REPORT No. 382

ELASTIC INSTABILITY OF MEMBERS HAVING SECTIONS COMMON IN AIRCRAFT CONSTRUCTION

By GEORGE W. TRAYER and H. W. MARCH [1]

SUMMARY

Two fundamental problems of elastic stability are discussed in this report, which was prepared by the Forest Products Laboratory [2] for publication by the National Advisory Committee for Aeronautics. In Part I formulas are given for calculating the critical stress at which a thin, outstanding flange of a compression member will either wrinkle into several waves or form into a single half wave and twist the member about its longitudinal axis. A mathematical study of the problem, which together with experimental work has led to these formulas, is given in an appendix. Results of tests substantiating the recommended formulas are also presented. In Part II the lateral buckling of beams is discussed. The results of a number of mathematical studies of this phenomenon have been published prior to this writing, but very little experimentally determined information relating to the problem has been available heretofore. Experimental verification of the mathematical deductions is supplied in this report.

INTRODUCTION

Designing for the greatest load with a given amount of material in a compression member generally leads to the distribution of material at the greatest possible distance from the neutral axis of the member. The extent to which such distribution can be carried is limited by the possibility of secondary failure. Compression members with relatively wide and thin outstanding parts may fail through local wrinkling or through twisting about the longitudinal axis at loads considerably less than those that would be expected to cause the more common failures of crushing for short lengths or flexure for longer lengths. When such a compression member does fail, a thin, outstanding element may either break up into several waves (wrinkle) or may buckle into a single half wave, depending upon the length and the torsional resistance offered by the member of which it forms a part. Such action has been observed for years. (References 2, 14, 15, 18, and 21.)

Again, the strength of a beam increases more rapidly with depth than with thickness, and consequently in aircraft, where weight is such an important matter, designers customarily use comparatively deep, narrow beams. The ratio of depth to breadth, however, has been kept within certain arbitrary or conventional limits in commercial practice, because of the well-known fact that a beam much deeper than it is wide may buckle laterally and twist before it will break by bending in a vertical plane. As a matter of fact, there is for each condition of loading and support a critical buckling load for such a beam just as there is a critical Euler load for a long column.

Either buckling or twisting or both are likely to occur in one member or another of an aircraft structure, and hence failure of a particular member may be either in a normal type of bending or compression resulting from the normal loading or through lateral buckling, wrinkling, or twisting under stresses having their origin in the normal loading. Means of estimating the stress at which elastic instability is likely to occur have therefore become necessary in the close designing of the present day, in order to provide against secondary failure. Realizing this, the Bureau of Aeronautics, Navy Department, financed an investigation of fundamental phases of elastic instability to be conducted by the Forest Products Laboratory. Wood was used in the experiments, not that the problem is limited to any one material, but because of the convenience with which test specimens can be made of wood.

The wrinkling and twisting problem has been investigated mathematically for homogeneous, isotropic materials, and useful results have been obtained, notably by Timoshenko. (References 17 and 21.) This report reviews the general theory, adds an analysis that applies to nonisotropic material such as wood, and discusses the diminution of the critical stress caused by the elastic giving of the material at the base of the flange. The exact mathematical approach to the problem leads to rather complicated results; through consideration of test data, however, these results can be simply expressed for problems of practical interest.

The allied problem of the lateral stability of deep beams has already been investigated rather fully from a mathematical standpoint. The results of such work have been published by Michell, Prandtl, Timoshenko,

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[2] Maintained at Madison, Wis., in cooperation with the University of Wisconsin.

and others. (References 9, 11, 13, 17, 20, and 23.) This report adds experimental verification of the results already obtained.

TEST MATERIAL

Test specimens were made of Sitka spruce cut in Oregon and shipped in log form to the Forest Products Laboratory where the wood was sawed into lumber, marked, and seasoned. As a result of this procedure the history of each piece and its location with respect to others in the same log were known. Part of the lumber was immediately kiln-dried after sawing and part was left to air-dry. Specimens were made from both the kiln-dried and the air-dried stock.

In selecting pieces for test specimens, the usual Army and Navy specifications were adhered to with an additional limitation as to knots and pitch pockets in that none was permitted, no matter how small.

The elastic properties of the material in the various planks from which the major test specimens were taken were determined by testing small control specimens cut from the same planks and so located as to be representative. In certain instances it was possible to accomplish the same result by cutting the control specimens from uninjured portions of the major test specimens after the main test had been completed. In other instances such properties as the stiffness in bending and the torsional rigidity of major test specimens were determined by a secondary test of the major specimens themselves either before or subsequent to the main instability test. In such secondary tests the stresses were kept well below the elastic limit and when they were made the usual control tests served only as a check.

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PART I

THE WRINKLING AND TWISTING OF COMPRESSION MEMBERS HAVING THIN, OUTSTANDING FLANGES

METHOD OF TEST

WRINKLING TESTS

Two principal series of wrinkling tests were made on compression members having thin, outstanding flanges. In one series, a number of specimens, all having a single flange of the same size, were tested under a compressive load and the half wave length and the load at which wrinkling started were recorded. The outstanding flanges were then reduced in width a given amount with the thickness left as before and the specimens were again tested. This procedure was continued until the widths had been so reduced that wrinkling did not occur.

In the other principal series of tests the width of flange was kept constant and the thickness was reduced after each test. Several specimens were used in order to obtain reliable averages for the half wave length and the wrinkling stress corresponding to each thickness. Figure 1 shows a specimen in the testing machine.

In addition to the two principal series of tests, a number of tests were made on built-up U, I, and +, sections under axial compression.

TWISTING TESTS

The set-up for the twisting tests is shown in Figure 2. Extension screws were attached to an ordinary 4-screw testing machine in which specimens up to several feet in length could then be handled. This set-up was used only to obtain maximum load. To obtain a load-twist curve, a 2-screw machine was used, one that could take specimens up to about 12 feet in length without the use of extension screws. A pointer approximately 3 feet in length was attached to one flange and in some instances to two flanges. As the column twisted, the end of the pointer passed over a plane table supported from the base of the testing machine and when increments of load were read by the operator at the balance beam the position of the pointer was marked and the load set opposite such marking.

Prior to the twisting test each specimen was tested in torsion in order to obtain the torsional rigidity of the member. The stresses were kept well within the elastic limit during this test.

ANALYSIS OF THE WRINKLING AND TWISTING PROBLEM

The failure of compression members that contain wide, outstanding parts, as illustrated in Figure 3, may be brought about through wrinkling of the outstanding parts themselves instead of through the normal failure of the member as a whole, if the outstanding parts are sufficiently thin. When such wrinkling occurs, the outstanding flange may either break up into a single half wave or into more than one, depending upon the torsional rigidity of the member and the fixity of the flanges. If an outstanding flange projects from a member that is high in torsional stiffness, wrinkling into several waves is likely to occur if the ratio of the outstanding width to the thickness of the flange is great. On the other hand, if the torsional stiffness is not great, the outstanding part or parts may form into single half waves and twist the member about its longitudinal axis. The critical values of the stresses at which one or the other type of buckling occurs are discussed in the following paragraphs.

WRINKLING

A mathematical approach to the wrinkling problem is given in the appendix, where it is shown that the critical value of the compressive stress p for a plate perfectly fixed along one edge, free along the opposite edge, and simply supported along the ends to which the load is applied is given by

$$p=kE\frac{h^2}{b^2} \tag{1}$$

in which h is the thickness of the plate, b its width, E the modulus of elasticity of the material, and k a coefficient depending upon the ratio of the length of plate a to the width b.

The appendix shows further that for structural steel the calculated minimum value of k is 1.16 and corresponds to a ratio of a to b of 1.6 or a multiple thereof. (Reference 21.) The theoretical formula for the minimum critical stress for steel would therefore be

$$p=1.16E\frac{h^2}{b^2} \tag{2}$$

The mathematical analysis, as already pointed out, at first assumes perfect fixity at the base of the outstanding flange, a condition probably never realized in actual practice. Consequently a critical stress much lower

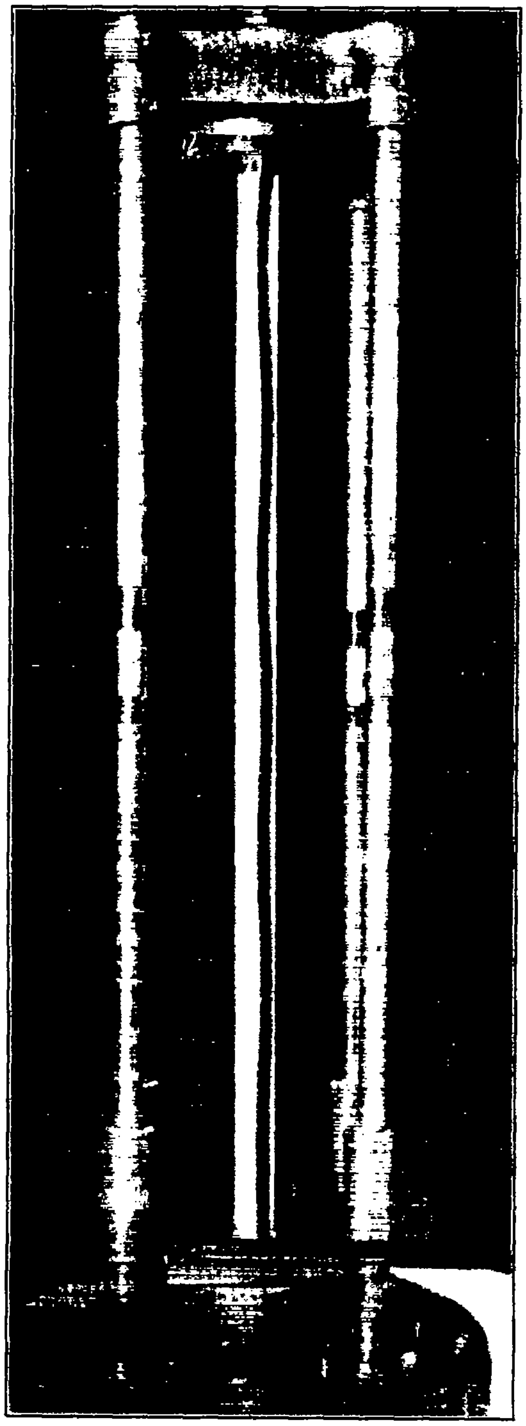

FIGURE 1.—The wrinkling under load in the testing machine of a compression member having a single thin, outstanding flange

than that predicted by the theory is to be expected. Roark, who used specimens like B and C of Figure 3, in which the outstanding flange was clamped between angles, found that the formula

$$p=0.6E\frac{h^2}{b^2} \tag{3}$$

represented his experimental results reasonably well. (References 14 and 15.) The reduction of the coefficient from 1.16 to 0.6 can be attributed to the lack of perfect fixity at the base of the flange. Even when an outstanding flange and the rigid back from which it projects are all in one piece, perfect fixity at the base

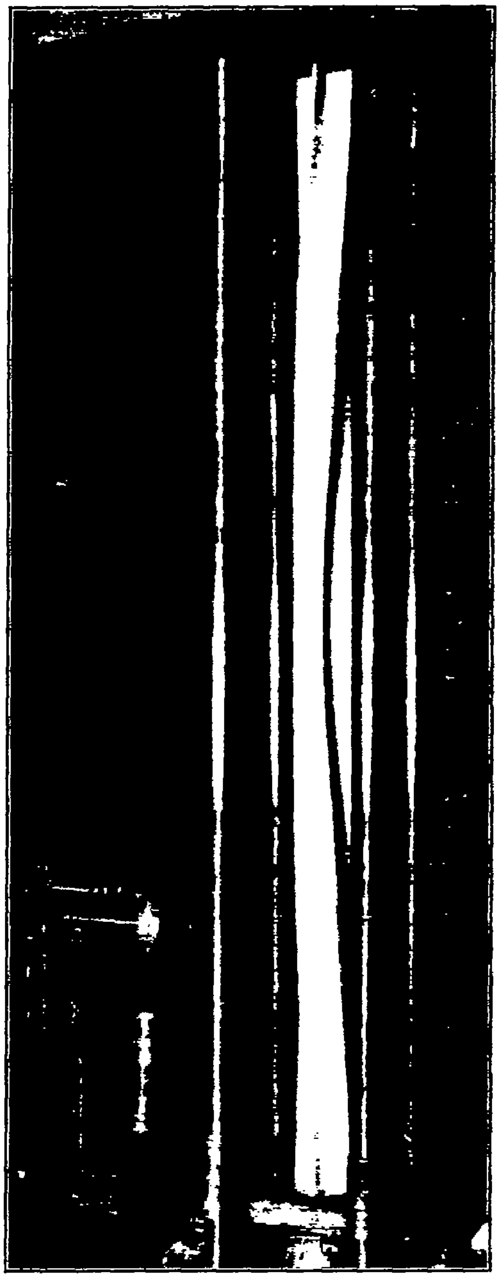

FIGURE 2.—The twisting under load in the testing machine of a compression member having several thin, outstanding flanges

of the flange can not be assumed. There is an elastic giving at the base of the plate and also in every device used in an attempt to obtain perfect fixity. Hence the exact coefficient that should be used for steel and other metals remains to be determined by experiment. A discussion of the situation for wood follows.

The appendix shows that, on the basis of the differential equation of a nonisotropic elastic plate, such as wood, a critical half wave length and a critical stress may be calculated. The same mathematical work, however, also shows that the values of the half wave length and the critical stress vary over a wide range as the inclination of the growth rings to the faces of the outstanding flange varies from 0° to 90°.

The fact that perfect fixity at the base of the flange, as at first assumed in the mathematical study, can not be obtained is true particularly of wood, which further complicates the problem. The stresses at the base of the flange resulting from the bending of the flange are acting perpendicularly to the grain of the wood, the direction in which wood is weakest.

The appendix shows that the critical stress for a quarter-sawn flange of spruce perfectly fixed at the edge is

$$p=0.228E\frac{h^2}{b^2}$$

For a similar flange with growth rings at 45° to the faces the critical stress is

$$p=0.117E\frac{h^2}{b^2}$$

Because of the elastic giving of the material at the base of the flange, however, there is a great reduction in the actual critical stress. Furthermore, this elastic giving tends to decrease the difference between the critical stresses for flanges with growth rings at 45° and 90°, respectively. Tests gave as the reduced coefficient 0.07 for spruce flanges, and the expression for the critical wrinkling stress then becomes

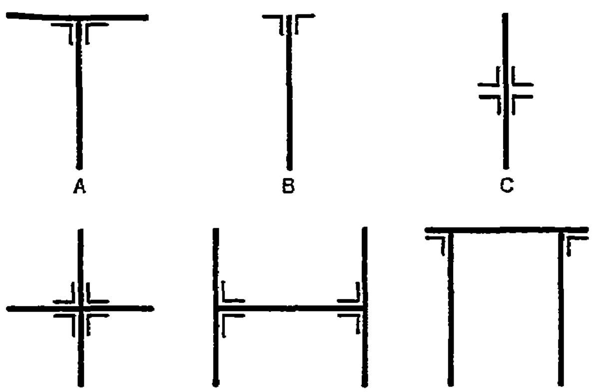

FIGURE 3.—Typical cross sections of compression members that have wide, thin, outstanding parts

$$p=0.07E\frac{h^2}{b^2} \tag{4}$$

Probably this coefficient may be applied to other species without appreciable error.

In Figure 4, in which wrinkling stress is plotted against the ratio of flange width to thickness, are shown the results of some actual tests. Each circle represents the average of from 4 to 18 values. The results have been adjusted by direct proportion to correspond to a modulus of elasticity along the grain of 1,600,000 pounds per square inch. The full line is the locus of the expression

$$p=0.07\times 1{,}600{,}000\frac{h^2}{b^2} \tag{5}$$

No record of the angle between the growth rings and the faces of the flange, the importance of which has been mentioned, was made at the time of test, but full-section blocks from many of the test specimens were saved and the angle was subsequently measured. The direction of the rings ranged from 45° to 90°, as it does in what may be called commercial edge-grain (quarter-sawn) stock. The test specimens, therefore, represent what would be found in actual practice. The variation in the test results is accounted for by the variation in the direction of the growth rings and the difficulty of determining accurately just when wrinkling started.

