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THE THEORY OF PROPELLERS

IV—THRUST, ENERGY, AND EFFICIENCY FORMULAS FOR SINGLE- AND DUAL-ROTATING PROPELLERS WITH IDEAL CIRCULATION DISTRIBUTION

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SUMMARY

Simple and exact expressions are given for the efficiency of single- and dual-rotating propellers with ideal circulation distribution as given by the Goldstein functions for single-rotating propellers and by the new functions for dual-rotating propellers from part I of the present series. The efficiency is shown to depend primarily on a defined load factor and, to a very small extent, on an axial loss factor. Tables and graphs are included for practical use of the results. The present paper is the fourth in a series on the theory of propellers.

INTRODUCTION

The thrust, the energy loss, and the efficiency of a propeller are given completely and uniquely by the condition of the wake far behind the propeller. Detailed knowledge of the propeller required to create the particular wake pattern is not needed; in fact, the propeller is not uniquely determined by the wake pattern. An element of lift may be transposed in a direction tangent to the vortex surface in such a manner as to maintain the identical vortex pattern far behind the propeller.

Several equivalent propellers may thus exist—all with the same vortex surface far behind the propeller. Such quantities as the diameter, the pitch, and the rate of advance of the surface of discontinuity far behind the propeller are therefore of a more fundamental significance in many respects than the similar quantities referring to the propeller. At any rate, it has been found convenient for the present theoretical treatment to consider all quantities as referring to the conditions of the ultimate wake. Only in the final stage of the actual design of the propeller are the interrelations of the propeller and the ultimate wake of concern. For the present investigation only knowledge of the ultimate wake is required. The thrust, the various energy losses, and the efficiency are dependent only on the ultimate wake.

The present paper is the fourth in a series on the theory of propellers. The first of the series (reference 1) deals with a set of new functions for the thrust distribution of dual-rotating propellers. The second (reference 2) concerns the axial interference velocity, and the third (reference 3) treats of the contraction of the propeller wake.

SYMBOLS

θ	angular coordinate on vortex sheet
R	tip radius of propeller
x	nondimensional radius in terms of tip radius
z	axial coordinate
p	number oi blades of propeller; also, pressure
Γ	circulation at radius x
ω	angular velocity of propeller
V	advance velocity of propeller
w	rearward displacement velocity of helical vortex surface (at infinity)
λ	advance ratio $\left(\frac{V+w}{\omega R}\right)$
$K(x)$	circulation function for single rotation $\left(\frac{p\Gamma\omega}{2\pi(V+w)w}\right)$
$K(x, \theta)$	circulation function for dual rotation
κ	mass coefficient $\left(2\int_0^1 K(x)x\,dx \text{ or } \frac{1}{\pi}\int_0^1\int_0^{2\pi} K(x, \theta)x\,d\theta\,dx\right)$
F	projected area of helix (at infinity)
S	control surface (at infinity)
σ	volume of wake region
ρ	density of fluid
v	interference velocity
v_z	axial interference velocity
v_r	radial interference velocity
v_t	tangential interference velocity
ϕ	velocity potential
T	thrust
H	pitch of wake helix $\left(2\pi\frac{V+w}{\omega}\right)$
ϵ	axial energy-loss factor
ϵ_r	radial energy-loss factor
ϵ_t	tangential energy-loss factor
E	energy loss in wake
η	efficiency $\left(\frac{TV}{TV+E}\right)$

c_s specific loading factor referred to wake at infinity $\left(\frac{2T}{F\rho V^2}\right)$

a apparent induced displacement velocity at propeller disk

MASS COEFFICIENT κ

In reference 1 the concept of a mass coefficient κ was introduced. By definition

$$\kappa=2\int_0^1 K(x)x\,dx$$

where

$$K(x)=\frac{p\Gamma\omega}{2\pi(V+w)w}$$

and x is the nondimensional radius of the wake. In more general terms to include also dual-rotating propellers, κ may be defined by

$$\kappa=\frac{1}{F}\int_F K(x,\,\theta)dS$$

where $K(x,\theta)$ is a function of both the radius x and the angle θ or the time t. The coefficient κ is thus the mean value of the circulation factor $K(x,\theta)$ over the area of the wake cross section.

