REPORT 1165

UNSTEADY OBLIQUE INTERACTION OF A SHOCK WAVE WITH A PLANE DISTURBANCE [1]

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SUMMARY

Analysis is made of the flow field produced by oblique impingement of weak plane disturbances of arbitrary profile on a plane normal shock. Three types of disturbance are considered:

(a) Sound wave propagating in the gas at rest into which the shock moves. The sound wave refracts either as a simple isentropic sound wave or as an attenuating isentropic pressure wave, depending on the angle between the shock and the incident sound wave. A stationary vorticity wave of constant pressure appears behind the shock.

(b) Sound wave overtaking the shock from behind. The sound wave reflects as a sound wave, and a stationary vorticity wave is produced.

(c) An incompressible vorticity wave stationary in the gas ahead of the shock. The incident wave refracts as a stationary vorticity wave, and either a sound wave or attenuating pressure wave is also produced.

Computations are presented for the first two types of incident wave, over the range of incidence angles, for shock Mach numbers of 1, 1.5, and ∞.

INTRODUCTION

The unsteady one-dimensional interaction of normal shock waves and disturbances, such as sound waves or other shock waves, has been studied quite thoroughly (an example is Kantrowitz' paper on shock stability, ref. 1). The steady interaction between normal shock waves and plane Mach waves has been treated by Adams (ref. 2).

The general class of unsteady flow problems is currently of increasing interest, in connection particularly with stability of high-speed aerodynamic and combustion processes. The effect of a shock passing through a flow field (or vice versa) is likely to be important in many applications. For example, a hot-wire anemometer intended to measure the fluctuating field of turbulence in a supersonic stream will actually measure the turbulence as modified by passage through the nearly steady bow shock of the probe.

Considering, for simplicity, that the flow interacting with a normal shock is a nonviscous field of weak disturbance, it may usually be considered irrotational and isentropic (such as produced by a moving slender body) and therefore can be imagined to be composed of a suitable array of sound waves. Another possible type of weak nonviscous disturbance would be a stationary, incompressible flow of variable vorticity (turbulence which is convected rapidly past the point of observation is commonly thought of in this way).

Either of these two types of flow may be represented as a linear combination of plane waves (each wave either a sound wave or a rotational wave, depending on the type of flow to be represented) of various amplitudes, wave lengths, and orientations. Thus, if the interactive effect of a shock and each constituent wave may be found by a linear analysis, the complete problem may in principle be solved by linear combination of the resulting flow fields behind the shock. The interaction between a turbulent field and a wind-tunnel screen or contraction, or both, has been successfully carried out in references 3 through 5 by this method.

The present report concerns the interaction of a normal shock met obliquely by a plane sound wave or by a convected plane vorticity wave. Since sound waves may impinge on a shock either from upstream or downstream, both cases are considered. The oblique interaction of a shock and weak vorticity wave is also treated in a current investigation by Ribner (ref. 6).

The shock is considered to be moving freely into gas nominally at rest (as in a shock tube, when wall effects are neglected). Of course, if the observer moves at a constant speed with the shock, the flow appears as that associated with a steady shock, under different stagnation conditions. The shock-tube point of view is adopted in order that there be no question of how the equivalent steady shock is "anchored"; that is, end effects on the shock are not contemplated.

GOVERNING UNSTEADY EQUATIONS

In the following paragraphs, the equations will be derived which pertain to the propagation of a plane normal shock wave through a gas at rest, as modified by the influence of a weak pattern of unsteady disturbance.

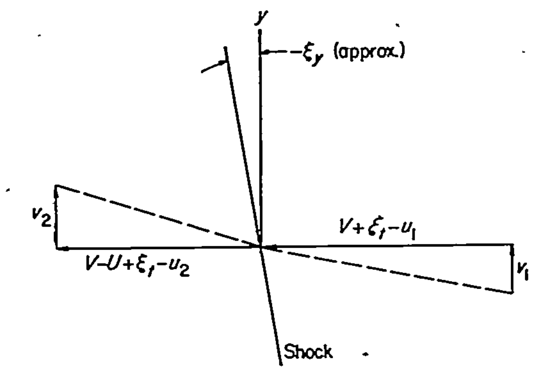
If the shock propagates without disturbance, its instantaneous position is $x_1=Vt$, in a coordinate system fixed in the fluid nominally at rest ahead of the shock (fig. 1). (All symbols are defined in appendix A.) The constant velocity of the shock front is V, and the corresponding constant velocity of the gas behind the shock is U, in this system. This one-dimensional motion is considered to be perturbed slightly by the presence of a weak field of unsteady plane flow. The velocity ahead of the shock is written as $u_1(x,y,t)$, $v_1(x,y,t)$; behind the shock as $U+u_2(x,y,t)$, $v_2(x,y,t)$. Pressure, density, and temperature are written as $P+p$, $R+\rho$,

[1] Supersedes NACA TN 2879, "Unsteady Oblique Interaction of a Shock Wave with a Plane Disturbance," by Franklin K. Moore, 1953.

and $\Theta+\theta$, respectively, where the capitalized symbols refer to the basic steady-shock motion, and the lower case to the unsteady disturbance. Throughout, subscripts 1 and 2 are used to specify conditions ahead of and behind the shock, respectively. As a result of the unsteady disturbance, the shock front itself undergoes a small unsteady displacement, given by $\xi(y,t)$. Thus, the position of the shock at any instant is $Vt+\xi(y,t)$.

SHOCK RELATIONS

Because of the rapidity with which changes occur across a shock wave, the disturbed shock may be regarded as behaving in a locally quasi-steady manner; that is, in a coordinate system fixed in the shock at each instant, the usual Rankine-Hugoniot relations apply. Because of the disturbance, the shock is slightly oblique in such a coordinate system (see sketch). Because the shock is only slightly oblique, the



shock relation concerning the product of velocity components normal to the shock front ahead of and behind the shock (ref. 7) may be written approximately as

$$(V-U+\xi_t-u_2)(V+\xi_t-u_1)=$$
$$2\,\frac{\gamma-1}{\gamma+1}\left[\frac{1}{2}(V+\xi_t-u_1)^2+\frac{\gamma J}{\gamma-1}(\Theta_1+\theta_1)\right] \quad \text{(1a)}$$

The assumption of a slightly oblique shock also provides that the equations of conservation of normal energy and mass, respectively, may be written:

$$\frac{1}{2}(V+\xi_t-u_1)^2+\frac{\gamma J}{\gamma-1}(\Theta_1+\theta_1)=$$
$$\frac{1}{2}(V-U+\xi_t-u_2)^2+\frac{\gamma J}{\gamma-1}(\Theta_2+\theta_2) \quad \text{(1b)}$$

$$(R_1+\rho_1)(V+\xi_t-u_1)=(R_2+\rho_2)(V-U+\xi_t-u_2) \quad \text{(1c)}$$

The remaining oblique shock relation states that the velocity component parallel to the shock is unaltered by passage through the shock. Because the shock is assumed to be only slightly oblique, this relation is, approximately,

$$v_1-V\xi_y=v_2-(V-U)\xi_y$$

or,

$$-\xi_y=\frac{v_2-v_1}{U} \quad \text{(1d)}$$

The shock relations for the basic undisturbed shock propagation are obtained from equations (1a), (1b), and (1c), with small quantities neglected:

$$(V-U)V=2\frac{\gamma-1}{\gamma+1}\left(\frac{1}{2}V^2+\frac{\gamma J}{\gamma-1}\Theta_1\right) \quad \text{(2a)}$$

$$\frac{1}{2}V^2+\frac{\gamma J}{\gamma-1}\Theta_1=\frac{1}{2}(V-U)^2+\frac{\gamma J}{\gamma-1}\Theta_2 \quad \text{(2b)}$$

$$R_1V=R_2(V-U) \quad \text{(2c)}$$

Terms of equations (1a), (1b), and (1c) which are of first order in small quantities yield the conditions which the disturbance field must satisfy at the shock:

$$(V-U)(\xi_t-u_1)+V(\xi_t-u_2)=2\frac{\gamma-1}{\gamma+1}\left[V(\xi_t-u_1)+\frac{\gamma J}{\gamma-1}\theta_1\right] \quad \text{(3a)}$$

$$V(\xi_t-u_1)+\frac{\gamma J}{\gamma-1}\theta_1=(V-U)(\xi_t-u_2)+\frac{\gamma J}{\gamma-1}\theta_2 \quad \text{(3b)}$$

$$\rho_1V+R_1(\xi_t-u_1)=\rho_2(V-U)+R_2(\xi_t-u_2) \quad \text{(3c)}$$

Equations (2), the state equation

$$\left.\begin{aligned} &P=JR\Theta \text{ (basic flow)} \\ &\frac{p}{P}=\frac{\rho}{R}+\frac{\theta}{\Theta} \text{ (perturbation flow)} \end{aligned}\right\} \quad \text{(4)}$$

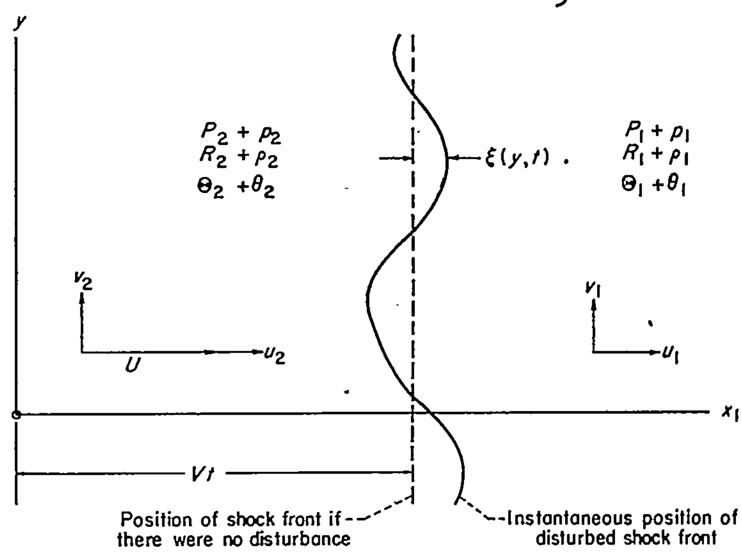


FIGURE 1.—Notation for shock wave propagating into a region of weak disturbance.

and the assumption that the incident flow ahead of the shock is isentropic permit equations (3) to be simplified, yielding the following set of disturbance shock relations (eq. (1d) is also included), which relate the disturbance fields ahead of and behind the shock:

$$\frac{\xi_t-u_2}{V}=B_1\frac{\xi_t-u_1}{V}+B_2\frac{p_1}{P_1} \tag{5a}$$

$$\frac{\rho_2}{R_2}=C_1\frac{\xi_t-u_1}{V}+C_2\frac{p_1}{P_1} \tag{5b}$$

$$\frac{p_2}{P_2}=D_1\frac{\xi_t-u_1}{V}+D_2\frac{p_1}{P_1} \tag{5c}$$

$$\frac{v_2}{V}=\frac{v_1}{V}-\frac{U}{V}\xi_y \tag{5d}$$

where the coefficients are constants depending on the Mach number of the undisturbed shock $M=V/a_1$:

$$\left.\begin{aligned}
B_1&=-\frac{2}{\gamma+1}\frac{1}{M^2}\left(1-\frac{\gamma-1}{2}M^2\right)\\
B_2&=\frac{\gamma-1}{\gamma+1}\frac{2}{\gamma M^2}\\
C_1&=\frac{2}{1+\frac{\gamma-1}{2}M^2}\\
C_2&=\frac{1}{\gamma}\left(1-\frac{\gamma-1}{1+\frac{\gamma-1}{2}M^2}\right)\\
D_1&=\frac{2\gamma M^2}{\gamma M^2-\frac{\gamma-1}{2}}\\
D_2&=\frac{M^2-\frac{\gamma-1}{2}}{\gamma M^2-\frac{\gamma-1}{2}}
\end{aligned}\right\} \tag{6}$$

EQUATIONS OF PLANE DISTURBANCE FIELD

In addition to the perturbation shock relations (eqs. (5)) which are concerned with the compatibility at the shock of weak disturbance fields ahead of and behind the shock, the equations satisfied by the disturbance fields themselves are required. The nonviscous equations of motion are written in a coordinate system at rest relative to the gas ahead of or behind the shock. That is, the following equations apply in the coordinate system of figure 1 for the disturbance field ahead of the shock, and in a coordinate system moving with velocity U, for the disturbance behind the shock. Subject to interpretation of the coordinate system, the same equations apply in both regions, and therefore the subscripts 1 and 2 are omitted for the time being. To first order in small quantities:

Momentum:

$$\left.\begin{aligned}
u_t&=-\frac{1}{R}p_x\\
v_t&=-\frac{1}{R}p_y
\end{aligned}\right\} \tag{7}$$

Continuity:

$$\rho_t+R(u_x+v_y)=0 \tag{8}$$

Energy:

$$c_vR\theta_t+P(u_x+v_y)=0 \tag{9}$$

State:

$$\frac{p}{P}=\frac{\rho}{R}+\frac{\theta}{\Theta} \tag{4}$$

These equations may be combined to yield the equation for the pressure disturbance:

$$\left[\frac{\partial^2}{\partial t^2}-a^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]p=0;\ a^2\equiv\frac{\gamma P}{R} \tag{10}$$

the equation for the entropy disturbance (which may be shown to be proportional to $(p/P)-\gamma(\rho/R)$):

$$\frac{\partial}{\partial t}\left(\frac{p}{P}-\gamma\frac{\rho}{R}\right)=0 \tag{11}$$

and the equation for the vorticity (v_x-u_y) of the disturbance flow:

$$\frac{\partial}{\partial t}(v_x-u_y)=0 \tag{12}$$

Thus, the pressure disturbance satisfies the wave equation (10), and any entropy variation (eq. (11)) or vorticity variation (eq. (12)) is steady, relative to the main flow. Any disturbance field satisfying these linear equations may be regarded as composed of two parts, one steady and the other unsteady, in a coordinate system at rest in the main flow. From equations (11) and (12), variation of entropy and vorticity may be assigned to the steady flow; and from equation (8), the associated velocity components satisfy the incompressible continuity equation. From equations (7), pressure variations must be assigned to the unsteady flow, satisfying the wave equation (10). The unsteady portion of the flow may then be regarded as produced by a pattern of sound waves. A weak nonviscous disturbance field may therefore be considered to include:

1. An unsteady, isentropic, irrotational disturbance, which may be regarded as produced by a pattern of sound waves, and
2. A steady rotational disturbance of constant pressure and (in general) variable entropy and density.

TYPES OF INITIAL DISTURBANCE CONSIDERED

The present analysis concerns the interaction of a shock wave with three types of initial plane disturbances:

A. SOUND WAVE OVERTAKEN BY SHOCK

The shock moves into a region in which a plane sound wave is propagating in a direction oblique to the direction of shock propagation (fig. 2(a)). Since the shock velocity is supersonic relative to the gas ahead, it will overtake the sound wave, whatever its direction of propagation. The solution for the interaction of such a wave with a shock may in principle be generalized by linear superposition to provide analysis of the passage of a shock through any isentropic field of small disturbance.

A general plane sound wave may be represented as follows (the particular profile of the wave need not be specified):

$$\left.\begin{aligned}\frac{u_1}{V}&=A_1 f\left(\frac{mx_1-ly+a_1t}{\lambda_1}\right)\\ \frac{v_1}{V}&=A_2 f\left(\frac{mx_1-ly+a_1t}{\lambda_1}\right)\\ \frac{p_1}{P_1}&=A_3 f\left(\frac{mx_1-ly+a_1t}{\lambda_1}\right)\\ \frac{\rho_1}{R_1}&=A_4 f\left(\frac{mx_1-ly+a_1t}{\lambda_1}\right)\end{aligned}\right\}\tag{12a}$$

where

$$l=\sin\psi_1;\ m=\cos\psi_1\tag{13}$$

and λ_1 is a length characterizing the scale of the disturbance. If the function f were a sine wave, λ_1 would be equivalent to the wave length. By equations (12), (11), and (7), respectively,

$$\left.\begin{aligned}lA_1&=-mA_2\\ A_3&=\gamma A_4\\ A_1&=-\frac{m}{\gamma M}A_3\end{aligned}\right\}\tag{12b}$$

The disturbance is of the type 1 discussed in the previous section (unsteady, isentropic, irrotational), and is longitudinal; that is, the fluctuating velocity component is in the direction of propagation of the sound wave.