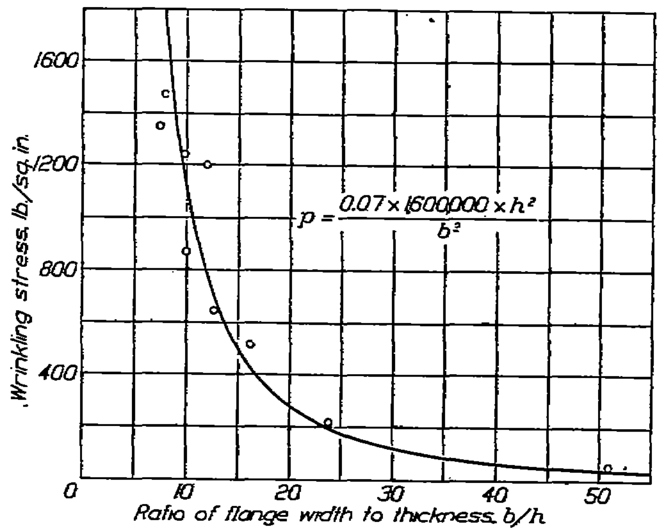

FIGURE 4.—The relation between the ratio of flange width to thickness and the wrinkling stress of thin, outstanding flanges

Since the phenomenon of wrinkling is one to avoid in good design, it is unnecessary to calculate the critical stress with extreme precision. Merely a fair approximation of the critical stress is sufficient to make sure that for the width and thickness of flange used the critical wrinkling stress will exceed the primary stress expected from the normal loads. Slightly superior design in this regard will seldom mean an appreciable sacrifice in load-weight ratio.

Length of outstanding flange.

The coefficient k in the expression for critical wrinkling stress is a minimum when the ratio of the length of plate a to the critical half wave length c is an integral number. If the plate is short and a/c is not an integer, the critical stress may be considerably greater than that given by the formula because the flange can not then break into the ideal half wave length. If the length is great, that is, if a/c is greater than 2 or 3, and the ratio a/c is not an integer, however, the critical stress will be only slightly above that given by the formula, since the plate can then break into a half wave length very close to the ideal. In either case, the formula will give values on the side of safety. For greater detail see Tables VI, VII, VIII, XIII, and XIV and Figures 23 and 24 in the appendix.

TWISTING

It is shown in the appendix that the critical buckling stress for a long steel plate simply supported along one side, free along the other side, and simply supported at the ends, to which load is applied, is given by

$$p=0.385E\frac{h^2}{b^2} \tag{6}$$

when Poisson's ratio is taken as 0.3.

For spruce the coefficient of equation (6) becomes 0.044 which, as explained in the discussion of equations (87), (88), and (89) of the appendix, is a good, average figure for this species. This value of the coefficient probably may also be applied to other species with sufficient accuracy. The critical stress is then given by

$$p=0.044E\frac{h^2}{b^2} \tag{7}$$

If a member with a section like D of Figure 3 is subjected to compressive stress, the outstanding flanges will usually form into a single half wave at a certain critical stress and in so doing will twist the member about its longitudinal axis. When such action occurs, the outstanding elements are essentially acting as plates simply supported on one side and free along the opposite side, and formulas (6) or (7) are used to calculate the critical stress.

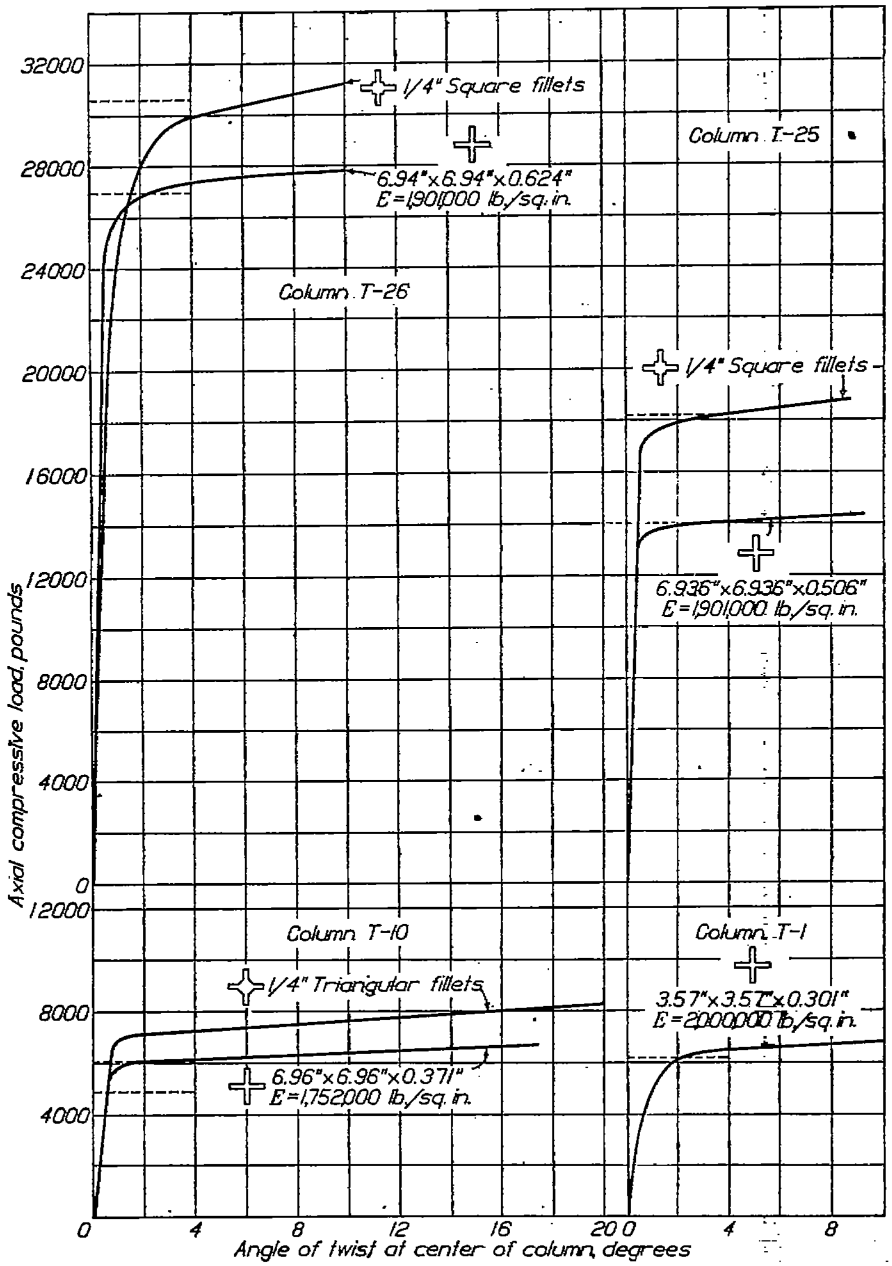

FIGURE 5.—The relation between angle of twist at the center of a column and axial compressive load for various cruciform cross sections

Members with I, H, or U sections, such as E and F of Figure 3, likewise may twist under compressive loads if the torsional rigidity of the section is not great. If the torsional rigidity is made large by using generous fillets or, as with a U section, by making the back considerably heavier than the legs, failure through wrinkling into several waves may be brought about and the critical stress in such cases must be computed by the formulas applying to that phenomenon.

Actually, the rigidity of the member may be such that failure will take place at a critical stress intermediate between the minimum twisting stress and the wrinkling stress, as pointed out in the appendix. It is extremely difficult, however, to calculate accurately the coefficient for the intermediate conditions. Consider for the moment wood members with a section like D of Figure 3. With no fillets at the junction of the four legs, the coefficient 0.044 was found to apply. As fillets were added, the critical stress increased in practically the same ratio as the torsional rigidity. A U section, such as F of Figure 3, will twist at a stress corresponding to a coefficient of 0.044 if the back and the legs are of the same thickness. If the thickness of the back is increased or if fillets are added, the critical stress will increase in about the same ratio as the torsional rigidity.

Hence the Forest Products Laboratory recommends that the critical twisting stress be first calculated for such sections as D, E, and F of Figure 3 on the supposition that no fillets are present and that all parts are of the same thickness. This stress should then be increased by multiplying it by the ratio of the torsional rigidity of the actual section to the torsional rigidity of the assumed section. This rule applies until the limiting critical stress corresponding to the coefficient 0.07 is reached.

In Figure 5 are shown a number of cruciform sections, some with and some without fillets. Wood columns having these sections failed through twisting about a longitudinal axis. Accompanying each section is a graph showing the relation between axial load and the angle of twist for the column corresponding to it. The horizontal dotted lines in these graphs are drawn at the critical loads calculated in accordance with the preceding recommendations.

For example, the critical stress for column T–25 (fig. 5) without fillets is given by

$$p=0.044\times 1{,}901{,}000\frac{(0.506)^2}{(3.215)^2}$$

$$=2{,}072 \text{ pounds per square inch.}$$

The area is 6.76 square inches and the critical load becomes

$$p=2{,}072\times 6.76=14{,}000 \text{ pounds.}$$

As a further illustration, the critical twisting stress for column T–25 (fig. 5) with ¼-inch square fillets is calculated thus:

The torsion constant K for the section without fillets is

$$K=2\times 0.318\times 6.936\times (0.506)^3=0.572.$$

For the section with fillets K must be calculated in three parts—the first part is the value K_1 for the square central portion of the column section, the dimensions of which are 1.006 inches on each edge; the second part is the total value K_2 for the four rectangles projecting from the square center; and the third part is the increase K_3 caused by the four junctions. (Reference 22, p. 26, and 1929 annual report, p. 696.) The junctions are treated as T junctions and the bar of each T is taken as half of the square center. The torsion constant is then the sum of the parts, which are calculated as follows:

$$K_1=\frac{(1.006)^4}{7.11} \qquad =0.144$$

$$K_2=2\times 0.315\times 2\times 2.965\times (0.506)^3=0.484$$

$$K_3=4\times 0.15\times (0.629)^4 \qquad =0.094$$

$$K \qquad \overline{=0.722}$$

Then

$$\frac{\text{Torsional rigidity with fillets}}{\text{Torsional rigidity without fillets}}=\frac{0.722}{0.572}=1.26.$$

Actual tests of the specimens, made prior to the twisting tests, yielded a ratio of 1.29.

$$p=2{,}072\times 1.26=2{,}610 \text{ pounds per square inch}$$

$$\text{Area with fillets}=7.01 \text{ square inches}$$

$$\text{Critical load } P=2{,}610\times 7.01=18{,}300 \text{ pounds.}$$

In figure 6 are shown a number of U sections of columns that failed through twisting about a longi-

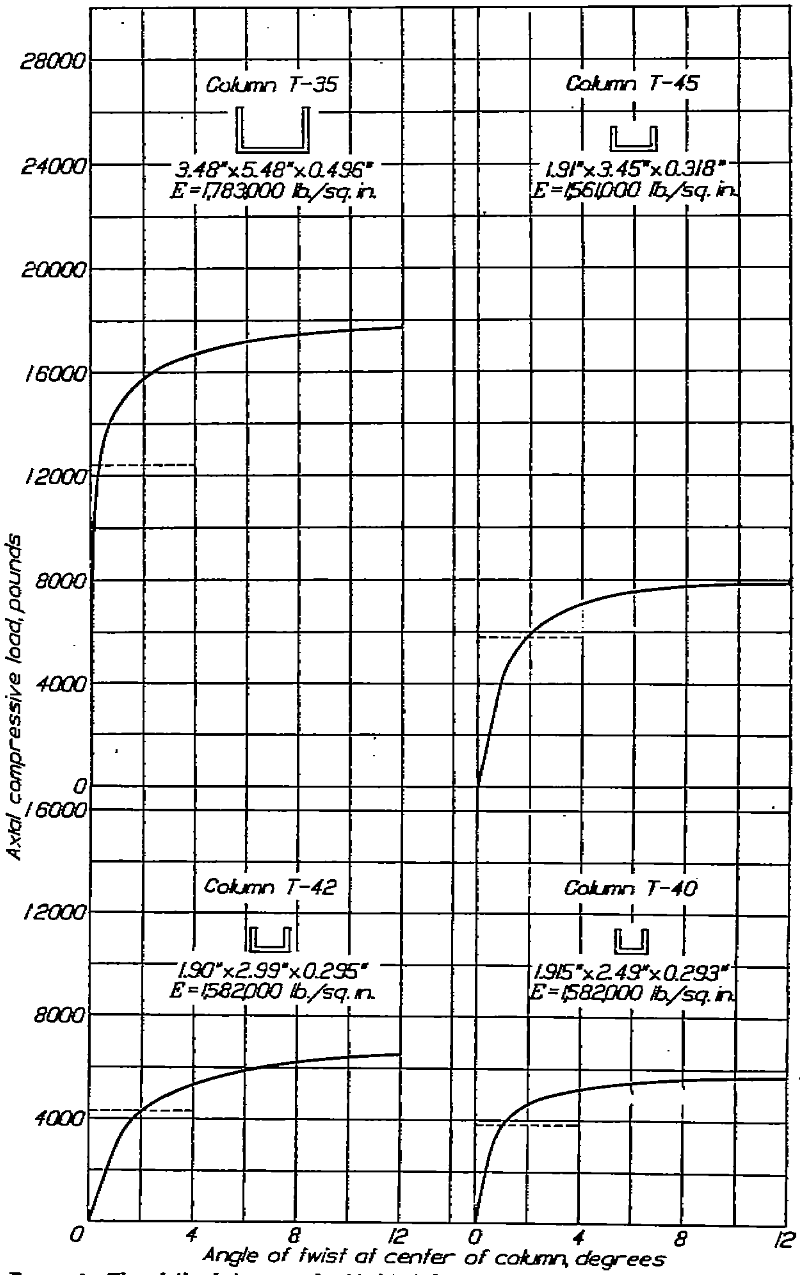

FIGURE 6.—The relation between angle of twist at the center of a column and axial compressive load for various channel cross sections

tudinal axis. Accompanying graphs show the relation between the axial load and the angle of twist. The horizontal dotted lines are drawn at the critical loads calculated by formula (7).

The agreement between tests results and calculated results as shown in Figures 5 and 6 is considered quite satisfactory.

Effect of length.

In arriving at the coefficients 0.385 for steel and 0.044 for spruce, which are used in the critical-stress formula for free twisting, the length of plate was assumed as several times the outstanding width. This assumption gives the lower limit for the critical stress. As the length is decreased to less than five or six times the width, these coefficients increase appreciably. Consequently, if the legs of a channel section, for example, are supported at intervals as by bracing and the distance between points of support is less than five or six times the width of the legs, the actual critical stress will be higher than that given by the proposed formulas.

CONCLUSIONS FOR PART I

Thin, outstanding flanges of compression members under load may buckle into several waves or may buckle into a single half wave, in which event they will tend to twist the member about its longitudinal axis.

If both the length and the torsional rigidity of the member are great such flanges will buckle into several waves (wrinkle) and the critical stress for spruce flanges is then given by

$$p=0.07E\frac{h^2}{b^2}$$

If the torsional rigidity of the member is not great, the thin, outstanding flanges will twist the member. Under such rigidity the flanges may be regarded as plates simply supported on three edges and free along the fourth edge. The critical stress for such a spruce plate is given by

$$p=0.044E\frac{h^2}{b^2}$$

Although the coefficients in the preceding formulas were obtained from the test of spruce flanges, the relations among the elastic constants for the various species are such that the coefficients may be expected to apply to all aircraft woods with safety.

Members having sections as shown in Figure 3 will twist under axial compression if the junction of the main elements is not strengthened with fillets. If generous fillets are used or if part of the main elements of the section are made heavier than the rest, the thin, outstanding elements may either wrinkle or twist the member, this depending upon the amount of torsional rigidity added. Elastic instability, therefore, may occur at a stress intermediate between the critical stresses corresponding to the coefficients 0.044 and 0.07. Intermediate critical stresses may be calculated by the rules given in this report.

Failure through local buckling can occur only when the critical stress is less than the stress required to cause primary failure.

Further conclusions, including calculated coefficients for steel, follow the mathematical appendix.

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PART II

THE LATERAL BUCKLING OF DEEP BEAMS

METHOD OF TEST

VARIATION OF FACTORS AFFECTING THE BUCKLING LOAD

In order to determine to what degree certain factors affect the critical load for lateral elastic instability of deep beams, tests were made in which all factors except one were held constant while the isolated factor was varied. In these tests the beams rested on two supports with their ends held vertical and clamped against lateral rotation but free to rotate in a longitudinal-vertical plane as the beam deflected. Load was applied at the center by means of the rod-and-bar framework shown in Figure 7.

The loading device consisted of five parts. A rod with an upset central portion passed through the beam at the neutral axis. The upset portion was threaded at each end so that the rod could be centered in the beam by means of two nuts, which were drawn snug against the sides of the beam during test. Slotted bars, the lateral positions of which were fixed by V's in the upset rod, connected each end of the rod to the ends of an evener bar and from the center of this evener bar a tiebar passed through the movable head of the testing machine and was pin-connected to it on the under side. All connections other than the pin connection mentioned were knife-edge.