It is shown in the following discussion that the momentum

$$\rho\int v\,d\sigma$$

contained in the space σ enclosed between two infinite planes perpendicular to the axis at a distance between them equal to the distance between successive surfaces of discontinuity is equal to

$$\kappa wF$$

The designation "mass coefficient" originates from this relation. The mass of air set in motion with a velocity w is of the cross section κF or, if the column F is considered to be in motion, it will attain the mean velocity κw; hence the term "mass coefficient" is used for κ. Reference 1 gives the circulation function $K(x)$ or $K(x,\theta)$ and the mass coefficient κ for all significant cases of single- and dual-rotating propellers. It may be seen later that the mass coefficient κ times w is not exactly identical with the thrust coefficient $\frac{1}{2}c_s$ because κ refers to a certain momentum and not to the thrust.

GENERAL CONSIDERATIONS OF PROPELLER WITH IDEAL CIRCULATION DISTRIBUTION

Expressions will be given for the thrust and energy loss for both single- and dual-rotating propellers. The discussion is restricted to the case of the ideal circulation distribution. In this case, the surface of discontinuity moves backward as a rigid surface at a constant rate of motion w. The equation of motion may be written in the form

$$p-p_0+\frac{1}{2}\rho v^2+\rho\frac{\partial\phi}{\partial t}=0$$

where the subscript *0* refers to the condition at infinity with the medium at rest. Because of the stipulation that the entire field moves backward as a rigid body, the relation

$$\phi=f(z-wt,\,x,\,\theta)$$

exists and, consequently,

$$\frac{\partial\phi}{\partial t}=-w\frac{\partial\phi}{\partial z}$$

Since $\frac{\partial\phi}{\partial z}=v_z$, it follows that

$$\frac{\partial\phi}{\partial t}=-wv_z$$

and the equation of motion for this type of rigid-pattern displacement flow is, in general,

$$p-p_0+\frac{1}{2}\rho v^2=\rho wv_z \tag{1}$$

CALCULATION OF THRUST

In calculating the thrust of the propeller, it is convenient to employ an imaginary control surface, which encloses the propeller but is infinitely distant from it. The control surface may be chosen as a cube with infinitely long sides and with the propeller at the center and the wake directed perpendicular to one wall S, crossing it in the middle and extending infinitely far beyond or outside the control surface. Let the center of the wake be the z-axis. By methods of classical mechanics, the instantaneous thrust is obtained as

$$T=\int[p-p_0+\rho(V+v_z)v_z]dS$$

Introducing equation (1) transforms this integral to

$$T=\rho\int\left[(V+w)v_z+v_z^2-\frac{1}{2}v^2\right]dS$$

Since the thrust may vary with time as is the case for the ideal dual-rotating propeller, an integration must also be performed with respect to time. This integration results in the expression

$$T=\frac{1}{t}\rho\int_0^t\int_S\left[(V+w)v_z+v_z^2-\frac{1}{2}v^2\right]dS\,dt$$

Since ϕ and therefore the velocities v and v_z are functions of $z-wt$, this integral may be obtained as a volume integral taken over a volume of an infinite cross section S and a length along the axis z equal to the distance between successive vortex sheets. This distance is

$$\frac{H}{p}=2\pi\frac{V+w}{\omega p}$$

where p is the number of blades. The integral thus becomes

$$T=\frac{1}{\frac{1}{p}H}\rho\int_\sigma\left[(V+w)v_z-\frac{1}{2}v^2+v_z^2\right]d\sigma$$

Now

$$\int_\sigma v_z d\sigma=\int_F\phi\,dS=\int_F\Gamma\,dS$$

where the surface integral is taken over one turn of the vortex surface with dS as the projection on the surface S perpendicular to the axis z. With

$$\Gamma=\frac{2\pi(V+w)w}{p\omega}K(x,\theta)$$

the following relation is obtained:

$$\frac{1}{\frac{1}{p}H}\int_\sigma v_z d\sigma=w\int_F K(x,\theta)dS$$

This relation may be transformed by the introduction of the mass coefficient κ (see reference 1), which is defined as the mean value of $K(x, \theta)$ over the projected wake area

$$\kappa=\frac{1}{F}\int_F K(x,\theta)dS$$

Then

$$\frac{1}{\frac{1}{p}H}\int_\sigma v_z d\sigma=\kappa wF$$

For the second integral occurring in the expression for the thrust, a similar treatment yields

$$\frac{1}{\frac{1}{p}H}\int_\sigma v^2 d\sigma=\kappa w^2F$$

Finally, by definition of a quantity ϵ, which may be recognized as the axial energy-loss factor, the third integral is

$$\frac{1}{\frac{1}{p}H}\int_\sigma v_z^2 d\sigma=\epsilon w^2F$$

This integral is obviously the expression for twice the axial energy loss contained in the volume σ between successive vortex sheets. Since $v_z^2<v^2$, it is evident that $\epsilon<\kappa$ and $\frac{\epsilon}{\kappa}<1$.

