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PROPULSION OF A FLAPPING AND OSCILLATING AIRFOIL

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SUMMARY

Formulas are given for the propelling or drag force experienced in a uniform air stream by an airfoil or an airfoil-aileron combination, oscillating in any of three degrees of freedom: vertical flapping, torsional oscillations about a fixed axis parallel to the span, and angular oscillations of the aileron about a hinge.

INTRODUCTION

It is the object of this paper to investigate theoretically the horizontal forces experienced by an airfoil or an airfoil-aileron combination n a uniform air stream made to execute flapping motion or to perform angular oscillations about a fixed axis parallel to the span. The problem treated is that of an infinite wing, or wing and aileron, performing steady sinusoidal oscillations in any of three degrees of freedom: vertical flapping at right angles to the direction of motion, oscillations about an arbitrary fixed axis parallel to its span, and oscillations of the aileron about a hinge.

The work of Wagner (reference 1) for calculating the distribution of vorticity in the wake of an airfoil in nonuniform motion appears as a starting point. A vortex wake is generated by the oscillatory motion, which in turn affects the entire nature of the forces experienced by the wing. Beautiful experimental checks of Wagner's theory of the manner in which the circulation builds up have been obtained by Farren and Walker. (Cf. reference 2, ch. 9 for a more detailed bibliography.) Birnbaum and Küssner (reference 3) have also attacked the problem of obtaining the lift forces on an oscillating wing by certain series expansions that are rather cumbersome to handle. Glauert (reference 4) has treated the case of an oscillating airfoil and has obtained expressions for the forces and moments that check with Wagner. Theodorsen (reference 5) has developed compact expressions for the lift and moments in the case of an airfoil-aileron combination of three independent degrees of freedom and has applied the results to an analysis of the wing-flutter problem. The foregoing references are concerned only with the lift forces, not with the horizontal forces; however, von Kármán and Burgers, who present in reference 2 a résumé of the work (to 1934) on nonuniform motion, calculate there the propulsion effect on a flapping wing. The present paper makes application of the compact formulas developed by Theodorsen and of the method outlined by von Kármán and Burgers to treat the propulsion on a wing oscillating in three independent degrees of freedom.

The assumptions underlying the theory are small amplitudes in the various degrees of freedom and a (infinitely) narrow width of the rectilinear vortex wake, as well as the usual assumption of a perfect fluid. Quantitative agreement with experimental values, which are not very abundant, can hardly be expected since the finite width of the wake is important with regard to considerations of the resistance; nevertheless the results can be useful for interpreting such experiments as exist on the so-called "Katzmayr effect" (reference 6) and for clearing up certain aerodynamic features of the nature of the flight of birds.[1] Experimental tests on an oscillating and flapping wing are being conducted at the present time by the N. A. C. A.

This paper is not concerned with the problem of *flutter*, which is an instability phenomenon that manifests itself in certain critical frequency ranges and is due to an interaction and feedback of energy because of *coupling* in the various degrees of freedom. (Cf. references 3 and 5.) Profile drag is to be considered as additive to the horizontal forces obtained.

FORCES AND MOMENTS ON AN OSCILLATING AIRFOIL

Consider an airfoil represented by the straight line of figure 1. The airfoil chord is of length $2b$ and (its mean position with b as reference unit length) is assumed to extend along the x axis from the leading edge $x=-1$ to the trailing edge $x=+1$. The coordinate $x=a$ represents the axis of rotation of the wing, $x=c$ the coordinate of the aileron hinge. The airfoil is assumed to undergo the following motions with small amplitudes: A vertical motion h of the entire wing, positive downward; a rotation about $x=a$ of angle of attack α, positive clockwise and measured by the direction of the velocity v at infinity and the instantaneous position of the wing; an aileron motion about the hinge $x=c$ of angle β,

[1] It is interesting to observe that the Katzmayr effect occurs in nature also in the motion of fish. See "The Physical Principles of Fish Locomotion," by E. G. Richardson, Jour. Exp. Biology, vol. XIII, no. 1, Jan. 1936, pp. 63–74.

measured with respect to the undeflected position of the wing itself.

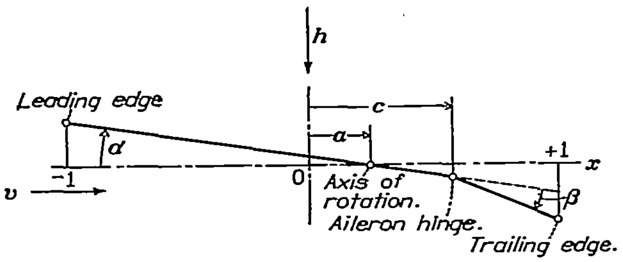

FIGURE 1.—Parameters of the airfoil-aileron combination.

Let us consider sinusoidal oscillations in the various degrees of freedom and use the complex-number notation

$$\left.\begin{aligned}\alpha&=\alpha_0e^{i(pt+\varphi_0)}\\ \beta&=\beta_0e^{i(pt+\varphi_1)}\\ h&=h_0e^{i(pt+\varphi_2)}\end{aligned}\right\} \tag{1}$$

The constants α_0, β_0, and h_0 represent the maximum amplitudes in the various degrees of freedom, φ_0, φ_1, and φ_2 are phase angles, and the parameter p determines the frequency of the oscillations. By means of the relation

$$p=\frac{kv}{b} \tag{2}$$

an important parameter k is defined, i. e., $k=pb/v$. It will be seen that $2\pi/k$ is the wave length between successive waves in the vortex wake in terms of the half-chord b as reference length.

The following three formulas for the lift and moments on an oscillating airfoil of three degrees of freedom are due to Theodorsen and are taken from reference 5:[2]

$$P=-\rho b^2(v\pi\dot{\alpha}+\pi\ddot{h}-\pi ba\ddot{\alpha}-vT_4\dot{\beta}-T_1b\ddot{\beta})-2\pi\rho vbC(k)Q \tag{3}$$

$$M_\alpha=-\rho b^2\left[\pi\left(\frac{1}{2}-a\right)vb\dot{\alpha}+\pi b^2\left(\frac{1}{8}+a^2\right)\ddot{\alpha}+T_{15}v^2\beta+T_{16}vb\dot{\beta}+2T_{13}b^2\ddot{\beta}-a\pi b\ddot{h}\right]+2\rho vb^2\pi\left(a+\frac{1}{2}\right)C(k)Q \tag{4}$$

$$M_\beta=-\rho b^2\left[T_{17}vb\dot{\alpha}+2T_{13}b^2\ddot{\alpha}+\frac{1}{\pi}v^2T_{18}\beta-\frac{1}{2\pi}vbT_{19}\dot{\beta}-\frac{1}{\pi}T_3b^2\ddot{\beta}-T_1b\ddot{h}\right]-\rho vb^2T_{12}C(k)Q \tag{5}$$

where

$$Q=v\alpha+\dot{h}+b\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{1}{\pi}T_{10}v\beta+\frac{b}{2\pi}T_{11}\dot{\beta}$$

