REPORT 1039

ON THE PARTICULAR INTEGRALS OF THE PRANDTL-BUSEMANN ITERATION EQUATIONS FOR THE FLOW OF A COMPRESSIBLE FLUID[1]

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SUMMARY

The particular integrals of the second-order and third-order Prandtl-Busemann iteration equations for the flow of a compressible fluid are obtained by means of the method in which the complex conjugate variables z and $\bar{z}$ are utilized as the independent variables of the analysis. The assumption is made that the Prandtl-Glauert solution of the linearized or first-order iteration equation for the two-dimensional flow of a compressible fluid is known. The forms of the particular integrals, derived for subsonic flow, are readily adapted to supersonic flows with only a change in sign of one of the parameters of the problem.

INTRODUCTION

For the past several years iteration methods have been increasingly applied to the solution of compressible-flow problems. The most useful method from the point of view of aeronautical applications and the one discussed in this report is based on small perturbations with respect to the undisturbed flow. The Prandtl-Glauert and Ackeret solutions in two-dimensional subsonic and supersonic flow, respectively, obtained by means of the linearization of the fundamental nonlinear differential equation for compressible flow, are presumed to be known and are taken as the initial steps in this iteration process. Higher-order solutions are then obtained by retaining appropriate powers and products of the perturbation quantities. This method of iteration has been variously labeled the Ackeret iteration process and the Prandtl-Busemann small perturbation method when limited to two-dimensional subsonic flow. The procedure has been extended in recent years to both two-dimensional and axisymmetrical supersonic-flow problems.

In a recent publication (reference 1), Van Dyke succeeded in obtaining by trial the particular integral of the nonhomogeneous second-order iteration equation for the velocity potential in supersonic flow. The general solution is then easily obtained by adding solutions of the homogeneous equation with proper regard to the boundary conditions at the surface of the solid and at infinity.

The purpose of the present report is to show a procedure by means of which the particular integrals of the higher-than-first-order iteration equations can be derived in a systematic manner. The explicit expressions obtained for the particular integrals of the second- and third-order iteration equations are believed to yield essentially the solution of the problem of high subsonic flow past an arbitrary two-dimensional profile, since it is never a difficult problem to supply the solutions of the homogeneous equation necessary for the fulfillment of the boundary conditions. It is noteworthy that the particular integrals, derived for subsonic flow, can be adapted to supersonic flow with simply a change in sign of one of the parameters.

FUNDAMENTAL EQUATIONS

The fundamental nonlinear differential equation governing the flow of a compressible fluid is

$$(c^2-u^2)\frac{\partial u}{\partial X}+(c^2-v^2)\frac{\partial v}{\partial Y}-uv\left(\frac{\partial v}{\partial X}+\frac{\partial u}{\partial Y}\right)=0 \tag{1}$$

where

X, Y rectangular Cartesian coordinates in flow plane

u, v fluid velocity components along X- and Y-axis, respectively

c local speed of sound

The condition for irrotational motion is that

$$\frac{\partial u}{\partial Y}=\frac{\partial v}{\partial X}$$

and leads to a velocity potential Φ defined by

$$\left.\begin{aligned} u&=\frac{\partial \Phi}{\partial X}\\ v&=\frac{\partial \Phi}{\partial Y}\end{aligned}\right\} \tag{2}$$

If the body is held fixed in a uniform stream of velocity U, the relation between the local speed of sound c and the speed of the fluid $\sqrt{u^2+v^2}$ is given for adiabatic processes by

$$\frac{c^2}{c_\infty^2}=1+\frac{\gamma-1}{2}M_\infty^2\left(1-\frac{u^2+v^2}{U^2}\right) \tag{3}$$

[1] Supersedes NACA TN 2159, "On the Particular Integrals of the Prandtl-Busemann Iteration Equations for the Flow of a Compressible Fluid" by Carl Kaplan, 1950.

where

c_∞ sound speed in undisturbed fluid

γ ratio of specific heats at constant pressure and constant volume

M_∞ Mach number of undisturbed stream (U/c_∞)