By use of these three expressions, the thrust may be written in the simple form

$$T=F\rho\left[(V+w)\kappa w+\epsilon w^2-\frac{1}{2}\kappa w^2\right]$$

$$=\rho F\kappa w\left[V+w\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)\right] \qquad (2)$$

CALCULATION OF ENERGY LOSS IN WAKE

By methods of classical mechanics, it can be shown that the expression for the instantaneous energy loss in the wake is

$$E=\int_S\left[\left(p-p_0+\frac{1}{2}\rho v^2\right)v_z+\frac{1}{2}\rho v^2V\right]dS$$

By use of relation (1) this expression becomes

$$E=\rho\int_S\left(v_z^2w+\frac{1}{2}v^2V\right)dS$$

By integrating over time and transferring to a volume integral as was done for the thrust, the expression is changed to

$$E=\rho\frac{1}{\frac{1}{p}H}\int_\sigma\left(v_z^2w+\frac{1}{2}v^2V\right)d\sigma$$

Replacing the integrals $\int_\sigma v_z^2d\sigma$ and $\int_\sigma v^2d\sigma$ as before gives finally

$$E=\rho F\kappa w^2\left(\frac{\epsilon}{\kappa}w+\frac{1}{2}V\right) \qquad (3)$$

EFFICIENCY

With the thrust and the energy loss known from equations (2) and (3), the efficiency defined as $\eta=\frac{TV}{TV+E}$ is given by

$$\eta=\frac{\rho F\kappa w\left[V+w\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)\right]V}{\rho F\kappa w\left[V+w\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)\right]V+\rho F\kappa w^2\left(\frac{\epsilon}{\kappa}w+\frac{1}{2}V\right)}$$

or

$$\eta=\frac{\left[V+w\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)\right]V}{(V+w)\left(V+\frac{\epsilon}{\kappa}w\right)}$$

With the introduction of a nondimensional quantity $\overline{w}=\frac{w}{V}$, the efficiency becomes finally

$$\eta=\frac{1+\overline{w}\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)}{(1+\overline{w})\left(1+\frac{\epsilon}{\kappa}\overline{w}\right)} \qquad (4)$$

which is the exact expression for the ideal efficiency of the heavily loaded single- or dual-rotating propeller. The efficiency is a function of the velocity ratio w/V infinitely far behind the propeller and is dependent on only one other parameter ϵ/κ, which is the ratio of the axial loss to the total loss.

By introducing the specific loading factor c_s in equation (2), the following expression is obtained:

$$\frac{1}{2}c_s=\frac{T}{F\rho V^2}=\kappa\overline{w}\left[1+\overline{w}\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)\right]$$

Substituting the quantity c_s in equation (4) gives

$$\eta=\frac{\left(\frac{1}{2}+A\right)\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)^2}{\left(\frac{\epsilon}{\kappa}+A\right)\left(\frac{1}{2}+\frac{1}{2}\frac{\epsilon}{\kappa}+\frac{\epsilon}{\kappa}A\right)} \qquad (5)$$

where

$$A^2=\frac{1}{4}+\frac{1}{2}\frac{c_s}{\kappa}\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)$$

This formula shows the efficiency as a function primarily of the parameter c_s/κ with a slight dependence on ϵ/κ. The efficiencies from the formulas given are plotted as functions of $\overline{w}$ and c_s/κ in figures 1 and 2, respectively; numerical values of the efficiencies are listed in tables I and II.