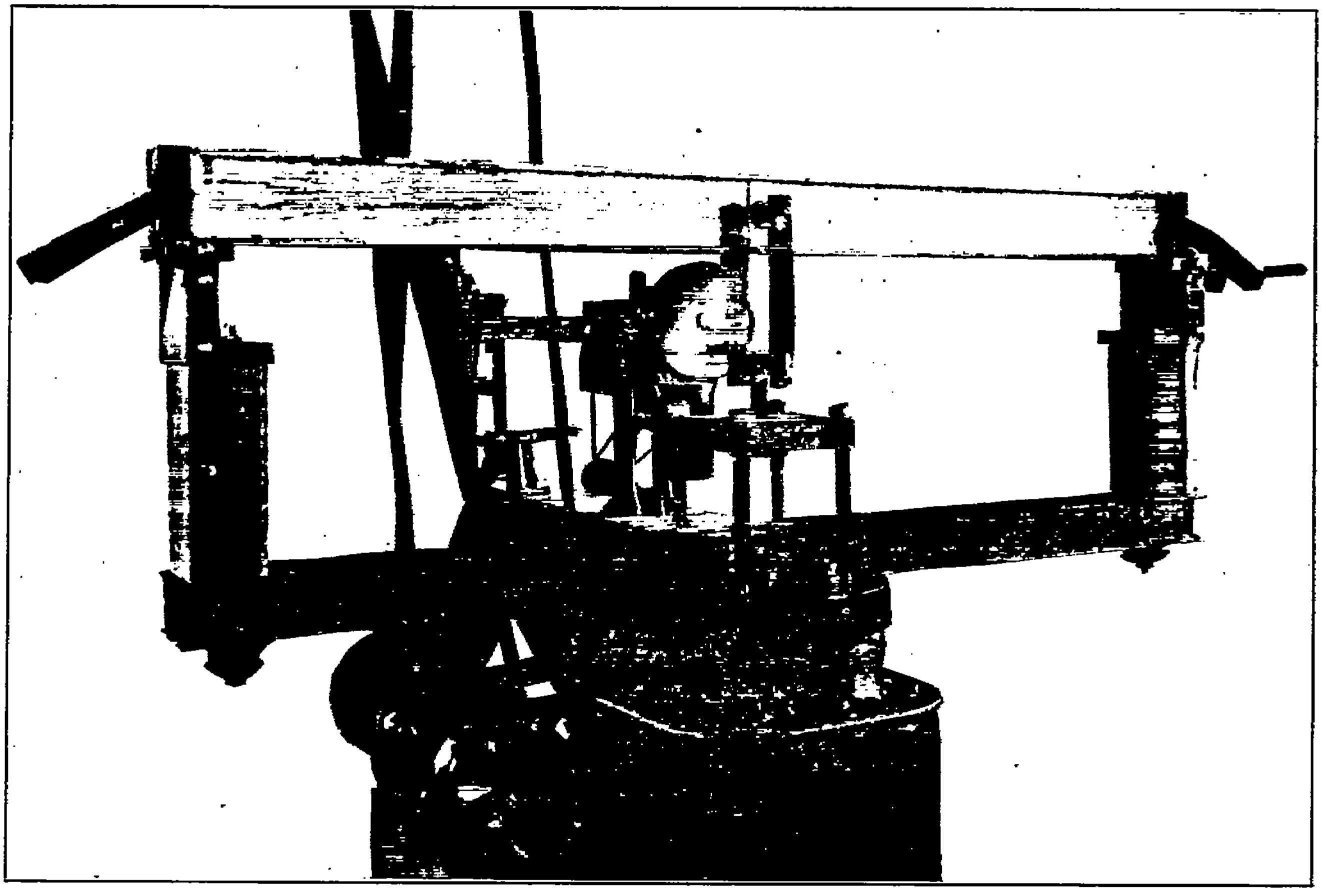

FIGURE 7.—The set-up of the test for lateral elastic instability of a single beam under center loading

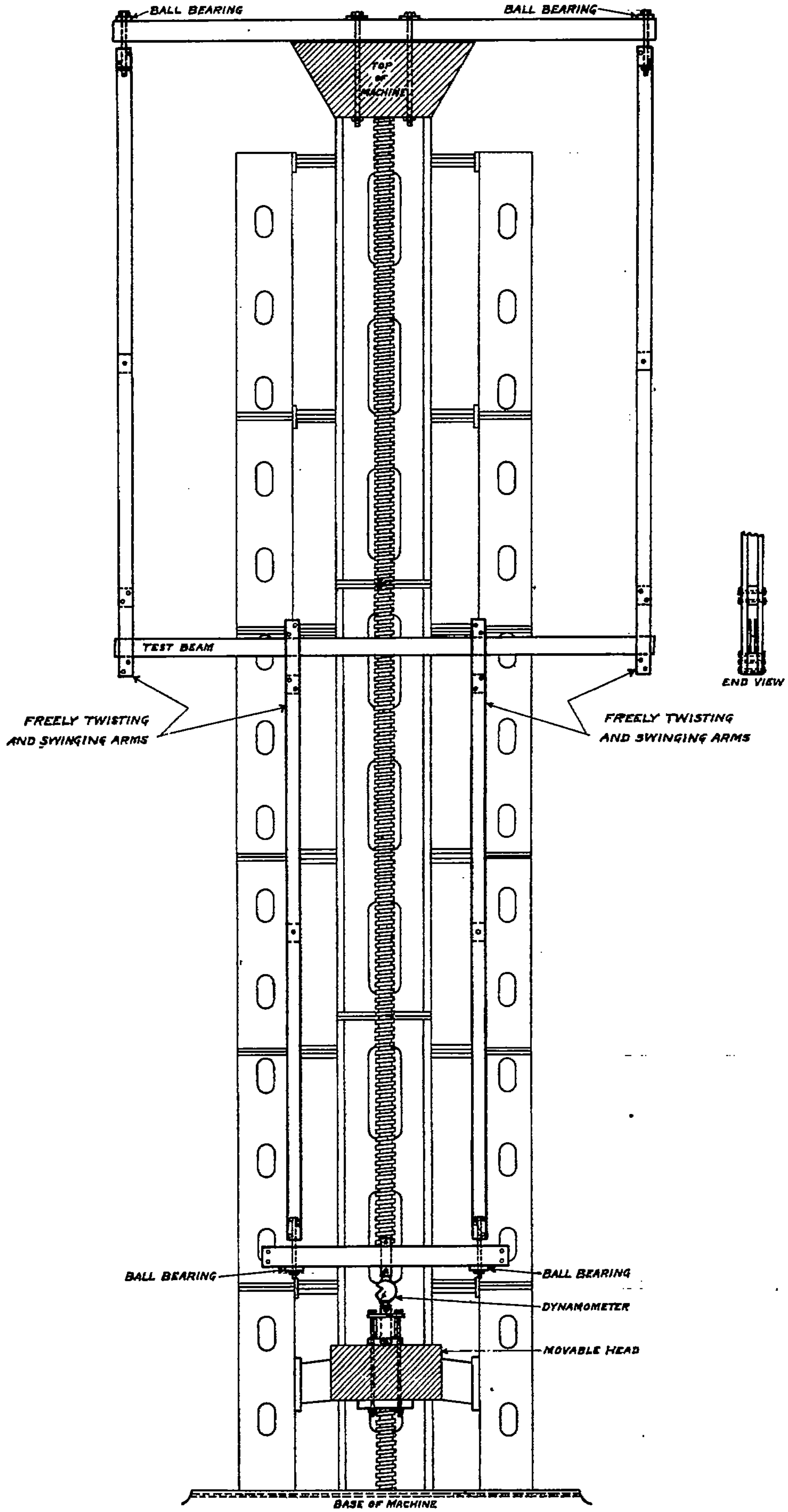

FIGURE 8.—The set-up of the test for critical buckling load under constant bending moment with the ends of the single beam held vertical and not restrained laterally

When load was applied by lowering the movable head, the beam could buckle freely to one side or the other. The set-up was not considered satisfactory until the beam buckled to one side and then to the other with the slightest adjustment of the rod by means of the two nuts.

TESTS OF SINGLE BEAMS UNDER VARIOUS LOADING CONDITIONS

Three different loading and fixity conditions were chosen to demonstrate the applicability of the formulas recommended for the calculation of critical buckling loads. These conditions were: First, constant bending moment with the ends of the beam held vertical and not restrained laterally; second, constant bending moment with the ends of the beam held vertical and restrained laterally; and third, a concentrated load at the center of a beam that rested on two supports with its ends both held vertical and restrained laterally.

Constant bending moment without lateral fixity was obtained by considering only the portion of a beam that was between two symmetrical loads. A total span of 14 feet was used and the two symmetrical load points were 60 inches apart. In order to permit the beam to swing freely, both supports and loads were applied through members, 16 feet long, that were free to swing and twist. The beams were wedged into these long members, which were slotted and of sufficient rigidity to hold the beams vertical. The two loading members were attached to an evener timber, which in turn was attached to the movable head of a testing machine with a tie bar. The set-up required head room of approximately 35 feet. A diagrammatic sketch of this set-up is shown in Figure 8.

Constant bending moment with lateral fixity was obtained by using a symmetrical 2-point loading and again considering only the portion of the beam between the loads. For this condition, improvised extension wings were put on a 30,000-pound capacity testing machine that permitted spans up to 16 feet. Load was applied at two symmetrical points, in some tests 5 and in other tests 6 feet apart. In order to obtain as complete lateral fixity as possible at the load points, lateral, horizontal, pin-connected tie rods were attached to the beam at intervals between the load points and the supports. In addition, pieces 1⅝ inches thick and about 6 inches deep were clamped to both sides of the beam from each load point outward and well toward the support. Figure 9 shows this assembly.

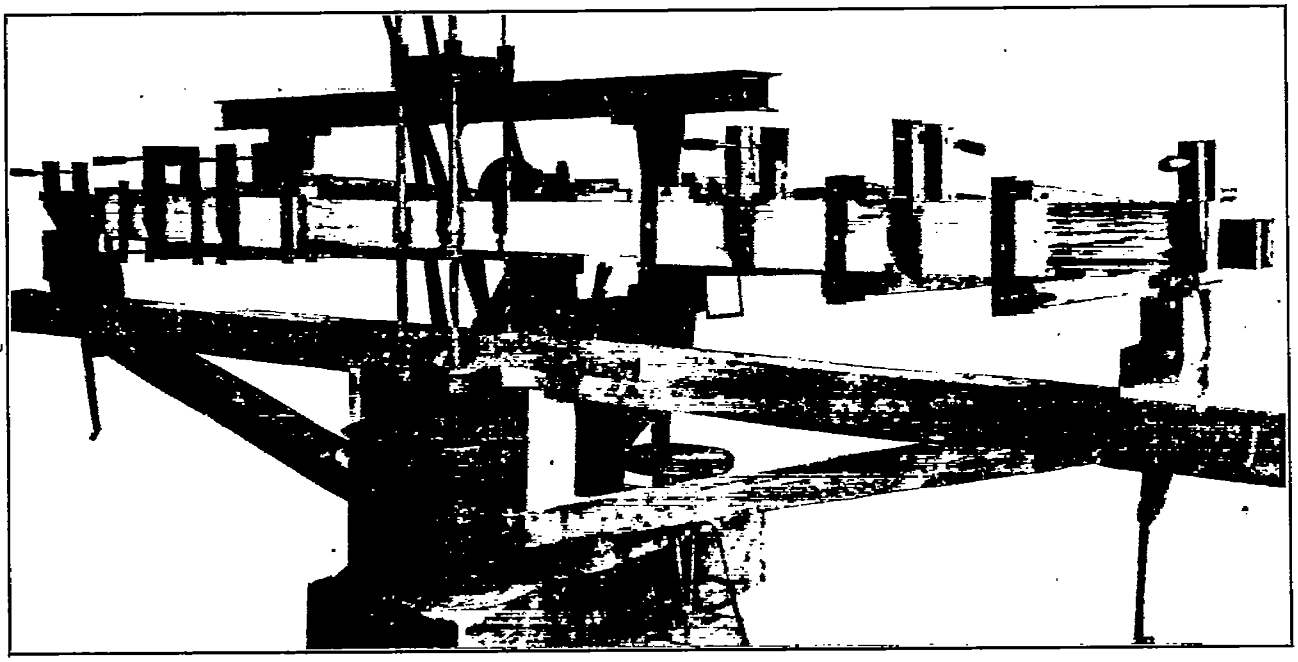

FIGURE 9.—The set-up of the test for critical buckling load under constant bending moment with the ends of the single beam held vertical and restrained laterally

The third method of test, namely, the application of a concentrated load at the center of a beam resting on two supports with its ends held vertical, was identical with the test procedure described under the heading, Variation of Factors Affecting the Buckling Load.

TESTS OF PANELS

Panels consisting of two beams held together with ribs were tested in two ways. The first method was to suspend the two beams on hanging supports 16 feet long and to apply load to each beam at two symmetrical points as just described for the testing of single beams under constant bending moment with ends held vertical and not restrained laterally. The second method was to support the two beams on four ball bearings and to apply a uniformly distributed load over the ribs themselves; in doing this strips were laid on the ribs upon which cans filled with sand were placed. The ends of the beams were held vertical at the supports but because of resting on ball bearings were not restrained laterally. Figure 10 shows a panel before test.

ANALYSIS OF THE LATERAL BUCKLING PROBLEM

A mathematical analysis of the lateral elastic instability of deep rectangular beams leads to the following general expression:

$$P=\frac{F\sqrt{EI_2GK}}{L^2} \qquad (8)$$

in which

P = the critical buckling load

E = the modulus of elasticity along the grain

I_2 = the moment of inertia about the principal vertical axis

G = the modulus of rigidity in torsion

K = the torsion constant of the section

L = the span

F = a constant depending upon the loading and fixity conditions.

(References 9, 11, 13, 17, 20, and 23.)

If b is taken as the width of beam and d the depth, I_2 in equation (8) becomes

$$I_2=\frac{db^3}{12}$$

and the torsion constant K is expressed as follows:

$$K=\beta\, db^3 \qquad (9)$$

in which β is a constant depending upon the ratio of d to b. Table I gives the values of β for various ratios of d to b.

TABLE I

THE FACTOR β FOR CALCULATING THE TORSIONAL RIGIDITY OF RECTANGULAR PRISMS

Ratio of sides, d/b	β	Ratio of sides, d/b	β
1.00	0.14058	2.25	0.24012
1.05	.14744	2.50	.24936
1.10	.15398	2.75	.25696
1.15	.16021	3.00	.26332
1.20	.16612	3.50	.27331
1.25	.17173	4.00	.28081
1.30	.17707	4.50	.28665
1.35	.18211	5.00	.29135
1.40	.18690	6.00	.29832
1.45	.19145	7.00	.30332
1.50	.19576	8.00	.30707
1.60	.20374	9.00	.30999
1.70	.21093	10.00	.31232
1.75	.21428	20.00	.32283
1.80	.21743	50.00	.32913
1.90	.22332	100.00	.33123
2.00	.22868	∞	.33333

Figure 12 shows the results of one representative series of these tests. The circles represent test values and the full line is the locus of equation (11). Again the agreement between actual test results and theory is considered good.

In the third series of tests, the span L was varied while all other factors were held constant. The buckling load for this condition reduced to

$$P=\frac{C_3}{L^2} \tag{12}$$

In Figure 13 are shown the results of two representative series of these tests. Again the circles represent actual test values and the full lines the respective loci of equation (12) for the two beams selected.