B. SHOCK OVERTAKEN BY SOUND WAVE

The sound wave propagates relative to the fluid behind the shock, in such a manner as to overtake the shock (fig. 2(b)). Thus, consideration will be restricted to cases for which $-a_2\cos\psi_{21}>V-U$. The initial disturbance may be specified in a manner similar to that employed for the preceding case.

The subscripts 1 and 2 have been introduced to denote the flow ahead of and behind the shock, respectively. In the present problem, the entire flow disturbance occurs behind the shock. The subscript 2 is therefore appropriate to both the incident and reflected waves, which will hereinafter be distinguished by second subscripts 1 and 2, respectively.

$$\left.\begin{aligned}\frac{u_{21}}{V}&=A_1 f\left(\frac{mx_2-ly+a_2t}{\lambda_{21}}\right)\\ \frac{v_{21}}{V}&=A_2 f\left(\frac{mx_2-ly+a_2t}{\lambda_{21}}\right)\\ \frac{p_{21}}{P_2}&=A_3 f\left(\frac{mx_2-ly+a_2t}{\lambda_{21}}\right)\\ \frac{\rho_{21}}{R_2}&=A_4 f\left(\frac{mx_2-ly+a_2t}{\lambda_{21}}\right)\end{aligned}\right\}\tag{14a}$$

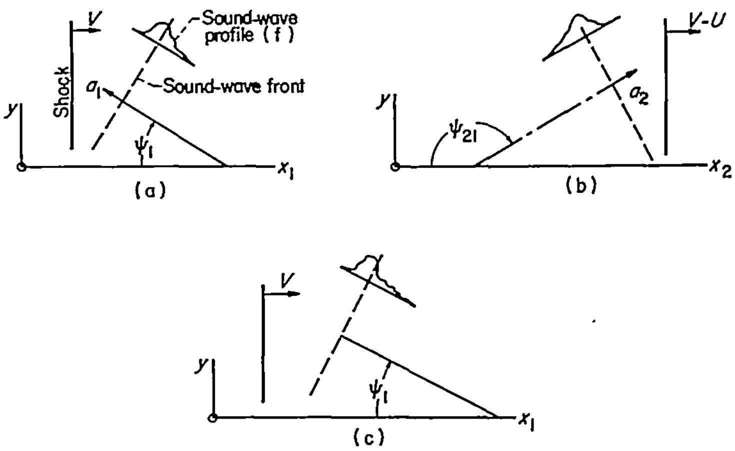
Pertinent equations of motion provide, as before,

$$\left.\begin{aligned}lA_1&=-mA_2\\ A_3&=\gamma A_4\\ A_1&=-m\frac{a_2}{\gamma V}A_3\end{aligned}\right\}\tag{14b}$$

The coordinate system x_2,y is fixed relative to the flow behind the shock.

C. STATIONARY VORTICITY WAVE OVERTAKEN BY SHOCK

The shock moves into a region occupied by a stationary plane shear disturbance of constant density, oblique relative to the shock front (fig. 2(c)). A system of such waves may



(a) Shock overtaking sound wave.
(b) Sound wave overtaking shock from behind.
(c) Shock overtaking stationary shear wave.

FIGURE 2.—Types of initial disturbance considered.

be employed to represent a turbulent field (refs. 3 and 4). Therefore, the effect of the passage of a shock through a single oblique shear wave may in principle be generalized by Fourier superposition to provide an analysis of the passage of turbulence through a normal shock.

The incident vorticity wave, of arbitrary profile, may be represented as follows:

$$\left.\begin{aligned}\frac{u_1}{V}&=A_1 f\left(\frac{mx_1-ly}{\lambda_1}\right)\\ \frac{v_1}{V}&=A_2 f\left(\frac{mx_1-ly}{\lambda_1}\right)\end{aligned}\right\}\tag{15}$$

From equation (9),

$$mA_1=lA_2\tag{16}$$

The disturbance is therefore a special case of the type 2 discussed in the previous section (a steady vorticity disturbance of constant pressure), and, by continuity, must be transverse; that is, the fluctuating velocity component is parallel to the plane of the shear waves. Since the wave is transverse, there may be a component of velocity disturbance parallel to the shock (perpendicular to the plane of fig. 2(b)) which may be of arbitrary amplitude.

This type of interaction (problem C) is treated in reference 6.

ANALYSIS OF INTERACTION BETWEEN SHOCK AND IMPOSED DISTURBANCE

THREE TYPES OF REFRACTION AT SHOCK

It has previously been shown that the equations of motion imply that any weak disturbance field may be divided into two parts: an unsteady isentropic irrotational field, and a steady vorticity disturbance. This point of view may be adopted with regard to the disturbance downstream of the shock, produced by the interaction.

Simple refracted sound wave—problem A.—In the case of a sound wave overtaken by a shock (problem A of the previous section), it would seem reasonable to expect that for ψ_1 near either 0 or π (fig. 2(a)), the isentropic part of the downstream field would simply be a refracted sound wave traveling away from the shock. This is indicated by the sequence of events shown in figure 3(a). At time t_1, the initial wave intersects the shock front at point P_1. At a later time $t_2=t_1+\delta t$, the sound wave has moved a distance $a_1\delta t$, the shock has moved a distance $V\delta t$, and the intersection occurs at point P_2. In the meantime, a cylindrical sound wave has been generated at point P_1 as a result of the shock interaction and expands with velocity a_2, while being convected with a velocity U. Thus, at time t_2, the effect of the intersection at t_1 is felt within a cylinder of radius $a_2\delta t$, with center at point Q_2. According to figure 3(a), an envelope is formed and may be identified as a simple refracted sound wave.

Attenuating refracted pressure wave—problem A.—Figure 3(a) is drawn for a rather small value of ψ_2. If ψ_1 is increased, there appears a critical value ψ_{cl} (fig. 3(b)) beyond which no envelope may be drawn. Thus, when $\psi_1>\psi_{cl}$, the influence of intersection P_1 is felt at P_2 before the intersection arrives at P_2. However, as ψ_1 is further increased, there appears another critical angle ψ_{cu} beyond which simple envelopes may again be drawn (fig. 3(d)) and simple sound wave refraction occurs.

When $\psi_{cl}<\psi_1<\psi_{cu}$, the downstream pressure disturbance cannot be a simple sound wave. The cylindrical sound waves produced by the interaction at the shock do not coalesce, but rather continue to expand independently, thus diminishing in strength as time progresses. Accordingly, the isentropic part of the downstream disturbance may be expected to die out at large distances downstream of the shock. This attenuating disturbance may, however, be expected to remain planar, because both the incoming disturbance and the shock are plane. This attenuating wave has been called a pressure wave rather than a sound wave, because, as will be shown subsequently, it does not propagate at the local velocity of sound.

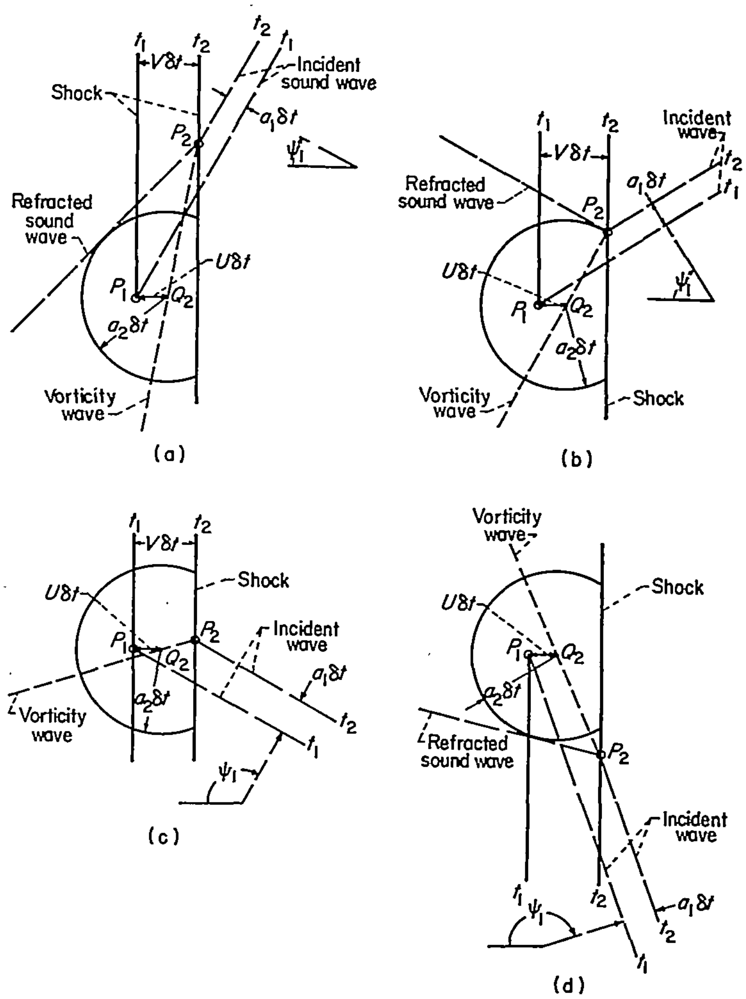
Expressions for ψ_{cl} and ψ_{cu} in terms of shock Mach number may be obtained from the following equation derived by inspection of figure 3(b):

$$a_2^2-(V-U)^2=(V\cot\psi_c+a_1\csc\psi_c)^2 \tag{17}$$

The solution of this equation is shown in figure 4, labeled "sound wave." The curves labeled "stationary vorticity wave" will be discussed subsequently. The curves both approach a value of 180° for $M=1$, and have a half-order singularity there. As $M\rightarrow\infty$, the curves become symmetric about $\psi_1=90°$, because a_1 becomes insignificant compared with V. The limiting value of ψ_{cl} is 67.8°.

Steady vorticity wave—problem A.—If a vorticity disturbance is created at the shock-disturbance intersection (fig. 3) and is thence convected with a velocity U, a vorticity wave appears along the line connecting P_2 and Q_2, whatever the value of ψ_1.

Thus, of the three types of refractions discussed, the "steady vorticity wave" always appears, in combination with either a "simple refracted sound wave," or an "attenuating pressure wave", depending on the angle ψ_1.



(a) $0<\psi_1<\psi_{cl}$.
(b) $\psi_1=\psi_{cl}$.
(c) $\psi_{cl}<\psi_1<\psi_{cu}$.
(d) $\psi_{cu}<\psi_1<\pi$.

FIGURE 3.—Formation of waves behind shock, because of interaction with sound wave (problem A).

Expectations for problem B.—In the case of the shock overtaken by a sound wave, it may be shown in constructing figures similar to figure 3 that a simple reflected sound wave will always occur, in conjunction, of course, with a steady vorticity wave.

Expectations for problem C.—In the case of the shock overtaking a stationary shear wave, sketches may be drawn similar to those presented in figure 3, except that the incident wave does not move in the time interval δt. The qualitative character of the downstream disturbance is expected to be the same as for problem A. Despite the fact that the initial disturbance is not a sound wave, sound waves are produced by the interaction and form envelopes when $0<\psi_1<\psi_{cl}$ or $\psi_{cu}<\psi_1<\pi$. In this case, the values of ψ_{cl} and ψ_{cu} are obtained from equation (17) with the term involving a_1 omitted. The solution, which is presented in figure 4, is symmetric about $\psi_1=90°$, approaches 90° at $M=1$ with a half-order singularity, and has the same asymptote at $M=\infty$ as in problem A. As in problem A, when $\psi_{cl}<\psi_1<\psi_{cu}$, an attenuating pressure wave occurs, and for all values of ψ_1, the steady vorticity wave appears.

The foregoing discussion may be summarized as follows: In case A, the incident sound wave refracts, as either a simple sound wave or as a more complicated attenuating pressure wave, and an oblique steady wave of variable entropy and vorticity appears. In case B, the incident sound wave reflects at the shock as a simple sound wave, and a steady vorticity wave appears. In case C, the initial "vorticity wave" refracts to form a stationary vorticity wave in which, because of the action of the shock, the entropy also varies. In addition, a sound wave, or pressure wave, is produced by the interaction.

SOLUTION OF PROBLEM A

In the problem of an oblique plane sound wave overtaken by the shock, there are two different solutions to be obtained—one for $0<\psi_1<\psi_{cl}$ and $\psi_{cu}<\psi_1<\pi$ and another for $\psi_{cl}<\psi_1<\psi_{cu}$. The first is the simpler and will be presented first.

Solution when $0<\psi_1<\psi_{cl}$ or $\psi_{cu}<\psi_1<\pi$.—The initial disturbance ahead of the shock is described in equations (12) and (14). The discussion of the preceding section has established that, in these ranges of ψ_1, the pressure variation behind the shock is associated with a simple sound wave. Accordingly, the disturbance pressure is written

$$\frac{p_2}{P_2}=Kf\left(\frac{\alpha x_2+\beta y+a_2 t}{\lambda_2}\right) \tag{18}$$

where K, α, β, and λ_2 are to be determined. In order that the pressure satisfy the wave equation (10),

$$\alpha^2+\beta^2=1 \tag{19}$$

Equation (18) is written on the assumption that the profile of the pressure disturbance carries through the shock undistorted, though its orientation, magnitude, and scale may change. This assumption may be regarded as a trial, the correctness of which is inferred from the self-consistency of the entire solution so obtained.

In view of the requirements of the shock relations (eqs. (5)), the arguments of the downstream pressure wave (eq. (18)) and the initial wave (eq. (12)) should match. At the shock, $x_1=Vt$ and $x_2=(V-U)t$. The matching requirement therefore is

$$\frac{[\alpha(V-U)+a_2]t+\beta y}{\lambda_2}=\frac{(mV+a_1)t-ly}{\lambda_1} \tag{20}$$

Equating coefficients of y and t yields

$$\left.\begin{aligned}\alpha&=\frac{1}{1-r}\left[\frac{\lambda_2}{\lambda_1}\left(m+\frac{1}{M}\right)-\frac{a_2}{V}\right]\\ \beta&=-l\frac{\lambda_2}{\lambda_1}\end{aligned}\right\} \tag{21}$$

where

$$r=U/V \tag{22}$$

Equations (19) and (21) yield a quadratic equation for λ_2/λ_1, the meaningful solution of which is

$$m\frac{\lambda_2}{\lambda_1}=\frac{\frac{a_2}{V}\left(1+\frac{1}{mM}\right)}{\left(1+\frac{1}{mM}\right)^2+n^2(1-r)^2}\left\{1+\sqrt{1-r\,\frac{1+n^2(1-r)^2\Big/\left(1+\frac{1}{mM}\right)^2}{1-\frac{\gamma-1}{\gamma+1}(1-r)}}\right\} \tag{23}$$

where

$$n=l/m=\tan\psi_1 \tag{24}$$

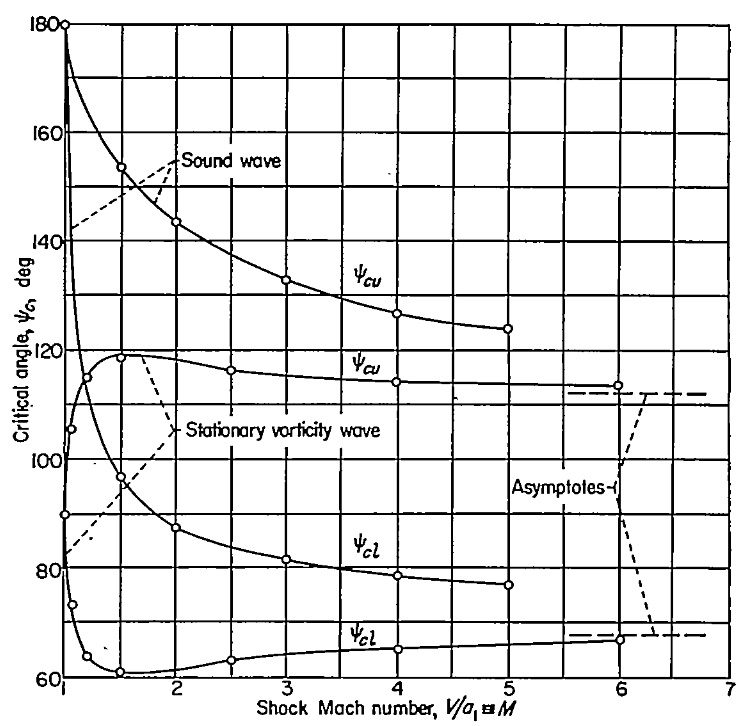


FIGURE 4.—Critical angle of incident wave for formation of refracted sound wave behind shock.