These equations are to be interpreted as follows: The real part of P denotes the lift force (positive downward) associated with the motion given by the real parts of (1); i. e., $\alpha=\alpha_0\cos(pt+\varphi_0)$, $\beta=\beta_0\cos(pt+\varphi_1)$, and $h=h_0\cos(pt+\varphi_2)$. The imaginary part of P denotes the lift force associated with the motions $\alpha=\alpha_0\sin(pt+\varphi_0)$, $\beta=\beta_0\sin(pt+\varphi_1)$, and $h=h_0\sin(pt+\varphi_2)$. Similarly M_α and M_β denote in complex form the moments (positive clockwise in fig. 1) about $x=a$ and $x=c$, respectively, due to the motions (1). (The mean value of α or β is considered zero. When the mean values are different from zero, the forces and moments arising from constant values α_m and β_m are to be added.) In equations (3), (4), and (5) there occur various symbols that have not yet been defined. The T's, i. e., T_1, T_3, T_4, etc., are constants defined completely by the parameters c and a (reference 5, p. 5). For reference they are listed in appendix I, where there is also given a collection of the symbols employed in the notation of this paper. The function $C(k)$ is a useful complex function of the parameter k (see (2)) and is given by

$$C(k)=F(k)+iG(k) \tag{6}$$

where

$$F=\frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

$$G=-\frac{Y_1Y_0+J_1J_0}{(J_1+Y_0)^2+(Y_1-J_0)^2}$$

Functions J_0, J_1, Y_0, and Y_1 are standard Bessel functions of the first and second kinds of argument k. Figure 2 and table I, which are taken from reference 5 (with certain minor changes), illustrate these functions.

In what follows we shall be interested only in one part of the preceding complex equations. It is an arbitrary matter whether to employ the real or imaginary parts. The choice made here is to treat the imaginary parts, and we write down for reference the imaginary parts of equations (1), (3), (4), and (5):

$$\left.\begin{aligned}\alpha&=\alpha_0\sin(pt+\varphi_0)\\ \beta&=\beta_0\sin(pt+\varphi_1)\\ h&=h_0\sin(pt+\varphi_2)\end{aligned}\right\} \tag{7}$$

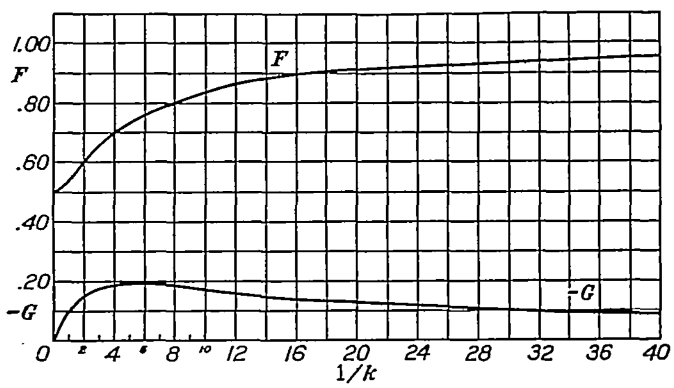

FIGURE 2.—The functions F and $-G$ against $1/k$.

[2] The writer wishes to record the fact that in order to establish a check on these general relations he has compared them with the widely varying expressions given by Wagner, Glauert, von Kármán and Burgers, and Küssner in their special cases (references 1 to 4). Identical agreement has resulted in all cases, except that in the case of Küssner's formulas a numerical check was made since an analytic check was not feasible. The numerical agreement was good except in the case of the wing-aileron combination where Küssner makes some rough approximations.

A recent paper by Cicala (reference 7) deserves mention. Cicala derives expressions for the lift and moment on an oscillating airfoil that seem to agree with the results of Theodorsen, although the method is somewhat more involved. The functions denoted by Cicala as λ' and λ'' correspond to 1-F and -G defined in equation (6).

$$
\begin{aligned}
P=&-\rho b^2[v\pi\alpha_0 p\cos(pt+\varphi_0)-\pi h_0p^2\sin(pt+\varphi_2)+\pi ba\alpha_0p^2\sin(pt+\varphi_0)\\
&-vT_4\beta_0p\cos(pt+\varphi_1)+T_1b\beta_0p^2\sin(pt+\varphi_1)]\\
&-2\pi\rho vbF\Big[v\alpha_0\sin(pt+\varphi_0)+h_0p\cos(pt+\varphi_2)+b\Big(\frac{1}{2}-a\Big)\alpha_0p\cos(pt+\varphi_0)\\
&+\frac{T_{10}}{\pi}v\beta_0\sin(pt+\varphi_1)+\frac{T_{11}}{2\pi}b\beta_0p\cos(pt+\varphi_1)\Big]\\
&-2\pi\rho vbG\Big[v\alpha_0\cos(pt+\varphi_0)-h_0p\sin(pt+\varphi_2)-b\Big(\frac{1}{2}-a\Big)\alpha_0p\sin(pt+\varphi_0)\\
&+\frac{T_{10}}{\pi}v\beta_0\cos(pt+\varphi_1)-\frac{T_{11}}{2\pi}b\beta_0p\sin(pt+\varphi_1)\Big]
\end{aligned}\tag{8}
$$

$$
\begin{aligned}
M_\alpha=&-\rho b^2\Big[\pi\Big(\frac{1}{2}-a\Big)vb\alpha_0p\cos(pt+\varphi_0)-\pi b^2\Big(\frac{1}{8}+a^2\Big)\alpha_0p^2\sin(pt+\varphi_0)\\
&+T_{15}v^2\beta_0\sin(pt+\varphi_1)+T_{16}vb\beta_0p\cos(pt+\varphi_1)\\
&-2T_{13}b^2\beta_0p^2\sin(pt+\varphi_1)+a\pi bh_0p^2\sin(pt+\varphi_2)\Big]\\
&+2\rho vb^2\pi\Big(a+\frac{1}{2}\Big)F\Big[v\alpha_0\sin(pt+\varphi_0)+h_0p\cos(pt+\varphi_2)\\
&+b\Big(\frac{1}{2}-a\Big)\alpha_0p\cos(pt+\varphi_0)+\frac{T_{10}}{\pi}v\beta_0\sin(pt+\varphi_1)\\
&+\frac{T_{11}}{2\pi}b\beta_0p\cos(pt+\varphi_1)\Big]+2\rho vb^2\pi\Big(a+\frac{1}{2}\Big)G\Big[v\alpha_0\cos(pt+\varphi_0)\\
&-h_0p\sin(pt+\varphi_2)-b\Big(\frac{1}{2}-a\Big)\alpha_0p\sin(pt+\varphi_0)+\frac{T_{10}}{\pi}v\beta_0\cos(pt+\varphi_1)\\
&-\frac{T_{11}}{2\pi}b\beta_0p\sin(pt+\varphi_1)\Big]
\end{aligned}\tag{9}
$$