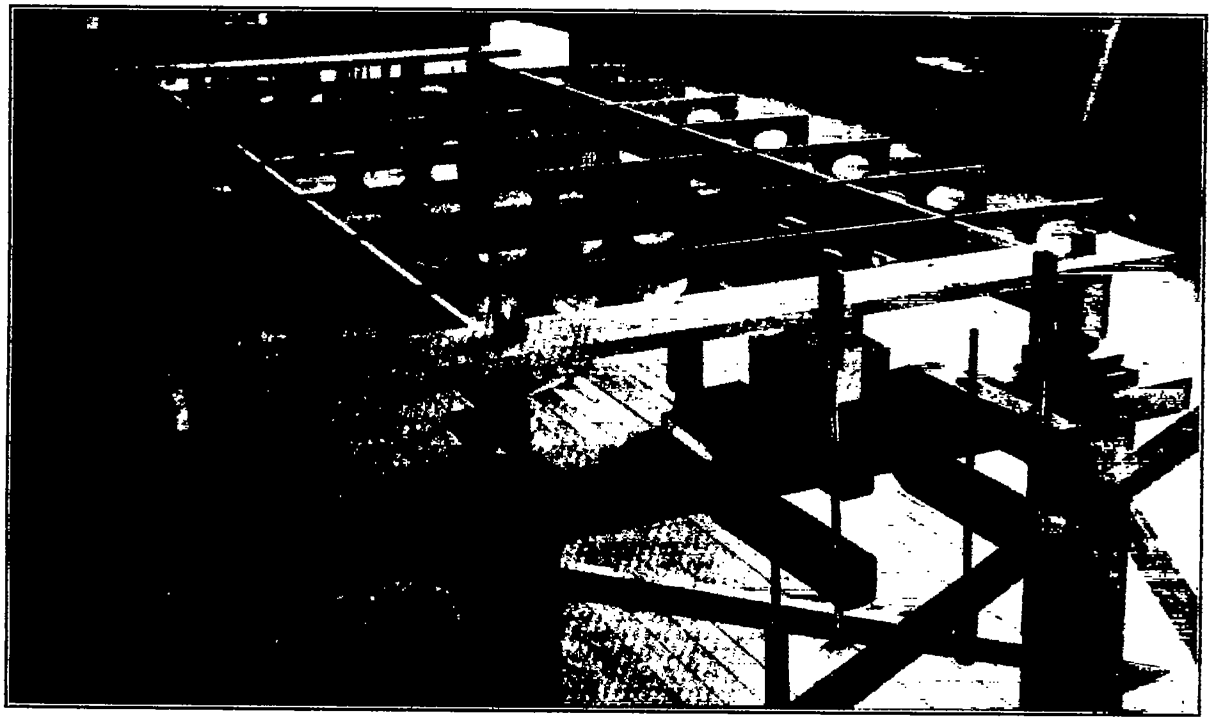

FIGURE 10.—A ribbed panel before test

In the first series of tests to check the relation of the various factors in the general equation, all factors except the depth of beam (d) were held constant. The buckling load then reduces to

$$P=C_1 d\sqrt{\beta} \tag{10}$$

in which C_1 is a constant. In Figure 11 are plotted the results of four series of tests in which d was varied while all other factors were held constant. The circles represent the actual loads and the full lines are loci of equation (10). The agreement is considered very satisfactory.

In the second series of tests, the width b was varied while all other factors were kept constant. The buckling load in this case becomes

$$P=C_2 b^3\sqrt{\beta} \tag{11}$$

The effect of the modulus of elasticity in bending could not be separated from that of the modulus of rigidity in torsion for the purpose of checking further the fundamental expression, because when one is changed the other changes with it, and therefore neither could be isolated. Moreover, it was impossible to ascertain experimentally with wood alone the importance of their combined effect on buckling load because the range over which their product varies is too limited. For steel, the modulus of rigidity in torsion is commonly taken as two-fifths of the modulus of elasticity in bending while for spruce it is in the neighborhood of one-fifteenth or one-sixteenth. Since some previous tests of steel beams have shown excellent agreement with critical values calculated by the formulas, it therefore appeared logical to assume that, if tests of wooden beams also checked values given by the formulas, the moduli of elasticity in bending and of rigidity in torsion are in their right relation in the formula. (References 6 and 9.)

Following are formulas that apply to rectangular beams under various loading and fixity conditions.

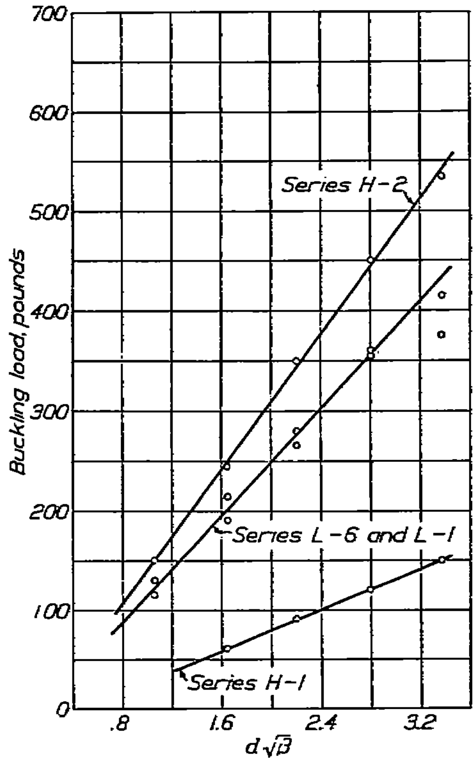

FIGURE 11.—The relation between the lateral buckling load and the depth of beam modified by a torsion correction factor ($d\sqrt{\beta}$), for deep, rectangular beams

In all cases the ends of the beam are assumed to be vertical. An end not restrained, in the terminology

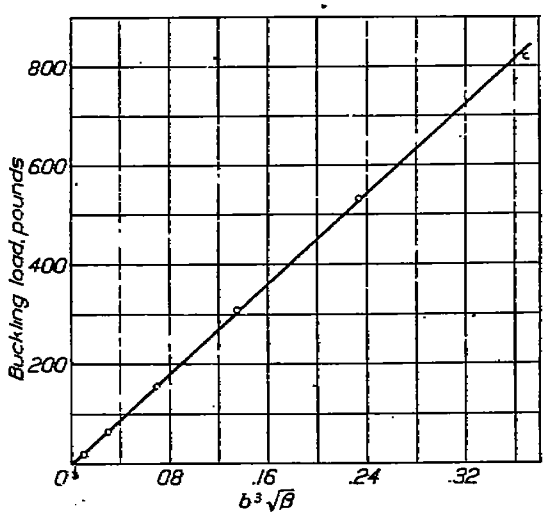

FIGURE 12.—The relation between the lateral buckling load and the cube of the width of beam modified by a torsion correction factor ($b^3\sqrt{\beta}$), for deep, rectangular beams

used, is held vertical but is not otherwise constrained, and an end restrained is both held vertical and clamped against lateral rotation. Figure 14 shows the lateral deflection of the longitudinal axis for three principal conditions of restraint.

CASE 1.—A thin, deep, rectangular beam under constant bending moment M, with its ends not restrained.

$$M=\frac{\pi\sqrt{EI_2GK}}{L}$$

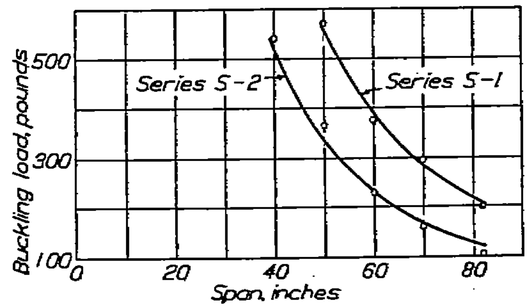

FIGURE 13.—The relation between the lateral buckling load and the span, for deep, rectangular beams

CASE 2.—The same as case 1 except that the ends are restrained.

$$M=\frac{2\pi\sqrt{EI_2GK}}{L}$$

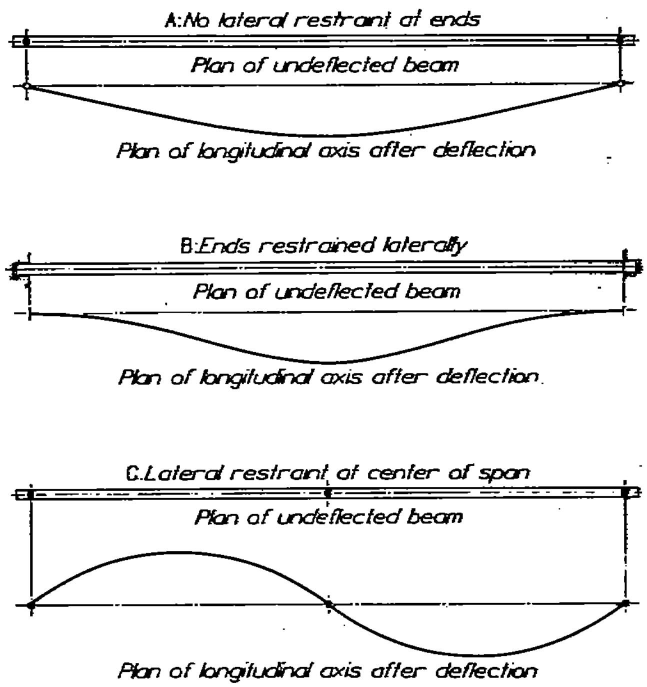

FIGURE 14.—The lateral deflection of the longitudinal axis of a single rectangular beam when the bending in a vertical plane becomes unstable and sidewise buckling occurs

CASE 3.—A thin, deep, rectangular cantilever with a concentrated load P at the end.

$$P=\frac{4\sqrt{EI_2GK}}{L^2}$$

CASE 4.—A thin, deep, rectangular cantilever with a uniformly distributed load W.

$$W=\frac{12.9\sqrt{EI_2GK}}{L^2}$$

CASE 5.—A thin, deep, rectangular beam supported at the ends and carrying a concentrated load P at the middle, with its ends not restrained.

$$P=\frac{16.9\sqrt{EI_2GK}}{L^2}$$

CASE 6.—The same as case 5 except that the ends are restrained.

$$P=\frac{25.9\sqrt{EI_2GK}}{L^2}$$

CASE 7.—A thin, deep, rectangular beam supported at its ends and carrying a uniformly distributed load W with its ends not restrained.

$$W=\frac{28.3\sqrt{EI_2GK}}{L^2}$$

CASE 8.—The came as case 7 except that the ends are restrained.

$$W=\frac{43.3\sqrt{EI_2GK}}{L^2}$$

CASE 9.—A thin, deep, rectangular beam subjected to a constant bending moment M and an axial thrust P', with its ends not restrained.

$$M=\frac{\pi\sqrt{EI_2GK}}{L}\sqrt{1-\frac{P'L^2}{\pi^2EI_2}}$$

CASE 10.—The same as case 9 except that the ends are restrained.

$$M=\frac{2\pi\sqrt{EI_2GK}}{L}\sqrt{1-\frac{P'L^2}{4\pi^2EI_2}}$$

CASE 11.—A thin, deep, rectangular beam supported at its ends and carrying both a uniformly distributed load W and a concentrated load P at the middle, with its ends not restrained.

$$\frac{PL^2}{16.9}+\frac{WL^2}{28.3}=\sqrt{EI_2GK}$$

Combinations of the preceding cases may be similarly expressed.

CASE 12.—A thin, deep, rectangular beam supported at its ends and carrying a concentrated load P at its middle, with lateral support as by tie-rods, at the middle, and the ends not restrained. Such a beam buckles laterally in two half waves. (Fig. 14, C.)

$$P=\frac{44.5\sqrt{EI_2GK}}{L^2}$$

BUCKLING FORMULAS FOR I BEAMS

The preceding formulas require modification when the beam has flanges, since the lateral flexure of the flanges then becomes important. Following are some of the results obtained by Timoshenko. (References 7, 16, 17, 18, and 20.) Two more symbols are introduced. Let

I_3 = the moment of inertia of *one* flange about the principal vertical axis

and let

$$\alpha^2=\frac{EI_3h^2}{2\ GKL^2}$$

CASE 13.—An I beam subjected to a constant bending moment M, with its ends not restrained.

$$M=\frac{\pi\sqrt{EI_2GK}}{L}\sqrt{1+\pi^2\alpha^2}$$

CASE 14.—The same as case 13 except that the ends are restrained.

$$M=\frac{2\pi\sqrt{EI_2GK}}{L}\sqrt{1+4\pi^2\alpha^2}$$

CASE 15.—The same as case 13 with the addition of an axial thrust P'.

$$M=\frac{\pi\sqrt{EI_2GK}}{L}\sqrt{1+\pi^2\alpha^2}\sqrt{1-\frac{P'L^2}{\pi^2EI_2}}$$

CASE 16.— The same as case 15 except that the ends are restrained.

$$M=\frac{2\pi\sqrt{EI_2GK}}{L}\sqrt{1+4\pi^2\alpha^2}\sqrt{4-\frac{P'L^2}{4\pi^2EI_2}}$$

CASE 17.—A cantilever I beam with a concentrated load P at the free end.

$$P=\frac{F\sqrt{EI_2GK}}{L^2}$$

in which values of F for reciprocal values of α^2, are:

$\frac{1}{\alpha^2}$:	0.1	1	2	4	8	12	16	24	32	40	∞
F:	44.3	15.7	12.2	9.8	8.0	7.2	6.7	6.2	5.9	5.6	4.0

CASE 18.—An I beam supported at its ends and carrying a uniformly distributed load W, with the ends not restrained.

$$W=\frac{F\sqrt{EI_2GK}}{L^2}$$

in which values of F, for reciprocal values of α^2 and for three differant placements of the load, are:

	$\frac{1}{\alpha^2}$:	0.4	4	8	16	32	48	64	96	160	320	∞
(1)	F:	143.2	53.0	42.6	36.3	32.6	31.5	30.5	29.8	29.2	28.6	28.3
(2)	F:	92.8	36.3	30.4	27.4	26.2	26.2	25.8	26.0	26.2	26.5	28.3
(3)	F:	221.6	78.2	59.4	43.1	40.7	38.1	36.0	34.4	32.6	31.0	28.3

The placements of the load on the beam, numbered to correspond with the values of F, are:

(1) Along the neutral axis.
(2) On the top.
(3) At the bottom.

CASE 19.—The same as case 18 except that the ends are restrained.

$$W=\frac{F\sqrt{EI_2GK}}{L^2}$$

$\frac{1}{\alpha^2}$:	0.4	4	8	16	32	96	128	200	400	∞
F:	488	160.8	119.2	91.2	73.0	58.0	55.8	53.4	51.2	43.3

CASE 20.—An I beam supported at its ends and carrying a concentrated load P at the middle, with the ends not restrained.

$$P=\frac{F\sqrt{EI_2GK}}{L^2}$$

	$\frac{1}{\alpha^2}$:	0.4	4	8	16	32	64	96	160	320	∞
(1)	F:	86.4	31.9	25.6	21.8	19.4	18.3	17.9	17.5	17.2	16.9
(2)	F:	51.4	20.2	17.0	15.4	14.9	14.9	15.0	15.4	15.7	16.9
(3)	F:	145.6	50.0	38.2	30.5	25.5	22.4	21.2	20.0	18.9	16.9

As in case 18 the load is applied:

(1) Along the neutral axis.
(2) On the top.
(3) At the bottom.

CASE 21.—The same as case 20 except that the ends are restrained.

$$P=\frac{F\sqrt{EI_2GK}}{L^2}$$

$\frac{1}{\alpha^2}$:	0.4	4	8	16	32	64	96	160	320	400	∞
F:	263	88.8	65.5	50.2	40.2	34.2	31.8	30.0	28.5	28.2	25.9

CASE 22.—An I beam supported at its ends and carrying a concentrated load P at the middle, with the ends not restrained, and the beam laterally supported at the middle, as when two parallel girders have a lateral connection between them at the middle of their span.

$$P=\frac{F\sqrt{EI_2GK}}{L^2}$$

$\frac{1}{\alpha^2}$:	0.4	4	8	16	32	96	128	200	400	∞
F:	466	154	114	86.4	69.2	54.6	52.4	49.8	47.4	44.5

CASE 23.—An I beam supported at its ends and carrying a distributed load W, with the ends not restrained, and the beam laterally supported at the middle of the span.

$$W=\frac{F\sqrt{EI_2GK}}{L^2}$$

	$\frac{1}{\alpha^2}$:	0.4	4	8	16	32	96	128	200	400	∞
(1)	F:	673	221	164	125.5	100.8	79.4	76.4	72.8	69.6	65.9
(2)	F:	586	194	145	112	91.2	73.7	71.5	68.9	66.8	65.9
(3)	F:	774	252	186	141	111.2	85.6	81.6	77.0	72.5	65.9

Again the load is applied:

(1) Along the neutral axis.
(2) On the top.
(3) At the bottom.

EXPERIMENTAL VERIFICATION OF THE BUCKLING FORMULAS

Time and funds were not available for the experimental verification of the formulas for all the loading and fixity conditions listed. Over 40 I and rectangular beams, however, were tested under the following conditions, which represent a considerable range for the fixity and the loading constant F.

CASE 1.—A rectangular beam subjected to a constant bending moment, with its ends not restrained.

CASE 2.—A rectangular beam subjected to a constant bending moment, with it sends restrained.

CASE 13.—An I beam subjected to a constant bending moment, with its ends not restrained.

CASE 14.—An I beam subjected to a constant bending moment, with its ends restrained.

CASE 21.—An I beam resting on two supports, with a concentrated load applied at the middle of the span, and the ends restrained.

The results are shown in Tables II, III, IV, and V. Since the exact fixity conditions assumed in the mathematical analyses are difficult of attainment, the agreement of test results with values given by the formula is remarkable. We consider this agreement, together with the agreement for a limited number of metal beams, conclusive proof that the formulas are applicable to beams under actual service conditions.