The other solution corresponds to a wave moving in the same direction as the shock and is rejected. (Inspection of fig. 3(a) shows that two families of envelopes might be indicated mathematically, and that only one is physically significant.) Thus, the inclination and scale of the pressure wave are fully determined, and only the magnitude K remains to be found.

The vorticity wave will also be assumed to have a profile given by the function f. In view of equation (12), its argument can be a function only of x_2 and y, and must furthermore match the argument of the incident wave at the shock, in order that the shock relations may be satisfied. At the shock, the argument of the incident wave is given by the right-hand side of equation (20). Also, $t=x_2/(V-U)$, at the shock. The argument of the vorticity wave must therefore be

$$\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1} \tag{25}$$

The inclination and scale of the vorticity wave are thus determined.

The density fluctuation behind the shock consists of two parts—one part associated with the pressure fluctuation to form the sound wave, its magnitude determined by equation (11); the other part associated with the vorticity wave. Thus,

$$\frac{\rho_2}{R_2}=\frac{K}{\gamma}f\left(\frac{\alpha x_2+\beta y+a_2t}{\lambda_2}\right)+Qf\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right) \tag{26}$$

Likewise, the velocity components each consist of two parts, the first associated with the sound wave and the second associated with the vorticity wave. Accordingly,

$$\frac{u_2}{V}=Ff\left(\frac{\alpha x_2+\beta y+a_2t}{\lambda_2}\right)+Gf\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right) \tag{27}$$

$$\frac{v_2}{V}=Hf\left(\frac{\alpha x_2+\beta y+a_2t}{\lambda_2}\right)+If\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right) \tag{28}$$

The requirement that all terms in the shock equations (5) have the same functional form and the same argument suggests that

$$\left.\begin{aligned}\xi_t&=VLf\left[\frac{(m+1/M)\,Vt-ly}{\lambda_1}\right]\\ -\xi_y&=\frac{n}{1+1/mM}Lf\left[\frac{(m+1/M)\,Vt-ly}{\lambda_1}\right]\end{aligned}\right\} \tag{29}$$

(Cross-differentiation shows that these two equations are compatible.)

The solution is completed by the algebraic determination of the various unknown constants. The coefficients F and H may be found in terms of K through equations (7); I may be found in terms of G from the incompressible continuity equation. The remaining unknowns K, Q, G, and L may be successively determined by use of the four shock relations (eqs. (5)) when it is recalled that the arguments of all quantities have been arranged to match at the shock. Neither the details of the remaining procedure nor of the final solution are particularly interesting, and therefore the analysis has been completed in appendix B. The numerical results will be discussed in a subsequent section.

Form of attenuating pressure wave when $\psi_{cl}<\psi_1<\psi_{cu}$.—It is intended to form the solution for this range of ψ_1 in essentially the same way as was done in the preceding paragraph. An essential step in that solution was the assumption that the refracted sound wave has the same profile as the initial disturbance. Therefore, in order to proceed with the analysis of the case $\psi_{cl}<\psi_1<\psi_{cu}$, it is first necessary to determine the form of the pressure disturbance just behind the shock and the manner in which it attenuates with distance behind the shock.

Tentatively, the pressure is written as a function of two variables only:

$$\frac{p_2}{P_2}=\frac{p_2}{P_2}(\eta,\zeta) \tag{30}$$

where

$$\left.\begin{aligned}\eta&\equiv-\frac{d}{\lambda_1}[x_2-(V-U)t]\\ \zeta&\equiv\frac{1}{\lambda_1}(\alpha x_2+\beta y+cVt)\end{aligned}\right\} \tag{31}$$

and the constants d, α, β, and c require determination. The variable η is proportional to distance behind the shock front, $-x_2+(V-U)t$. Neither the undisturbed shock front nor the incident wave has curvature. Therefore, it is expected that along any one of a family of planes moving with constant velocity, any variation of pressure would be due solely to the attenuation associated with distance behind the shock. This consideration leads to the definition of the second variable ζ, such that the equation ζ=constant defines a plane moving obliquely with a constant velocity.

The wave equation (10) is satisfied if

$$\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}\right)\frac{p_2}{P_2}=0 \tag{32a}$$

$$\alpha=-\frac{V^2}{a_2^2}(1-r)c \tag{32b}$$

$$\frac{V^2}{a_2^2}[d^2(1-r)^2-c^2]=d^2-(\alpha^2+\beta^2) \tag{32c}$$

The boundary conditions to be applied in solving Laplace's equation (32a) are

$$\frac{p_2}{P_2}(\infty,\zeta)=0 \tag{33}$$

and a condition (eq. (5c)) at the shock providing compatibility with the initial disturbance. In the following discussion, this information will be used to infer a likely form for the pressure wave.

Part of the downstream velocity variation is associated with the pressure to form an isentropic irrotational flow.

For this part of the disturbance field, a velocity potential $\varphi(\eta,\zeta)$ may therefore be defined such that

$$u_2=\varphi_{x_2}=-\frac{d}{\lambda_1}\varphi_\eta+\frac{\alpha}{\lambda_1}\varphi_\zeta \tag{34a}$$

$$v_2=\varphi_y=\frac{\beta}{\lambda_1}\varphi_\zeta \tag{34b}$$

In view of equations (7) and (34),

$$\frac{\partial}{\partial t}\text{grad }\varphi=-\frac{1}{R_2}\text{grad }p_2;\ \frac{p_2}{R_2}=-\varphi_t$$

Therefore,

$$\frac{p_2}{R_2}=-\frac{d}{\lambda_1}V(1-r)\varphi_\eta-\frac{c}{\lambda_1}V\varphi_\zeta \tag{34c}$$

In the case of the simple sound wave, compatibility at the shock was obtained by supposing that the profile of the disturbance (the function f) carried through the shock undistorted. In the present case, the corresponding assumption would be that, just behind the shock at $\eta=0$, the various disturbance quantities each contain a term proportional to $f(\zeta)$ and that ζ should therefore match the argument of the initial disturbance at the shock. From equation (34b), if $v_2(0,\zeta)$ is to contain a term proportional to $f(\zeta)$, then the potential must contain a term proportional to

$$\varphi^{(1)}=-\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\tan^{-1}\frac{\tau-\zeta}{\eta}d\tau$$

which satisfies equation (32a) and hence the wave equation, satisfies boundary condition (33), and has the property that $\varphi_\zeta^{(1)}(0,\zeta)=f(\zeta)$. This solution may be regarded as the result of a distribution of singularities along the plane of the shock ($\eta=0$). In the skew coordinate system η,ζ, these singularities may be identified as potential-flow vortices. Therefore, from equations (34a) and (34c), u_2 and p_2 would each contain a term linear in $f(\zeta)$ at the shock and, in addition, a term linear in

$$g(\zeta)\equiv\varphi_\eta^{(1)}(0,\zeta)=\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}\frac{f(\tau)d\tau}{\tau-\zeta} \tag{35a}$$

where $P.V.$ denotes the Cauchy principal value of an improper integral.

Of course, v_2 would likely contain a term linear in $g(\zeta)$ at the shock also, and therefore the potential would have another part

$$\varphi^{(2)}=-\frac{1}{2\pi}\int_{-\infty}^{\infty}f(\tau)ln[\eta^2+(\tau-\zeta)^2]d\tau$$

satisfying equations (32a) and (33), and having the properties:

$$\varphi_\zeta^{(2)}(0,\zeta)=g(\zeta);\ \varphi_\eta^{(2)}(0,\zeta)=-f(\zeta) \tag{35b}$$

This solution may be regarded as a distribution of potential-flow sources along the plane of the shock in the η,ζ system.

Thus, the quantities associated with the attenuating pressure wave may tentatively be written in the following form:

$$\frac{p_2(\eta,\zeta)}{P_2}=K^{(1)}\Phi^{(1)}(\eta,\zeta)+K^{(2)}\Phi^{(2)}(\eta,\zeta) \tag{36}$$

where

$$\left.\begin{aligned}\Phi^{(1)}&\equiv\varphi_\zeta^{(1)}=-\varphi_\eta^{(2)}=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\frac{\eta}{\eta^2+(\tau-\zeta)^2}d\tau\\ \Phi^{(2)}&\equiv\varphi_\zeta^{(2)}=\varphi_\eta^{(1)}=\frac{1}{\pi}\int_{-\infty}^{\infty}f(\tau)\frac{\tau-\zeta}{\eta^2+(\tau-\zeta)^2}d\tau\end{aligned}\right\} \tag{37}$$

and the constants $K^{(1)}$ and $K^{(2)}$ require determination. At the shock, equations (35) and (36) provide that:

$$\frac{p_2(0,\zeta)}{P_2}=K^{(1)}f(\zeta)+K^{(2)}g(\zeta)$$

Examples.—(a) If it happens that $f(\zeta)=\sin 2\pi\zeta$, then $g(\zeta)=\cos 2\pi\zeta$, and

$$\Phi^{(1)}=e^{-2\pi\eta}\sin 2\pi\zeta$$

$$\Phi^{(2)}=e^{-2\pi\eta}\cos 2\pi\zeta$$

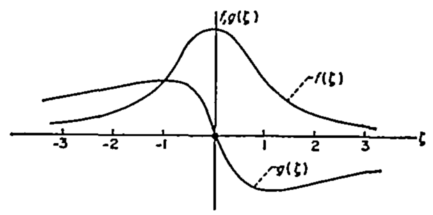
Thus,

$$\frac{p_2}{P_2}=e^{-2\pi\eta}[K^{(1)}\sin 2\pi\zeta+K^{(2)}\cos 2\pi\zeta] \tag{38}$$

and, therefore, the disturbance undergoes a phase shift in passing through the shock and subsequently attenuates exponentially with distance behind the shock.

(b) If $f(\zeta)=(1+\zeta^2)^{-1}$, then $g(\zeta)=-\zeta(1+\zeta^2)^{-1}$ (see accompanying sketch), and

$$\frac{p_2}{P_2}=\frac{1}{(1+\eta)^2+\zeta^2}[K^{(1)}(1+\eta)-K^{(2)}\zeta]$$



Solution when $\psi_{cl}<\psi_1<\psi_{cu}$.—The form of the pressure wave has been adduced in the previous paragraph (eq. (36)). The quantities d, α, β, and c may be found by using equations (32) and the requirement that the disturbance function f have the same argument ahead of and behind the shock. At the shock, $x_1=Vt$, $x_2=(V-U)t$, this requirement leads to the equations

$$\beta=-l \tag{39a}$$

$$\alpha=\frac{m+1/M-c}{1-r} \tag{39b}$$

Equations (32b) and (39b) yield

$$c=\frac{m+1/M}{1-\frac{V^2}{a_2^2}(1-r)^2} \tag{39c}$$

and equation (32c) may be solved for d:

$$d=\sqrt{\frac{-\frac{V^2}{a_2^2}c^2+\alpha^2+l^2}{1-\frac{V^2}{a_2^2}(1-r)^2}} \tag{39d}$$

The vorticity wave is expected to involve a linear combination of the profile functions f and g, just behind the shock, and is not expected to change its form subsequently, because it is not time-dependent in the x_2,y coordinate system and therefore cannot attentuate, as does the pressure wave, or otherwise change character. A matching procedure at the shock yields, as previously, the argument given in expression (25).

The remaining analysis parallels that following expression (25) in the paragraph Solution when $0<\psi_1<\psi_{cl}$ or $\psi_{cu}<\psi_1<\pi$.

$$\frac{\rho_2}{R_2}=\frac{1}{\gamma}\frac{p_2}{P_2}+Q^{(1)}f\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right)+Q^{(2)}g\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right) \tag{40}$$

$$\frac{u_2}{V}=F^{(1)}\Phi^{(1)}+F^{(2)}\Phi^{(2)}+G^{(1)}f\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right)+ G^{(2)}g\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right) \tag{41}$$

$$\frac{v_2}{V}=H^{(1)}\Phi^{(1)}+H^{(2)}\Phi^{(2)}+I^{(1)}f\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right)+ I^{(2)}g\left(\frac{\frac{m+1/M}{1-r}x_2-ly}{\lambda_1}\right) \tag{42}$$

$$\frac{\xi_t}{V}=L^{(1)}f\left[\frac{(m+1/M)Vt-ly}{\lambda_1}\right]+L^{(2)}g\left[\frac{(m+1/M)Vt-ly}{\lambda_1}\right] \tag{43}$$

$$-\xi_v=\frac{n}{1+1/Mm}\left\{L^{(1)}f\left[\frac{m+1/M)Vt-ly}{\lambda_1}\right]+ L^{(2)}g\left[\frac{(m+1/M)Vt-ly}{\lambda_1}\right]\right\} \tag{44}$$

The various unknown constants remain to be determined algebraically through equations (7), the incompressible continuity equation, and the shock equations (5), as before. These equations suffice to determine a greater number of constants than were required in the previous case because the functions f, g, $\Phi^{(1)}$, and $\Phi^{(2)}$ and their derivatives form two separate groups of functions whose coefficients may be separately equated. Details of this procedure are provided in appendix B.

SOLUTION OF PROBLEM B

The analysis of problem B (sound wave overtaking shock from behind) is identical in all essential respects to that of problem A when only a simple refracted sound wave is involved. The only differences which arise are the slightly different matching of arguments at the shock and a slightly different form assumed by the shock relations (5).

The equation (18) is adopted in the present case to represent the reflected sound wave. Matching the argument of this expression with that of the initial disturbance (eq. (14)) yields

$$\frac{1}{\lambda_{22}}\{[\alpha V(1-r)+a_2]t+\beta y\}=\frac{1}{\lambda_{21}}\{[mV(1-r)+a_2]t-ly\} \tag{45}$$

whence,

$$\alpha=\frac{1}{1-r}\left\{\frac{\lambda_{22}}{\lambda_{21}}\left[m(1-r)+\frac{a_2}{V}\right]-\frac{a_2}{V}\right\} \tag{46a}$$

$$\beta=-l\frac{\lambda_{22}}{\lambda_{21}} \tag{46b}$$

Equations (19) and (46) provide a quadratic equation for $\lambda_{22}/\lambda_{21}$, the useful solution being:

$$\frac{\lambda_{22}}{\lambda_{21}}=\frac{\frac{a_2^2}{V^2}-(1-r)^2}{(1-r)^2+2\frac{a_2}{V}m(1-r)+\frac{a_2^2}{V^2}} \tag{46c}$$

The other solution is $\lambda_{22}/\lambda_{21}=1$, corresponding to the incident wave itself.

The right side of equation (45) is used as the argument in the expressions for ξ_t and ξ_v:

$$\left.\begin{aligned}\xi_t&=VLf\{[mV(1-r)+a_2]t-ly\}\\ \xi_v&=-\frac{n}{1-r+\frac{a_2}{mV}}Lf\{[mV(1-r)+a_2]t-ly\}\end{aligned}\right\} \tag{47}$$

The right side of equation (45) also yields the argument of the vorticity wave, when the substitution $t=x_2/(V-U)$ is made:

$$\frac{\left[m+\frac{a_2}{V(1-r)}\right]x_2-ly}{\lambda_{21}} \tag{48}$$

Equations (26), (27), and (28) may be adopted to complete the description of the flow, except that expression (48) must be used for the argument of the vorticity wave. The analysis is completed in appendix B.

SOLUTION OF PROBLEM C

The only differences between problem A and problem C (stationary vorticity wave overtaken by shock) involve the shock equations (5) and the matching of arguments at the shock. The difference in matching is due to the fact that in the present case, the disturbance is stationary, while in problem A, the disturbance moves with velocity a_1. Accordingly, when $0<\psi_1<\psi_{cl}$, equations (18), (21), (23), and (25) through (29) apply directly to the present case if the quantity $1/M$ is omitted wherever it appears explicitly. When $\psi_{cl}<\psi_1<\psi_{cu}$, equations (36), (37), and (39) through (44) may also be adopted, again by omitting terms proportional to $1/M$. The remaining details of the analysis are provided in appendix B.

