prism as the function Ψ was with such a prism in equation (10), page 12, that is, let

$$\Psi'=\psi'-\frac{1}{2}(\xi^2+\eta^2),$$

where ψ' is a function conjugate to ϕ'. It follows that

$$X_z=\tau\kappa\frac{\partial\Psi'}{\partial\eta},$$

$$Y_z=\tau\kappa\left(\gamma\frac{\partial\Psi'}{\partial\eta}-\delta\frac{\partial\Psi'}{\partial\xi}\right).$$

If we now express these components in terms of the variables x and y, using equations (9), and let

$$\Psi' (\xi, \eta) = \Psi' (\delta x, y - \gamma x) = \Psi (x, y),$$

we obtain simply

$$X_z = \tau\kappa \frac{\partial \Psi}{\partial y}, \qquad Y_z = -\tau\kappa \frac{\partial \Psi}{\partial x}. \tag{21}$$

It follows that the curves, $\Psi(x,y) = \text{constant}$, are lines of shearing stress and that the intensity of the shearing stress at any point is equal to

$$\tau\kappa \frac{\partial \Psi}{\partial v}, \tag{22}$$

v denoting the normal at the point in question to the curve $\Psi(x, y) = \text{constant}$, which passes through that point.

Applications to certain nonisotropic prisms with simple cross sections.

To take a typical example, let us suppose that the material is wood. It will be assumed that the plane $X'OZ$, Figure 12, is parallel to the annual rings which are considered to lie in planes. The moduli G_1 and G_2 (equations (3)) are sometimes called the tangential and the radial moduli, respectively.

(*a*) The Circle:

Let the axes OX' and OX coincide so that $\alpha = 0$. After the transformation (9) the circle becomes an ellipse with the semi-axes

$$a\sqrt{\frac{G_2}{G_1}} \text{ and } a.$$

On letting $\alpha = 0$ in equation (20) the modulus of rigidity of the transformed elliptic section is found to be

$$\overline{G} = G_1\sqrt{\frac{G_1}{G_2}}.$$

The torsional rigidity of the original circular cylinder is equal to that of the transformed isotropic elliptic cylinder. We find (p. 15).

$$C = \pi \frac{G_1 G_2}{G_1 + G_2} a^4. \tag{23}$$

On comparing this result (equation (23)) with that on page 21, we see that the torsional rigidity of the given nonisotropic circular cylinder with moduli G_1 and G_2 is equal to the torsional rigidity of an equal isotropic circular cylinder with the modulus,

$$\frac{2G_1G_2}{G_1 + G_2}.$$

It has sometimes been erroneously assumed that this quantity is the mean modulus for a section of any shape.

(*b*) The Ellipse:

The annual rings make an angle α with the X-axis. The section of the transformed cylinder is obtained by using equations (9). The transformed section is an ellipse whose axis can be found. Entering these axes and the modulus as given by (20) in the expression for the torque of an elliptic cylinder, page 15, we find as the torsional rigidity of the transformed section and consequently that of the original section

$$C = \frac{m^3 G_1 G_2}{\kappa + m^2 \mu} \pi b^4, \tag{24}$$

where $m=\frac{a}{b}$ and κ and μ are given by equation (5). If

$$\frac{a}{b}=2 \text{ and } G_1=0.9G_2,$$

we obtain

$$C=\frac{7.2\pi b^4G_2}{4.6\sin^2\alpha+4.9\cos^2\alpha}.$$

Denote by C_0 and C_{90}, respectively, the values of C when $\alpha=0°$ and $90°$, respectively. Then

$$\frac{C_{90}}{C_0}=1.065$$

and if

$$\frac{a}{b}=3, \qquad \frac{C_{90}}{C_0}=1.087.$$

If $G_1=0.8G_2$, we find that

$$\frac{C_{90}}{C_0}=1.142 \text{ when } \frac{a}{b}=2,$$

and

$$\frac{C_{90}}{C_0}=1.195 \text{ when } \frac{a}{b}=3.$$

The torsional rigidity of an elliptic cylinder in which the annual rings are perpendicular to the major axis is greater than that of an equal cylinder of the same material with the rings parallel to the major axis.