$$
\begin{aligned}
M_\beta=&-\rho b^2\Big[T_{17}vb\alpha_0p\cos(pt+\varphi_0)-2T_{13}b^2\alpha_0p^2\sin(pt+\varphi_0)+\frac{1}{\pi}v^2T_{18}\beta_0\sin(pt+\varphi_1)\\
&-\frac{1}{2\pi}vbT_{19}\beta_0p\cos(pt+\varphi_1)+\frac{1}{\pi}T_3b^2\beta_0p^2\sin(pt+\varphi_1)+T_1bh_0p^2\sin(pt+\varphi_2)\Big]\\
&-\rho vb^2T_{12}F\Big[v\alpha_0\sin(pt+\varphi_0)+h_0p\cos(pt+\varphi_2)+b\Big(\frac{1}{2}-a\Big)\alpha_0p\cos(pt+\varphi_0)\\
&+\frac{T_{10}}{\pi}v\beta_0\sin(pt+\varphi_1)+\frac{T_{11}}{2\pi}b\beta_0p\cos(pt+\varphi_1)\Big]-\rho vb^2T_{12}G\Big[v\alpha_0\cos(pt+\varphi_0)\\
&-h_0p\sin(pt+\varphi_2)-b\Big(\frac{1}{2}-a\Big)\alpha_0p\sin(pt+\varphi_0)+\frac{T_{10}}{\pi}v\beta_0\cos(pt+\varphi_1)\\
&-\frac{T_{11}}{2\pi}b\beta_0p\sin(pt+\varphi_1)\Big]
\end{aligned}\tag{10}
$$

In addition to these equations we will need the expression for the force on the aileron. This equation is obtained in complex form as (use formulas on pp. 5–8, reference 5)

$$
\begin{aligned}
P_\beta=&-\rho b^2\Big(-vT_4\dot{\alpha}-T_4\ddot{h}+bT_9\ddot{\alpha}-\frac{1}{2\pi}vT_5\dot{\beta}-\frac{b}{2\pi}T_2\ddot{\beta}\Big)\\
&-2\rho bv\sqrt{1-c^2}\Big[\frac{b}{2}(1-c)\dot{\alpha}+\frac{1}{\pi}v\sqrt{1-c^2}\beta+\frac{b}{2\pi}(1-c)T_{10}\dot{\beta}\Big]\\
&-2\rho vbT_{20}C(k)Q
\end{aligned}
$$

And the imaginary part is

$$
\begin{aligned}
P_\beta=&-\rho b^2\Big[-vT_4\alpha_0p\cos(pt+\varphi_0)+T_4h_0p^2\sin(pt+\varphi_2)-bT_9\alpha_0p^2\sin(pt+\varphi_0)\\
&-\frac{1}{2\pi}vT_5\beta_0p\cos(pt+\varphi_1)+\frac{b}{2\pi}T_2\beta_0p^2\sin(pt+\varphi_1)\Big]-2\rho bv\sqrt{1-c^2}\Big[\frac{b}{2}(1-c)\alpha_0p\cos(pt+\varphi_0)\\
&+\frac{1}{\pi}v\sqrt{1-c^2}\beta_0\sin(pt+\varphi_1)+\frac{b}{2\pi}(1-c)T_{10}\beta_0p\cos(pt+\varphi_1)\Big]-2\rho vbT_{20}F\Big[v\alpha_0\sin(pt+\varphi_0)\\
&+h_0p\cos(pt+\varphi_2)+b\Big(\frac{1}{2}-a\Big)\alpha_0p\cos(pt+\varphi_0)+\frac{T_{10}}{\pi}v\beta_0\sin(pt+\varphi_1)\\
&+\frac{T_{11}}{2\pi}b\beta_0p\cos(pt+\varphi_1)\Big]-2\rho vbT_{20}G\Big[v\alpha_0\cos(pt+\varphi_0)-h_0p\sin(pt+\varphi_2)\\
&-b\Big(\frac{1}{2}-a\Big)\alpha_0p\sin(pt+\varphi_0)+\frac{T_{10}}{\pi}v\beta_0\cos(pt+\varphi_1)-\frac{T_{11}}{2\pi}b\beta_0p\sin(pt+\varphi_1)\Big]
\end{aligned}\tag{11}
$$

Following the method of von Kármán and Burgers (reference 2), the average horizontal force will be determined in two ways: (1) by the energy formula given in equation (12), and (2) by the force formula given in equation (13). The agreement of the results of the two methods will thus furnish a check on the work.

ENERGY FORMULA

$$\overline{W}=\overline{E}+\overline{P}_x v \tag{12}$$

where $\overline{W}$ represents the average work done in unit time in maintaining the oscillations (7) against the forces and moments (8), (9), and (10); $\overline{E}$ represents the average increase in kinetic energy in unit time in the vortex wake and; $\overline{P}_x v$ denotes average work done in unit time by the propulsive force P_x.[3]

FORCE FORMULA

$$P_x=\pi\rho S^2+\alpha P+\beta P_\beta \tag{13}$$

where P_x is the propelling force; α and β are given in (7); P in (8), and P_β in (11); S is obtained from the relation $S=\lim_{x\to -1}\frac{1}{2}\gamma\sqrt{x+1}$ where γ is the vorticity distribution. The value of S is finite, since γ is infinite in the order of $\frac{1}{\sqrt{x+1}}$ at the leading edge $x=-1$, and is given in equation (25) and derived in appendix II.[4]

We proceed first to evaluate $\overline{W}$ in equation (12). The instantaneous rate at which work is done in maintaining the oscillations is

$$\dot{W}=-(P\dot{h}+M_\alpha\dot{\alpha}+M_\beta\dot{\beta})$$

For the average work done in unit time we have

$$\overline{W}=-\frac{p}{2\pi}\int_0^{\frac{2\pi}{p}}(P\dot{h}+M_\alpha\dot{\alpha}+M_\beta\dot{\beta})dt \tag{14}$$

On employing equations (7) to (10) and performing the indicated integrations, we obtain after some lengthy but elementary reductions

$$\overline{W}=\pi\rho b^2\frac{p^3}{k}(B_1h_0^2+B_2\alpha_0^2+B_3\beta_0^2+2B_4\alpha_0h_0+2B_5\beta_0h_0+2B_6\alpha_0\beta_0) \tag{15}$$

where

$$B_1=F$$

$$B_2=b^2\left\{\frac{1}{2}\left(\frac{1}{2}-a\right)-\left(a+\frac{1}{2}\right)\left[F\left(\frac{1}{2}-a\right)+\frac{G}{k}\right]\right\}$$

$$B_3=b^2\left[-\frac{T_{19}}{4\pi^2}+\frac{T_{12}}{2\pi}\left(\frac{T_{11}}{2\pi}F+\frac{T_{10}}{\pi}\frac{G}{k}\right)\right]$$

$$B_4=\frac{b}{2}\left[\left(\frac{1}{2}-2aF+\frac{G}{k}\right)\cos(\varphi_2-\varphi_0)-\left(\frac{F}{k}-G\right)\sin(\varphi_2-\varphi_0)\right]$$