TABLE I

PROPELLER EFFICIENCY

$$\left[\eta=\frac{1+\overline{w}\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)}{(1+\overline{w})\left(1+\frac{\epsilon}{\kappa}\overline{w}\right)}\right]$$

	η						
$\overline{w}$ \ ϵ/κ	0	1/100	1/10	1/5	2/5	3/5	1
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.05	.9762	.9762	.9760	.9759	.9757	.9754	.9751
.10	.9545	.9545	.9541	.9537	.9528	.9520	.9504
.15	.9348	.9347	.9338	.9329	.9311	.9294	.9263
.20	.9167	.9165	.9150	.9135	.9105	.9077	.9028

TABLE II

PROPELLER EFFICIENCY (IN SERIES FORM)

$$\left[\eta=1-\frac{1}{4}\left(\frac{c_s}{\kappa}\right)+\frac{3}{16}\left(\frac{c_s}{\kappa}\right)^2-\frac{1}{16}\left[\frac{5}{2}+2\frac{\epsilon}{\kappa}-\left(\frac{\epsilon}{\kappa}\right)^2\right]\left(\frac{c_s}{\kappa}\right)^3+\ldots\right]$$

	η						
c_s/κ \ ϵ/κ	0	1/100	1/10	1/5	2/5	3/5	1
0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
.1	.9767	.9767	.9767	.9767	.9767	.9766	.9766
.2	.9563	.9562	.9561	.9560	.9559	.9558	.9557
.3	.9377	.9376	.9373	.9370	.9366	.9362	.9360
.4	.9200	.9199	.9192	.9186	.9174	.9166	.9160
.5	.9023	.9022	.9009	.8995	.8973	.8958	.8945

It is of some interest to give the exact expression for η as a function of $\overline{w}$ and of c_s/κ in the form of infinite power series. These series are

$$\eta=1-\frac{1}{2}\overline{w}+\frac{1}{2}\left(1-\frac{\epsilon}{\kappa}\right)\overline{w}^2-\frac{1}{2}\left(1-\frac{\epsilon}{\kappa}-\frac{\epsilon^2}{\kappa^2}\right)\overline{w}^3+\ldots \tag{6}$$

and

$$\eta=1-\frac{1}{4}\frac{c_s}{\kappa}+\frac{3}{16}\left(\frac{c_s}{\kappa}\right)^2-\frac{1}{16}\left(\frac{5}{2}+2\frac{\epsilon}{\kappa}-\frac{\epsilon^2}{\kappa^2}\right)\left(\frac{c_s}{\kappa}\right)^3+\ldots \tag{7}$$

The very small dependence of η on ϵ/κ, particularly in equation (7), is evident.

INDUCED VELOCITY AT PROPELLER

If the efficiency is written in the form

$$\eta=\frac{1}{1+a}$$

the quantity a gives the apparent induced displacement velocity at the propeller disk. From equation (4)

$$a=\frac{\frac{1}{2}\overline{w}+\frac{\epsilon}{\kappa}\overline{w}^2}{1+\overline{w}\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)} \tag{8}$$

or, in power series in $\overline{w}$,

$$a=\frac{1}{2}\overline{w}-\frac{1}{2}\left(\frac{1}{2}-\frac{\epsilon}{\kappa}\right)\overline{w}^2+\frac{1}{2}\left(\frac{1}{4}-\frac{\epsilon^2}{\kappa^2}\right)\overline{w}^3+\ldots \tag{9}$$

or, in terms of c_s/κ,

$$a=\frac{1}{4}\frac{c_s}{\kappa}-\frac{1}{8}\left(\frac{c_s}{\kappa}\right)^2+\frac{1}{16}\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)\left(\frac{5}{2}-\frac{\epsilon}{\kappa}\right)\left(\frac{c_s}{\kappa}\right)^3+\ldots \tag{10}$$

FINAL REMARKS

Exact formulas for the ideal efficiencies have been developed. The efficiency is given simply as

$$\eta=f_1\left(\overline{w},\frac{\epsilon}{\kappa}\right)$$

or

$$\eta=f_2\left(\frac{c_s}{\kappa},\frac{\epsilon}{\kappa}\right)$$

It has been shown by series developments and graphs that the dependence of η on the parameter ϵ/κ, which is the axial loss in terms of the total loss, is very small, particularly in the second formula. For this reason the numerical value of ϵ/κ need not be known to a high degree of accuracy. It can be shown that ϵ/κ is approximately equal to κ and, for most practical purposes, this approximation is sufficient. On the other hand, the formulas are exact and the value of ϵ must be obtained to the degree of exactness actually desired. Values of ϵ for single-rotating two-blade propellers are given as an example in the appendix.

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LANGLEY FIELD, VA., *October 12, 1944*.