(c) THE RECTANGLE:

Let α, the angle between the annual rings and the X-axis, equal zero.

The equations of transformation (9) become

$$\xi=\sqrt{\frac{G_2}{G_1}}x,$$

$$\eta=y.$$

The rectangle with sides $2a$ and $2b$ is transformed into another rectangle with sides

$$2a\sqrt{\frac{G_2}{G_1}} \text{ and } 2b.$$

The modulus of rigidity of the transformed isotropic rectangular prism is, in accordance with (20)

$$G_1\sqrt{\frac{G_1}{G_2}}.$$

Then by the formula on page 15 the torsional rigidity of the transformed section is

$$C=G_1ab^3\left\{\frac{16}{3}-\lambda\frac{b}{a}\sqrt{\frac{G_1}{G_2}}\right\}, \tag{25}$$

in which λ is to be taken from Table I by replacing the ratio of the sides by

$$\frac{a}{b}\sqrt{\frac{G_2}{G_1}}.$$

This result is in direct agreement with that of St. Venant, who obtained formulas for the cases in which $\alpha=0°$ and $\alpha=90°$.

If $\alpha=90°$, we find that the rectangle is transformed into a rectangle with sides

$$2a\sqrt{\frac{G_1}{G_2}} \text{ and } 2b$$

and that the modulus of rigidity of the transformed isotropic prism is

$$G_2\sqrt{\frac{G_2}{G_1}}.$$

Entering these results in the formula for an isotropic rectangular prism on page 15, we again obtain St. Venant's result for this case.

If $G_1=0.8G_2$ the torsional rigidity of a quarter-sawn board whose sides are in the ratio 3 to 1 is 18 per cent greater than that of a plain-sawn board of the same dimensions.

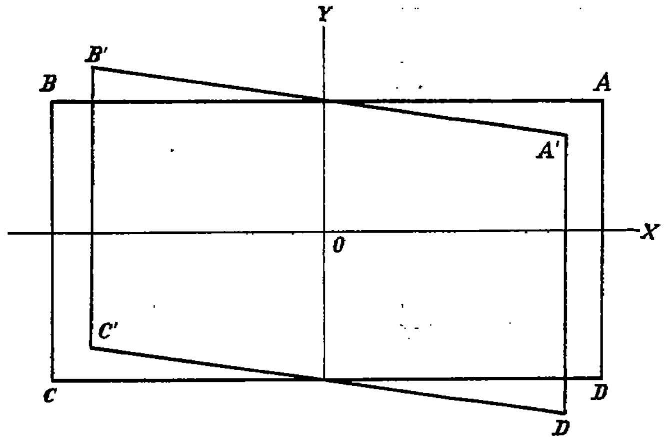

FIGURE 13

In general, the rectangle with sides $2a$ and $2b$ and vertices $ABCD$ is transformed into a parallelogram $A'B'C'D'$ whose vertices are at the points

$$(\delta a,\ b-\gamma a),\ (-\delta a,\ b+\gamma a),\ (-\delta a, -b+\gamma a), \text{ and } (\delta a, -b-\gamma a) \text{ respectively.}$$

(Fig. 13.) The sides are

$$A'B'=2a\sqrt{\gamma^2+\delta^2} \text{ and}$$

$$A'D'=2b$$

The modulus of rigidity of the prism of transformed section is given by equation (20). The acute angle between adjacent sides of the parallelogram is found from the equation

$$\tan\theta=\frac{\delta}{\gamma}$$

The torsional rigidity of the transformed isotropic prism whose section is a parallelogram is calculated by the approximate formula

$$C=\frac{A^4}{41J}\overline{G}$$

where A is the area of the section and J is its polar moment of inertia. This formula is quite accurate at the extremes $\alpha=0°$ and $\alpha=90°$ if the ratio of the sides is 3 to 1. The use of this approximate formula to compute the torsional rigidity of the transformed prisms appears to be justified, since the angles of the parallelograms into which the rectangular sections are transformed differ but little from right angles. If the ratio of the sides of the rectangle is different from 3 to 1 the factor 41 in the denominator should be replaced by a different number so chosen that the formula gives results that agree well with the exact values for $\alpha=0°$ and $\alpha=90°$.