A REPORTED DISAGREEMENT WITH EXPERIMENTAL RESULTS

The only experimental record of tests with wood that has come to the attention of the present authors is an undergraduate thesis that has been published as National Advisory Committee for Aeronautics Technical Note 232, "The Lateral Failure of Spars." In this note a wide difference between actual and theoretical results is reported, the statement being made that actual loads ranged from one-half to one-fifth the loads calculated by the formula applying to the test conditions. Examination of this note, however, leads to the conclusion that the theoretical formulas were not correctly applied in two respects, as follows:

1. The coefficient 16.9, which the authors of the note used, applies only to the conditions of case 5 of the present report. Their loading conditions, however, were those of case 12, which requires a coefficient of 44.5. In addition, the ends of the test beam were under light lateral restraint, which would increase the coefficient to about 50.

2. It appears that they used the moment of inertia about the principal horizontal axis instead of that about the principal vertical axis.

Only part of the test results reported could be checked, since in several instances the beams were stressed beyond the elastic limit and stress-strain curves with which to modify the modulus of elasticity were not available, yet proper work-up of their experimental data gives results that check with precision the theoretical results.

TABLE II

THE CONSTANT BENDING MOMENT REQUIRED TO CAUSE LATERAL BUCKLING AND TWISTING OF THIN, DEEP, RECTANGULAR BEAMS HAVING THEIR ENDS UNRESTRAINED LATERALLY BUT HELD VERTICALLY ALTHOUGH FREE TO ROTATE IN A LONGITUDINAL-VERTICAL PLANE

STRESSES WITHIN THE ELASTIC LIMIT

1	2	3	4	5	6	7	8	9	10	11
Beam	Nominal dimensions (inches)	EI_2 by—		GK by—		L	Buckling moment by—			
							Calculation from columns—			Test
		Calculation	Test	Calculation	Test		3 and 5	3 and 6	4 and 6	
R-102	¾ by 6	398,000		75,050	78,000	60	9,070	9,225		10,480
R-105	1 by 6	680,000		218,800	256,300	60	20,200	21,870		18,850
R-107	1 by 6	984,000	938,000	175,900	161,300	60	21,800	20,850	20,380	22,620
R-109	1¼ by 6	1,744,000	1,667,000	328,800	373,200	60	39,620	42,230	41,380	43,200
R-111	1½ by 6	3,830,000	3,638,000	547,000	496,000	60	70,700	67,300	70,300	67,500

STRESSES BEYOND THE ELASTIC LIMIT

1	2	3	4	5	6	7
Beam	Nominal dimensions (inches)	Corrected $E'I_2$	GK by test	L	Buckling moment by—	
					Calculation from columns 3 and 4	Test
R-110	1½ by 4½	1,630,000	389,600	60	41,730	42,350
R-112	2 by 6	4,670,000	1,059,000	60	116,250	102,800
R-113	1½ by 4½	2,105,000	355,000	60	45,250	42,350
R-114	2 by 6	5,980,000	1,088,000	60	133,500	96,250

All calculations were made with a slide rule.

E=modulus of elasticity as determined from control tests increased 11 per cent to correct for shear distortion.
E'=secant modulus of elasticity as obtained from a stress-strain curve.
I_2=moment of inertia of a beam about its principal vertical axis.
G=modulus of rigidity.
K=torsion constant for the section.
L=length subjected to constant moment.

TABLE III

THE CONSTANT BENDING MOMENT REQUIRED TO CAUSE LATERAL BUCKLING AND TWISTING OF THIN, DEEP I BEAMS HAVING THEIR ENDS UNRESTRAINED BUT HELD VERTICALLY ALTHOUGH FREE TO ROTATE IN A LONGITUDINAL-VERTICAL PLANE

1	2	3	4	5	6	7	8	9	10
Beam	Nominal dimensions (inches)	h^2	EI_1 by calculation	EI_2 by—		GK by test	L	Buckling moment by—	
				Calculation	Test			Calculation from columns 6 and 7	Test
I-10	2 by 6 by ¾ flange by ¾ web	35.40	921,000	2,140,000	2,290,000	124,100	60	32,550	34,510
I-11	1½ by 6 by ½ flange by ¾ web	35.76	294,000	1,006,000	1,006,000	126,300	60	19,710	21,000
I-12	2¾ by 5¼ by ⅞ flange by ⅝ web	26.73	2,709,000	5,710,000	5,730,000	151,500	60	63,100	75,000
I-13	2 by 7 by 5/16 flange by 5/16 web	48.58	412,000	862,000	878,000	12,910	60	9,870	13,450
I-14	1½ by 5 by ⅜ flange by ⅜ web	24.60	225,000	492,000	529,000	15,330	60	5,770	5,350
I-15	1½ by 6 by ⅜ flange by ⅜ web	36.00	242,500	536,000	545,000	17,150	60	6,590	8,860
I-16	1 by 6 by ½ flange by ½ web	35.71	80,600	262,000	332,000	32,250	60	5,750	6,210
I-17	1 by 6 by ⅜ flange by ⅜ web	35.58	59,050	158,000	221,000	13,450	60	3,159	2,970
I-18	2 by 7 by ⅜ flange by ⅜ web	48.58	503,000	1,070,000	1,022,000	18,400	60	12,070	15,120
I-19	1 by 6 by 7/16 flange by 7/16 web	35.95	71,200	206,000	219,500	22,820	60	3,945	4,590
I-22	2¾ by 5¼ by ⅞ flange by ⅝ web	27.67	3,209,000	5,880,000	5,955,000	150,000	60	66,700	65,500

All calculations were made with a slide rule.

h=height of beam.
E=modulus of elasticity as determined from control tests increased 11 per cent to correct for shear distortion.
I_1=moment of inertia of 1 flange about the principal vertical axis of the beam.
I_2=moment of inertia of a beam about its principal vertical axis.
G=modulus of rigidity.
K=torsion constant for the section.
L=length subjected to constant moment.

TABLE IV

THE CONSTANT BENDING MOMENT REQUIRED TO CAUSE LATERAL BUCKLING OF THIN, DEEP BEAMS HAVING THEIR ENDS RESTRAINED LATERALLY AND HELD VERTICALLY ALTHOUGH FREE TO ROTATE IN A LONGITUDINAL-VERTICAL PLANE

RECTANGULAR BEAMS

1	2	3	4	5	6	7	8	9	10	11
Beam	Nominal dimensions (inches)	EI_2 by—		GK by—		L	Buckling moment by—			
							Calculation from columns—			Test
		Calculation	Test	Calculation	Test		3 and 5	3 and 6	4 and 6	
R-101	¾ by 6	430,200		76,600		72	15,850			14,850
R-102	¾ by 6	398,000		75,050	78,000	60	18,120	18,470		16,740
R-107	1 by 6	984,000	938,000	175,900	161,300	60	43,600	41,720	40,730	36,050

I BEAMS

1	2	3	4	5	6	7	8	9	10
Beam	Nominal dimensions (inches)	h^2	EI_1 by calculation	EI_2 by—		GK by test	L	Buckling moment by—	
				Calculation	Test			Calculation from columns 6 and 7	Test
I-10	2 by 6 by ¾ flange by ¾ web	35.40	921,000	2,140,000	2,290,000	124,100	60	81,650	55,900
I-11	1½ by 6 by ¼ flange by ¾ web	35.76	294,000	1,006,000	1,006,000	126,300	60	45,100	35,150
I-13	2 by 7 by 5/16 flange by 5/16 web	48.58	412,000	862,000	878,000	12,910	60	34,420	25,330
I-15	1½ by 6 by ⅜ flange by ⅜ web	36.00	242,500	536,000	545,000	17,150	60	19,700	14,180
I-16	1 by 6 by ¼ flange by ¼ web	35.71	80,600	262,000	332,000	32,250	60	13,220	11,350
I-17	1 by 6 by ⅜ flange by ⅜ web	35.58	59,050	153,000	221,000	13,450	60	7,780	7,700
I-18	2 by 7 by ⅜ flange by ⅜ web	48.58	503,000	1,070,000	1,022,000	18,400	60	41,340	29,430
I-19	1 by 6 by 7/16 flange by 7/16 web	35.95	71,200	206,000	219,500	22,320	60	9,360	8,370
I-20	1½ by 6 by 7/16 flange by 7/16 web	35.76	241,300	551,000	547,000	27,270	60	21,180	16,610
I-21	1½ by 6 by ⅜ flange by ⅜ web	24.50	197,000	425,000	562,500	13,650	60	15,750	11,490

All calculations were made with a slide rule.

h=height of beam.
E=modulus of elasticity as determined from control tests increased 11 per cent to correct for shear distortion.
I_1=moment of inertia of 1 flange about the principal vertical axis of the beam.
I_2=moment of inertia of a beam about its principal vertical axis.
G=modulus of rigidity.
K=torsion constant for the section.
L=length subjected to constant moment.

TABLE V

THE CONCENTRATED CENTER LOAD REQUIRED TO CAUSE LATERAL BUCKLING OF THIN, DEEP I BEAMS SUPPORTED AT EACH END WITH THE ENDS RESTRAINED LATERALLY AND HELD VERTICALLY ALTHOUGH FREE TO ROTATE IN A LONGITUDINAL-VERTICAL PLANE

1	2	3	4	5	6	7	8	9	10	11
Beam	Nominal dimensions (inches)	h^2	EI_1 by calculation	EI_2 by calculation	GK by calculation	L	α^2	F	Buckling load by—	
									Calculation	Test
I-1	1½ by 6 by ⅜ flange by ⅜ web	35.05	189,800	423,000	15,040	82	0.0328	40.8	492	400
B-1	1 by 6 by ¼ flange by ¼ web	34.69	74,800	236,200	23,390	82	.0063	30.6	338	405
B-2	1½ by 6 by ⅜ flange by ⅜ web	34.22	184,500	412,000	13,010	82	.0360	41.8	455	450
C-1	2 by 7 by 5/16 flange by 5/16 web	47.61	269,100	562,000		82				550
C-2	1½ by 5 by ⅜ flange by ⅜ web	24.40	137,100	299,500	10,080	82	.0247	37.3	305	307
A-2	1½ by 5 by ⅜ flange by ⅜ web	24.21	139,400	414,200	12,770	82	.0268	38.5	416	400
6-2-65	2 by 6 by ⅞ flange by 7/16 web	34.57	339,500	718,000	22,960	82	.0380	42.4	810	720
I-4	1½ by 5 by ⅜ flange by ⅜ web	24.21	190,500	414,800	12,680	82	.0271	36.9	418	410
I-5	1 by 6 by 7/16 flange by 7/16 web	35.05	71,500	212,000	20,220	82	.0092	31.1	302	320
I-6	2 by 6 by 7/16 flange by 7/16 web	35.05	552,000	1,171,000	26,150	82	.0550	47.8	1,244	960

All calculations were made with a slide rule.

h=height of beam.
E=modulus of elasticity as determined from control tests increased 11 per cent to correct for shear distortion
I_1=moment of inertia of 1 flange about the principal vertical axis of the beam.
I_2=moment of inertia of a beam about its principal vertical axis.
G=modulus of rigidity.
K=torsion constant for the section.
L=span.
$\alpha^2=\frac{EI_1 h^2}{2GKL^2}$.
F=multiplying factor in the lateral buckling formula, dependent upon α^2

STRESSES BEYOND THE ELASTIC LIMIT

The calculation of a critical load that produces a fiber stress beyond the elastic limit is possible by means of the preceding formulas if the modulus for inelastic deformation is known. Although this modulus is a variable beyond the elastic limit, it may be obtained from a stress-strain diagram. Figure 15 shows such a diagram for a spruce beam in bending. From it the required modulus, for a stress either below or above the elastic limit, may be determined by means of the formula:

$$\text{Modulus of elasticity} = 130.7 \frac{\text{stress}}{k \times \text{strain}} \tag{13}$$

in which k is a constant that need not be evaluated when Figure 15 is available. The modulus below the elastic limit will be called E in this report and that above will be called E'. Although both depend upon the slope of the line connecting the origin with the stress-strain curve at the particular stress in question, E' is usually spoken of as the secant modulus.

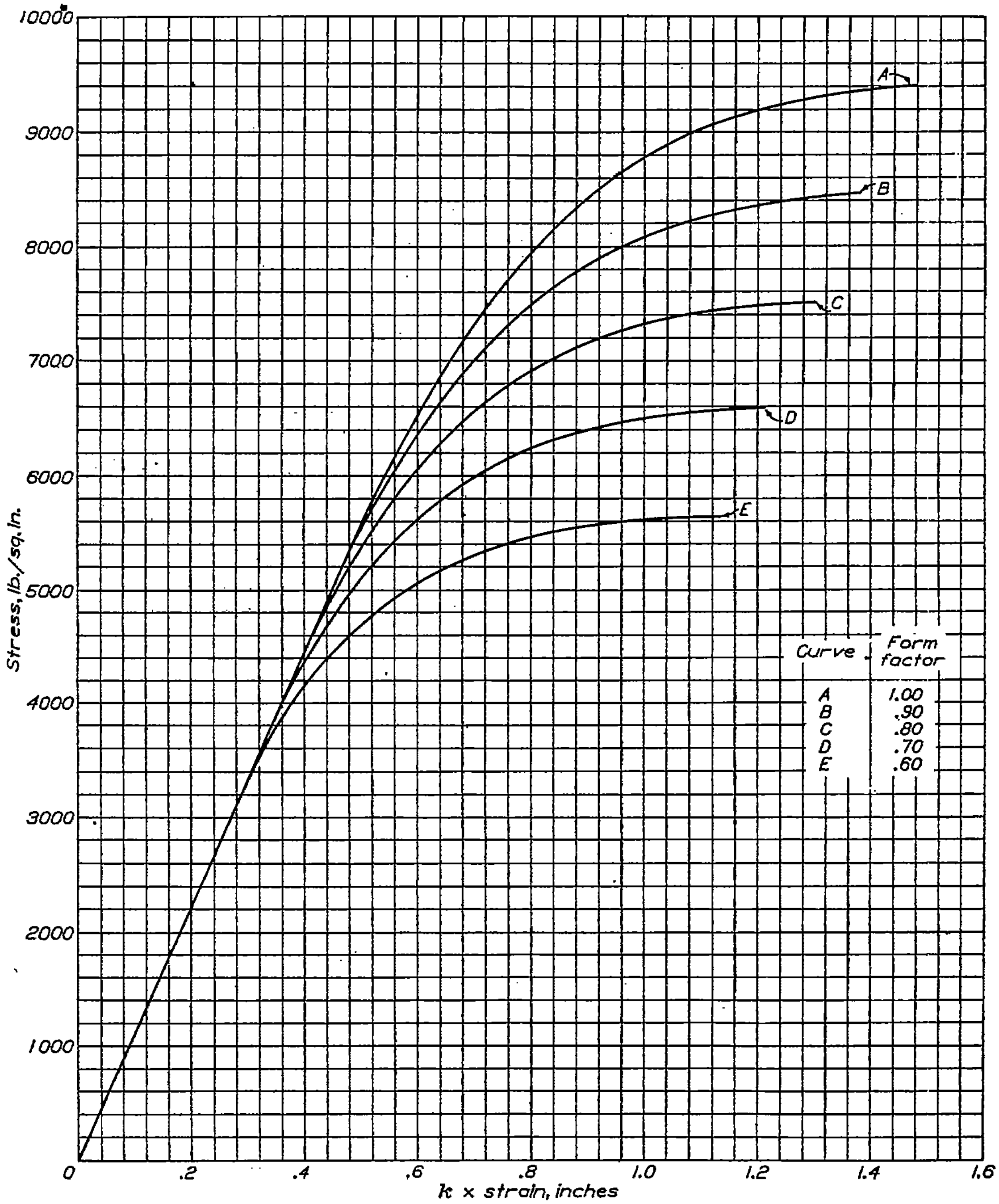

FIGURE 15.—Stress-strain curves for spruce beams. Values taken from these curves are for use in the equation: Modulus of elasticity $=130.7 \frac{\text{stress}}{k \times \text{strain}}$

The formula proposed by Karman and advocated by Timoshenko for calculating E',

$$E' = \frac{4E \times E_1}{(\sqrt{E} + \sqrt{E_1})^2}, \tag{14}$$

in which E_1 is the tangent modulus on the compression side of the beam and E is the initial modulus, can not be used for wood. (Reference 16.) It can not apply to wood because when the maximum load in bending is reached the stress-strain curve for the compression fibers has turned downward, which means that E_1 has become negative. In fact, before the maximum load is reached the tangent to the stress-strain curve for the compression fibers has become horizontal, which means that the formula would give the beam no stiffness, whereas it actually is still resisting an increasing load.