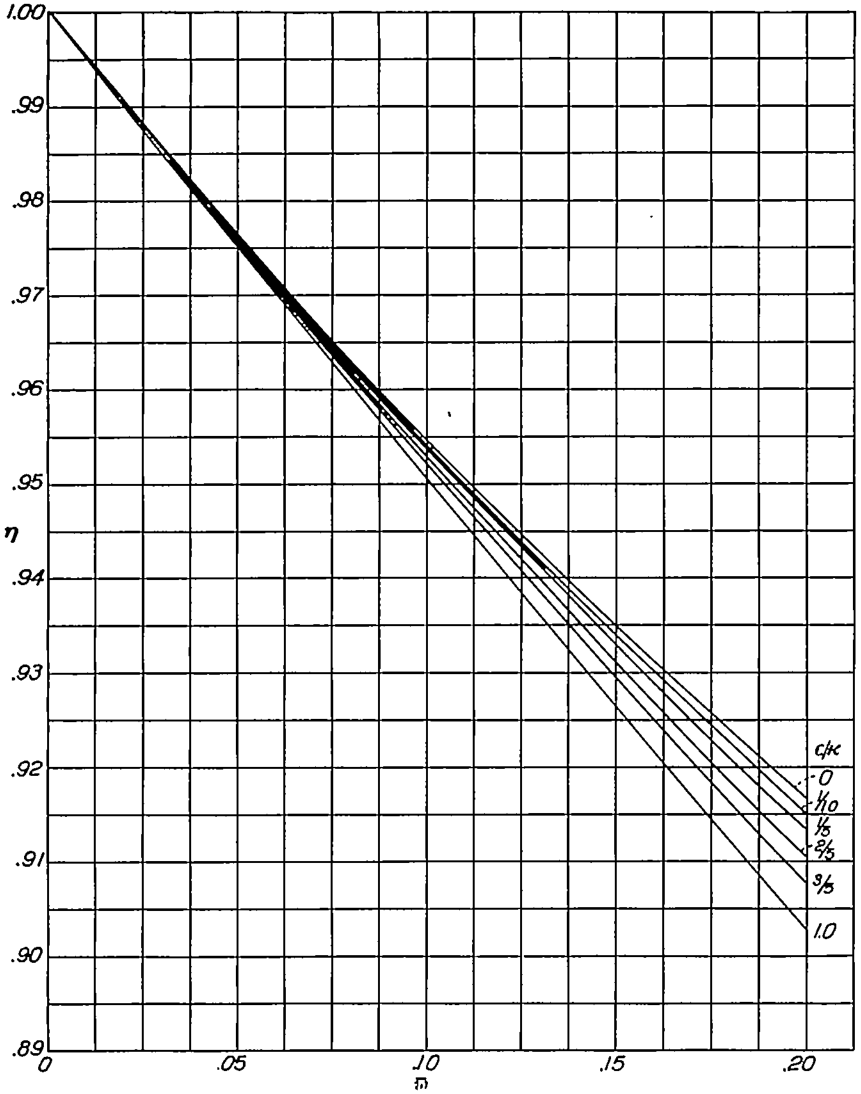

FIGURE 1.—Propeller efficiency.

$$\eta=\frac{1+\bar{w}\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)}{(1+\bar{w})\left(1+\frac{\epsilon}{\kappa}\bar{w}\right)}$$