With the introduction of a characteristic length l as unit of length and the undisturbed stream velocity U as unit of velocity, the quantities X, Y, u, v, and Φ for the remainder of the analysis denote, respectively, the nondimensional quantities X/l, Y/l, u/U, v/U, and Φ/Ul, while c and c_∞ retain their original meanings. By means of equations (2), equations (1) and (3) then become, respectively,

$$\left(\frac{c^2}{c_\infty{}^2}-M_\infty{}^2u^2\right)\Phi_{XX}+\left(\frac{c^2}{c_\infty{}^2}-M_\infty{}^2v^2\right)\Phi_{YY}-2M_\infty{}^2\Phi_X\Phi_Y\Phi_{XY}=0 \tag{4}$$

and

$$\frac{c^2}{c_\infty{}^2}=1+\frac{\gamma-1}{2}M_\infty{}^2[1-(u^2+v^2)] \tag{5}$$

where the subscripts X and Y denote partial differentiations with respect to the designated variables.

In order to obtain the iteration equations based on small perturbations of the undisturbed stream, the assumption is made that the velocity potential Φ can be expanded in the form

$$\Phi=X+\Phi_1+\Phi_2+\Phi_3+\ .\ .\ . \tag{6}$$

For the purpose of defining and controlling the iteration procedure, the function Φ_{n+1} and its derivatives are then regarded as small compared with the preceding approximation Φ_n and its derivatives.

From equations (2) and (6),

$$u=\Phi_X=1+\Phi_{1X}+\Phi_{2X}+\Phi_{3X}+\ .\ .\ .$$

and

$$v=\Phi_Y=\Phi_{1Y}+\Phi_{2Y}+\Phi_{3Y}+\ .\ .\ .$$

When these expressions for u, v are introduced into equation (4), together with the expression for $c^2/c_\infty{}^2$ given by equation (5), and the powers and products of Φ_n and their derivatives are grouped according to the assumptions of the small perturbation method, the following iteration equations for the first three approximations Φ_1, Φ_2, and Φ_3 result:

$$(1-M_\infty{}^2)\Phi_{1XX}+\Phi_{1YY}=0 \tag{7}$$

$$(1-M_\infty{}^2)\Phi_{2XX}+\Phi_{2YY}=M_\infty{}^2[(\gamma+1)\Phi_{1X}\Phi_{1XX}+(\gamma-1)\Phi_{1X}\Phi_{1YY}+2\Phi_{1Y}\Phi_{1XY}] \tag{8}$$

$$(1-M_\infty{}^2)\Phi_{3XX}+\Phi_{3YY}=M_\infty{}^2\Big\{\left(\frac{1}{2}\Phi_{1X}{}^2+\Phi_{2X}\right)[(\gamma+1)\Phi_{1XX}+(\gamma-1)\Phi_{1YY}]+\frac{1}{2}\Phi_{1Y}{}^2[(\gamma-1)\Phi_{1XX}+(\gamma+1)\Phi_{1YY}]+\Phi_{1X}[(\gamma+1)\Phi_{2XX}+(\gamma-1)\Phi_{2YY}]+2(\Phi_{1X}\Phi_{1Y}\Phi_{1XY}+\Phi_{2Y}\Phi_{1XY}+\Phi_{1Y}\Phi_{2XY})\Big\} \tag{9}$$

For slender bodies, the first few steps of this iteration process may be expected to yield an accurate result with the exception of a small region in the neighborhood of a stagnation point. Even at stagnation points, the iteration method has been shown to represent correctly the effect of compressibility (reference 2). The accuracy of the calculations obviously depends upon the number of terms determined, each additional term reducing the region of inaccuracy in the neighborhood of a stagnation point.

The iteration equations (7), (8), and (9) may be put into more familiar forms by the introduction of a new set of independent variables x and y, where

$$\left.\begin{aligned}x&=X\\ y&=Y\sqrt{1-M_\infty{}^2}\end{aligned}\right\} \tag{10}$$

Thus, for $M_\infty<1$, equation (7) is transformed into a Laplace equation; whereas equations (8) and (9) are transformed into Poisson equations with the right-hand sides composed of, respectively, double products and triple products of previously determined perturbation quantities. It is further assumed that the solution of equation (7) is available. This initial step in the approximation to the exact nonlinear solution is usually easily obtained, as it represents the Prandtl-Glauert approximation (reference 3, appendix B). The purpose of the present report is then to derive explicit expressions for the particular solutions of the second- and third-order iteration equations (8) and (9).