As previously mentioned, an initial disturbance of type C may have a third fluctuating velocity component parallel to the shock, which might be represented as follows:

$$\frac{w_1}{V}=A_5f\left(\frac{mx_1-ly}{\lambda_1}\right)$$

(see eqs. (15)). The amplitude A_5 is arbitrary, within the limitations of linear analysis. This disturbance will pass through the shock unaffected and become part of the steady vorticity wave behind the shock. Thus,

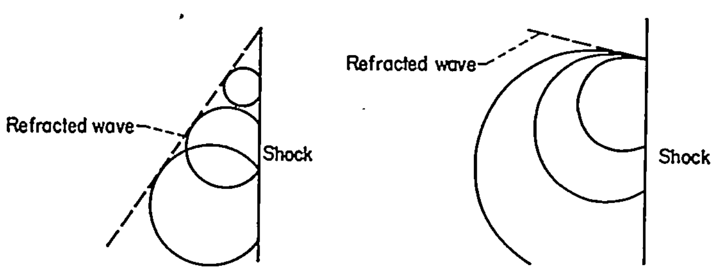
$$\frac{w_2}{V}=A_5 f\left(\frac{\frac{m}{1-r}x_2-ly}{\lambda_1}\right)$$

The shock front itself will not be affected by this purely transverse disturbance.

SINGULARITIES AT ψ_{cl} AND ψ_{cu}

Equation (23), which applies in the refracted sound-wave solution, contains a radical which vanishes at ψ_{cl} and ψ_{cu}, and becomes imaginary when $\psi_{cl}<\psi_1<\psi_{cu}$. It may be shown that the quantity under the radical vanishes with a nonzero slope. Accordingly, the quantity λ_2/λ_1, though finite, has a half-order singularity at ψ_{cl} and ψ_{cu}. Furthermore, this quantity is involved in all the formulas characterizing the refracted sound wave (see appendix B). In the range $\psi_{cl}\leq\psi_1\leq\psi_{cu}$, the attenuation coefficient d (eq. (39d)) vanishes with half-order singularities at ψ_{cl} and ψ_{cu} and similarly affects the remainder of the analysis.

The reason for this singular behavior may be inferred from figure 3. According to figure 3(a), when $0<\psi_1<\psi_{cl}$, the refracted sound wave is the envelope of an imagined succession of cylindrical waves, as shown in the following sketch.



At ψ_{cl}, however, according to figure 3(b), the cylindrical waves all meet at a common point of tangency, as shown in the sketch. Therefore, successive waves reinforce at one point, giving a singularity of the flow. This singularity, of course, depends on the fact that the theory is linear. An exact analysis would presumably show steep, though not singular, flow gradients.

This situation is similar to that arising in the linearized analysis of compressible flow about bodies: as the Mach number of the flow approaches 1, the Mach waves have a common point of tangency at the nose, and the wave drag shows a reciprocal half-order singularity in Mach number. In the present case, the physical quantities remain finite, but have infinite rates of variation with ψ_1.

The preceding discussion applies to problems A and C, but not to B.

RESULTS AND DISCUSSION

In the following paragraphs, the results (presented in graphical form) will be described for problems A and B. Results for problem C are presented in reference 6. The solution for each problem has essentially three elements:

1. Disturbance of the shape of the shock front
2. Characteristics of the isentropic pressure wave behind the shock
3. Characteristics of the steady vorticity wave behind the shock.

Computations have been carried out for three Mach numbers, 1, 1.5, and infinity. Of course, the case $M=1$ is really degenerate, because the shock is then a weak sound wave, and therefore the interaction with the incident disturbance is obtained by linear superposition, the initial disturbance passing through the "shock" with no change. Also, $M=1$ is a singular point, because the range of angles of incidence of initial disturbance for which the attenuating pressure wave appears in problems A and C vanishes with a half-order singularity (fig. 4), and in problem B, because the incident wave is unable to overtake the "shock" moving with sonic velocity.

The results will show that the critical angles ψ_{cl} and ψ_{cu} are also singular points. In many instances there are not a sufficient number of points near the singular points for which computations have been made, so that not all the curves can be faired with complete confidence. For this reason, the computed points are shown circled, so that the basis for the fairing will be clear in each case.

No mention will be made of the temperature disturbance behind the shock, which may be obtained directly from equation (4) if the pressure and density disturbances are known.

All disturbance quantities found by the linearized analysis of the present report will be proportional to the intensity of the incident wave. Therefore, results are divided by the pressure amplitude of the incident wave A_3 (see eqs. (12a) and (14a)).

PROBLEM A—SHOCK OVERTAKING SOUND WAVE

1. Shock front disturbance.—In figure 5 are shown the amplitudes $L^{(1)\prime}\equiv L^{(1)}/A_3$ and $L^{(2)\prime}\equiv L^{(2)}/A_3$ of the incremental velocity of the shock front, due to the interaction. From equations (29), (43), and (44) these amplitudes are associated, respectively, with the functions f (which defines the profile shape of the incident wave) and g (which is an additional profile function arising when $\psi_{cl}<\psi_1<\psi_{cu}$) to give the actual incremental velocity. The variations with ψ_1 are quite extreme, particularly at the critical angles, where, in fact, there are half-order singularities. The variations with Mach number are equally severe.

When $\psi_1=0$ (incident wave moving parallel to and toward the shock), the figure shows that the shock front is retarded by a pressure wave. In the case $M=1$, this is because the velocity in an incident compressive sound wave, relative to which the shock (really a weak compression wave) propagates as it moves through the disturbance, is directed against the shock. When the incident compression wave moves in the same direction as the shock ($\psi_1=\pi$), the shock front is speeded up for low shock Mach numbers; at $M=1$, this is true because the incremental velocity due to the incident wave is in the same direction as the "shock" movement. Whether $\psi_1=0$ or π, there is a smaller accelerating effect due

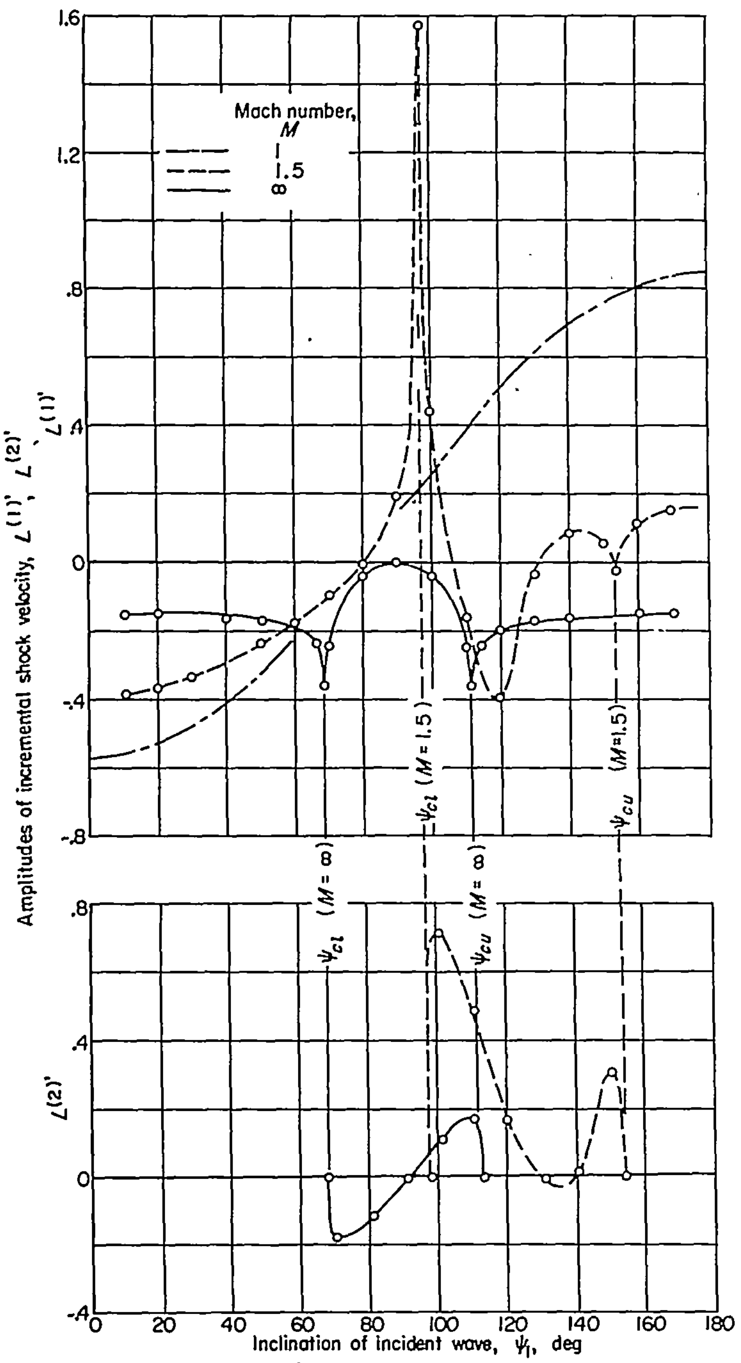


FIGURE 5.—Problem A: shock front disturbance (shock overtaking sound wave).

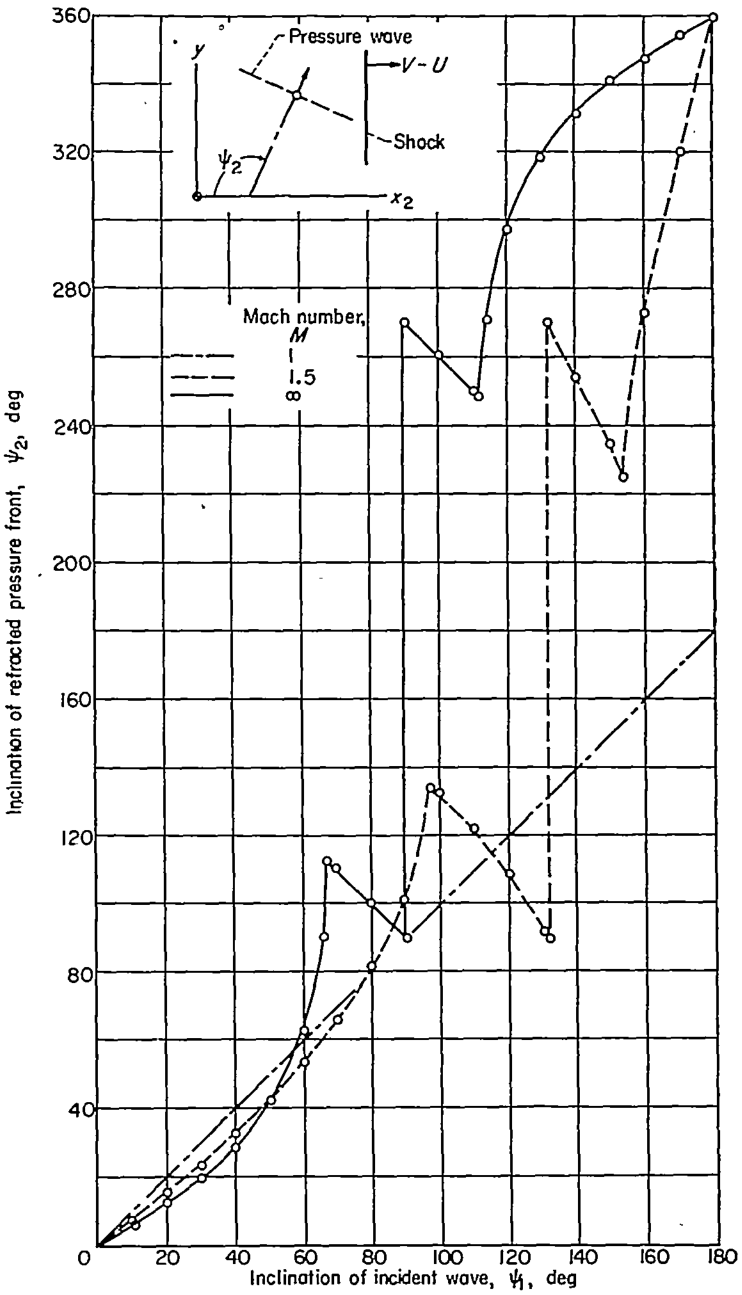


FIGURE 6.—Problem A: inclination of refracted pressure front (shock overtaking sound wave).

to the higher velocity of sound in the incident compression wave—that is why the curve for $M=1$ is not perfectly antisymmetrical about $\psi_1=\pi/2$.

When $M=\infty$, the curves are symmetrical about $\psi_1=\pi/2$ because the incremental sound-wave velocity is vanishingly small compared with the shock velocity.

For each value of M there is a value of ψ_1 for which the shock intersects the incident sound wave permanently at one point on the traveling wave, and the problem becomes essentially steady, so that the increment in shock velocity vanishes (though a steady displacement occurs). This happens when $a_1=-mV$ or $\psi_1=\sec^{-1}(-M)$, yielding 131.8° when $M=1.5$, and 90° when $M=\infty$, and is the case of steady interaction of a Mach wave and a normal shock which has been treated by Adams (ref. 2).

2. **Characteristics of pressure wave.**—(a) Inclination of refracted pressure front: Figure 6 shows the angle between the directions of propagation of the shock and the refracted pressure wave behind the shock. In view of equations (18) and (31), this quantity is given by the equation

$$\psi_2=\cot^{-1}(-\alpha/\beta)$$

Of course, when $\psi_{cl}<\psi_1<\psi_{cu}$, this wave is not a sound wave, and the inclination shown refers to a front parallel to which physical quantities depend only on distance behind the shock. Outside the range $\psi_{cl}<\psi_1<\psi_{cu}$, the pressure wave is a sound wave. When $M=\infty$, the curve is symmetrical about $\psi_1=\pi/2$.

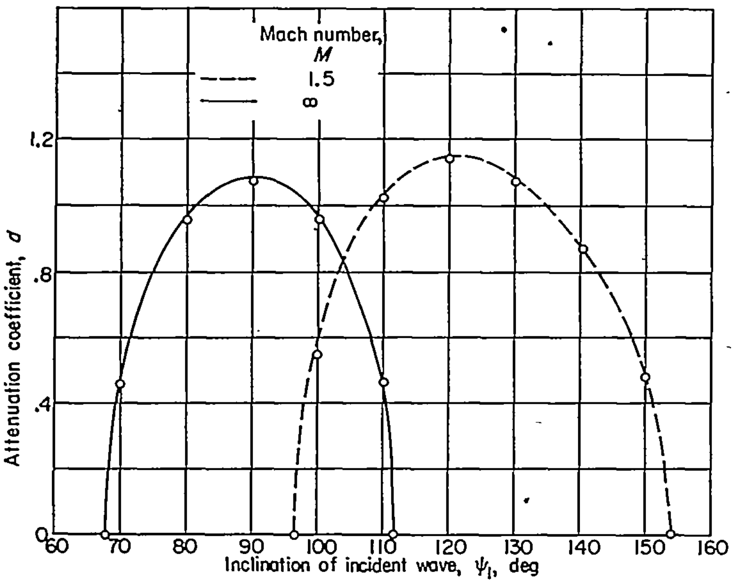


FIGURE 7.—Problem A: attenuation of pressure wave behind shock (shock overtaking sound wave).

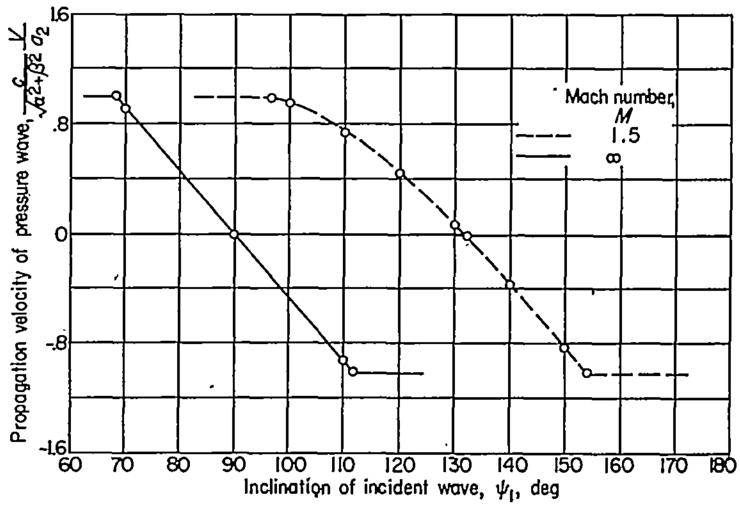


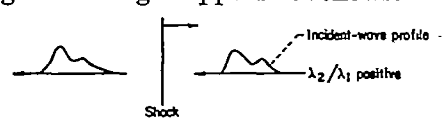
FIGURE 8.—Problem A: propagation velocity of pressure wave behind shock (shock overtaking sound wave).