Whatever the method used, more than one trial will have to be made in the calculation of the critical stress because E' is not known until the stress is known. In calculating critical loads by simply substituting E' in the formulas that were developed on the assumption that the elastic limit was not passed, two further assumptions are made, as follows:

1. Passing the elastic limit does not affect the torsion modulus G.
2. The decrease in E is constant along the span.

In investigating critical loads, four rectangular beams were subjected to a constant bending moment that produced lateral buckling at a fiber stress beyond the elastic limit. The results appear in Table II. The corrected values of EI_2 given in the table were obtained by multiplying the secant modulus E' by the moment of inertia I_2 of the cross section about its principal vertical axis. The calculated critical bending moment for the first beam listed in the second part of the table (R–110) is about 1½ per cent lower than the test value, while the calculated values for the second (R–112) and the third (R–113) beams are respectively 13 and 7 per cent higher than the test values. The second (R–112) and the fourth (R–114) beams, which were of the same size, were made from adjacent planks cut from the same log. Control tests showed the material in R–114 to be slightly superior. Consequently its low test bending moment is difficult to account for unless the beam had become slightly warped before test, in which event the actual stress at failure would be higher than the calculated stress and the value of E' lower than that used.

LOAD NOT APPLIED ALONG THE NEUTRAL AXIS

The development of the buckling formulas is greatly simplified by the assumption that the load is applied along the neutral axis of the beam, and in aircraft work usually no material error will normally be introduced by assuming such an application of the load. In a few of the cases for which formulas are given, coefficients are also given for load applied along the neutral axis, on the compression flange, and on the tension flange of the beam. For the development of the formulas for a load placed above or below the neutral axis, attention is again directed to the work of Timoshenko and to advanced texts on strength of materials or applied elasticity. (References 7, 12, and 18.)

BUCKLING OF BEAMS TIED TOGETHER WITH RIBS

When two thin, deep beams are tied together with ribs, in addition to carrying whatever direct load is normally placed upon them the ribs will act to prevent lateral buckling of the beams. Very often, though, when the direct load is transferred to the beams from the ribs, the ribs may be laboring to sustain the load already upon them and consequently may have no

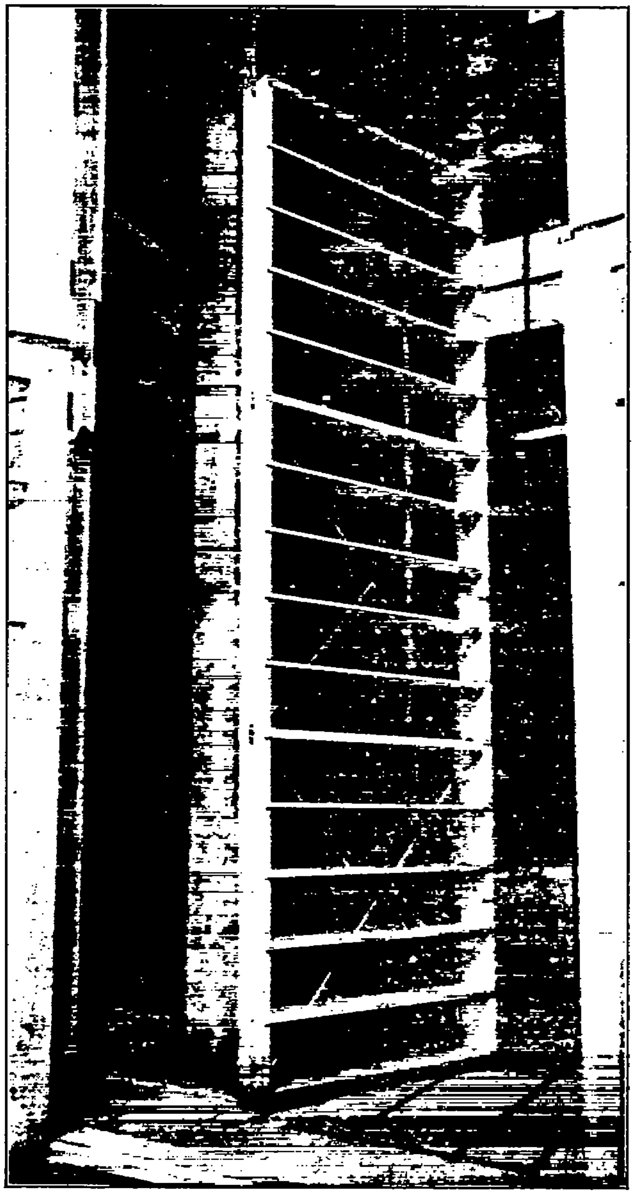

FIGURE 16.—The test of a panel to show that the tendency of an axially loaded single spar to buckle is transmitted by the ribs to an unloaded single spar

reserve strength left for any extra load that a tendency of the beam to buckle would produce.

The first panel test was made to demonstrate the fact that the tendency of an axially loaded spar to buckle is transmitted by the ribs to the unloaded spar. For this test there was made a panel consisting of two 1⅛ by 6 inch spars spaced 55 inches center to center, four compression ribs spaced 55 inches, and drag wires in the three bays. No ribs were put in between the compression ribs. Axial load was applied to but one spar, which deflected alternately in and out between compression ribs as the beam of Figure 14, C, deflected. The test was stopped at a load of 12,750 pounds with the panel still uninjured. The deflections were increasing rapidly at that time, and apparently the load was very near its maximum. Auxiliary ribs were then put in between adjacent compression ribs, four in each bay. Figure 16 shows the completed panel ready for test. Axial load was again applied to but one spar. The test was stopped at a load of 29,000 pounds, which was very near the maximum.

Under the conditions of the second test, in which all ribs were in place, the two spars act as one, the lateral rigidity of the panel being the combined rigidities of the two spars. Similar tests were made by the Engineering Division of the War Department, Air Service, at McCook Field with identical results.

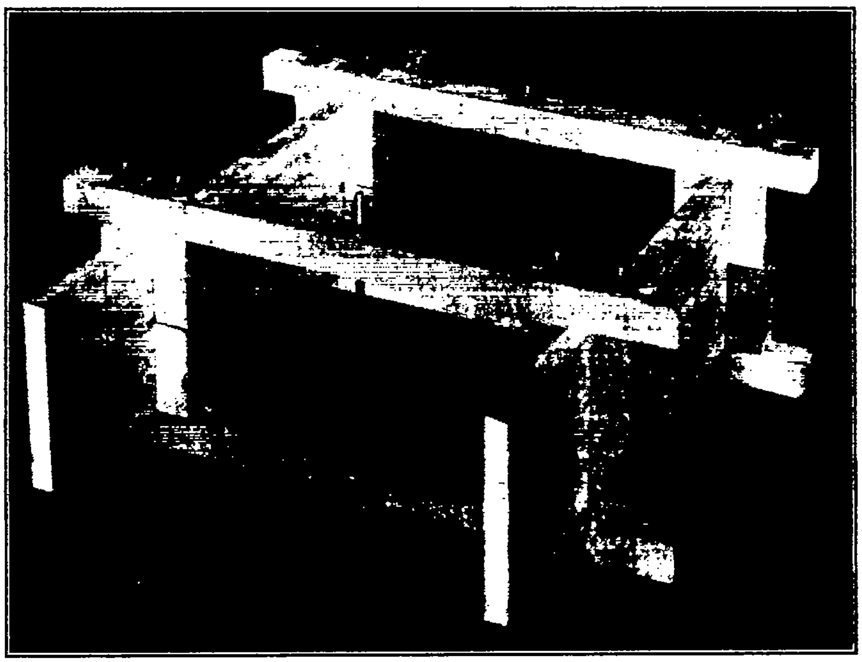

FIGURE 17.—Wing ribs for which the degree of attachment of the ribs to the beams is adjustable

In the next panel tests the beams were subjected to bending, and load was applied directly to them and not to the ribs. Two 1 by 6 inch rectangular beams subjected to a constant bending moment over 60 inches of their length were tied together with four ribs spaced 12 inches center to center in the 60 inch bay. Constant moment was applied by using the apparatus shown in Figure 8, except that double the number of support and load rods were used. The ribs that tied the two beams together were as shown in Figure 17; they were held in place simply by the friction under the heads of the bolts, the holes for which were slotted. Obviously, if the bolts were not drawn tight the beams could buckle very easily, while if they were drawn tight twisting was practically prevented. The evener bar was not pin-connected to the movable head in this test but was rigidly attached to it, so that if one beam stopped taking load more was thrown upon the other. When the panel was assembled the bolts holding the cleats along one beam were drawn up tightly, while those along the other beam were not. The beam supported by the less rigid cleats quit taking load at a moment of 35,530 inch-pounds, while the one with the more rigid cleats did not buckle until it was subjected to a moment of 53,620 inch-pounds. The results show what may happen when the ribs start to fail. Incidentally, had the beams been held so as to restrict bending to a vertical plane, each should have carried 65,550 inch-pounds and had they been free to buckle laterally each was calculated to sustain 21,270 inch-pounds.

The next panel tested was similar except that the ribs were glued to the flanges. Load was applied to the beams as before, and failure occurred when each beam was subjected to a moment of 55,600 inch-pounds. The calculated bending moment for each with bending confined to a vertical plane was 62,800 inch-pounds.

The third and final step was the test of single bays with load applied to the ribs alone. (Fig. 10.) The panels were 8 feet between supports and the beams 36 inches center to center. Seven ribs of the lightened plywood type, rectangular in form, extending 12¾ inches beyond each beam and spaced 12 inches apart, tied the two beams together. The ends of the beams rested on thrust bearings and were held vertical during test. Roller bearings under the ball bearings at one support permitted movement as the beam deflected. Thin strips 7 feet 5 inches long, notched at the ribs, were laid on the ribs, and cans filled with sand were placed on them.

For this fixity and loading the beams, which were rectangular and ½ by 4 inches in cross section, should have buckled laterally at approximately 91 pounds each if unsupported by the ribs. If bending had been confined to a vertical plane, 970 pounds should have been required to break each beam. The ribs when supported laterally should have been good for 250 to 300 pounds. The preceding values are calculated ones.

The two beams were supported at the center by a cross timber resting on two jack screws, with the ribs supported only by the beams. A load of 735 pounds was put on the panel and the screws lowered. The beams remained in a vertical plane throughout their length. The timber was again brought up against the two beams to relieve the load and more load was added. No buckling occurred at 1,155 pounds when the screws were lowered. Again the two beams were supported at the center by the cross timber and more load was applied. A total of 1,370 pounds was sustained by the ribs with the beams still supported at the center. This load, however, was approaching the maximum for the ribs. When the screws were again lowered the ribs did not have sufficient additional strength to resist the tendency of the beams to buckle and they gave way.

The two beams, which were uninjured in this test, were again used in a second panel. This second panel was like the first in every respect, but the loading was somewhat different. In place of the notched 7-foot 5-inch loading strips, short smooth strips that extended over two and three ribs alternately were used. Instead of having the long strips with their notches hold the tops of the ribs in line, strips ⅛-inch thick and 2 inches wide were laid flat along each side of each rib and tacked at the ends and center to the short loading strips. In this test, as in the first, the lower chords of the ribs were unsupported. Because the short loading strips permitted freer lateral play in the beams, this panel failed at a lower load than the first. A maximum load of 900 pounds was obtained, at which load the lower part of the ribs buckled until the ribs lay almost flat against the loading strips.

In the third and final test of this series the bottoms as well as the tops of the ribs were held in line and the same beams were used again. Ten rows of 1½-inch commercial cotton tape were run parallel to the spars and sewed to the ribs. Two diagonal pieces on both top and bottom were then sewed to the parallel rows. Although this taping was hardly comparable with wing covering, it held the ribs in line quite well. The short loading strips of the previous test were again used in addition to the tape.

As previously stated, the lateral buckling load of each spar when it was unsupported was calculated as 91 pounds, which is 182 pounds for the panel. The load required to break each one if bending had been confined to a vertical plane was 970 pounds or 1,940 pounds for the panel. Failure occurred at a total load of 1,470 pounds, at which one beam buckled badly and collapsed. The ribs had started to buckle somewhat, which permitted the one beam to buckle out of a vertical plane. Greater strength of the ribs or greater torsional rigidity of the spar would have prevented this buckling and twisting. A box beam of the same strength in bending, for example, would not have buckled at this same load.

The nose of an airplane wing helps to hold the front or deeper spar in line and the wing covering keeps the ribs in line. With this support, fairly large ratios of depth to breadth may be used if the ribs are made with just a little surplus strength.

Some years ago, after the test of a great many beams in connection with a study of form factors, the suggestion was made that the ratio of the moment of inertia about the principal horizontal axis to the moment of inertia about the principal vertical axis be kept low, below 25 if possible. A further suggestion was that when this value was exceeded special attention should be given to the factors that insure lateral rigidity. (Reference 10, p. 16, and 1923 annual report, p. 390.) As a result of the present experiments, the Forest Products Laboratory has learned what factors are involved in the lateral buckling load and has concluded that no arbitrary ratio for the moments of inertia can properly be set and that such a method of design should not be used.

In previous tests it was practically impossible to prevent the buckling of I beams having a moment-of-inertia ratio of 39. In the panel with the 1 by 6 inch beams just mentioned, for which the moment-of-inertia ratio is 36, the maximum moment was approximately 89 per cent of the moment that would have been required to cause failure had bending been confined to a vertical plane, and even this percentage value could not have been obtained if it had not been for the excess strength of the ribs. In the third test of the last panel, which had ½ by 4 inch beams and for which the moment ratio is 64, the maximum load was approximately 76 per cent of the load required to cause failure had bending been confined to a vertical plane.

In all of the recent tests it is probable that the beams were receiving less lateral support than the beams in an ordinary wing panel would receive and the end fixity was less than that which obtains in the usual drag bay. With a more or less rigid nose, such as one of plywood or metal, and ribs slightly over strength, beams with moment-of-inertia ratios considerably in excess of 25 can be counted upon for their full bending strength.

CONCLUSIONS FOR PART II

Deep beams may fail through buckling laterally and twisting at loads much less than those calculated by means of the usual beam formula.

There is for each fixity and loading condition a critical lateral buckling load for a deep beam just as there is a critical load for a column.

A mathematical analysis of the problem for various loading and fixity conditions leads to formulas that contain the dimensions of the beam, the modulus of elasticity along the grain, the modulus of rigidity in torsion, the span, and a constant depending upon the loading and fixity conditions.

Experimental results confirm the practical applicability of these formulas.

When one spar of an airplane wing or other panel is subjected to an axial load and the other spar and the ribs are not loaded, the lateral rigidity of the whole combination is the sum of the lateral rigidities of the two spars.

When two deep beams fastened together with ribs are subjected to bending, lateral buckling of the beams may or may not be prevented. When one or both of such beams are heavily stressed and in need of lateral support, the ribs, if they are not stronger than is necessary to carry the load upon them, can not carry the extra load that is induced by the tendency of the beams to buckle.

A fairly rigid nose and ribs slightly overstrength will permit the use of aircraft wing beams that have a relatively large ratio of moment of inertia about the principal horizontal axis to that about the principal vertical axis.

No arbitrary moment-of-inertia ratio can be used with certainty. Each particular case must be studied individually and lateral support must be provided in accordance with the tendency of the beam to buckle laterally rather than to bend in a vertical plane.

This investigation was undertaken as a study in aircraft design. The conclusions, however, are of general application, even though some of them for convenience are worded as if they applied only to aircraft.

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APPENDIX

A MATHEMATICAL STUDY OF THE ELASTIC STABILITY OF THIN, OUTSTANDING FLANGES UNDER COMPRESSION

INTRODUCTION

In discussing the stability of a column or other compression member having one or more thin, outstanding flanges, it is necessary to consider not only the conditions for the stability of the column as a whole but also the stability of the flanges themselves. The problem of the stability of such a flange is essentially that of the stability of a rectangular plate simply supported along the ends to which the load is applied, free along one of the other edges, and on the remaining edge either simply supported, imperfectly fixed, or perfectly fixed, depending upon the nature of the section. Timosheno has discussed this problem in considerable detail for plates of isotropic material. (References 17 and 21.) In the following appendix his methods will be extended to plates composed of a nonisotropic material, such as wood, which will be considered to have three mutually perpendicular planes of elastic symmetry. His analysis for isotropic plates will also be summarized and some further conclusions drawn.

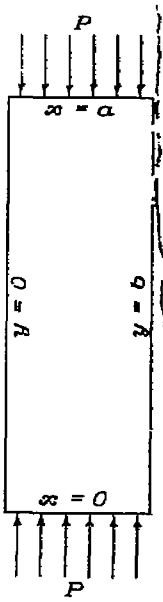

FIGURE 18.—A rectangular plate under a uniform compressive load on two opposite edges

EXACT METHOD; BASE OF FLANGE PERFECTLY FIXED

DIFFERENTIAL EQUATION FOR THE DEFLECTION OF A FLANGE OF NONISOTROPIC MATERIAL UNDER A COMPRESSIVE LOAD

A plate of thickness h, Figure 18, is considered to lie in the XY-plane and to be bounded by the lines $x=0$, $x=a$, $y=0$, and $y=b$. Uniform compressive loads P per unit length of edge, parallel to the X-axis, are applied to the edges $x=0$ and $x=a$, which are simply supported. The edge $y=b$ is free while the edge $y=0$ is either simply supported, partially fixed, or perfectly fixed.