CALCULATION OF THE PARTICULAR INTEGRAL OF THE SECOND-ORDER ITERATION EQUATION

By introducing the independent variables x and y defined by equation (10), the second-order iteration equation (8) becomes

$$\Phi_{2xx}+\Phi_{2yy}=2M_\infty{}^2[(1+\sigma)\Phi_{1x}\Phi_{1xx}+\Phi_{1y}\Phi_{1xy}] \tag{11}$$

where

$$\sigma=\frac{\gamma+1}{2}\frac{M_\infty{}^2}{\beta^2}$$

$$\beta^2=1-M_\infty{}^2$$

and where use has been made of Laplace's equation

$$\Phi_{1xx}+\Phi_{1yy}=0$$

The procedure for obtaining the particular integrals of the higher-order iteration equations is based on the use of the complex conjugate variables z and $\bar{z}$ as independent variables. Thus,

$$z=x+iy$$

$$\bar{z}=x-iy$$

and the equivalence of operators

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial z}+\frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y}=i\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \bar{z}}\right)$$

$$\frac{\partial^2}{\partial x^2}=\frac{\partial^2}{\partial z^2}+2\,\frac{\partial^2}{\partial z\,\partial \bar{z}}+\frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial^2}{\partial y^2}=-\frac{\partial^2}{\partial z^2}+2\,\frac{\partial^2}{\partial z\,\partial \bar{z}}-\frac{\partial^2}{\partial \bar{z}^2}$$

Then equation (7) for Φ_1 becomes

$$4\,\frac{\partial^2\Phi_1}{\partial z\,\partial \bar{z}}=0 \tag{12}$$

The most general real solution of this equation is

$$\Phi_1=\frac{1}{2}\,[w_1(z)+\bar{w}_1(\bar{z})] \tag{13}$$

or

$$\Phi_1=\mathrm{R.P.}\,w_1(z)=\mathrm{R.P.}\,\bar{w}_1(\bar{z})$$

where $w_1(z)$ is an arbitrary analytic function of z, $\bar{w}_1(\bar{z})$ is its conjugate complex, and where the symbol R.P. stands for "real part of." The imaginary part of $w_1(z)$ is a function ψ_1, say, related to Φ_1 by means of the Cauchy-Riemann equations and hence also satisfies Laplace's equation. The function ψ_1 does not represent the stream function of the actual compressible flow and does not appear in the final expressions of the particular integrals. The following relations will be found useful and are easily verified:

$$\Phi_1=\frac{1}{2}\,(w_1+\bar{w}_1)$$

$$\Phi_{1x}=\frac{1}{2}\,(w_{1z}+\bar{w}_{1\bar{z}})$$

$$\Phi_{1y}=\frac{i}{2}\,(w_{1z}-\bar{w}_{1\bar{z}})$$

$$\Phi_{1xy}=\frac{i}{2}\,(w_{1zz}-\bar{w}_{1\bar{z}\bar{z}})$$

$$\Phi_{1xx}=\frac{1}{2}\,(w_{1zz}+\bar{w}_{1\bar{z}\bar{z}})$$

Then

$$2\Phi_{1x}\Phi_{1xx}=\mathrm{R.P.}\,(w_{1z}+\bar{w}_{1\bar{z}})\,w_{1zz}$$

$$2\Phi_{1y}\Phi_{1xy}=-\mathrm{R.P.}\,(w_{1z}-\bar{w}_{1\bar{z}})\,w_{1zz}$$

and equation (11) for the second approximation Φ_2 becomes

$$\Phi_{2z\bar{z}}=\frac{1}{4}\,M_\infty^2\,\mathrm{R.P.}\,[\sigma w_{1z}w_{1zz}+(2+\sigma)\,\bar{w}_{1\bar{z}}w_{1zz}]$$

$$=\frac{1}{4}\,M_\infty^2\,\mathrm{R.P.}\left[\frac{\sigma}{2}\,(w_{1z}^2)_z+(2+\sigma)\,(w_{1z}\bar{w}_{1\bar{z}})_z\right]$$

If Φ_2 is defined to be the real part of a nonanalytic function $w_2(z,\bar{z})$, then

$$w_{2z\bar{z}}=\frac{1}{4}\,M_\infty^2\left[\frac{\sigma}{2}\,(w_{1z}^2)_z+(2+\sigma)\,(w_{1z}\bar{w}_{1\bar{z}})_z\right] \tag{14}$$