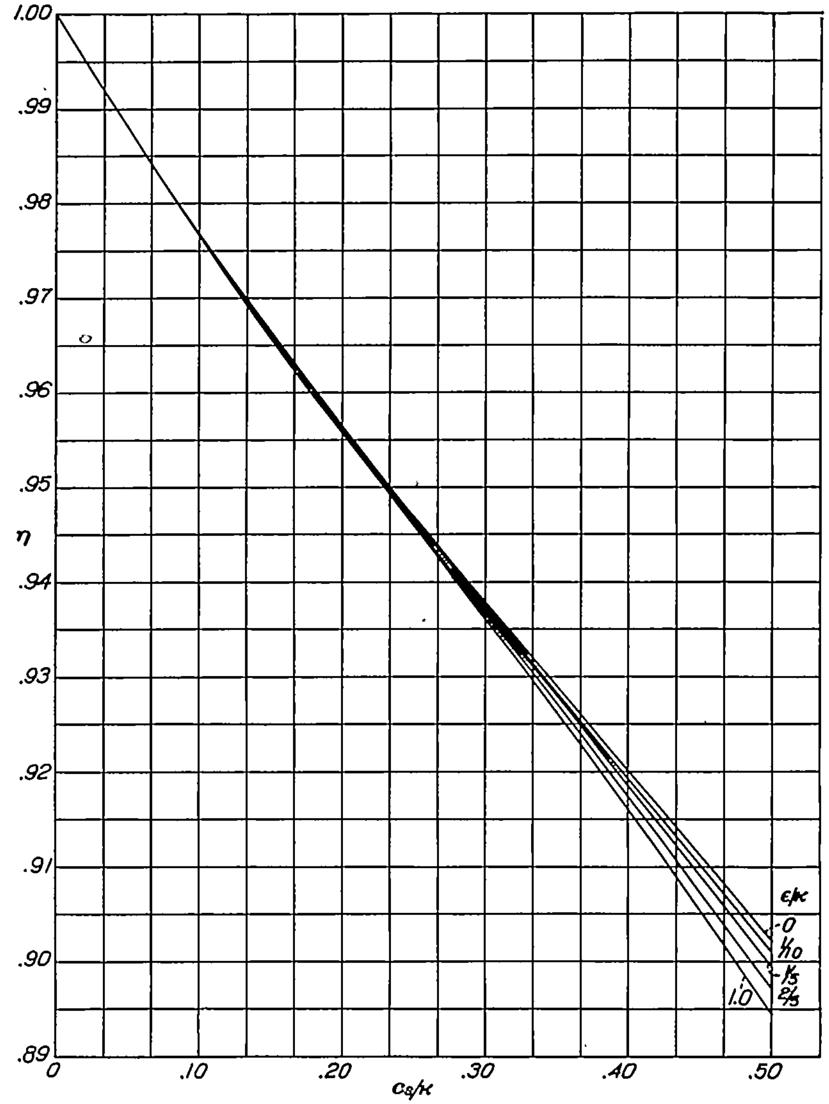

FIGURE 2.—Propeller efficiency.

$$\eta=\frac{\left[1+\sqrt{1+\frac{c_s}{\kappa}\left(1+2\frac{\epsilon}{\kappa}\right)}\right]\left(\frac{1}{2}+\frac{\epsilon}{\kappa}\right)^2}{\left[\frac{\epsilon}{\kappa}+\frac{1}{2}\sqrt{1+\frac{c_s}{\kappa}\left(1+2\frac{\epsilon}{\kappa}\right)}\right]\left[1+\frac{\epsilon}{\kappa}+\frac{\epsilon}{\kappa}\sqrt{1+\frac{c_s}{\kappa}\left(1+2\frac{\epsilon}{\kappa}\right)}\right]}$$

APPENDIX

DETERMINATION OF AXIAL LOSS FACTOR ϵ

It is seen from the efficiency formulas that the axial loss ratio ϵ/κ enters as a parameter. Since the dependence is very small, it is sufficient to know an approximate value of ϵ/κ. It is concluded from the following discussion that the loss ratio ϵ/κ is only slightly greater than the numerical value of the mass coefficient κ, since this relation holds for the known case of an infinite number of blades and shows reasonable agreement also in the case of a two-blade propeller for $\lambda=\frac{1}{2}$. Until the loss ratio has been obtained by direct calculation, the practice of putting $\frac{\epsilon}{\kappa}=\kappa$ is considered satisfactory for all purposes.

Axial, tangential, and radial loss factors are defined for single-rotating propellers by

$$\epsilon=\frac{1}{w^2F}\int_S v_z^2dS$$

$$\epsilon_t=\frac{1}{w^2F}\int_S v_t^2dS$$

and

$$\epsilon_r=\frac{1}{w^2F}\int_S v_r^2dS$$

respectively. Further, the total loss factor is given as

$$\epsilon+\epsilon_t+\epsilon_r=\kappa$$

In figure 3, $\frac{d\epsilon}{d(x^2)}$, $\frac{d\epsilon_t}{d(x^2)}$, and $\frac{d\epsilon_r}{d(x^2)}$ are plotted against x^2 for the case of a two-blade propeller with $\lambda=\frac{1}{2}$. These plots were made by using the functions and constants given by Goldstein for the velocities v_z, v_t, and v_r. The curves, upon integration, yield the values

$$\epsilon=0.0925$$

$$\epsilon_t=0.0768$$

$$\epsilon_r=0.0932$$

The sum of these three,

$$\kappa=\epsilon+\epsilon_t+\epsilon_r=0.2625$$

is very nearly equal to the value of κ obtained from figure 3 in part I (reference 1). It is noted that the radial loss, which has been neglected in all previous discussions of this subject, is the largest of the three losses.