(*d*) THE EQUILATERAL TRIANGLE:

By the transformation (9) the equilateral triangle (fig. 14) with vertices A, B, and C at

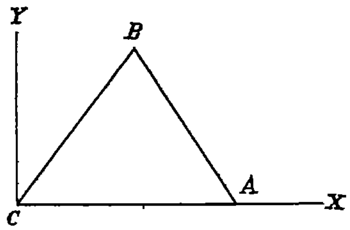

FIGURE 14

the points

$$(a, 0), \left(\frac{a}{2}, \frac{a}{2}\sqrt{3}\right), \text{ and } (0, 0)$$

is transformed into a triangle with vertices at

$$(\delta a, -\gamma a), \left[\frac{\delta a}{2}, (\sqrt{3}-\gamma)\frac{a}{2}\right], \text{ and } (0, 0),$$

respectively. Table VII gives the lengths of the new sides $C'B'$ and $C'A'$ and their included angle $C' = A'C'B'$ corresponding to various values of the angle α, α being the angle made by the planes of the annual rings with the X-axis, for wood. It was assumed that

$$G_1 = 0.8G_2$$

TABLE VII

α	δ	γ	Modulus	$C'B'/a$	$C'A'/a$	C'	C/K
°						° ′	
0	1.118	0	0.716G_2	1.031	1.118	57 08	0.1928
7½	1.115	.032	.719G_2	1.016	1.116	58 25	.1937
15	1.101	.062	.734G_2	1.000	1.103	59 52	.192
22½	1.077	.085	.772G_2	.984	1.080	61 21	.1924
30	1.052	.102	.808G_2	.970	1.056	62 45	.1924
45	.994	.111	.906G_2	.950	1.001	64 45	.1918

The torsional rigidity C used in the last column of Table VII was computed by the formula

$$C = \frac{A^4}{45J}\overline{G}, \tag{27}$$

where A, J and $\overline{G}$ have the same meaning as on page 42. This formula, which is exact for the equilateral triangle, was thought to be sufficiently accurate for the computation of the rigidity of the slightly distorted transformed sections. According to the computed values the torsional rigidity does not vary appreciably with the angle α made by a plane of symmetry (the plane of the annual rings, for wood) with one of the bases. This result is not surprising in view of the symmetry of the section.

THE SOAP-FILM METHOD OF SOLVING THE TORSION PROBLEM FOR PRISMS OF NONISOTROPIC MATERIAL

The torsion problem for a prism of nonisotropic material having perpendicular longitudinal planes of elastic symmetry has been reduced by the linear transformation (9) to the torsion problem for an isotropic prism of a transformed section and a given modulus of elasticity. The soap-film method may accordingly be used when the transformed section is such that a rigorous mathematical solution of the torsion problem for this section is not available.

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It follows from (19) in the way in which (15), page 12, was obtained from (6), page 11, that the torsional rigidity of the original prism, which is the same as that of the transformed prism, is given by

$$C = \frac{2\kappa}{\delta} \int\int \Psi' \, d\xi \, d\eta. \tag{28}$$

This means that the torsional rigidity of the prism is proportional to the volume inclosed by the surface $z = \Psi' (\xi, \eta)$ and the plane $z = 0$, the function $\Psi' (\xi, \eta)$ vanishing on the boundary of the transformed section. Now $\Psi' (\xi, \eta)$ is proportional to the ordinates of a soap film stretched over an opening of the shape of the transformed section of the prism, the film being under a uniform excess pressure on one side. The proportionality factor can be determined from (28) and the previous discussion of the soap-film method.

The contour lines of the soap film stretched over the transformed section are the curves $\Psi' (\xi, \eta) = \text{constant}$. These curves when transformed by (9) become the curves $\Psi (x, y) = \text{constant}$, the lines of shearing stress of the original nonisotropic prism. This follows immediately from equations (21). From the distance between adjacent curves $\Psi (x, y) = \text{constant}$, we can, in accordance with (22), estimate the intensity of the shearing stress at a given point.