This equation can be integrated immediately by inspection and yields the general solution

$$w_2=\frac{1}{4}\,M_\infty^2\left[\frac{\sigma}{2}\,\bar{z}w_{1z}^2+(2+\sigma)\,\bar{w}_1w_{1z}+F(z)\right] \tag{15}$$

where, because only the real part Φ_2 of w_2 is of interest, only one arbitrary analytic function $F(z)$ need be included. The function $F(z)$ satisfies Laplace's equation and is so chosen as to satisfy the required boundary conditions at the surface of the body and at infinity. The part of the expression on the right-hand side of equation (15), excluding the arbitrary function $F(z)$, is the particular integral of equation (14) and may be expressed in real form in the following manner:

Suppose

$$F(z)=-\frac{\sigma}{2}\,zw_{1z}^2+(2+\sigma)w_1w_{1z}+f(z) \tag{16}$$

where $f(z)$ is again an arbitrary analytic function of z.

Then with the aid of the relation

$$w_{1z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right)(\Phi_1+i\psi_1)=\Phi_{1x}-i\Phi_{1y}$$

where use has been made of the Cauchy-Riemann conditions

$$\Phi_{1x}=\psi_{1y}$$

$$\Phi_{1y}=-\psi_{1x}$$

the expression for Φ_2, obtained from equation (15), becomes

$$\Phi_2=\frac{1}{4}M_\infty^2\,[-2\sigma y\Phi_{1x}\Phi_{1y}+2\,(2+\sigma)\,\Phi_1\Phi_{1x}+\mathrm{R.P.}\,f(z)] \tag{17}$$

The expression on the right-hand side of this equation, excluding R.P. $f(z)$, namely,

$$\Phi_2=M_\infty^2\left[\left(1+\frac{\sigma}{2}\right)\Phi_1-\frac{1}{2}\,\sigma y\Phi_{1y}\right]\Phi_{1x} \tag{18}$$

corresponds precisely to the particular integral obtained by Van Dyke (reference 1) for two-dimensional supersonic flow with σ replaced by $-\sigma$, where for supersonic flow the definition of σ is

$$\sigma=\frac{\gamma+1}{2}\frac{M_\infty^2}{M_\infty^2-1}$$

It is rather noteworthy that the particular integral of the second-order iteration equation (8) can be obtained for both subsonic and supersonic flows by simply interchanging the sign of the parameter σ.

CALCULATION OF THE PARTICULAR INTEGRAL OF THE THIRD-ORDER ITERATION EQUATION

In this section, the particular integral of equation (9) involving only Φ_1, Φ_2, and their derivatives is derived. For this purpose, the variables x,y and the parameter σ are introduced. Equation (9) then takes the following form:

$$\Phi_{3xx}+\Phi_{3yy}=2M_\infty^2\Big\{(1+\sigma)(\Phi_{1xx}\Phi_{2x}+\Phi_{1x}\Phi_{2xx})+[2\beta^2(1+\sigma)-1]\Phi_{1x}\Phi_{1y}\Phi_{1xy}+(1+\sigma)\left[2\beta^2(1+\sigma)-\frac{3}{2}\right]\Phi_{1xx}\Phi_{1x}^2+$$
$$\frac{1}{2}(\sigma\beta^2-1)\Phi_{1xx}\Phi_{1y}^2+\Phi_{1xy}\Phi_{2y}+\Phi_{1y}\Phi_{2xy}\Big\} \tag{19}$$

Use is again made of the complex conjugate variables z and $\bar{z}$ as independent variables. Thus,