$$B_5=\frac{b}{2}\left[\left(-\frac{T_4}{2\pi}+\frac{T_{11}+T_{12}}{2\pi}F+\frac{T_{10}}{\pi}\frac{G}{k}\right)\cos(\varphi_2-\varphi_1)-\left(\frac{T_{10}}{\pi}\frac{F}{k}+\frac{T_4}{\pi}G\right)\sin(\varphi_2-\varphi_1)\right]$$

$$\begin{aligned}B_6=\frac{b^2}{2}\Bigg\{&\left[\frac{T_{11}}{4\pi}-\left(\frac{1}{2}-a\right)\frac{T_4}{2\pi}+\left(\frac{T_4}{2\pi}-\frac{T_{11}+T_{12}}{2\pi}a\right)F-\left(\left(a+\frac{1}{2}\right)\frac{T_{10}}{\pi}-\frac{T_{12}}{2\pi}\right)\frac{G}{k}\right]\cos(\varphi_1-\varphi_0)\\&+\left[\frac{T_{15}}{2\pi}\frac{1}{k}-\left(\left(a+\frac{1}{2}\right)\frac{T_{10}}{\pi}+\frac{T_{12}}{2\pi}\right)\frac{F}{k}+\left(\frac{T_{11}+T_{12}}{4\pi}-\frac{T_4}{\pi}a\right)G\right]\sin(\varphi_1-\varphi_0)\Bigg\}\end{aligned}$$

In order to calculate $\overline{E}$ in equation (12), we need the expression for the vorticity in the wake. The magnitude of the vorticity in the wake is given in complex form by

$$U=U_0e^{i\varphi}e^{ip\left(t-\frac{(x-1)b}{v}\right)} \tag{16}$$

where $U_0e^{i\varphi}$ is a complex quantity determined in (19). (Cf. reference 5, p. 8, in which x instead of $x-1$ is used in the exponent.) From the definition of the circulation about the airfoil as the integral of the vorticity in the wake we have in complex form,

$$\Gamma=\int_1^\infty Udx=-\frac{i}{k}U_0e^{i\varphi}e^{ipt} \tag{17}$$

Also from reference 5, equation (8), the condition for smooth flow at the trailing edge leads to the relation

$$\frac{1}{2\pi}\int_1^\infty\sqrt{\frac{x+1}{x-1}}Udx=v\alpha+\dot{h}+b\left(\frac{1}{2}-a\right)\dot{\alpha}+\frac{T_{10}}{\pi}v\beta+b\frac{T_{11}}{2\pi}\dot{\beta}=Q \tag{18}$$

Combining (17) and (18) we may write

$$\Gamma=2\pi Q-\int_1^\infty\left(\sqrt{\frac{x+1}{x-1}}-1\right)Udx$$

On equating coefficients of e^{ipt} on both sides of this relation and solving for the quantity $U_0e^{i\varphi}$ (for the evaluation of the definite integral in terms of Bessel functions see reference 5, p. 8), we obtain

$$U_0e^{i\varphi}=-4(A+iB)(J+iK)e^{-ik} \tag{19}$$

where,

$$J=\frac{J_1+Y_0}{D},\ K=\frac{Y_1-J_0}{D},$$

$$D=(J_1+Y_0)^2+(Y_1-J_0)^2,\ J^2+K^2=\frac{1}{D}$$

and

$$\left.\begin{aligned}A&=v\alpha_0\cos\varphi_0-h_0p\sin\varphi_2-b\left(\frac{1}{2}-a\right)\alpha_0p\sin\varphi_0+\frac{T_{10}}{\pi}v\beta_0\cos\varphi_1-\frac{T_{11}}{2\pi}b\beta_0p\sin\varphi_1\\B&=v\alpha_0\sin\varphi_0+h_0p\cos\varphi_2+b\left(\frac{1}{2}-a\right)\alpha_0p\cos\varphi_0+\frac{T_{10}}{\pi}v\beta_0\sin\varphi_1+\frac{T_{11}}{2\pi}b\beta_0p\cos\varphi_1\end{aligned}\right\} \tag{20}$$

[3] When the energy released in the wake in unit time is less than the work required in unit time to maintain the oscillations, i. e., $\overline{E}<\overline{W}$, then $\overline{P}_x$ is positive and is a true propelling force. When $\overline{E}>\overline{W}$, then $\overline{P}_x$ is negative and denotes not *propulsion* but *resistance* or *drag*.

[4] Formula (13) is obtained by a slight extension of the method of reference 2, pp. 305-306. The "suction" force $\pi\rho S^2$ arising from the infinite vorticity at the leading edge is explained in reference 2 (pp. 52 and 306) along lines laid down by Grammel and Cisotti. (See also reference 8, pp. 135 and 203.) The fact that this infinity occurs implies that the ideal flow for an infinitely thin wing is unrealizable. We are regarding this case, however, as a limiting one of a wing that is rounded and smooth at the leading edge and sharp at the trailing edge

When the imaginary part of U is denoted by γ, which is the only part of interest, the vorticity in the wake is given by

$$\gamma=A_0 \cos kx+B_0 \sin kx \tag{21}$$

where

$$\frac{1}{4}A_0=(BK-AJ) \sin pt-(AK+BJ) \cos pt$$

$$\frac{1}{4}B_0=(BK-AJ) \cos pt+(AK+BJ) \sin pt$$

The induced vertical velocity at a great distance x downstream is

$$w_z=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\gamma(x')}{x-x'}dx'=\frac{1}{2}(A_0 \sin kx-B_0 \cos kx)$$

The difference in potential at points of the x axis in the wake is

$$\phi_2-\phi_1=b\int \gamma dx=\frac{b}{k}(A_0 \sin kx-B_0 \cos kx)$$

and the kinetic energy in the wake (per unit length) at a point x along the surface of discontinuity far from the airfoil is[5]

$$E_1=\frac{1}{2}\rho w_z(\phi_2-\phi_1) \tag{22}$$

$$=\frac{1}{2}\rho\frac{b}{2k}(A_0 \sin kx-B_0 \cos kx)^2$$

$$=4\rho\frac{b}{k}[(BK-AJ) \cos (pt+kx)+(AK+BJ) \sin (pt+kx)]^2$$

The mean value of E_1 with respect to time is independent of x and is given by

$$\overline{E}_1=\frac{p}{2\pi}\int_0^{2\pi/p}E_1dt=\frac{2\rho b}{kD}(A^2+B^2)$$