(b) Coefficient of attenuation of pressure wave: Figure 7 shows the coefficient d which appears in equations (31) and (39d). This quantity vanishes at ψ_{cl} and ψ_{cu}, indicating that the sound-wave solution and attenuating wave solution meet continuously at the critical angles. When M is either 1.5 or ∞, the maximum value of d is about 1. This implies a rather rapid attenuation—if $f(\zeta)=\sin 2\pi\zeta$, it has been shown (eqs. (38)) that the wave attenuates as $exp(-2\pi\eta)$. From the definition of η (eq. (31)), when $d\approx 1$, the attenuation factor becomes e^{-1} at a distance behind the shock approximately equal to $1/2\pi$ times the wave length of the incident sound wave.

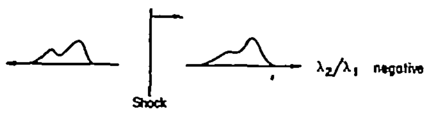
(c) Propagation velocity of pressure wave when $\psi_{cl}<\psi_1<\psi_{cu}$: Figure 8 shows the quantity c of equations (31) and (39c), combined with other quantities to give propagation velocity as a fraction of the speed of sound a_2. Since the solution when $\psi_{cl}<\psi_1<\psi_{cu}$ meets the sound-wave solution continuously, the propagation velocity is equal to the speed of sound at ψ_{cl} and ψ_{cu}. The change of sign of c is taken into account in figure 6 by the 180° shift of direction shown at the angle for which $c=0$.

(d) Ratio of scales of pressure waves behind and ahead of shock: Figure 9 shows the quantity λ_2/λ_1 (eq. (23)), which was defined only for the sound-wave solution. However, inspection of equation (31) shows that the equivalent quantity when $\psi_{cl}<\psi_1<\psi_{cu}$ is $(\alpha^2+\beta^2)^{-1/2}$.

The reversal of sign of λ_2/λ_1 signifies a reversal of the direction of propagation of the pressure wave relative to the shape of the incident wave. For example, when $\psi_1=0$, the incoming and outgoing waves might appear as follows:



whereas when $\psi_1=\pi$, they would appear thus:



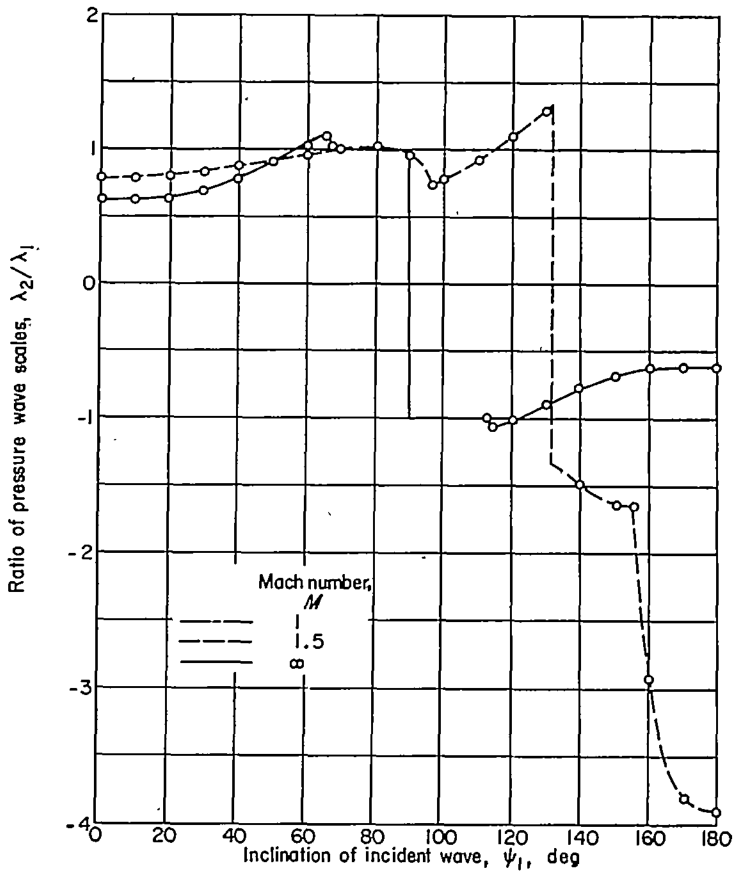


FIGURE 9.—Problem A: ratio of scales of pressure wave upstream and downstream of shock (shock overtaking sound wave).

The difference between these two cases consists of a difference in sign in the arguments of the refracted wave in the two cases, and arises formally in the present analysis as a change in sign of λ_2/λ_1.

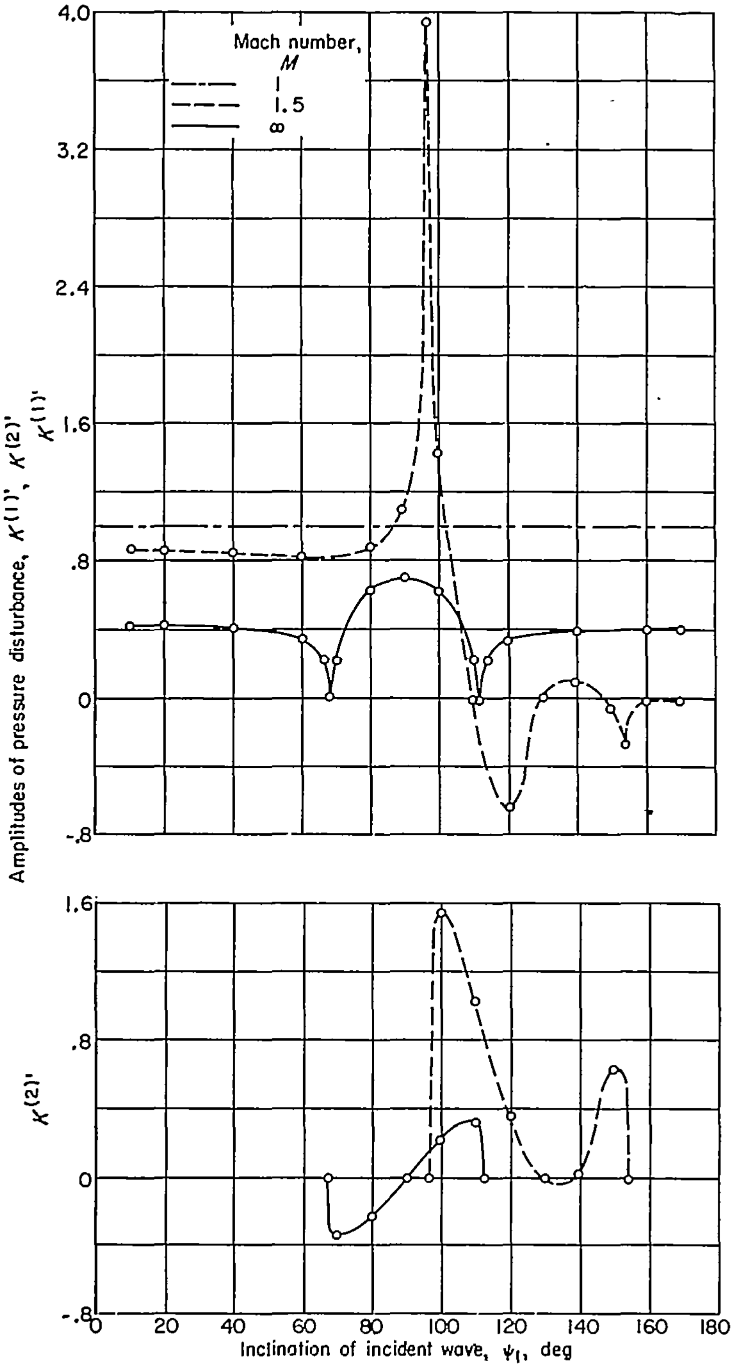


FIGURE 10.—Problem A: amplitudes of pressure disturbance behind shock (shock overtaking sound wave).

When $M=1.5$, the magnitude of λ_2/λ_1 is greater for $\psi_1=\pi$ than for $\psi_1=0$, because, when the incident wave is traveling in the same direction as the shock, the shock requires a longer time to traverse the incident wave than when the two waves travel in opposing directions.

(e) Amplitudes of pressure disturbance behind shock: Figure 10 shows the coefficients (of f, or of $\Phi^{(1)}$ and $\Phi^{(2)}$ when $\psi_{cl}<\psi_1<\psi_{cu}$) which describe the pressure wave behind the shock: $K^{(1,2)'}\equiv K^{(1,2)}/A_3$ (see eqs. (18) and (36)). As in figure 5, the flow is shown to be singular at the two critical angles and to vary markedly with both ψ_1 and M. At $M=1.5$, near $\psi_1=\pi$, the refracted sound wave is seen to be very weak.

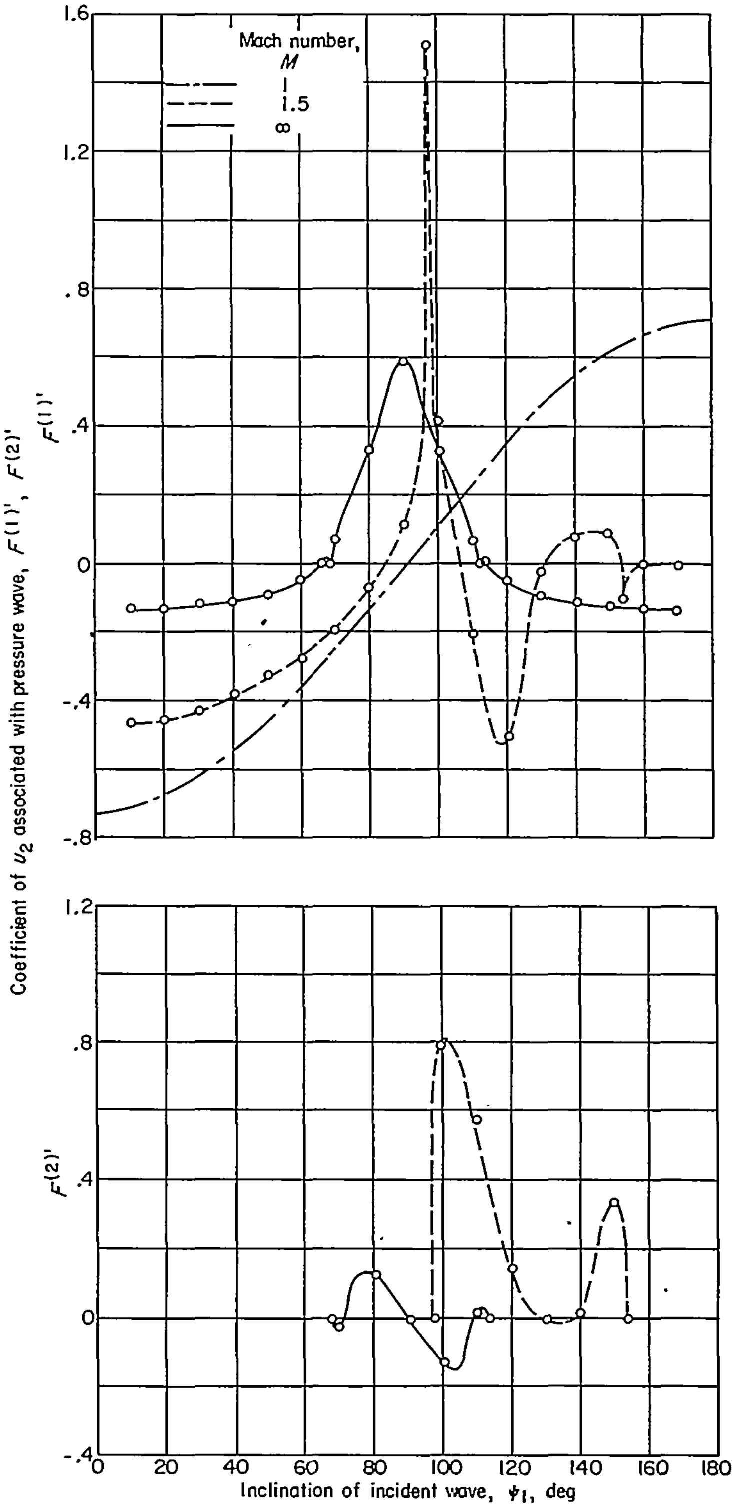


FIGURE 11.—Problem A: coefficient for part of u_2 associated with pressure wave (shock overtaking sound wave).

Since the pressure wave is isentropic, the coefficients of the corresponding part of the density variation are equal to $1/\gamma$ times the pressure coefficient.

(f) and (g) Coefficients for velocity components in pressure wave: Figures 11 and 12 show the coefficients for the velocity components associated with the isentropic pressure wave behind the shock (eqs. (27), (28), (41), and (42)). The velocity resultant is longitudinal with respect to the direction of propagation of the pressure wave, except when $\psi_{cl}<\psi_1<\psi_{cu}$.

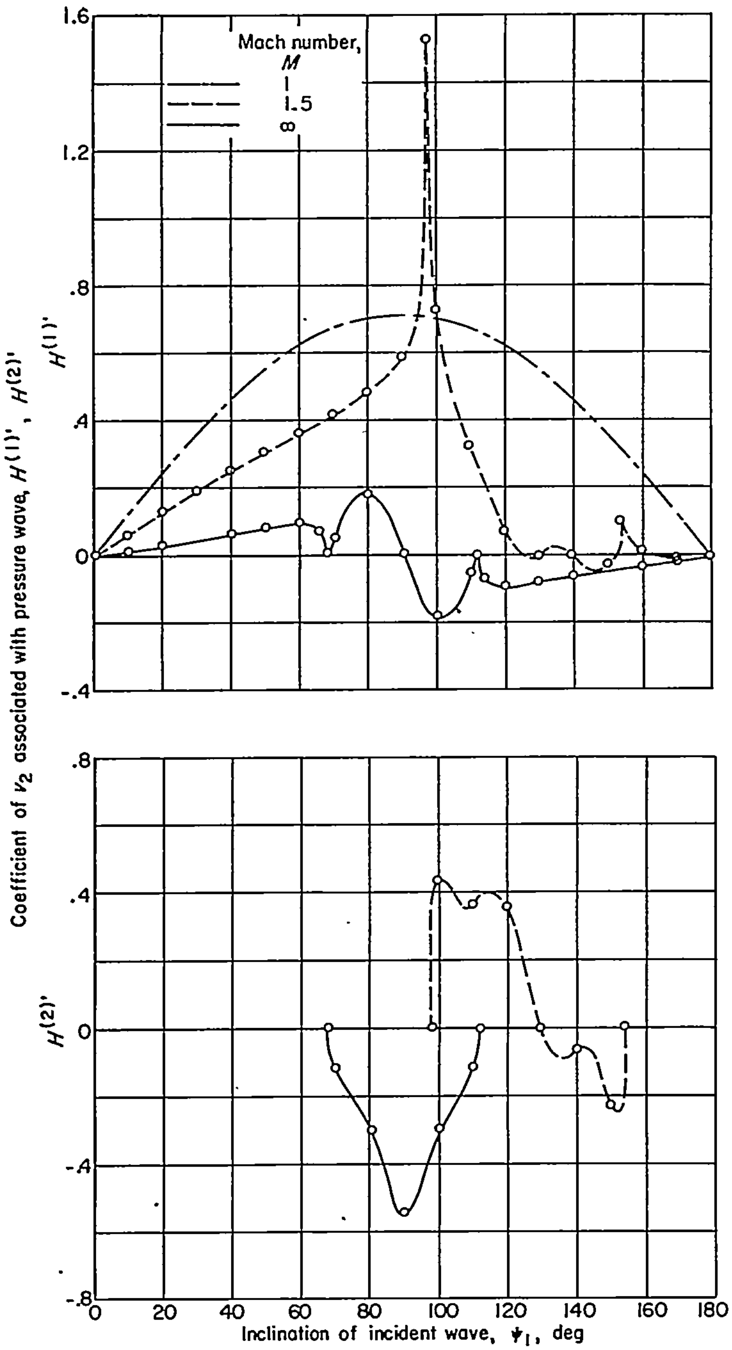


FIGURE 12.—Problem A: coefficient for part of v_2 associated with pressure wave (shock overtaking sound wave).