In the case of an infinite number of blades, the formulas for $\frac{d\epsilon}{d(x^2)}$ and $\frac{d\epsilon_t}{d(x^2)}$ can be integrated explicitly to give

$$\epsilon=1+\frac{\lambda^2}{1+\lambda^2}-2\lambda^2\log\left(1+\frac{1}{\lambda^2}\right)$$

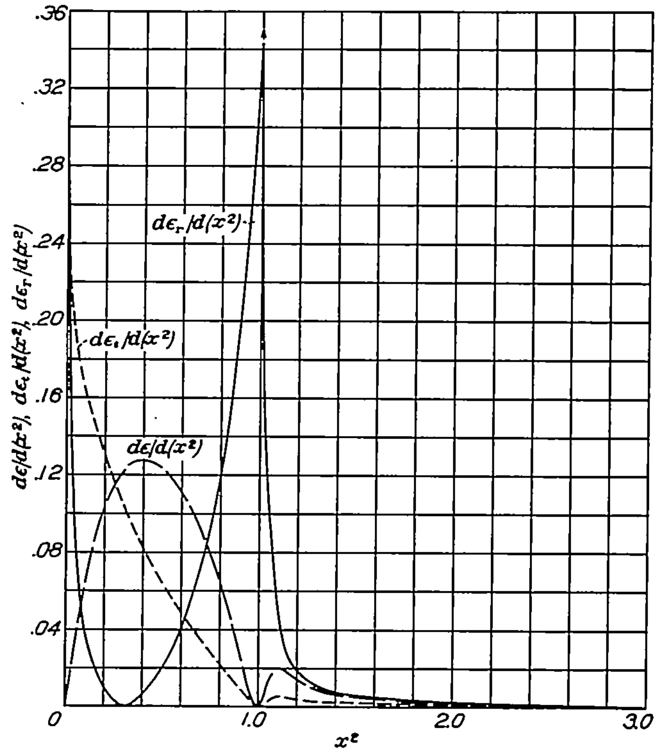

FIGURE 3.—Distribution of axial, tangential, and radial energy losses for two-blade propeller with $\lambda=\frac{1}{2}$.

$$\epsilon_t=-\frac{\lambda^2}{1+\lambda^2}+\lambda^2\log\left(1+\frac{1}{\lambda^2}\right)$$

and

$$\epsilon_r=0$$

The total energy loss is then

$$\kappa=\epsilon+\epsilon_t=1-\lambda^2\log\left(1+\frac{1}{\lambda^2}\right)$$

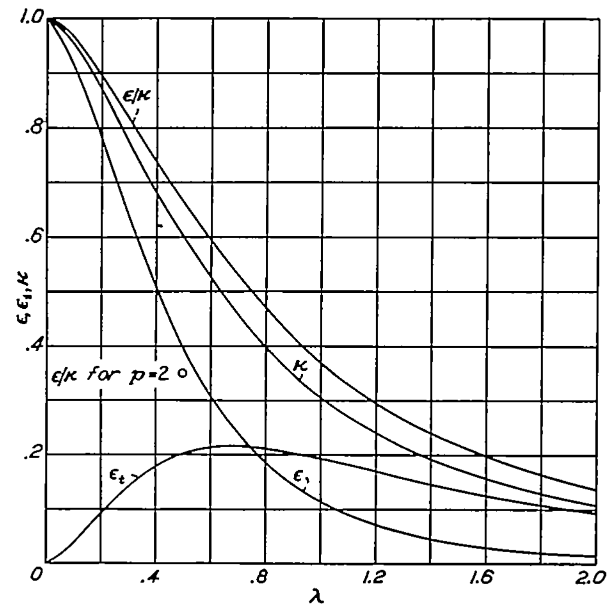

FIGURE 4.—Loss functions for an infinite number of blades.

The functions ϵ, ϵ_t, and κ are plotted against λ in figure 4. A plot of the function ϵ/κ is also shown in figure 4. The value of ϵ/κ for a two-blade propeller with $\lambda=\frac{1}{2}$, which is obtained from the given data, is shown by a point in figure 4. It is noted that, for the case of an infinite number of blades, ϵ/κ is slightly greater than κ; whereas, for the two-blade propeller with $\lambda=\frac{1}{2}$, $\frac{\epsilon}{\kappa}=0.35$ and $\kappa=0.2625$. The quantity ϵ/κ may therefore be tentatively estimated as somewhat greater than κ.

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