$$\Phi_2=\text{R.P.}\,w_2(z,\bar{z})$$
$$\Phi_{2x}=\text{R.P.}\,(w_{2z}+w_{2\bar{z}})$$
$$\Phi_{2y}=\text{R.P.}\,i(w_{2z}-w_{2\bar{z}})$$
$$\Phi_{2xx}=\text{R.P.}\,(w_{2zz}+2w_{2z\bar{z}}+w_{2\bar{z}\bar{z}})$$
$$\Phi_{2xy}=\text{R.P.}\,i(w_{2zz}-w_{2\bar{z}\bar{z}})$$
$$\Phi_{1xx}\Phi_{1x}^2=\text{R.P.}\,\frac{1}{4}(w_{1z}+\overline{w}_{1\bar{z}})^2w_{1zz}$$
$$\Phi_{1xx}\Phi_{1y}^2=-\text{R.P.}\,\frac{1}{4}(w_{1z}-\overline{w}_{1\bar{z}})^2w_{1zz}$$
$$\Phi_{1xx}\Phi_{2x}+\Phi_{1x}\Phi_{2xx}=\text{R.P.}\,\frac{1}{2}(w_{1zz}+\overline{w}_{1\bar{z}\bar{z}})(w_{2z}+w_{2\bar{z}})+\text{R.P.}\,\frac{1}{2}(w_{1z}+\overline{w}_{1\bar{z}})(w_{2zz}+2w_{2z\bar{z}}+w_{2\bar{z}\bar{z}})$$
$$\Phi_{1xy}\Phi_{2y}=-\text{R.P.}\,\frac{1}{2}(w_{1zz}-\overline{w}_{1\bar{z}\bar{z}})(w_{2z}-w_{2\bar{z}})$$
$$\Phi_{1y}\Phi_{2xy}=-\text{R.P.}\,\frac{1}{2}(w_{1z}-\overline{w}_{1\bar{z}})(w_{2zz}-w_{2\bar{z}\bar{z}})$$
$$\Phi_{1x}\Phi_{1y}\Phi_{1xy}=-\text{R.P.}\,\frac{1}{4}(w_{1z}^2-\overline{w}_{1\bar{z}}^2)w_{1zz}$$

Then equation (19), with $\Phi_3=\text{R.P.}\,w_3(z,\bar{z})$, can be written as follows:

$$w_{3z\bar{z}}=\frac{1}{4}(1-\beta^2)\sigma(w_{2z}w_{1zz}+w_{2zz}w_{1z}+w_{2\bar{z}}\overline{w}_{1\bar{z}\bar{z}}+w_{2\bar{z}\bar{z}}\overline{w}_{1\bar{z}})+\frac{1}{4}(1-\beta^2)(2+\sigma)(w_{2z}\overline{w}_{1\bar{z}\bar{z}}+w_{2zz}\overline{w}_{1\bar{z}}+w_{2\bar{z}}w_{1zz}+w_{2\bar{z}\bar{z}}w_{1z})+$$
$$\frac{1}{16}(1-\beta^2)\sigma[\beta^2(1+2\sigma)-(1-2\sigma)]w_{1z}^2w_{1zz}+\frac{1}{16}(1-\beta^2)[\beta^2(4+5\sigma+2\sigma^2)+\sigma(3+2\sigma)]\overline{w}_{1\bar{z}}^2w_{1zz}+$$
$$\frac{1}{8}(1-\beta^2)[\beta^2(2+5\sigma+2\sigma^2)+(-2+\sigma+2\sigma^2)]w_{1z}\overline{w}_{1\bar{z}}w_{1zz}$$

Introducing the expressions for $w_2(z,\bar{z})$ and its derivatives with respect to z and $\bar{z}$ from equation (15) yields for $w_{3z\bar{z}}$ the following equation:

$$w_{3z\bar{z}}=\frac{1}{16}(1-\beta^2)^2\sigma^2\bar{z}(w_{1z}^2w_{1zz})_z+\frac{1}{16}(1-\beta^2)^2\sigma(2+\sigma)(\bar{z}\overline{w}_1)_{\bar{z}}(w_{1z}w_{1zz})_z+\frac{1}{16}(1-\beta^2)^2\sigma(w_{1z}F_z)_z+\frac{1}{16}(1-\beta^2)\sigma[\beta^2(2+\sigma)+$$
$$(4+3\sigma)]\overline{w}_{1\bar{z}}(w_{1z}^2)_z+\frac{1}{32}(1-\beta^2)^2\sigma(2+\sigma)(w_{1z}^2)_z(\bar{z}\overline{w}_{1\bar{z}})_{\bar{z}}+\frac{1}{16}(1-\beta^2)^2(2+\sigma)^2w_{1zz}(\overline{w}_1\overline{w}_{1\bar{z}})_{\bar{z}}+\frac{1}{16}(1-\beta^2)^2(2+\sigma)(\overline{w}_{1\bar{z}}F_z)_{z\bar{z}}+$$
$$\frac{1}{32}(1-\beta^2)^2(2+\sigma)^2(\overline{w}_1^2)_{\bar{z}}w_{1zzz}+\frac{1}{96}(1-\beta^2)\sigma^2(5+3\beta^2)(w_{1z}^3)_z+\frac{1}{32}(1-\beta^2)[\beta^2(8+10\sigma+3\sigma^2)+\sigma(6+5\sigma)]\overline{w}_{1\bar{z}\bar{z}}w_{1z}^2 \tag{20}$$