3. Characteristics of steady vorticity wave behind shock.—(a) Inclination of the steady vorticity wave: Figure 13 shows the inclination ψ_3 of the vorticity front behind the shock. From equation (25),

$$\psi_3 \equiv \cot^{-1}\left[\frac{1+1/mM}{n(1-r)}\right]$$

(b) Ratio of scale of vorticity wave to that of incident sound wave: From equations (12a) and (25), this is

$$(l\sqrt{\cot^2\psi_3+1})^{-1}=\frac{\sin\psi_3}{l}$$

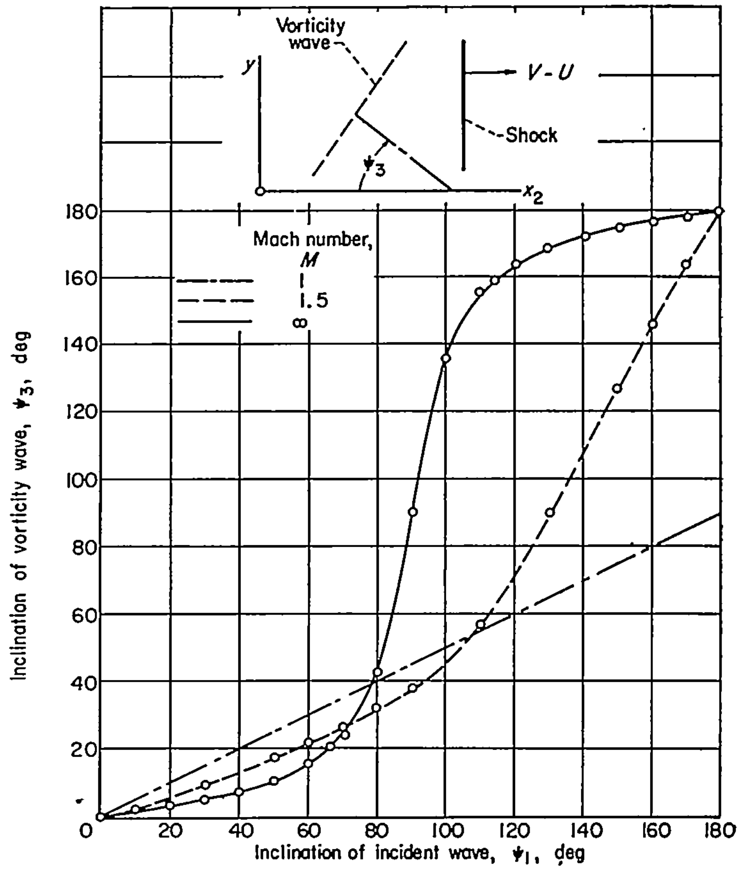


FIGURE 13.—Problem A: inclination of vorticity wave behind shock (shock overtaking sound wave).

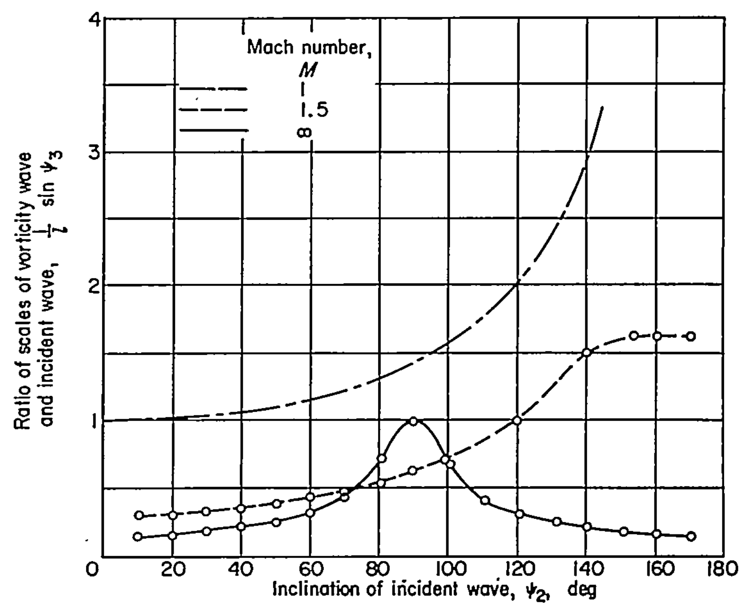


FIGURE 14.—Problem A: ratio of scales of vorticity wave behind shock to that of incident sound wave (shock overtaking sound wave).

which is shown in figure 14. When $M=1$ and $\psi_1=\pi$, the scale ratio goes to ∞ because the "shock" is unable to overtake the incident wave.

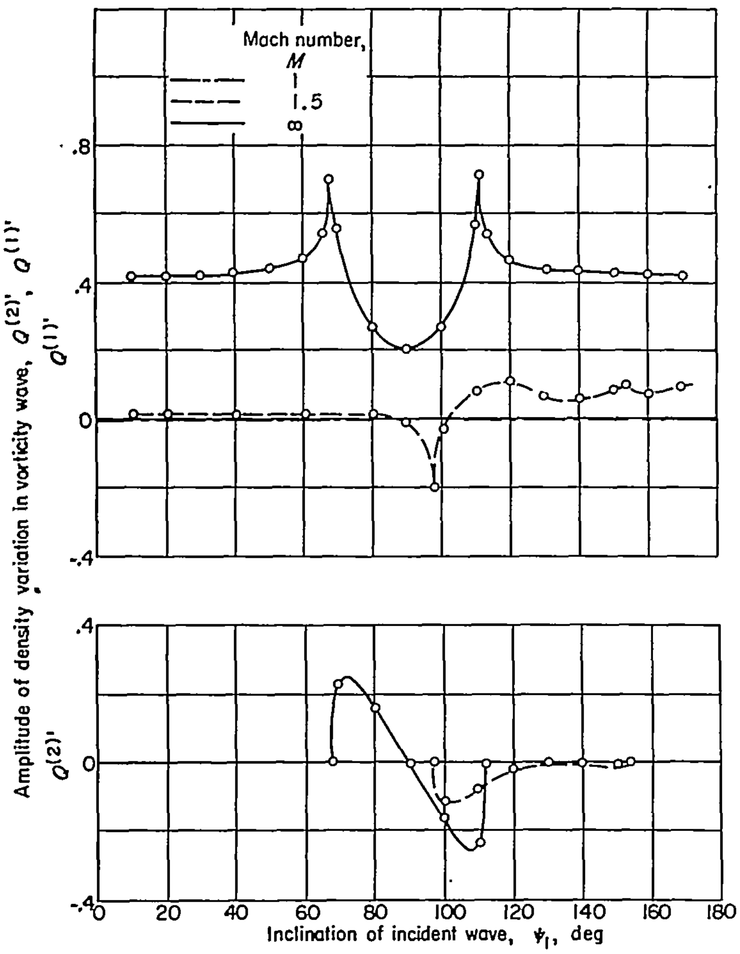


FIGURE 15.—Problem A: amplitude of density variation in vorticity wave behind shock (shock overtaking sound wave).

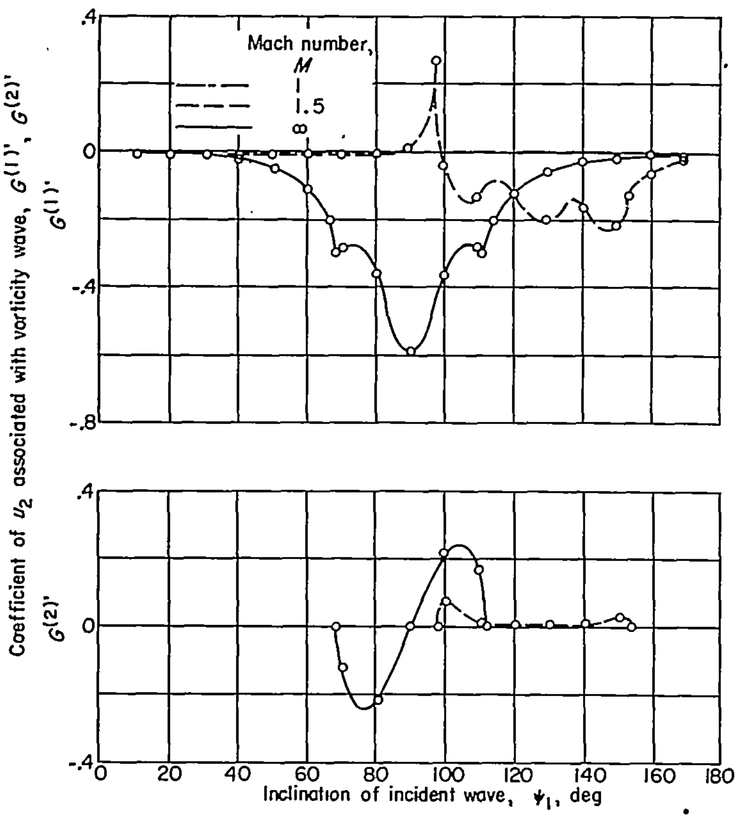


FIGURE 16.—Problem A: coefficient for part of u_2 associated with vorticity wave (shock overtaking sound wave).

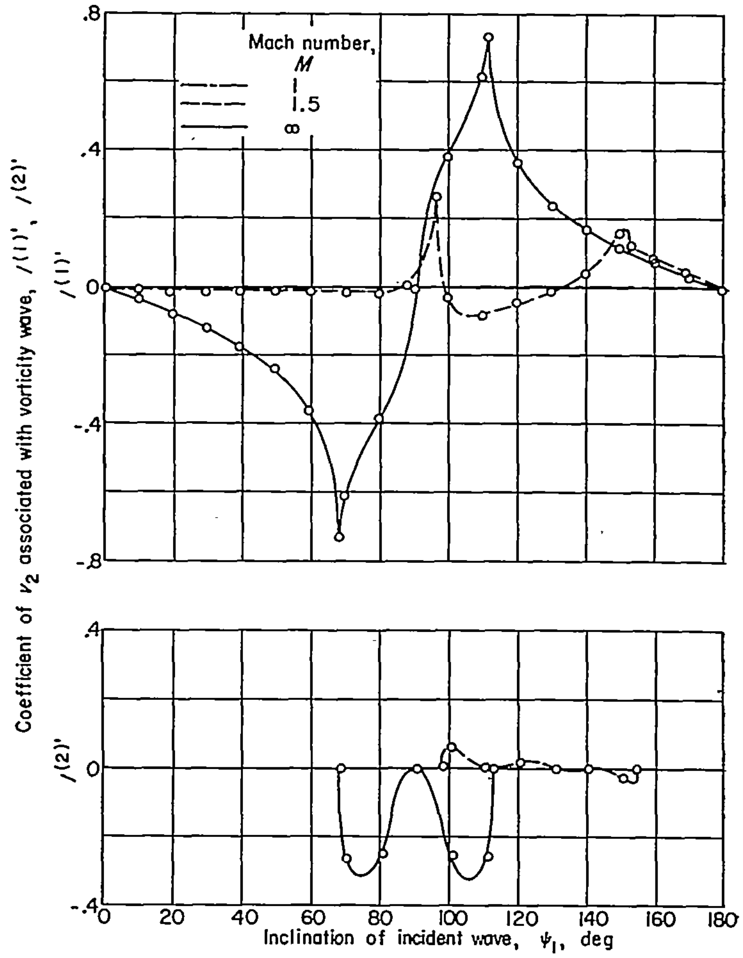


FIGURE 17.—Problem A: coefficients for part of v_2 associated with vorticity wave (shock overtaking sound wave).

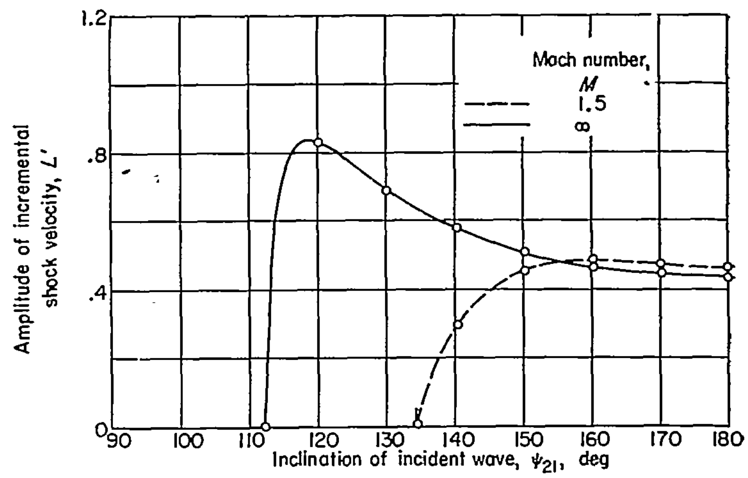


FIGURE 18.—Problem B: shock-front disturbance (sound wave overtaking shock).

(c) Amplitude of density variation in vorticity wave behind shock: Figure 15 shows $Q^{(1)\prime}$ and $Q^{(2)\prime}$ (eqs. (26) and (40)). Apparently, when M is of order 1.5, the vorticity wave is very weak.

(d) and (e) Coefficients for velocity components in vorticity wave: Figures 16 and 17 show the coefficients for the transverse velocity field associated with the vorticity wave

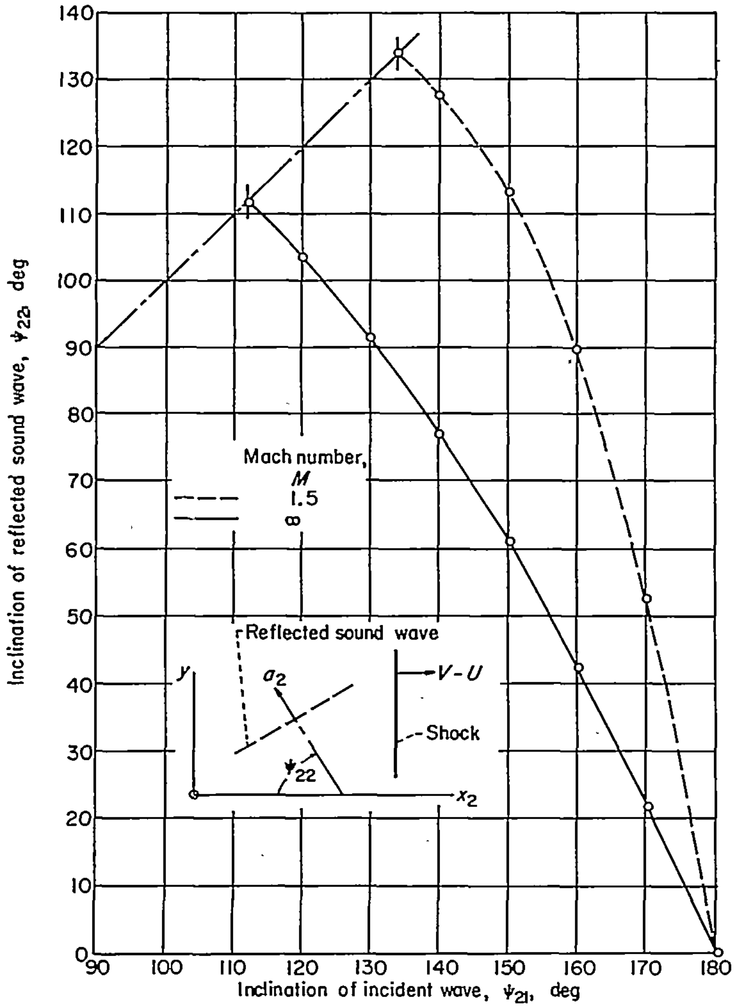


FIGURE 19.—Problem B: inclination of reflected sound wave (sound wave overtaking shock).

PROBLEM B—SOUND WAVE OVERTAKING SHOCK FROM BEHIND

Consistent with the previous analysis, computations have been carried out only for values of ψ_{21} (incident-wave inclination) sufficiently close to 180° that the component of propagation velocity in the direction of the motion of the shock is greater than the velocity of the shock relative to the fluid behind. These values of ψ_1 are 134.5° and 112.2° for M=1.5 and ∞, respectively.

1. **Shock-front disturbance.**—Figure 18 shows the amplitude of the incremental shock-front velocity $L'=L/A_3$. When ψ_{21} is near 180°, an incident pressure wave displaces the shock ahead.

2. **Characteristics of reflected sound wave.**—The downstream pressure wave in problem B is always a simple sound wave.

(a) Inclination of reflected wave: The inclination $\psi_{22}=\cot^{-1}(-\alpha/\beta)$ (see eq. (18)) is shown in figure 19. In effect, the incident and reflected waves coalesce into a single wave at the critical angles.

(b) Ratio of scales of reflected and incident sound waves: This ratio is given by equation (46c) and is shown in figure 20. At $\psi_{21}=\pi$, the scale ratio is greater when M=1.5 than when M=∞, just as in problem A, and for the same reason. At the critical angles, the ratio becomes 1, because the incident and reflected waves coalesce.