And, finally, the average value of the increase in energy in the field in unit time is

$$\overline{E}=v\overline{E}_1=\frac{2\rho bv}{kD}(A^2+B^2)$$

or also

$$\overline{E}=\pi\rho b^2\frac{p^3}{k}[C_1h_0^2+C_2\alpha_0^2+C_3\beta_0^2+2C_4\alpha_0h_0+2C_5\beta_0h_0+2C_6\alpha_0\beta_0] \tag{23}$$

where

$$C_1=\frac{2}{\pi kD}$$

$$C_2=\frac{2b^2}{\pi kD}\left[\frac{1}{k^2}+\left(\frac{1}{2}-a\right)^2\right]$$

$$C_3=\frac{2b^2}{\pi kD}\left[\left(\frac{T_{10}}{\pi k}\right)^2+\left(\frac{T_{11}}{2\pi}\right)^2\right]$$

$$C_4=\frac{2b}{\pi kD}\left[-\frac{1}{k}\sin (\varphi_2-\varphi_0)+\left(\frac{1}{2}-a\right)\cos (\varphi_2-\varphi_0)\right]$$

$$C_5=\frac{2b}{\pi kD}\left[-\frac{T_{10}}{\pi}\frac{1}{k}\sin (\varphi_2-\varphi_1)+\frac{T_{11}}{2\pi}\cos (\varphi_2-\varphi_1)\right]$$

$$C_6=\frac{2b^2}{\pi kD}\left[\left(\frac{T_{10}}{\pi}\frac{1}{k^2}+\left(\frac{1}{2}-a\right)\frac{T_{11}}{2\pi}\right)\cos (\varphi_1-\varphi_0)+\frac{1}{k}\left(\frac{T_{10}}{\pi}\left(\frac{1}{2}-a\right)-\frac{T_{11}}{2\pi}\right)\sin (\varphi_1-\varphi_0)\right]$$

Equation (12) now defines $\overline{P}_xv$ and hence $\overline{P}_x$. We have

$$\overline{P}_xv=\overline{W}-\overline{E}$$

or

$$\overline{P}_x=\pi\rho bp^2[A_1h_0^2+A_2\alpha_0^2+A_3\beta_0^2+2A_4\alpha_0h_0+2A_5\beta_0h_0+2A_6\alpha_0\beta_0] \tag{24}$$

where from equations (15) and (23)

$$A_1=B_1-C_1,\ A_2=B_2-C_2, \text{ etc.}$$

We shall now proceed to the direct calculation of P_x from (13). The value of S is derived in appendix II and in complex form is given by

$$S=\frac{\sqrt{2}}{2}\left[2C(k)Q-b\dot{\alpha}-\frac{2}{\pi}\sqrt{1-c^2}v\beta+\frac{T_4}{\pi}b\dot{\beta}\right]$$

Again we shall use only the imaginary part of this expression which is

$$S=\frac{\sqrt{2}}{2}(M \sin pt+N \cos pt) \tag{25}$$

where

$$M=2F\left[v\alpha_0 \cos \varphi_0-h_0p \sin \varphi_2-b\left(\frac{1}{2}-a\right)\alpha_0p \sin \varphi_0+\frac{T_{10}}{\pi}v\beta_0 \cos \varphi_1-\frac{T_{11}}{2\pi}b\beta_0p \sin \varphi_1\right]$$

$$-2G\left[v\alpha_0 \sin \varphi_0+h_0p \cos \varphi_2+b\left(\frac{1}{2}-a\right)\alpha_0p \cos \varphi_0+\frac{T_{10}}{\pi}v\beta_0 \sin \varphi_1+\frac{T_{11}}{2\pi}b\beta_0p \cos \varphi_1\right]$$

$$+b\alpha_0p \sin \varphi_0-\frac{2}{\pi}\sqrt{1-c^2}\,v\beta_0 \cos \varphi_1-\frac{T_4}{\pi}b\beta_0p \sin \varphi_1$$

[5] The expression $\frac{1}{2}\rho w_z(\phi_2-\phi_1)$ is actually equal to $\frac{1}{2}\rho\phi\frac{\partial\phi}{\partial n}$ taken along the surface of discontinuity (the x axis) where $\phi=\phi_2-\phi_1$ and $\frac{\partial\phi}{\partial n}=w_z$. The latter expression is equal to $\frac{1}{2}\rho\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]$ taken over a proper space interval, i. e., represents the kinetic energy in a certain volume.

and

$$N=2F\left[v\alpha_0 \sin \varphi_0+h_0p \cos \varphi_2+b\left(\frac{1}{2}-a\right)\alpha_0p \cos \varphi_0+\frac{T_{10}}{\pi}v\beta_0 \sin \varphi_1+\frac{T_{11}}{2\pi}b\beta_0p \cos \varphi_1\right]$$
$$+2G\left[v\alpha_0 \cos \varphi_0-h_0p \sin \varphi_2-b\left(\frac{1}{2}-a\right)\alpha_0p \sin \varphi_0+\frac{T_{10}}{\pi}v\beta_0 \cos \varphi_1-\frac{T_{11}}{2\pi}b\beta_0p \sin \varphi_1\right]$$
$$-b\alpha_0p \cos \varphi_0-\frac{2}{\pi}\sqrt{1-c^2}\,v\beta_0 \sin \varphi_1+\frac{T_4}{\pi}b\beta_0p_0 \cos \varphi_1$$

The mean value of $\pi\rho S^2$ with respect to time is

$$\frac{p}{2\pi}\int_0^{2\pi/p}\pi\rho S^2dt=\frac{\pi\rho}{4}(M^2+N^2)$$

This expression becomes, after a considerable number of terms cancel,

$$\pi\rho\overline{S_1^2}=\pi\rho bp^2(a_1h_0^2+a_2\alpha_0^2+a_3\beta_0^2+2a_4\alpha_0h_0+2a_5\beta_0h_0+2a_6\alpha_0\beta_0) \tag{26}$$