In the derivation of this expression for $w_{3_{z\bar z}}$, free use was made of the fact that insofar as the real part Φ_3 of w_3 is concerned, terms of the nature $g(z)\bar h(\bar z)$ and $\bar g(\bar z)h(z)$ are equivalent. It is important to note that this type of operation leaves unaltered the real part Φ_3 of w_3. Since Φ_3 is the quantity sought in the calculations, changes in the imaginary part of w_3 are of no consequence in the final results.

Equation (20) can be integrated immediately by inspection and yields the following result:

$$
\begin{aligned}
w_3=&\frac{1}{32}(1-\beta^2)^2\sigma^2\bar z^2 w_{1z}^2 w_{1zz}+\frac{1}{16}(1-\beta^2)^2\sigma\bar z w_{1z}F_z+\frac{1}{16}(1-\beta^2)^2\sigma(2+\sigma)\bar z\bar w_1 w_{1z}w_{1zz}+\frac{1}{16}(1-\beta^2)\sigma[\beta^2(2+\sigma)+(4+3\sigma)]\bar w_1 w_{1z}^2+\\
&\frac{1}{32}(1-\beta^2)^2\sigma(2+\sigma)\bar z\bar w_{1\bar z}w_{1z}^2+\frac{1}{16}(1-\beta^2)^2(2+\sigma)^2\bar w_1\bar w_{1\bar z}w_{1z}+\frac{1}{16}(1-\beta^2)^2(2+\sigma)(F\bar w_{1\bar z}+F_z\bar w_1)+\frac{1}{32}(1-\beta^2)^2(2+\sigma)^2\bar w_1^2 w_{1zz}+\\
&\frac{1}{96}(1-\beta^2)\sigma^2(3\beta^2+5)\bar z w_{1z}^3+\frac{1}{32}(1-\beta^2)[\beta^2(8+10\sigma+3\sigma^2)+\sigma(6+5\sigma)]\bar w_{1\bar z}\int w_{1z}^2\,dz
\end{aligned}
\tag{21}
$$

Equation (21) is the particular integral of equation (20). The most general solution is obtained by adding an arbitrary analytic function $G(z)$, satisfying the homogeneous or Laplace equation $G_{z\bar z}=0$. An arbitrary function of $\bar z$, customarily included in the general solution, need not be considered here because only the real part Φ_3 of w_3 is of interest. In fact, the omitted arbitrary function is the complex conjugate $\bar G(\bar z)$.

In order to obtain the desired form of Φ_3 (the real part of w_3) from equation (21), $F(z)$ is replaced by its expression given in equation (16), and the real part of $f(z)$ in equation (17) is replaced by

$$
\text{R.P.}\, f(z)=\frac{4}{M_\infty^2}\Phi_2+2\sigma y\Phi_{1x}\Phi_{1y}-2(2+\sigma)\Phi_1\Phi_{1x}
$$

The final form of the particular integral of the third-order iteration equation (9) then becomes