(c) Amplitude of reflected sound wave: The pressure

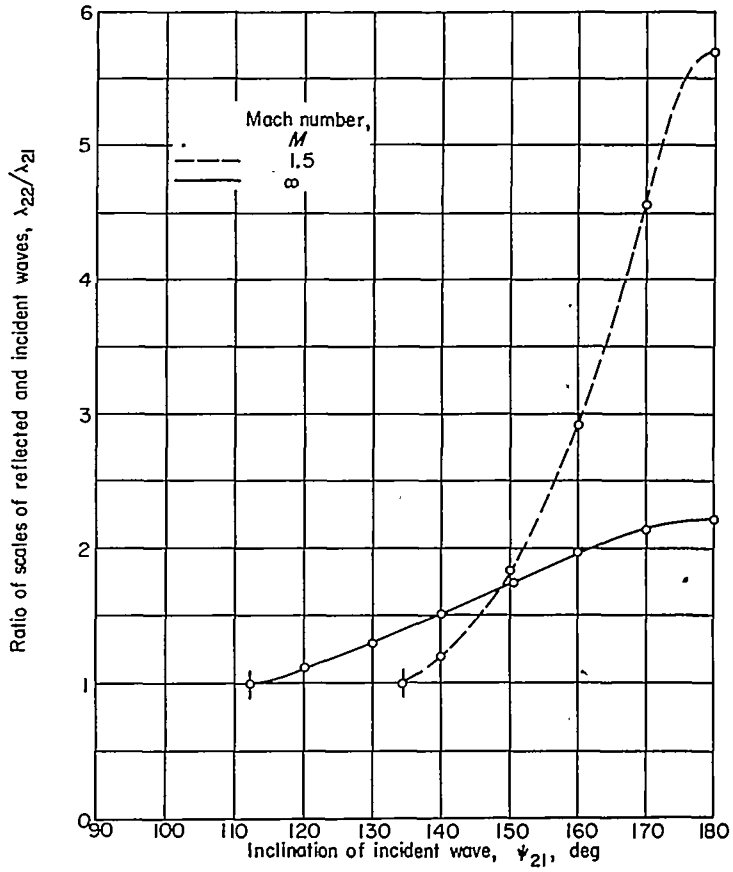


FIGURE 20.—Problem B: ratio of scales of reflected and incident sound waves (sound wave overtaking shock).

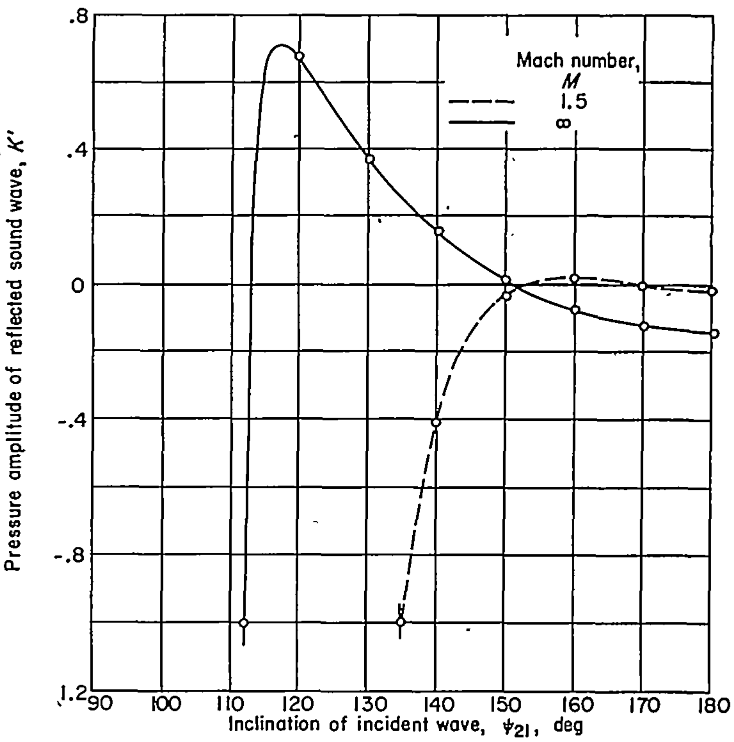


FIGURE 21.—Problem B: amplitude of reflected sound wave (sound wave overtaking shock).

amplitude $K'\equiv K/A_3$ (eq. (18)) is shown in figure 21. At $\psi_{21}=\pi$, an incident compression wave reflects as an expansion wave with a strength which is greater the higher the Mach number, but always less than that of the incident wave.

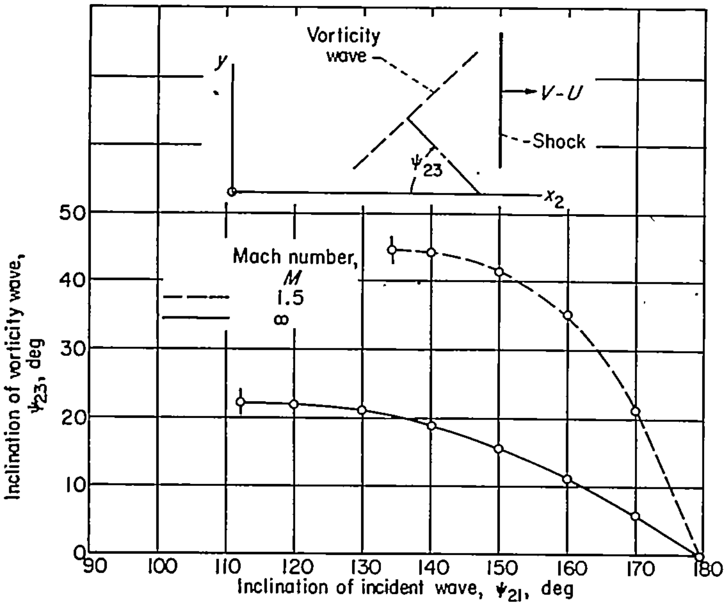


FIGURE 22.—Problem B: inclination of vorticity wave produced behind shock (sound wave overtaking shock).

The velocity components associated with the reflected sound wave are obtained simply from K' and are therefore not plotted.

3. **Characteristics of steady vorticity wave.**—The coefficient Q' of the density fluctuation in the vorticity wave is simply proportional to L' (combining eqs. (B26) and (B27) of appendix B) and is therefore not shown in a figure.

$$\text{At } M=1.5,\ Q'=-0.147L'$$

$$\text{At } M=\infty,\ Q'=-1.43L'$$

(a) Inclination of vorticity wave: From equation (47), the inclination is given by

$$\psi_{23}=\cot^{-1}\left\{\frac{1}{n}\left[1+\frac{a_2/V}{m(1-r)}\right]\right\}$$

which is plotted in figure 22.

(b) Ratio of scale of vorticity wave to that of incident sound wave: As in problem A, this ratio is given by $(1/l)\sin\psi_{23}$ and is shown in figure 23. As for the reflected sound wave, the scale ratio is larger for M nearer one.

(c) and (d) Velocity variations in vorticity wave: Coefficients G' and I' of the transverse velocity fluctuation (eqs. (27) and (28)) in the vorticity wave are shown in figures 24 and 25.

4. **Wave reflection at critical angle.**—The analysis and figures show that at the critical angle the incident and reflected sound waves coalesce to form a single sound wave. This statement may be interpreted to mean that a sound wave incident at the critical angle reflects as a steady vorticity wave only. Therefore, the shock disturbance and vorticity wave characteristics may be expressed in terms of the pressure amplitude of a single incident sound wave of strength $A_3+K=A_3(1+K')$.

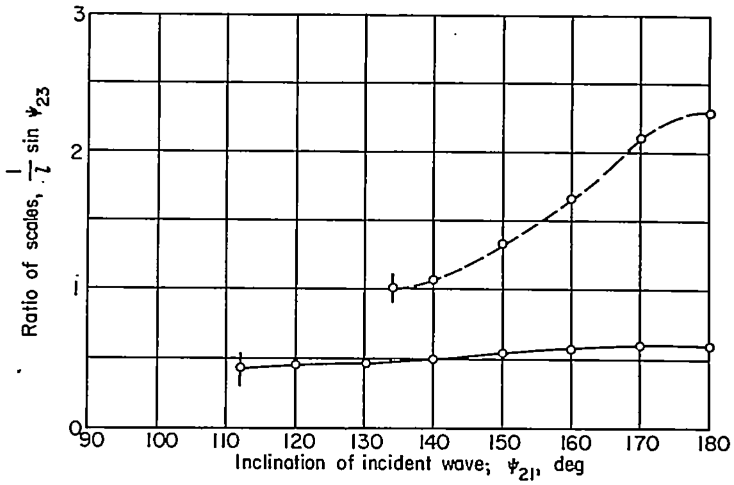


FIGURE 23.—Problem B: ratio of scales of vorticity wave behind shock and incident sound wave (sound wave overtaking shock).

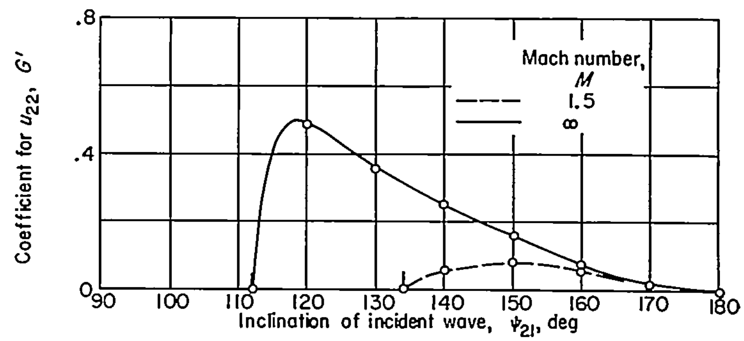


FIGURE 24.—Problem B: coefficient for part of u_{22} associated with vorticity wave (sound wave overtaking shock).

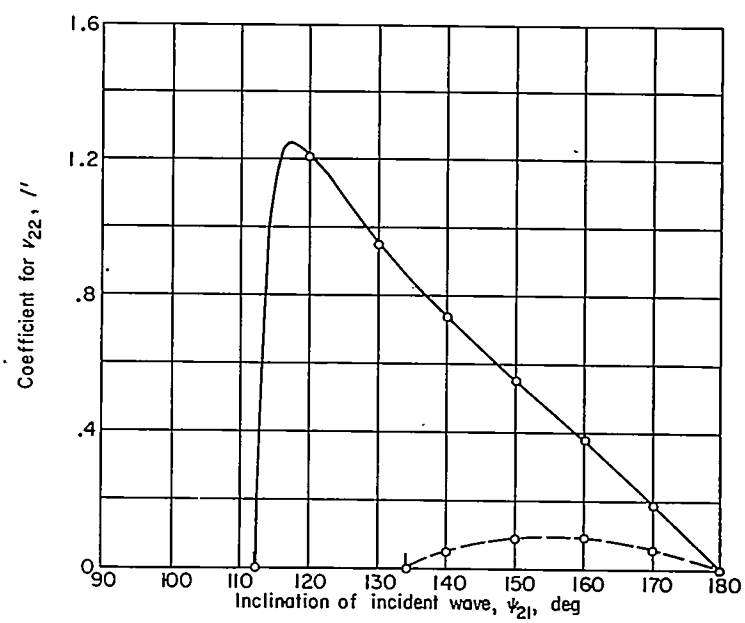


FIGURE 25.—Problem B: coefficient for part of v_{22} associated with vorticity wave (sound wave overtaking shock).

CONCLUDING REMARKS

In principle, the interaction of a shock with any weak flow field may be obtained by first constructing the initial flow field as a linear combination of plane waves of varying strength and orientation. From the present analysis the interaction of each constituent wave with the shock may be found. Assembling the resulting waves behind the shock then would yield the desired solution. Regrettably, the formulas for the interaction depend on the angle of incidence in a rather complicated way and it would in general be difficult to evaluate explicitly integrals in which these formulas are used for the distribution functions. Numerical procedures could be used for this purpose, though a technique would be required for dealing with the singularities at the critical angles ψ_{cl} and ψ_{cu}.

The nature of the solution of problem B perhaps requires a clarification, in that the angles of incidence ψ_{21} have been restricted to the range for which the sound wave overtakes the shock. For purposes of superposition, however, all incidence angles must be considered. Inspection of figures 6 to 12 shows that a sound wave of any incidence angles between 0 and π may be identified either as an incident wave or as a reflected wave, in the sense of the analysis. For purposes of linear superposition, the distinction between incident and reflected waves is of no significance. The point of view may be adopted that whatever its angle, a constituent wave of the initial flow has associated with it another sound wave of another angle and a steady vorticity wave. The notion of cause and effect is not needed.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, *February 8, 1954.*

APPENDIX A

NOTATION

The following symbols are used in this report:

A_1, A_2, A_3, A_4	coefficients of incident disturbance (eqs. (12a), (14a), or (15))
a	velocity of sound
B_1, B_2	coefficients in shock relations (eqs. (6))
C_1, C_2	coefficients in shock relations (eqs. (6))
c	dimensionless velocity of propagation of pressure wave behind shock (eq. (31) or (39c))
c_v	specific heat at constant volume
D_1, D_2	coefficients in shock relations (eqs. (6))
d	coefficient of attenuation of pressure wave behind shock (eq. (31) or (39d))
F	coefficient of part of u_2 associated with pressure wave (eq. (27) or (41))
f	profile function of incident wave (eqs. (12a), (14a), or (15))
G	coefficient of part of u_2 associated with vorticity wave (eq. (27) or (41))
g	additional profile function appearing behind shock $\left(=\frac{1}{\pi}P.V.\int_{-\infty}^{\infty}f(\tau)(\tau-\zeta)^{-1}d\tau, \text{ eq. (35a)}\right)$
H	coefficient of part of v_2 associated with pressure wave (eq. (28) or (42))
h_n	functions involved in solution when $\psi_{cl}<\psi_1<\psi_{cu}$ (eqs. (B19), appendix B)
I	coefficient of part of v_2 associated with vorticity wave (eq. (28) or (42))
J	gas constant (eq. (4))
K	coefficient of p_2 (eq. (18) or (36))
L	coefficient for $\xi(y,t)$ (eqs. (29) or (43))
l	$\sin\psi_1$ or $\sin\psi_{21}$
M	Mach number of shock ($=V/a_1$)
m	$\cos\psi_1$ or $\cos\psi_{21}$
n	$\tan\psi_1$ or $\tan\psi_{21}$
P	mean static pressure
p	perturbation in static pressure
Q	coefficient for density fluctuation in vorticity wave (eq. (26) or (40))
R	mean gas density
r	ratio of U to V
t	time
U	mean velocity of gas behind shock (fig. 1)
u	perturbation of velocity component in x-direction (fig. 1)
V	mean velocity of propagation of shock in gas at rest (fig. 1)
v	perturbation of velocity component in y-direction
x_1, x_2	coordinates measured in the direction of the shock propagation, relative to which the gas is (on the average) at rest ahead of and behind the shock, respectively (figs. 1 and 2)
y	coordinate orthogonal to x_1 or x_2 (figs. 1 and 2)
α, β	functions defining pressure front (eqs. (21) or (39))
γ	ratio of specific heats (≈ 1.4 for air)
ζ	variable upon which pressure wave depends when $\psi_{cl}<\psi_1<\psi_{cu}$ (eqs. (31))
η	variable upon which pressure wave depends when $\psi_{cl}<\psi_1<\psi_{cu}$ (eqs. (31))
Θ	mean static temperature
θ	fluctuation in static temperature
λ	scale of pressure wave (eq. (23))
ξ	displacement of shock front (fig. 1)
ρ	fluctuation in gas density
σ, χ	functions appearing in solution for L (eqs. (B9) or (B31), appendix B)
Φ	function associated with pressure wave when $\psi_{cl}<\psi_1<\psi_{cu}$ (eqs. (37))
φ	velocity potential associated with pressure wave when $\psi_{cl}<\psi_1<\psi_{cu}$
ψ_1, ψ_{21}	angles of inclination of incident waves (fig. 2)

ψ_2, ψ_{22} angles of inclination of pressure wave behind shock (figs. 6 or 19)

ψ_3, ψ_{23} angles of inclination of vorticity wave behind shock (figs. 13 or 22)

ψ_{cl}, ψ_{cu} lower and upper bounds, respectively, of the range of ψ_1 for which the attenuating pressure wave appears (eq. (17))

Subscripts:

Subscript notation for partial differentiation has been used where convenient.