The case in which the edge $y=0$ is perfectly fixed, a case which rarely or never occurs in practice, is first treated for both isotropic and nonisotropic material, making use of the differential equation for the deflection of the plate from its plane and of appropriate boundary conditions. A simpler approximate method based on energy considerations is then applied to the same case and the results are compared and found to check in a satisfactory manner. The approximate method is then applied to the case in which the edge in question is only partially fixed, the case in which the edge is simply supported appearing as a limiting form of partial fixity.

The differential equation satisfied by the deflection w is obtained from the following differential equations connecting the stress resultants T, S, and N and the stress couples G and H acting upon an elementary portion of the plate with edges dx and dy. (Reference 8; art. 326, equations (24), (25), (26), and art. 331, equations (45) and (46).) The notation used is that of Love. (Reference 8, art. 294.)

$$\begin{aligned}
&\frac{\partial T_1}{\partial x}-\frac{\partial S_2}{\partial y}-N_1\frac{\partial^2 w}{\partial x^2}-N_2\frac{\partial^2 w}{\partial x\partial y}+X'=0\\
&\frac{\partial S_1}{\partial x}+\frac{\partial T_2}{\partial y}-N_1\frac{\partial^2 w}{\partial x\partial y}-N_2\frac{\partial^2 w}{\partial y^2}+Y'=0 \qquad (15)\\
&\frac{\partial N_1}{\partial x}+\frac{\partial N_2}{\partial y}+T_1\frac{\partial^2 w}{\partial x^2}-S_2\frac{\partial^2 w}{\partial x\partial y}+S_1\frac{\partial^2 w}{\partial x\partial y}+T_2\frac{\partial^2 w}{\partial y^2}+Z'=0.
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial H_1}{\partial x}-\frac{\partial G_2}{\partial y}+N_2+L'=0\\
&\frac{\partial G_1}{\partial x}+\frac{\partial H_2}{\partial y}-N_1+M'=0 \qquad (16)\\
&G_1\frac{\partial^2 w}{\partial y\partial x}-G_2\frac{\partial^2 w}{\partial x\partial y}+H_1\frac{\partial^2 w}{\partial x^2}+H_2\frac{\partial^2 w}{\partial y^2}+S_1+S_2=0.
\end{aligned}$$

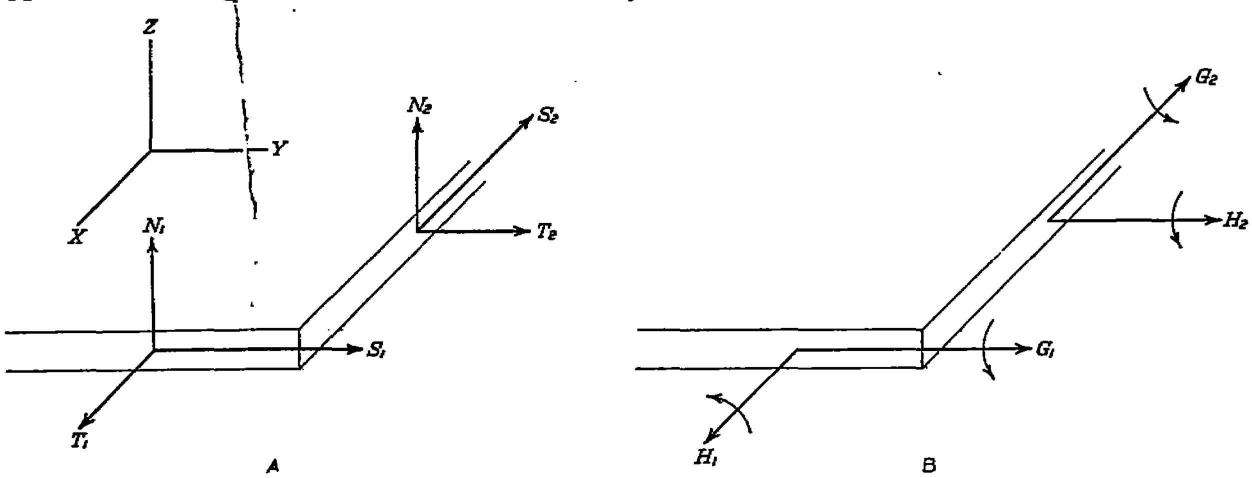

FIGURE 19.—(A) Stress resultants and (B) stress couples on an element of a plate

In equations (15) and (16) $X'=Y'=Z'=M'=N'=0$, since the components of the external force per unit area and of the external couple are zero.

To calculate $T_1 \cdot\cdot\cdot\cdot\cdot\cdot H_2$ it is necessary to express the components of stress $X_x \cdot\cdot\cdot\cdot\cdot\cdot X_y$ in terms of the deflection w and the elastic constants. (Figure 20.) The displacements u and v are given with sufficient accuracy by

$$\left.\begin{aligned} u &= -z\frac{\partial w}{\partial x} \\ v &= -z\frac{\partial w}{\partial y}. \end{aligned}\right\} \tag{17}$$

The components of strain are

$$\begin{aligned} e_{xx} &= \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} \\ e_{yy} &= \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \\ e_{xy} &= \frac{\partial u}{\partial y}+\frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \tag{18}$$

For a more extensive discussion of the components of strain, see article 329 of reference 8.

Assume that the material of the plate, wood, has three mutually perpendicular planes of elastic symmetry. (Reference 8, arts. 110 and 111.) Denote by

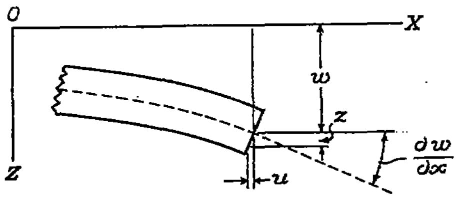

FIGURE 20.—Components of displacement in terms of deflection

E_x, E_y, and E_z Young's moduli in the directions x, y, and z, respectively, by σ_{xy} Poisson's ratio associated with contraction parallel to the Y-axis and stress parallel to the X-axis, and by μ_{xy} the modulus of rigidity corresponding to the directions x and y. The stress components X_x, Y_y, and X_y are then given by

$$\begin{aligned} X_x &= \frac{E_x}{1-\sigma_{xy}\sigma_{yx}}(e_{xx}+\sigma_{yx}e_{yy}) \\ &= \frac{-E_x z}{1-\sigma_{xy}\sigma_{yx}}\left(\frac{\partial^2 w}{\partial x^2}+\sigma_{yx}\frac{\partial^2 w}{\partial y^2}\right) \\ Y_y &= \frac{-E_y z}{1-\sigma_{xy}\sigma_{yx}}\left(\frac{\partial^2 w}{\partial y^2}+\sigma_{xy}\frac{\partial^2 w}{\partial x^2}\right) \\ X_y &= \mu_{xy}\, e_{xy} = -2\mu_{xy} z\frac{\partial^2 w}{\partial x \partial y}. \end{aligned} \tag{19}$$

By definition

$$G_1 = \int_{-h/2}^{h/2} X_x z dz.$$

(Reference 8, art. 294.) Then

$$G_1 = -D_1\left(\frac{\partial^2 w}{\partial x^2}+\sigma_{yx}\frac{\partial^2 w}{\partial y^2}\right), \tag{20}$$

where

$$D_1 = \frac{E_x h^3}{12(1-\sigma_{xy}\sigma_{yx})}. \tag{21}$$

In like manner

$$G_2 = -D_2\left(\frac{\partial^2 w}{\partial y^2}+\sigma_{xy}\frac{\partial^2 w}{\partial x^2}\right), \tag{22}$$

where

$$D_2 = \frac{E_y h^3}{12(1-\sigma_{xy}\sigma_{yx})}. \tag{23}$$

Further, from their definitions,

$$H_1 = -H_2 = M\frac{\partial^2 w}{\partial x \partial y}, \tag{24}$$

where

$$M = \mu_{xy}\frac{h^3}{6}. \tag{25}$$

In the last of equations (16) the quantities G_1, G_2, H_1, and H_2, which are expressed by (20), (22), and (24) in terms of second partial derivatives of w, are each multiplied by second derivatives of w. Each of these derivatives may be considered small and the product of two of them negligible. It follows that

$$S_1 = -S_2. \tag{26}$$

From the first two equations (16) and equations (20), (22), and (24) it is found that

$$\begin{aligned} N_1 &= -D_1\left(\frac{\partial^3 w}{\partial x^3}+\sigma_{yx}\frac{\partial^3 w}{\partial x \partial y^2}\right) - M\frac{\partial^3 w}{\partial x \partial y^2} \\ N_2 &= -D_2\left(\frac{\partial^3 w}{\partial y^3}+\sigma_{xy}\frac{\partial^3 w}{\partial x^2 \partial y}\right) - M\frac{\partial^3 w}{\partial x^2 \partial y}. \end{aligned} \tag{27}$$

It is clear from their definitions and (19) that S_1 and S_2 are small. (Reference 8, art. 294.) Also from its definition and equation (18) T_2 is small. Equation (26) and the first two of equations (15) are satisfied approximately by taking

$$S_1 = S_2 = T_2 = 0 \tag{28}$$

and

$$T_1 = \text{constant} = -P \tag{29}$$

where P is the load per unit length of the loaded edges. The third of equations (15), on making use of (27), (28), and (29), then gives the differential equation of the plate:

$$\begin{aligned} &-D_1\left(\frac{\partial^4 w}{\partial x^4}+\sigma_{yx}\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) - M\frac{\partial^4 w}{\partial x^2 \partial y^2} \\ &-D_2\left(\frac{\partial^4 w}{\partial y^4}+\sigma_{xy}\frac{\partial^4 w}{\partial x^2 \partial y^2}\right) - M\frac{\partial^4 w}{\partial x^2 \partial y^2} - P\frac{\partial^2 w}{\partial x^2} = 0. \end{aligned}$$

Or

$$D_1\frac{\partial^4 w}{\partial x^4}+2K\frac{\partial^4 w}{\partial x^2 \partial y^2}+D_2\frac{\partial^4 w}{\partial y^4}+P\frac{\partial^2 w}{\partial x^2} = 0 \tag{30}$$

where

$$K = \frac{D_1\sigma_{yx}+D_2\sigma_{xy}+2M}{2}. \tag{31}$$

BOUNDARY CONDITIONS

On the simply supported edges $x=0$ and $x=a$ of Figure 18,

$$w=0 \tag{32}$$

and

$$G_1=0. \tag{32a}$$

The last condition requires that

$$\frac{\partial^2 w}{\partial x^2}+\sigma_{yx}\frac{\partial^2 w}{\partial y^2}=0. \tag{33}$$

On the fixed edge, $y=0$,

$$w=0 \tag{34}$$

and

$$\frac{\partial w}{\partial y}=0. \tag{35}$$

On the free edge, $y=b$,

$$G_2=0$$

and

$$N_2+\frac{\partial H_2}{\partial x}=-D_2\left(\frac{\partial^3 w}{\partial y^3}+\sigma_{xy}\frac{\partial^3 w}{\partial x^2\partial y}\right)-2M\frac{\partial^3 w}{\partial x^2\partial y}=0.$$

Rewriting these conditions for the edge $y=b$,

$$\frac{\partial^2 w}{\partial y^2}+\sigma_{xy}\frac{\partial^2 w}{\partial x^2}=0 \tag{36}$$

and

$$\frac{\partial^3 w}{\partial y^3}+(2-\sigma)\frac{\partial^3 w}{\partial x^2\partial y}=0, \tag{37}$$

where

$$2-\sigma=\sigma_{xy}+\frac{2M}{D_2},$$

that is,

$$\sigma=\frac{(2-\sigma_{xy})E_y-4\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_y}. \tag{38}$$

SOLUTION OF THE DIFFERENTIAL EQUATION

Conditions (32) and (33) are satisfied by

$$w=\sin\frac{m\pi x}{a}f(y)=\sin\lambda x\, f(y). \tag{39}$$

It will be convenient to replace $m\pi/a$ by π/c, for if the flange breaks up into more than a single half wave each portion of length $a/m=c$ may be considered as a plate of length c simply supported at its ends. We shall accordingly interpret λ as given by the equation

$$\lambda=\frac{\pi}{c}$$

where c may be either the entire length of the flange or a portion of this length, as circumstances require.

In accordance with (30) $f(y)$ in (39) must satisfy an ordinary linear differential equation of the fourth order. Its solution can be written

$$f(y)=C_1e^{-\alpha y}+C_2e^{\alpha y}+C_3\cos\beta y+C_4\sin\beta y, \tag{40}$$

where

$$\alpha^2=\frac{\lambda}{D_2}\left[\lambda^2(K^2-D_1D_2)+D_2P+\frac{K\lambda^2}{D_2}\right]^{\frac{1}{2}}$$
$$\beta^2=\frac{\lambda}{D_2}\left[\lambda^2(K^2-D_1D_2)+D_2P-\frac{K\lambda^2}{D_2}\right]^{\frac{1}{2}}. \tag{41}$$

Conditions (34) and (35) are satisfied if the constants in (40) are so related that we may write

$$f(y)=A(\cos\beta y-\cosh\alpha y)+B\left(\sin\beta y-\frac{\beta}{\alpha}\sinh\alpha y\right). \tag{42}$$

The substitution of (39) combined with (42) in the conditions (36) and (37) leads to the equations:

$$A[(\beta^2+\sigma_{xy}\lambda^2)\cos\beta b+(\alpha^2-\sigma_{xy}\lambda^2)\cosh\alpha b]+$$
$$B[(\beta^2+\sigma_{xy}\lambda^2)\sin\beta b+\frac{\beta}{\alpha}(\alpha^2-\sigma_{xy}\lambda^2)\sinh\alpha b]=0 \tag{43}$$

and

$$A[\beta(\beta^2+2\lambda^2-\sigma\lambda^2)\sin\beta b-\alpha(\alpha^2-2\lambda^2+\sigma\lambda^2)\sinh\alpha b]+$$
$$B[-\beta(\beta^2+2\lambda^2-\sigma\lambda^2)\cos\beta b-\beta(\alpha^2-2\lambda^2+\sigma\lambda^2)\cosh\alpha b]=0. \tag{44}$$

In (44) note that after some reduction

$$\beta^2+(2-\sigma)\lambda^2=\alpha^2-\sigma_{xy}\lambda^2$$
$$\alpha^2-(2-\sigma)\lambda^2=\beta^2+\sigma_{xy}\lambda^2.$$

In this reduction the following relations were used:

$$\alpha^2-\beta^2=\frac{2K\lambda^2}{D_2},\quad 2-\sigma=\sigma_{xy}+\frac{2M}{D_2},\quad\text{and } E_y\sigma_{xy}=E_x\sigma_{yx}.$$

(Reference 1, p. 104.) Using the abbreviations

$$t=\beta^2+\sigma_{xy}\lambda^2$$
$$s=\alpha^2-\sigma_{xy}\lambda^2, \tag{45}$$

the equations (43) and (44) can be written in the form

$$A[t\cos\beta b+s\cosh\alpha b]+B[t\sin\beta b+\frac{\beta}{\alpha}s\sinh\alpha b]=0$$
$$A[\beta s\sin\beta b-\alpha t\sinh\alpha b]+B[-\beta s\cos\beta b-\beta t\cosh\alpha b]=0. \tag{46}$$

In order that solutions of the system (46) other than $A=0$ and $B=0$ may exist, that is, that a solution different from zero of the differential equation (30) of the form (39) may exist, it is necessary and sufficient that the determinant of the coefficients of A and B in (46) vanish. The result of equating the determinant to zero is, after some reduction,

$$2ts+(t^2+s^2)\cos\beta b\cosh\alpha b=\left(\frac{\alpha^2t^2-\beta^2s^2}{\alpha\beta}\right)\sin\beta b\sinh\alpha b. \tag{47}$$

Multiplying this equation through by b^4, the terms can be arranged so that α and β occur only in the combinations αb and βb. We then write (equations (41))

$$\alpha b=(\sqrt{UV}+V)^{\frac{1}{2}}$$
$$\beta b=(\sqrt{UV}-V)^{\frac{1}{2}}, \tag{48}$$

where

$$V=\pi^2\frac{b^2}{c^2}\frac{K}{D_2}, \tag{49}$$

and

$$U=\frac{1}{D_2K}\left[\pi^2\frac{b^2}{c^2}(K^2-D_1D_2)+D_2Pb^2\right]. \tag{50}$$

GENERAL EXPRESSION FOR CRITICAL STRESS

By assigning a value to the ratio c/b the quantity V is determined. The corresponding value of U can then be found by solving equation (47). The value of the critical stress

$$p=\frac{P}{h}$$

corresponding to this value of c/b can then be found from equation (50). From (50) it follows that

$$p=\frac{1}{b^2h}\left[KU-\pi^2\frac{b^2}{c^2}\left(\frac{K^2}{D_2}-D_1\right)\right]$$

$$=\left[\frac{K}{E_xh^3}U-\pi^2\frac{b^2}{c^2}\frac{1}{E_xh^3}\left(\frac{K^2}{D_2}-D_1\right)\right]E_x\frac{h^2}{b^2}.$$

Or

$$p=kE_x\frac{h^2}{b^2}. \tag{51}$$

ELASTIC CONSTANTS OF SPRUCE

The elastic constants to be used in the computation depend upon the orientation of the planes of elastic symmetry of the wood in the plate. It will be assumed throughout the discussion that the grain of the wood is parallel to the X-axis, the direction in which the compressive load is applied. Two cases for the direction of the growth rings of the wood will be considered, one in which the rings are perpendicular to the faces of the plate and another in which they make an angle of 45° with the faces.