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APPENDIX B

THE GRIFFITH AND TAYLOR FORMULAS FOR TORQUE AND STRESS

Method of calculating torsional rigidity.

The method of Griffith and Taylor, which gives fairly accurate results for many sections, is summarized in this appendix. For a comparison with our results see the discussion in the body of the report.

We may write for any section

$$T = GK\frac{\theta}{L},$$

in which

T = the twisting moment.

G = the modulus of rigidity.

K = the torsion constant.

$\frac{\theta}{L}$ = the unit angle of twist.

For a circle

$$K = \frac{\pi r^4}{2},$$

which may be written

$$K = \frac{\pi r^2}{2} \times r^2 = \frac{A}{2} r^2,$$

in which A = the area. In general, then, let us assume that

$$K = \frac{A}{2} C^2,$$

in which C is called the equivalent torsional radius of the section. To determine the twist for a given moment, we must then find C for the section in question. Checking C for an equilateral triangle against C for its inscribed circle, we find that, while the area for the triangle is 65 per cent greater, C is only 10 per cent greater; this shows that projecting corners add but little to torsional stiffness. The first step, then, in getting the correct C for the section under consideration is to round off any projecting corners with an arc of suitable radius. The radius of such an arc depends upon the angle through which the tangent to the boundary turns in passing around such a corner, and also upon the radius of the largest circle that can be drawn in the section touching the boundary at more than two points. Let α be the angle through which the tangent passes in turning a corner; for the corner of a square it is 90 degrees, for the apex of an equilateral triangle it is 120 degrees, and so on. Let b equal the radius of the largest circle that can be drawn in the section and call r the radius of the arc for rounding off the corner. Table VIII gives the ratio of r to b.

TABLE VIII

$\frac{\alpha^\circ}{180^\circ}$	$\frac{r}{b}$	$\frac{\alpha^\circ}{180^\circ}$	$\frac{r}{b}$
0.0	1.000	0.6	0.375
.1	.930	.7	.270
.2	.850	.8	.210
.3	.750	.9	.170
.4	.625	1.0	.155
.5	.500		

In this way we make a new figure with all the outward corners rounded off.

Let

A_1 = the area of the new figure.

P_1 = the perimeter of the new figure.

Then our first approximation of C is

$$C_1=\frac{2A_1}{P_1}$$

Our second approximation is obtained as follows:

Let

A = the area of the original section.

P = the perimeter of the original section.

b = the radius of the largest inscribed circle.

$=\frac{2A}{P}.$

Then the square of C_1 as obtained by the first approximation must be multiplied by a factor λ taken from Table IX.

TABLE IX

$\frac{b}{h}$	λ	$\frac{b}{h}$	λ
1.00	1.000	0.70	0.897
.95	.998	.65	.848
.90	.994	.60	.793
.85	.984	.55	.732
.80	.966	.50	.667
.75	.938		

We have then

$$K=\frac{A}{2}\lambda C_1^2,$$

$$=\frac{A}{2}\lambda\left(\frac{2A_1}{P_1}\right)^2.$$

Sections in which more than one circle touching the boundary in three points can be drawn require special treatment. They must be divided into component sections. A value of C for each component is then calculated and the results added to obtain a C for the whole section. In dividing a section into component parts, the following rule is used: Imagine a circle of varying radius to move inside the section. There may be several positions where the circle and the boundary have three or more points of contact, and between each pair of such positions there will be a position of the circle where its radius is a minimum. Draw the division lines through the points of contact of these minimum circles. When the section includes long, narrow portions bounded by lines parallel or nearly so, such as the web of an I beam, the division lines should be drawn at a distance from the commencement of the parallel portion equal to half its thickness. (For further details see Figure 15 and the calculation it accompanies.) Slightly tapering flanges of I beams should also be treated in this way. When such divisions have been made the preceding method is applied to each part separately, and the results are added. Two important instructions must be remembered, however: *In the perimeter of each component part do not include the division lines, and do not round corners formed by the junction of a division line and a line of the original boundary.*

As an illustration of the method, the calculation of the torsion constant for I beam No. 7 of Table IV, page 30, follows:

The first step is to divide the section into seven component parts as shown in Figure 15. The division lines are placed by moving half the thickness away from the commencement of

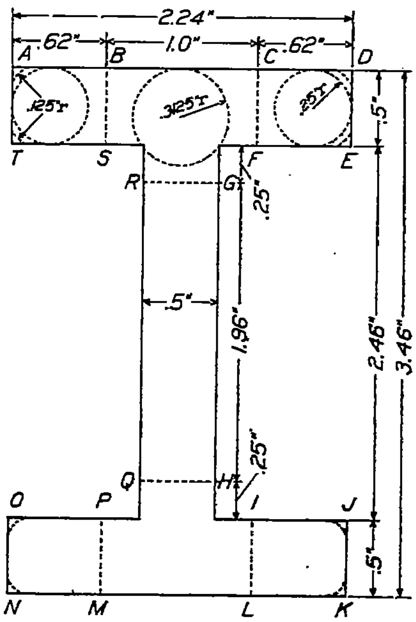

FIGURE 15.—Section of test specimen I-7 showing component simple sections

parallel portions. The next step is to round off the corners. The tangent turns through 90 degrees at all corners. Therefore, in accordance with Table VIII $\frac{r}{b}$ is equal to one-half. Table X gives the remainder of the calculations. All calculations were made with a 20-inch slide rule.

TABLE X

Part	A	P	$\frac{2A}{P}$	b	$\frac{b}{h}$	λ	A_1	P_1	$\frac{2A_1}{P_1}$	K
B, C, F, G, R, S	0.625	2.00	0.625	0.3125	0.500	0.667	0.625	2.00	0.625	0.0814
Q, H, I, L, M, P										.0814
A, B, S, T	.310	1.74	.3564	.250	.701	.899	.3033	1.633	.3715	.0192
C, D, E, F										.0192
I, J, K, L										.0192
O, P, M, N										.0192
R, G, H, Q	.980	3.92	.500	.250	.500	.667	.980	3.92	.500	.0817
Total K										.3213

Method of calculating stress.

Two formulas are given for calculating maximum stress. For compound sections, the formulas may be applied to each component part separately. Where there are no reentrant angles, the following formula is used:

$$q=\frac{2bG\tau}{1+\frac{\pi^2 b^4}{A^2}}\left[1+0.15\left(\frac{\pi^2 b^4}{A^2}-\frac{b}{\rho}\right)\right]$$

in which τ is the unit angle of twist $\frac{\theta}{L}$ and ρ is the radius of curvature of the boundary at the point in question. The maximum stress will usually occur at one of the points of contact of the largest inscribed circle. An exception may occur if the boundary is more concave at some other part than at these points of contact.

When the twisting moment is known and the angle of twist is not, τ may be obtained, of course, from—

$$T=KG\tau.$$

Where the boundary is concave, the following formula is recommended:

$$q=\frac{2bG\tau}{1+\frac{\pi^2 b^4}{A^2}}\left[1+\left\{0.118\log_e\left(1-\frac{b}{\rho}\right)-0.238\frac{b}{\rho}\right\}\tanh\frac{2\alpha}{\pi}\right]$$

in which α is the angle turned through by the tangent in turning around the reentrant portion. It must be remembered that for reentrant angles α is *negative.*

APPENDIX C

DESIGN VALUES FOR AIRPLANE MATERIAL

Recommended design values for wood for use in connection with the formulas of this report are given herewith. For metal, the allowable shearing stress values at present specified should be used for q except where better data are now available (as in Technical Note Number 189 of the National Advisory Committee for Aeronautics). For steels for which values are not now available, 10,000 pounds per square inch added to half the ultimate tensile strength gives a value that may be used for the ultimate shearing stress in torsion. The values for wood follow.

Spruce, $G=\frac{E}{15.5}=\frac{1{,}300{,}000}{15.5}=84{,}000$ pounds per square inch.

Spruce, 45° plywood, $G_1=5G=420{,}000$ pounds per square inch.

Spruce, $q=1{,}000$ pounds per square inch.

Spruce, 45° plywood, $q=2{,}370$ pounds per square inch.