where

$$a_1=F^2+G^2$$
$$a_2=b^2\left\{(F^2+G^2)\left[\frac{1}{k^2}+\left(\frac{1}{2}-a\right)^2\right]+\frac{1}{4}-\left(\frac{1}{2}-a\right)F-\frac{1}{k}G\right\}$$
$$a_3=b^2\left\{(F^2+G^2)\left[\left(\frac{T_{10}}{\pi k}\right)^2+\left(\frac{T_{11}}{2\pi}\right)^2\right]+\frac{1-c^2}{\pi^2k^2}+\frac{T_4^2}{4\pi^2}\right.$$
$$\left.+F\left(\frac{-2T_{10}\sqrt{1-c^2}}{\pi^2k^2}+\frac{T_4T_{11}}{2\pi^2}\right)+\frac{G}{k}\left(\frac{T_4T_{10}}{\pi^2}+\frac{T_{11}\sqrt{1-c^2}}{\pi^2}\right)\right\}$$
$$a_4=b\left\{(F^2+G^2)\left[-\frac{1}{k}\sin(\varphi_2-\varphi_0)+\left(\frac{1}{2}-a\right)\cos(\varphi_2-\varphi_0)\right]-\frac{F}{2}\cos(\varphi_2-\varphi_0)+\frac{G}{2}\sin(\varphi_2-\varphi_0)\right\}$$
$$a_5=b\left\{(F^2+G^2)\left[-\frac{T_{10}}{\pi k}\sin(\varphi_2-\varphi_1)+\frac{T_{11}}{2\pi}\cos(\varphi_2-\varphi_1)\right]+\frac{F}{2}\left[\frac{2\sqrt{1-c^2}}{\pi k}\sin(\varphi_2-\varphi_1)+\frac{T_4}{\pi}\cos(\varphi_2-\varphi_1)\right]\right.$$
$$\left.+\frac{G}{2}\left[\frac{2\sqrt{1-c^2}}{\pi k}\cos(\varphi_2-\varphi_1)-\frac{T_4}{\pi}\sin(\varphi_2-\varphi_1)\right]\right\}$$
$$a_6=b^2\left\{(F^2+G^2)\left(\left[\frac{T_{10}}{\pi k^2}+\frac{T_{11}}{2\pi}\left(\frac{1}{2}-a\right)\right]\cos(\varphi_1-\varphi_0)+\left[\frac{T_{10}}{\pi}\left(\frac{1}{2}-a\right)-\frac{T_{11}}{2\pi}\right]\frac{1}{k}\sin(\varphi_1-\varphi_0)\right)\right.$$
$$+\frac{F}{2}\left[\frac{-2\sqrt{1-c^2}}{\pi k^2}+\frac{T_4}{\pi}\left(\frac{1}{2}-a\right)-\frac{T_{11}}{2\pi}\right]\cos(\varphi_1-\varphi_0)-\left[\frac{2\sqrt{1-c^2}}{\pi}\left(\frac{1}{2}-a\right)+\frac{T_4+T_{10}}{\pi}\right]\frac{1}{k}\sin(\varphi_1-\varphi_0)$$
$$+\frac{G}{2}\left[\frac{-2\sqrt{1-c^2}}{\pi k^2}+\frac{T_4}{\pi}\left(\frac{1}{2}-a\right)+\frac{T_{11}}{2\pi}\right]\sin(\varphi_1-\varphi_0)+\left[\frac{2\sqrt{1-c^2}}{\pi}\left(\frac{1}{2}-a\right)+\frac{T_4-T_{10}}{\pi}\right]\frac{1}{k}\cos(\varphi_1-\varphi_0)$$
$$\left.+\frac{\sqrt{1-c^2}}{2\pi}\frac{1}{k}\sin(\varphi_1-\varphi_0)-\frac{T_4}{4\pi}\cos(\varphi_1-\varphi_0)\right\}$$

We proceed to calculate the average values of the terms αP and βP_β in (13) by employing equations (7), (8), and (11). There results

$$\overline{\alpha P}=\pi\rho bp^2(b_2\alpha_0^2+2b_4\alpha_0h_0+2b_6\alpha_0\beta_0) \tag{27}$$

where

$$b_2=b^2\left[-\frac{a}{2}-\frac{F}{k^2}+\left(\frac{1}{2}-a\right)\frac{G}{k}\right]$$
$$b_4=\frac{b}{2}\left[\left(\frac{1}{2}+\frac{G}{k}\right)\cos(\varphi_2-\varphi_0)+\frac{F}{k}\sin(\varphi_2-\varphi_0)\right]$$
$$b_6=\frac{b^2}{2}\left[\left(-\frac{T_1}{2\pi}-\frac{F}{k^2}\frac{T_{10}}{\pi}+\frac{G}{k}\frac{T_{11}}{2\pi}\right)\cos(\varphi_1-\varphi_0)\right.$$
$$\left.+\left(-\frac{T_4}{2\pi}+F\frac{T_{11}}{2\pi}+\frac{G}{k}\frac{T_{10}}{\pi}\right)\frac{1}{k}\sin(\varphi_1-\varphi_0)\right]$$

Also

$$\overline{\beta P_\beta}=\pi\rho bp^2(c_3\beta_0^2+2c_5\beta_0h_0+2c_6\alpha_0\beta_0) \tag{28}$$

where

$$c_3=b^2\left[-\frac{T_2}{2\pi^2}-\frac{1-c^2}{\pi^2k^2}-\frac{F}{k^2}\frac{T_{10}T_{20}}{\pi^2}+\frac{G}{k}\frac{T_{11}T_{20}}{2\pi^2}\right]$$
$$c_5=\frac{b}{2}\left[\left(-\frac{T_4}{2\pi}+\frac{T_{20}}{\pi}\frac{G}{k}\right)\cos(\varphi_2-\varphi_1)+\frac{T_{20}}{\pi}\frac{F}{k}\sin(\varphi_2-\varphi_1)\right]$$
$$c_6=\frac{b^2}{2}\left\{\left[\frac{T_9}{\pi}-\frac{T_{20}}{\pi}\frac{F}{k^2}+\frac{T_{20}}{\pi}\left(\frac{1}{2}-a\right)\frac{G}{k}\right]\cos(\varphi_1-\varphi_0)\right.$$
$$+\left[\frac{T_4-(1-c)\sqrt{1-c^2}}{2\pi}-\frac{T_{20}}{\pi}\left(\frac{1}{2}-a\right)F\right.$$
$$\left.\left.-\frac{T_{20}}{\pi}\frac{G}{k}\right]\frac{1}{k}\sin(\varphi_1-\varphi_0)\right\}$$

Finally from (13) the average propulsive force is

$$\overline{P}_x=\pi\rho b p^2[a_1h_0^2+(a_2+b_2)\alpha_0^2+(a_3+c_3)\beta_0^2 +2(a_4+b_4)\alpha_0h_0+2(a_5+c_5)\beta_0h_0 +2(a_6+b_6+c_6)\alpha_0\beta_0] \tag{29}$$

In order that equations (24) and (29) agree we must have that

$$\begin{aligned} A_1&=a_1\\ A_2&=a_2+b_2\\ A_3&=a_3+c_3\\ A_4&=a_4+b_4\\ A_5&=a_5+c_5\\ A_6&=a_6+b_6+c_6 \end{aligned} \tag{30}$$

Each of these relations may be reduced to an identity, e. g., consider A_1 and a_1. From (15), (23), and (26)

$$A_1=B_1-C_1=F-\frac{2}{\pi kD}$$

$$a_1=F^2+G^2$$

In order that $A_1=a_1$ the following relation must hold

$$F=F^2+G^2+\frac{2}{\pi kD} \tag{31}$$

To show that this is true note that

$$F=\frac{J_1(J_1+Y_0)+Y_1(Y_1-J_0)}{(J_1+Y_0)^2+(Y_1-J_0)^2}=\frac{J_1^2+Y_1^2+J_1Y_0-J_0Y_1}{D}$$

$$F^2+G^2=(F+iG)\ (F-iG)=\frac{J_1^2+Y_1^2}{D}$$

(cf. reference 5, p. 8) and from a well-known property of the Bessel functions,

$$J_1Y_0-J_0Y_1=\frac{2}{\pi k}$$

Hence equation (31) follows.

By the use of the relation (31) and the definitions of the various T's given in the appendix, it can be verified that the remaining relations in (30) are also identities.