$$
\begin{aligned}
\Phi_3=&-\frac{1}{2}(1-\beta^2)\sigma y(\Phi_{1x}\Phi_{2y}+\Phi_{1y}\Phi_{2x})+\frac{1}{2}(1-\beta^2)(2+\sigma)(\Phi_{1x}\Phi_2+\Phi_1\Phi_{2x})+\\
&\frac{1}{8}(1-\beta^2)^2\sigma^2y^2\Phi_{1xx}(\Phi_{1x}^2-\Phi_{1y}^2)-\frac{1}{4}(1-\beta^2)^2\sigma^2y^2\Phi_{1x}\Phi_{1y}\Phi_{1xy}+\frac{1}{48}(1-\beta^2)\sigma[(6+5\sigma)+3(-2+\sigma)\beta^2]y\Phi_{1y}^3+\\
&\frac{1}{16}(1-\beta^2)\sigma[(10-\sigma)-(10+7\sigma)\beta^2]y\Phi_{1x}^2\Phi_{1y}+\frac{1}{4}(1-\beta^2)^2\sigma(2+\sigma)y\Phi_1(\Phi_{1x}\Phi_{1xy}+\Phi_{1y}\Phi_{1xx})+\\
&\frac{1}{16}(1-\beta^2)[(-16-10\sigma+\sigma^2)+(16+22\sigma+7\sigma^2)\beta^2]\Phi_1\Phi_{1x}^2-\frac{1}{16}(1-\beta^2)\sigma[(6+5\sigma)+3(2+\sigma)\beta^2]\Phi_1\Phi_{1y}^2-\\
&\frac{1}{8}(1-\beta^2)^2(2+\sigma)^2\Phi_1^2\Phi_{1xx}+\frac{1}{16}(1-\beta^2)[\sigma(6+5\sigma)+(8+10\sigma+3\sigma^2)\beta^2]\Phi_{1x}\int[(\Phi_{1x}^2-\Phi_{1y}^2)dx+2\Phi_{1x}\Phi_{1y}\,dy]
\end{aligned}
\tag{22}
$$

The corresponding expression for Φ_3 for supersonic flow is obtained by simply replacing σ by $-\sigma$ with

$$
\sigma=\frac{\gamma+1}{2}\,\frac{M_\infty^2}{M_\infty^2-1}
$$

and β^2 by $-\beta^2$ with $\beta^2=M_\infty^2-1$. The physical plane variables X and Y are easily inserted into both equations (18) and (22) by means of the transformation equations (10),

$$
x=X
$$

$$
y=\beta Y
$$

It is pointed out that the forms of the two particular integrals, equations (18) and (22), derived in this report are identical for both subsonic and supersonic flow. The apparent differences are caused by a change in sign of the parameter β^2. Thus, β^2 and σ are positive for both subsonic and supersonic flow. Actually, of course, the functions represented by Φ_1, Φ_2, Φ_3, . . . are different for the two types of flow. For subsonic flow, these functions are derived from analytic and nonanalytic functions of z and $\bar z$; whereas for supersonic flow, they involve the real "characteristics" variables $x\pm\beta y$.

Note that the last term of the expression on the right-hand side of equation (22) contains the indefinite integral

$$
I=\int[(\Phi_{1x}^2-\Phi_{1y}^2)dx+2\Phi_{1x}\Phi_{1y}\,dy]
\tag{23}
$$

It is obvious from the corresponding complex integral in equation (21) that the integrand of equation (23) is an exact differential. This fact can also be easily verified with the help of Laplace's equation

$$\Phi_{1xx}+\Phi_{1yy}=0$$

Thus,

$$\frac{\partial}{\partial y}(\Phi_{1x}^2-\Phi_{1y}^2)=\frac{\partial}{\partial x}(2\Phi_{1x}\Phi_{1y}) \tag{24}$$

Equation (24) represents the necessary and sufficient condition that the integrand of equation (23) be an exact differential. Further, according to the theory of exact differential equations, the integral I may be expressed as follows:

$$I=\int^x M\,dx+\int\left(N-\frac{\partial}{\partial y}\int^x M\,dx\right)dy \tag{25}$$

where

$$M=\Phi_{1x}^2-\Phi_{1y}^2$$

$$N=2\Phi_{1x}\Phi_{1y}$$

and where by $\int^x M\,dx$ is meant the result of integrating $M\,dx$ with y considered constant. The expression within the parentheses, namely

$$N-\frac{\partial}{\partial y}\int^x M\,dx$$

is a function of y only. This statement can be verified as follows: Thus,

$$\frac{\partial}{\partial x}\left(N-\frac{\partial}{\partial y}\int^x M\,dx\right)=\frac{\partial N}{\partial x}-\frac{\partial}{\partial y}\frac{\partial}{\partial x}\int^x M\,dx \tag{26}$$

and because y is considered constant in the process of integrating $M\,dx$, it is clear that

$$\frac{\partial}{\partial x}\int^x M\,dx=M$$

Hence the right-hand side of equation (26) is $\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}$, which vanishes because of the condition for the existence of an exact differential.

Note that in general it is simpler to perform the complex integration $\int w_{1z}^2 dz$ rather than transform to a real integral and then perform the integration.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *May 29, 1950.*

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