1 conditions ahead of shock

2 conditions behind shock

Double subscripts 21 and 22 (problem B), incident and reflected waves, respectively.

Superscripts:

(1) coefficients associated with f at shock

(2) coefficients associated with g at shock

Primed coefficients are referred to intensity of incident wave. (A_3 in problems A and B, $\sqrt{A_1^2+A_2^2}$ in problem C).

Primes are also used to denote ordinary differentiation of f and g with respect to their arguments.

APPENDIX B

COMPLETION OF INTERACTION ANALYSIS

PROBLEM A

The solution of problem A is found in two separate ranges of ψ_1, for which the solution involves a refracted sound wave in one case and a refracted attenuating pressure wave in the other. The analysis will be completed in that order.

1. $0<\psi_1<\psi_{cl}$, $\psi_{cu}<\psi_1<\pi$.—Equations (27) and (28) are substituted into equations (7), and coefficients of f' are equated, yielding:

$$\left.\begin{aligned} F&=-\frac{a_2}{\gamma V}K\alpha \\ H&=-\frac{a_2}{\gamma V}K\beta \end{aligned}\right\} \tag{B1}$$

When equations (26), (27), and (28) are substituted into continuity equation (8), the quantities associated with the sound wave combine to satisfy continuity. The terms associated with the vorticity wave then must satisfy the incompressible continuity equation ($u_x+v_y=0$) because the corresponding density term is time-independent. Therefore, equating coefficients of f' yields

$$I=\frac{1+1/Mm}{n(1-r)}G \tag{B2}$$

The shock equations remain to be satisfied; it should be noted that the arguments of f for the various quantities have been matched at the shock, so that coefficients of f may be equated. From equations (5a), (27), (29), and (12a),

$$G=L-F-B_1(L-A_1)-B_2A_3 \tag{B3}$$

From equations (5b), (5c), (5d), respectively, with the necessary substitutions made,

$$Q=C_1(L-A_1)+C_2A_3-\frac{K}{\gamma} \tag{B4}$$

$$K=D_1(L-A_1)+D_2A_3 \tag{B5}$$

$$L=\frac{1+1/Mm}{nr}(H+I-A_2) \tag{B6}$$

The terms H and I in equation (B6) may be expressed in terms of L by using equations (B1), (B2), (B3), and (B5). The resulting equation for L yields the result

$$L'\equiv\frac{L}{A_3}=\frac{-\frac{a_2}{\gamma V}\beta\left(D_1\frac{m}{\gamma M}+D_2\right)+\frac{1+\frac{1}{Mm}}{n(1-r)}\left[\frac{a_2}{\gamma V}\alpha\left(D_1\frac{m}{\gamma M}+D_2\right)-\left(B_1\frac{m}{\gamma M}+B_2\right)\right]-\frac{l}{\gamma M}}{\frac{rn}{1+\frac{1}{Mm}}+\frac{a_2}{\gamma V}\beta D_1-\frac{1+\frac{1}{Mm}}{n(1-r)}\left(1+\frac{a_2}{\gamma V}\alpha D_1-B_1\right)} \tag{B7}$$

or, from equations (6), (21), and (23):

$$L'=-\frac{m}{\gamma M}\frac{\left[1+\frac{\gamma+1}{4}\frac{M}{m}\left(\frac{3-\gamma}{\gamma+1}+\frac{\gamma-1}{2}r\right)\right]\chi+\frac{\gamma+1}{4}\left(1-2\frac{\gamma-1}{\gamma+1}\frac{n}{\sigma}-\frac{n\sigma}{1-r}\right)(1-r)}{1+\chi-\frac{\gamma+1}{4}r\left(1+\frac{\sigma^2}{1-r}\right)} \tag{B8}$$

where

$$\left.\begin{aligned} \sigma&\equiv\frac{n(1-r)}{1+\frac{1}{Mm}} \\ \text{and} \quad \chi&\equiv\left(1-\frac{\gamma+1}{2}r\frac{1+\sigma^2}{1+\frac{\gamma-1}{2}r}\right)^{1/2} \end{aligned}\right\} \tag{B9}$$

and where use is made of the results from equations (2), (6), and (22) that

$$r=\frac{2}{\gamma+1}\left(1-\frac{1}{M^2}\right)$$

$$\left(\frac{a_2}{V}\right)^2=(1-r)\left(1+\frac{\gamma-1}{2}r\right)$$

When L' is known, the remaining coefficients follow from equations (B1) through (B5).

2. $\psi_{cl}<\psi_1<\psi_{cu}$.—Equations (36) and (41) substituted into (7) give

$$F^{(1)}\Phi_t^{(1)}+F^{(2)}\Phi_t^{(2)}=-\frac{a_2^2}{\gamma V}(K^{(1)}\Phi_x^{(1)}+K^{(2)}\Phi_{x_2}^{(2)}) \quad \text{(B10)}$$

From equations (31) and (37),

$$\Phi_t^{(1,2)}=(V-U)\frac{d}{\lambda_1}\Phi_\eta^{(1,2)}+\frac{cV}{\lambda_1}\Phi_\zeta^{(1,2)}$$

$$\Phi_{x_2}^{(1,2)}=-\frac{d}{\lambda_1}\Phi_\eta^{(1,2)}+\frac{\alpha}{\lambda_1}\Phi_\zeta^{(1,2)}$$

Equations (37) show that

$$\Phi_\eta^{(1)}=\Phi_\zeta^{(2)};\ \Phi_\zeta^{(1)}=-\Phi_\eta^{(2)}$$

Accordingly, equation (B10) may be written:

$$(1-r)dF^{(1)}\Phi_\eta^{(1)}-cF^{(1)}\Phi_\eta^{(2)}+(1-r)dF^{(2)}\Phi_\eta^{(2)}+cF^{(2)}\Phi_\eta^{(1)}$$
$$=-\frac{a_2^2}{\gamma V^2}[-dK^{(1)}\Phi_\eta^{(1)}-\alpha K^{(1)}\Phi_\eta^{(2)}-dK^{(2)}\Phi_\eta^{(2)}+\alpha K^{(2)}\Phi_\eta^{(1)}]$$

By equating coefficients of $\Phi_\eta^{(1)}$ and $\Phi_\eta^{(2)}$ separately,

$$\left.\begin{aligned}(1-r)dF^{(1)}+cF^{(2)}&=\frac{a_2^2}{\gamma V^2}(dK^{(1)}-\alpha K^{(2)})\\(1-r)dF^{(2)}-cF^{(1)}&=\frac{a_2^2}{\gamma V^2}(dK^{(2)}+\alpha K^{(1)})\end{aligned}\right\} \quad \text{(B11)}$$

Similarly, equations (36), (42), and (7) yield:

$$\left.\begin{aligned}(1-r)dH^{(1)}+cH^{(2)}&=-\frac{a_2^2}{\gamma V^2}\beta K^{(2)}\\(1-r)dH^{(2)}-cH^{(1)}&=\frac{a_2^2}{\gamma V^2}\beta K^{(1)}\end{aligned}\right\} \quad \text{(B12)}$$

When equations (41) and (42) are substituted into equation (8), and coefficients of f' and g' are equated, the following equations are obtained:

$$\left.\begin{aligned}I^{(1)}&=\frac{1+\frac{1}{Mm}}{n(1-r)}G^{(1)}\\I^{(2)}&=\frac{1+\frac{1}{Mm}}{n(1-r)}G^{(2)}\end{aligned}\right\} \quad \text{(B13)}$$

The shock conditions remain to be applied. Equations (5a), (5b), (5c), and (5d) yield, respectively, the following four pairs of equations, when the coefficients of f and g are separately equated:

$$\left.\begin{aligned}G^{(1)}&=L^{(1)}-F^{(1)}-B_1(L^{(1)}-A_1)-B_2A_3\\G^{(2)}&=L^{(2)}-F^{(2)}-B_1L^{(2)}\end{aligned}\right\} \quad \text{(B14)}$$

$$\left.\begin{aligned}Q^{(1)}&=C_1(L^{(1)}-A_1)+C_2A_3-K^{(1)}/\gamma\\Q^{(2)}&=C_1L^{(2)}-K^{(2)}/\gamma\end{aligned}\right\} \quad \text{(B15)}$$

$$\left.\begin{aligned}K^{(1)}&=D_1(L^{(1)}-A_1)+D_2A_3\\K^{(2)}&=D_1L^{(2)}\end{aligned}\right\} \quad \text{(B16)}$$

$$\left.\begin{aligned}L^{(1)}&=\frac{1+1/Mm}{nr}(H^{(1)}+I^{(1)}-A_2)\\L^{(2)}&=\frac{1+1/Mm}{nr}(H^{(2)}+I^{(2)})\end{aligned}\right\} \quad \text{(B17)}$$

Equations (B11) through (B17) may be combined to yield the solutions for $L^{(1)}$ and $L^{(2)}$

$$\left.\begin{aligned}L^{(1)\prime}&\equiv\frac{L^{(1)}}{A_3}=\frac{h_1h_2-\left(\frac{m}{\gamma M}+\frac{D_2}{D_1}\right)h_3^2}{h_2^2+h_3^2}\\L^{(2)\prime}&\equiv\frac{L^{(2)}}{A_3}=h_3\frac{h_1+\left(\frac{m}{\gamma M}+\frac{D_2}{D_1}\right)h_2}{h_2^2+h_3^2}\end{aligned}\right\} \quad \text{(B18)}$$

where

$$\left.\begin{aligned}h_1&\equiv-\frac{1-r}{r}\frac{1}{\sigma}\left[\left(\frac{m}{\gamma M}B_1+B_2\right)\frac{1}{\sigma}+\frac{l}{\gamma M}\right]\\h_2&\equiv1-\frac{1-r}{r}\frac{1-B_1}{\sigma^2}\\h_3&\equiv-\frac{2d(1-r)}{l\sigma\left(1+\frac{\gamma-1}{2}r\right)}\end{aligned}\right\} \quad \text{(B19)}$$

Equations (B16) are substituted into equations (B11), which may be solved to yield

$$\left.\begin{aligned}F^{(1)\prime}&=h_4\left(L^{(1)\prime}+\frac{m}{\gamma M}+\frac{D_2}{D_1}\right)-h_5L^{(2)\prime}\\F^{(2)\prime}&=h_4L^{(2)\prime}+h_5\left(L^{(1)\prime}+\frac{m}{\gamma M}+\frac{D_2}{D_1}\right)\end{aligned}\right\} \quad \text{(B20)}$$

where

$$\left.\begin{aligned}h_4&\equiv\frac{4}{\gamma+1}\frac{\sigma^2}{(1+\sigma^2)}\\h_5&\equiv\frac{2rd\sigma(1-r)}{l(1+\sigma^2)}\frac{V^2}{a_2^2}\end{aligned}\right\} \quad \text{(B21)}$$

Similarly,

$$\left.\begin{aligned}H^{(1)\prime}&=\frac{1}{\sigma}h_4\left(L^{(1)\prime}+\frac{n}{\gamma M}+\frac{D_2}{D_1}\right)+\sigma h_5L^{(2)\prime}\\H^{(2)\prime}&=\frac{1}{\sigma}h_4L^{(2)\prime}-\sigma h_5\left(L^{(1)\prime}+\frac{n}{\gamma M}+\frac{D_2}{D_1}\right)\end{aligned}\right\} \quad \text{(B22)}$$

The remaining quantities follow from equations (B14) and (B15).

PROBLEM B

The solution of problem B is restricted to angles ψ_{21} for which $-m>\frac{V}{a_2}(1-r)$, and involves a reflected sound wave and vorticity wave behind the shock. The momentum equations (7) provide

$$\left.\begin{aligned} F&=-\frac{a_2}{\gamma V}\alpha K \\ H&=-\frac{a_2}{\gamma V}\beta K \end{aligned}\right\} \tag{B23}$$

and the continuity equation (8) provides that

$$I=\frac{1}{l}\left(m+\frac{a_2/V}{1-r}G\right) \tag{B24}$$

All perturbation quantities vanish ahead of the shock, and shock relations (5a), (5b), (5c), and (5d) yield, respectively,

$$G=L(1-B_1)-F-A_1 \tag{B25}$$

$$Q=C_1L-A_4-K/\gamma \tag{B26}$$

$$K=D_1L-A_3 \tag{B27}$$

$$L=\frac{m(1-r)+a_2/V}{lr}(H+I+A_2) \tag{B28}$$

By solving the foregoing set of equations for L,

$$L'\equiv\frac{L}{A_3}=\frac{\gamma+1}{2\gamma}\left(1+\frac{\gamma-1}{2}r\right)\frac{\chi}{1+\chi-\frac{\gamma+1}{4}r\left(1+\frac{\sigma^2}{1-r}\right)} \tag{B29}$$

where

$$\left.\begin{aligned} \sigma&\equiv\frac{n}{1+\frac{a_2/V}{m(1-r)}} \\ \chi&\equiv-(l\sigma+m)\sqrt{\frac{1-r}{1+\frac{\gamma-1}{2}r}} \end{aligned}\right\} \tag{B30}$$

The remaining quantities follow from equations (B23) through (B27).

PROBLEM C

The analysis of this problem parallels that of problem A.

1. $0<\psi_1<\psi_{cl}$, $\psi_{cu}<\psi_1<\pi$.—The equations of motion (7) and (8) provide

$$F=-\frac{a_2}{\gamma V}K\alpha$$

$$H=-\frac{a_2}{\gamma V}K\beta$$

$$I=\frac{G}{n(1-r)}$$

The shock equations (5) give

$$G=L-F-B_1(L-A_1)$$

$$Q=C_1(L-A_1)-K/\gamma$$

$$K=D_1(L-A_1)$$

$$L=\frac{1}{rn}(H+I-A_2)$$

These equations may be solved to yield

$$L'\equiv\frac{L}{\sqrt{A_1^2+A_2^2}}=l\left[1-\frac{\gamma+1}{4}\frac{r(1-n\sigma)}{1+\chi-\frac{\gamma+1}{4}r\left(1+\frac{\sigma^2}{1-r}\right)}\right]$$

where, in this case,

$$\left.\begin{aligned} \sigma&\equiv n(1-r) \\ \chi&\equiv\left(1-\frac{\gamma+1}{2}r\frac{1+\sigma^2}{1+\frac{\gamma-1}{2}r}\right)^{1/2} \end{aligned}\right\} \tag{B31}$$

2. $\psi_{cl}<\psi_1<\psi_{cu}$. The analysis is essentially identical to the corresponding part of problem A, except that the constants B_2, C_2, and D_2 do not appear in the present case, and the Mach number M does not appear explicitly. Except for these differences, equations (B14) through (B17) may be carried over to the present case. The solutions for $L^{(1)}$ and $L^{(2)}$ may be written as

$$L^{(1)\prime}\equiv\frac{L^{(1)}}{\sqrt{A_1^2+A_2^2}}=\frac{h_1h_2+lh_3^2}{h_2^2+h_3^2}$$

$$L^{(2)\prime}\equiv\frac{L^{(2)}}{\sqrt{A_1^2+A_2^2}}=h_3\frac{h_1-lh_2}{h_2^2+h_3^2}$$

The definitions of the h's follow those of problem A (eqs. (B19) and (B21)), with the exceptions that

$$h_1\equiv\frac{1-r}{r}\frac{1}{\sigma}\left(\frac{l}{\sigma}-m\right)$$

and, in this case, $\sigma\equiv n(1-r)$. The solutions for the F's and H's are:

$$F^{(1)\prime}\equiv\frac{F^{(1)}}{\sqrt{A_1^2+A_2^2}}=h_4(L^{(1)\prime}-l)-h_5L^{(2)\prime}$$

$$F^{(2)\prime}\equiv\frac{F^{(2)}}{\sqrt{A_1^2+A_2^2}}=h_4L^{(2)\prime}+h_5(L^{(1)\prime}-l)$$

$$H^{(1)\prime}\equiv\frac{H^{(1)}}{\sqrt{A_1^2+A_2^2}}=\frac{1}{\sigma}h_4(L^{(1)\prime}-l)+\sigma h_5L^{(2)\prime}$$

$$H^{(2)\prime}\equiv\frac{H^{(2)}}{\sqrt{A_1^2+A_2^2}}=\frac{1}{\sigma}h_4L^{(2)\prime}-\sigma h_5(L^{(1)\prime}-l)$$

The remaining quantities follow as before.

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