It may be of interest to consider the special cases of one degree of freedom. Let the motion of the wing consist only of the vertical motion $\bar{h}$ at right angles to the direction of flight, i. e., flapping motion. The propelling force is then [6]

$$\overline{P}_x=\pi\rho b p^2h_0^2(F^2+G^2) \tag{32}$$

Consider in this case the ratio

$$\frac{\overline{P}_xv}{\overline{W}}=\frac{\text{energy of propulsion}}{\text{total energy}}$$

$$=\frac{F^2+G^2}{F} \tag{33}$$

This function, shown in figure 3, represents the theoretical efficiency of the flapping motion (unity=100 percent). It is observed that a propelling force exists in the entire range of $1/k$, the efficiency being 50 percent for infinitely rapid oscillations and 100 percent for infinitely slow flapping.

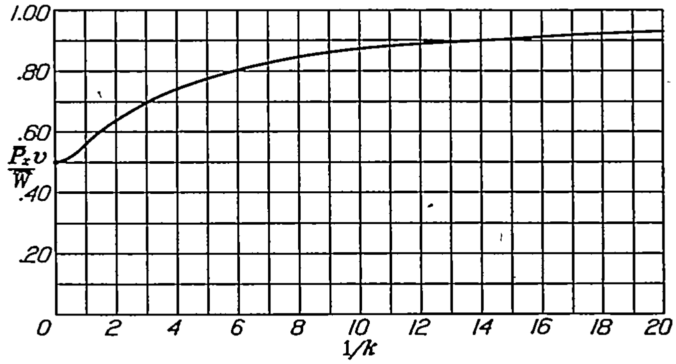

FIGURE 3.—The ratio of energy of propulsion to the energy required to maintain the oscillations $\left(\frac{\overline{P}_xv}{\overline{W}}\right)$ as a function of $1/k$ for the case of pure flapping.

For the special case of angular oscillations about a alone ($h=0$, $\beta=0$) the horizontal force is

$$\overline{P}_x=\pi\rho b p^2 b^2\alpha_0^2\left\{(F^2+G^2)\left[\frac{1}{k^2}+\left(\frac{1}{2}-a\right)^2\right] +\frac{1}{2}\left(\frac{1}{2}-a\right)-\frac{F}{k^2}-\left(\frac{1}{2}-a\right)\frac{G}{k}\right\} \tag{34}$$

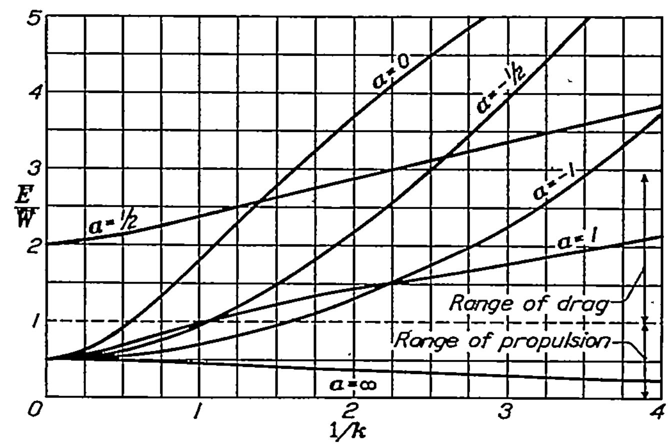

FIGURE 4.—The ratio of the energy dissipated in the wake to the energy required to maintain the oscillations ($\overline{E}/\overline{W}$) as a function of $1/k$ for the case of pure angular oscillations about $x=a$.

In figure 4 there is shown the ratio $\overline{E}/\overline{W}$, in this case for several positions of the axis of rotation. These curves give the ratio of the energy per unit time released in the wake to the work per unit time required to maintain the oscillations. In the range of values $0<\overline{E}/\overline{W}<1$, $\overline{P}_x$ is positive and denotes a thrust or propelling force; for other values it is negative and denotes a drag force.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *May 4, 1936.*

[6] This result agrees with the formula of von Kármán and Burgers (reference 2, p. 306). The expressions of reference 2 denoted by

$$b_1=1+A_1-\lambda A_1(Q-S)+\lambda A_2(P-C)$$
$$b_2=A_2-\lambda A_2(Q-S)-\lambda A_1(P-C)$$

reduce in our notation simply to $2F$ and $2G$, respectively.

APPENDIX I

NOTATION

α, angle of attack (fig. 1).

β, aileron angle (fig. 1).

h, vertical distance (fig. 1).

$\dot{\alpha}=\frac{d\alpha}{dt}$, $\ddot{\alpha}=\frac{d^2\alpha}{dt^2}$, etc.

α_0, β_0, h_0, φ_0, φ_1, φ_2, amplitudes and phase angles of the oscillations (equation (7)).

b, half chord, used as reference length.

x, coordinate in direction of airfoil chord.

t, time.

v, velocity of the general motion in direction of x axis.

p, 2π times the frequency of the oscillations.

k, *reduced* frequency (equation (2)). The wave length between successive waves in the vortex wake is $2\pi b/k$.

a, coordinate of axis of rotation (fig. 1).

c, coordinate of aileron hinge (fig. 1).

i, imaginary unit $\sqrt{-1}$.

e, base of natural logarithms.

ρ, mass density of air.

P, lift force on airfoil (+downward in fig. 1).

M_α, moment on airfoil about a (+clockwise in fig. 1).

M_β, moment on aileron about c (+clockwise in fig. 1).

P_β, lift force on aileron (+downward in fig. 1).

$C(k)$, F, G, J_0, J_1, Y_0, Y_1, J, K, D, Bessel functions of the argument k. (Cf. equations (6) and (19), fig. 2, and table I.)

$\overline{W}$, average work done in unit time in maintaining the oscillations.

$\overline{E}$, average increase of kinetic energy in the wake in unit time.

$\overline{P}_x$, average force in the direction of the x axis (+propulsion, −drag).

$A_1 \ldots A_6$, $B_1 \ldots B_6$, $C_1 \ldots C_6$, $a_1, \ldots a_6$, b_2, b_4, b_6, c_3, c_5, c_6, coefficients.

Q, defined by equation (18).

A, B, defined by equation (20).

A_0, B_0, defined by equation (21).

M, N, defined by equation (25).

U, distribution of vorticity in the wake in complex form (equation (16)).

$U_0 e^{i\varphi}$, coefficient of U, given in equation (19).

γ, imaginary part of U.

S, defined by equation (13); see also appendix II.

Γ, circulation about the airfoil, defined by equation (17).

DEFINITIONS OF THE T's

$$T_1=-\frac{1}{3}(2+c^2)\sqrt{1-c^2}+c\cos^{-1}c$$

$$T_2=c(1-c^2)-(1+c^2)\sqrt{1-c^2}\cos^{-1}c+c(\cos^{-1}c)^2$$

$$[T_2=T_4(T_{11}+T_{12})]$$

$$T_3=-\frac{1}{8}(1-c^2)(5c^2+4)+\frac{1}{4}c(7+2c^2)\sqrt{1-c^2}\cos^{-1}c-\left(\frac{1}{8}+c^2\right)(\cos^{-1}c)^2$$

$$T_4=c\sqrt{1-c^2}-\cos^{-1}c$$

$$T_5=-(1-c^2)+2c\sqrt{1-c^2}\cos^{-1}c-(\cos^{-1}c)^2$$

$$T_6=T_2$$

$$T_7=\frac{1}{8}c(7+2c^2)\sqrt{1-c^2}-\left(\frac{1}{8}+c^2\right)\cos^{-1}c$$

$$T_8=-\frac{1}{3}(1+2c^2)\sqrt{1-c^2}+c\cos^{-1}c=-\frac{1}{3}(1-c^2)^{\frac{3}{2}}-cT_4$$

$$T_9=\frac{1}{2}\left[\frac{1}{3}(1-c^2)^{\frac{3}{2}}+aT_4\right]$$

$$T_{10}=\sqrt{1-c^2}+\cos^{-1}c$$

$$T_{11}=(2-c)\sqrt{1-c^2}+(1-2c)\cos^{-1}c$$

$$T_{12}=(2+c)\sqrt{1-c^2}-(1+2c)\cos^{-1}c$$

$$[T_{12}-T_{11}=2T_4]$$

$$T_{13}=-\frac{1}{2}(T_7+(c-a)T_1)$$

$$T_{14}=\frac{1}{16}+\frac{1}{2}ac$$

$$T_{15}=T_4+T_{10}=(1+c)\sqrt{1-c^2}$$

$$T_{16}=T_1-T_8-(c-a)T_4+\frac{1}{2}T_{11}$$

$$\left[T_{16}+T_{17}=-\left(\frac{1}{2}-a\right)T_4+\frac{1}{2}T_{11}\right]$$

$$T_{17}=-2T_9-T_1+\left(a-\frac{1}{2}\right)T_4$$

$$T_{18}=T_5-T_4T_{10}$$

$$T_{19}=T_4T_{11}$$

$$T_{20}=-\sqrt{1-c^2}+\cos^{-1}c$$

$$[T_{20}=T_{10}-2\sqrt{1-c^2}]$$

APPENDIX II

EVALUATION OF S (EQUATION (25))

From reference 5 (p. 7) we have that the condition for smooth flow at the *trailing edge* is obtained from the equation

$$\frac{\partial}{\partial x}(\varphi_\Gamma+\varphi_\alpha+\varphi_{\dot h}+\varphi_{\dot\alpha}+\varphi_\beta+\varphi_{\dot\beta})_{x=1}=0 \qquad (1)$$

where the φ's are as follows (a $\pm$ sign is to be prefixed to each φ, $+$ for the upper surface, $-$ for the lower surface):

$$\frac{\partial\varphi_\Gamma}{\partial x}=\frac{1}{2\pi}\int_1^\infty\sqrt{\frac{x_0^2-1}{1-x^2}}\frac{1}{x_0-x}Udx_0$$

$$\varphi_\alpha=v\alpha b\sqrt{1-x^2}$$

$$\varphi_{\dot h}=\dot h b\sqrt{1-x^2}$$

$$\varphi_{\dot\alpha}=\dot\alpha b^2\left(\frac{1}{2}x-a\right)\sqrt{1-x^2}$$

$$\varphi_\beta=\frac{1}{\pi}v\beta b[\sqrt{1-x^2}\cos^{-1}c-(x-c)\log N]$$

$$\varphi_{\dot\beta}=\frac{1}{2\pi}\dot\beta b^2[\sqrt{1-c^2}\sqrt{1-x^2}+(x-2c)\sqrt{1-x^2}\cos^{-1}c$$
$$-(x-c)^2\log N]$$

where

$$N=\frac{1-cx-\sqrt{1-c^2}\sqrt{1-x^2}}{x-c}$$

Condition (1) leads to the relation (cf. (18))

$$\frac{1}{2\pi}\int_1^\infty\sqrt{\frac{x_0+1}{x_0-1}}\,Udx_0=v\alpha+\dot h+b\left(\frac{1}{2}-a\right)\dot\alpha+\frac{T_{10}}{\pi}v\beta$$
$$+b\frac{T_{11}}{2\pi}\dot\beta=Q \qquad (2)$$

The *leading-edge* vorticity may be written as

$$2\frac{\partial}{\partial x}(\varphi_\Gamma+\varphi_\alpha+\varphi_{\dot h}+\varphi_{\dot\alpha}+\varphi_\beta+\varphi_{\dot\beta})_{x=-1}=\frac{2S}{\sqrt{1+x}}$$

On substituting for the φ's, making use of relation (2) and of equation XI, reference 5, which is

$$C(k)=\frac{\int_1^\infty\frac{x_0}{\sqrt{x_0^2-1}}e^{-ikx_0}dx_0}{\int_1^\infty\frac{x_0+1}{\sqrt{x_0^2-1}}e^{-ikx_0}dx_0}$$

there results

$$S=\frac{\sqrt{2}}{2}[2C(k)Q-b\dot\alpha-\frac{2}{\pi}\sqrt{1-c^2}\,v\beta+\frac{T_4}{\pi}b\dot\beta]$$

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TABLE I.—VALUES OF THE BESSEL FUNCTIONS

k	$1/k$	J_0	J_1	Y_0	Y_1	D	F	$-G$
∞	0	----	----	----	----	0	0.5000	0
10	1/10	−0.2459	0.0435	0.0557	0.2490	.2548	.5006	.0124
6	1/6	.1506	−.2767	−.2882	−.1750	.4251	.5017	.0206
4	1/4	−.3971	−.0660	−.0169	.3979	.6389	.5037	.0305
2	1/2	.2239	.5767	.5104	−.1070	1.2913	.5129	.0577
1	1	.7652	.4401	.0883	.7812	2.6706	.5394	.1003
.8	1 1/4	.8463	.3688	−.0868	.9780	3.4076	.5541	.1165
.6	1 2/3	.9120	.2867	−.3085	−1.2604	4.7198	.5788	.1378
.5	2	.9385	.2423	−.4446	−1.4714	5.8486	.5979	.1507
.4	2 1/2	.9604	.1960	−.6060	−1.7808	7.6823	.6250	.1650
.3	3 1/3	.9776	.1483	−.8072	−2.2929	11.130	.6650	.1793
.2	5	.9900	.0995	−1.0810	−3.3235	19.570	.7276	.1886
.1	10	.9975	.0499	−1.5342	−6.460	57.810	.8320	.1723
.05	20	.9994	.0250	−1.979	−12.8	194.26	.9090	.1305
.025	40	.9999	.0125	−2.430	−25.6	713.4	.9545	.0872
.01	100	1.000	.0050	−3.006	−63.7	4195	.9824	.0482
0	∞	1.000	0	−∞	−∞	∞	1.000	0