In the first case (fig. 21) Young's moduli E_x, E_y, and E_z are equal to E_L, E_R, and E_T, respectively, the subscripts L, R, and T denoting the longitudinal, radial, and tangential moduli, respectively. The values for these and other elastic constants for spruce were taken from a report of the British Aeronautical Research Committee. (Reference 1, p. 105.) The values are:

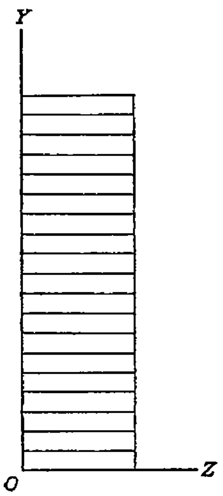

FIGURE 21.—The cross section of a quarter-sawn flange

$E_L=1.95\times10^6$	$\sigma_{LR}=0.45$
$E_R=0.13\times10^6$	$\sigma_{LT}=0.539$
$E_T=0.07\times10^6$	$\sigma_{RT}=0.559$
$\mu_{LR}=0.104\times10^6$	$\sigma_{RL}=0.03$
$\mu_{LT}=0.072\times10^6$	$\sigma_{TL}=0.0194$
$\mu_{RT}=0.005\times10^6$	$\sigma_{TR}=0.301$

In the second case, when the growth rings make an angle of 45° with the faces of the plate (fig. 22), the elastic constants $E_y \cdots\cdots\cdots \mu_{xy}$ can be computed from those just given by the following formulas:

$$\frac{1}{E_y}=\frac{1}{4E_R}+\frac{1}{4E_T}+\frac{1}{4\mu_{RT}}-\frac{\sigma_{RT}}{2E_R},$$

$$\sigma_{yx}=\frac{E_y}{2E_L}(\sigma_{LR}+\sigma_{LT}),$$

$$\sigma_{xy}=\frac{E_L}{E_y}\sigma_{yx},$$

$$\mu_{xy}=\frac{2\mu_{LT}\mu_{LR}}{\mu_{LT}+\mu_{LR}}.$$

(Reference 8, art. 111.) It is then found that

$$E_y=0.01875\times10^6$$
$$\sigma_{yx}=0.00475$$
$$\sigma_{xy}=0.494$$
$$\mu_{xy}=0.0851\times10^6.$$

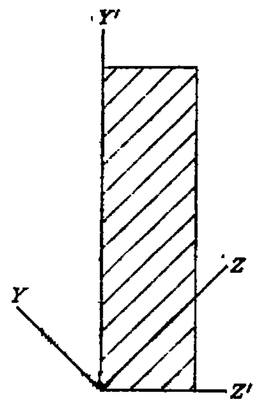

FIGURE 22.—The cross section of a wood flange the growth rings of which make an angle of 45° with the faces

CRITICAL STRESS FOR A FLANGE OF SPRUCE

Values of k in equation (51), the equation for critical stress, which result from solving equation (47) for the cases of growth rings perpendicular to the faces of the flange and at 45° to the faces, are given in Tables VI and VII, respectively.

TABLE VI

THEORETICAL CONSTANTS FOR FLANGED COMPRESSION MEMBERS OF SPRUCE HAVING THE GROWTH RINGS PERPENDICULAR TO THE FACES OF THE FLANGE, CALCULATED BY THE EXACT MATHEMATICAL METHOD

c/b	U	k
3.40	15.42	0.228540
3.30	15.18	.228346
3.25	14.95	.228057
3.20	14.87	.228376
3.10	14.58	.229166
3.00	14.30	.230577

The minimum critical stress for growth rings perpendicular to the faces of the flange occurs when the half wave length is 3.25 times the outstanding width of the flange. This critical stress is equal to 0.228 E_xh^2/b^2. Ordinarily the length of the column is such that the flange can not break up into segments the length of which is exactly 3.25 times the outstanding width. Under such a condition the stress will be increased as the values in the table indicate. Considerable increases would be found for considerable departures from the optimum value of the ratio c/b. Such departures occur only when the column is so short that its length is less than two or three times the optimum half wave length.

TABLE VII

THEORETICAL CONSTANTS FOR FLANGED COMPRESSION MEMBERS OF SPRUCE HAVING THE GROWTH RINGS AT AN ANGLE OF 45° WITH THE FACES OF THE FLANGE, CALCULATED BY THE EXACT MATHEMATICAL METHOD

c/b	U	k
5.34	14.83	0.117308
5.20	14.78	.117119
5.10	14.76	.117113
5.00	14.74	.117118
4.90	14.73	.117227
4.80	14.75	.117539

Consideration of Tables VI and VII shows that the theoretical critical stress is considerably less when the growth rings make an angle of 45° with the faces of the flange (fig. 22) than when they make an angle of 90° (fig. 21). The chief factor in determining the variation in the critical stress with variation in the angle between rings and faces is the ratio E_y/E_x. This ratio is nearly constant when the angle made by the rings with the faces of the flange lies between 20° and 70°, and hence the results for rings at an angle of 45° may be taken to apply over this range. When the rings are parallel to the faces of the flange, however, the minimum critical stress is found by an approximate method given later in this report to be

$$0.164\ E_x\ h^2/b^2$$

for a flange with a perfectly fixed edge. This critical stress is intermediate between those for flanges with the rings at angles of 45° and of 90° with the faces.

The theoretical critical stress for a flange with a perfectly fixed edge is not attained in practice because the condition of perfect fixity at the base of the flange is not realized. Later in this report it will be pointed out more in detail that as the fixity at the base of the flange decreases the variation of the critical stress with inclination of growth rings becomes smaller and ultimately, as the fixity continues to diminish, the critical stress for a flange with rings parallel to its faces becomes less than that for a similar flange with rings at 45°, which in turn is always less than that for a similar flange with range at 90°.

DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS FOR A FLANGE OF ISOTROPIC MATERIAL

The preceding analysis is an extension to flanges of nonisotropic material of the method that Timoshenko used in discussing flanges of isotropic material. (Reference 17, p. 350.) When the material is isotropic the differential equation (30) becomes

$$\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}+\frac{P}{C}\frac{\partial^2 w}{\partial x^2}=0 \tag{52}$$

where

$$C=\frac{Eh^3}{12(1-\sigma^2)}. \tag{53}$$

The boundary conditions are given by equations (32) to (37) after σ_{xy} and σ_{yx} have been replaced with σ. The differential equation and the boundary conditions are then those used by Timoshenko. The critical load is determined by solving equation (47) where t and s are given by (45) with $\sigma_{xy}=\sigma$ and where α and β are given by (48) with

$$V=\pi^2\frac{b^2}{c^2} \tag{54}$$

and

$$U=\frac{P}{C}b^2. \tag{55}$$

CRITICAL STRESS FOR A FLANGE OF ISOTROPIC MATERIAL

The values of U corresponding to various values of the ratio c/b as calculated by Timoshenko are given in Table VIII for flanges of isotropic material. In the third column of this table appear the values of k in the formula

$$p=kE\frac{h^2}{b^2},$$

where p is the critical stress. This formula is obtained at once from equation (55) by noting that

$$p=\frac{P}{h}\text{ and }C=\frac{Eh^3}{12(1-\sigma^2)}.$$

In the computations σ was taken as 0.25.

TABLE VIII

THEORETICAL CONSTANTS FOR FLANGED COMPRESSION MEMBERS OF ISOTROPIC MATERIAL, CALCULATED BY THE EXACT MATHEMATICAL METHOD AND WITH POISSON'S RATIO TAKEN AS 0.25.

c/b	U	k
1.0	16.76	1.490
1.1	15.41	1.370
1.2	14.47	1.286
1.3	13.88	1.234
1.4	13.45	1.196
1.5	13.20	1.173
1.6	13.13	1.167
1.635	13.11	1.165
1.7	13.15	1.169
1.8	13.24	1.177
1.9	13.43	1.194
2.0	13.67	1.215
2.2	14.35	1.276
2.4	15.21	1.352

In Table VIII the critical stress is least when the half wave length is equal to 1.635 times the width of the outstanding flange. If a, the total length, is either less than $1.635b$ or somewhat greater than this amount the critical stress will be greater, as Table VIII shows. As a increases toward twice the ideal half wave length the critical stress begins to diminish, reaching the same minimum value at $a=3.27b$ as at $a=1.635b$. When the column is long in comparison with the width of the outstanding flange (the length three or more times the width) the flange will break up into waves the half length of which is approximately $1.635b$, and the critical stress will then differ but little from that for this ideal half wave length.

APPROXIMATE METHOD

DISCUSSION

Approximate results were obtained by Timoshenko with a method that is based upon energy relationships and that is an important extension of a method used by Bryan. (References 3, 4, 5, 19, 20, and 21.) The deflection of the plate (fig. 18) is expressed as a sum of terms of the form

$$w=A_1\phi_1(x,\,y)+A_2\phi_2(x,\,y)+\ldots. \tag{56}$$

the functions ϕ_1, ϕ_2, ... being chosen to satisfy the boundary conditions as nearly as possible and the coefficients A_1, A_2, ... being arbitrary. This expression for the deflection w is then substituted in the integral representing the energy of deformation of the plate. The result is a function of the arbitrary constants A_1, A_2, ... The energy is then equated to the work done by the compressive load P per unit length acting on the edges $x=0$ and $x=a$. The result is an equation that can be solved for P in terms of the arbitrary constants A_1, A_2, ... The ratios A_2/A_1, A_3/A_1, ... are then chosen in such a way as to make P a minimum. If the resulting stress,

$$p=\frac{P}{h}$$

where h is the thickness of the plate, is less than the stress for primary failure of the column of which the plate is a member, the plate will fail by buckling at the critical stress p. For a full discussion of the method, with examples of its application to simple cases, see Timoshenko's paper. (Reference 19.)

The energy of deformation of the plate, under the assumption that the stress components X_z, Y_z, and Z_z are negligible, is given by

$$V=\frac{1}{2}\int_0^a\int_0^b\int_{-\frac{h}{2}}^{\frac{h}{2}}(X_xe_{xx}+Y_ye_{yy}+X_ye_{xy})\,dz\,dy\,dx. \tag{57}$$

Substituting the values of the strain components given in (18) and those of the stress components given in (19) for nonisotropic material having three mutually perpendicular planes of elastic symmetry, the result is

$$V=\frac{h^3}{24}\int_0^a\int_0^b\left\{\frac{1}{1-\sigma_{xy}\sigma_{yx}}\left[E_x\left(\frac{\partial^2w}{\partial x^2}\right)^2+E_y\left(\frac{\partial^2w}{\partial y^2}\right)^2+2E_x\sigma_{yx}\frac{\partial^2w}{\partial y^2}\frac{\partial^2w}{\partial x^2}\right]+4\mu_{xy}\left(\frac{\partial^2w}{\partial x\partial y}\right)^2\right\}dy\,dx. \tag{58}$$

For isotropic material this becomes

$$V=\frac{C}{2}\int_0^a\int_0^b\left\{\left(\frac{\partial^2w}{\partial x^2}+\frac{\partial^2w}{\partial y^2}\right)^2-2(1-\sigma)\left[\frac{\partial^2w}{\partial x^2}\frac{\partial^2w}{\partial y^2}-\left(\frac{\partial^2w}{\partial x\partial y}\right)^2\right]\right\}dy\,dx. \tag{59}$$

The work done by a compressive load P per unit length of edge, applied to the edges $x=0$ and $x=a$ (fig. 18) is given by

$$T=\frac{P}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2dy\,dx. \tag{60}$$

In what follows, the integrations with respect to x in (58), (59), and (60) will be performed between the limits 0 and c, where c is the half wave length of the deformed surface. In certain cases c will be equal to a, while in others it will be a fractional part of a.

BASE OF FLANGE PERFECTLY FIXED

The assumed deflection (equation (56)) will be taken as

$$w=\{A_1(6b^2y^2-4by^3+y^4)+A_2(y^5-10b^2y^3+20b^3y^2)\}\sin\frac{\pi x}{c}. \tag{61}$$

The functions of y in the first and second terms of (61) represent respectively the deflection of a cantilever fixed at the end $y=0$ under a uniform load and under a load that is proportional to y. Timoshenko in treating the isotropic plate by this method chose other functions. (Reference 21, p. 405.) It is not apparent that either choice possesses any particular advantages over the other.

Flange of nonisotropic material.

Entering (61) in (58) it follows that for nonisotropic material

$$V=\frac{h^3A_1{}^2b^9E_x}{48c^3(1-\sigma_{xy}\sigma_{yx})}[d_0+d_1z+d_2z^2] \tag{62}$$

in which, letting

$$\rho=\frac{c^2}{b^2}, \tag{63}$$

$$\begin{aligned} d_0&=2.311\pi^4+\pi^2\left[41.15\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}-3.432\sigma_{yx}\right]\rho+28.8\frac{E_y}{E_x}\rho^2\\ d_1&=16.788\pi^4+\pi^2\left[303.4\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}-26.30\sigma_{yx}\right]\rho+208.3\frac{E_y}{E_x}\rho^2\\ d_2&=30.488\pi^4+\pi^2\left[559.7\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}-50.16\sigma_{yx}\right]\rho+377.2\frac{E_y}{E_x}\rho^2 \end{aligned} \tag{64}$$

and

$$z=\frac{A_2}{A_1}b. \tag{65}$$

From (60) and (61) it follows that

$$T=\frac{P\pi^2}{4c}A_1^2b^9(c_0+c_1z+c_2z^2) \tag{66}$$

where

$$c_0=2.311,\ c_1=16.788,\ \text{and}\ c_2=30.488. \tag{67}$$

Equating T and V as given by (62) and (66) and solving for $p=P/h$,

$$p=\frac{1}{12\pi^2(1-\sigma_{xy}\sigma_{yx})\rho}\left(\frac{d_0+d_1z+d_2z^2}{c_0+c_1z+c_2z^2}\right)E_x\frac{h^2}{b^2}. \tag{68}$$

The critical stress p will be a minimum if z is the larger of the roots of

$$\frac{dp}{dz}=0.$$

Equation (68) may be written

$$p=kE_x\frac{h^2}{b^2}. \tag{69}$$

The calculation outlined assumes the ratio c/b to be given and determines the critical stress for this ratio. By calculating k for a series of ratios c/b the ideal half wave length is found as that which makes the critical stress a minimum.

In Tables IX, X, and XI, the values of k for certain values of the ratio c/b are given for flanges of spruce, the growth rings being respectively perpendicular to the faces of the flange, inclined to them at an angle of 45°, and parallel to them. The elastic constants for spruce given earlier in this appendix were used in the calculations. For rings parallel to the faces, we note that

$$E_x=1.95\times10^6$$
$$E_y=0.07\times10^6$$
$$\mu_{xy}=0.072\times10^6$$
$$\sigma_{xy}=0.539$$
$$\sigma_{yx}=0.0194.$$

(Reference 1, p. 105.)

TABLE IX

THEORETICAL CONSTANTS FOR FLANGES OF SPRUCE, UNDER LONGITUDINAL COMPRESSION, THAT HAVE THE GROWTH RINGS PERPENDICULAR TO THE FACES OF THE FLANGE, CALCULATED BY THE APPROXIMATE METHOD

c/b	k
3.2	0.228356
3.3	.228256
3.4	.228719

TABLE X

THEORETICAL CONSTANTS FOR FLANGES OF SPRUCE, UNDER LONGITUDINAL COMPRESSION, THAT HAVE THE GROWTH RINGS AT AN ANGLE OF 45° WITH THE FACES OF THE FLANGE, CALCULATED BY THE APPROXIMATE METHOD

c/b	k
5.1	0.116812
5.2	.116806
5.3	.117021

TABLE XI

THEORETICAL CONSTANTS FOR FLANGES OF SPRUCE, UNDER LONGITUDINAL COMPRESSION, THAT HAVE THE GROWTH RINGS PARALLEL TO THE FACES OF THE FLANGE, CALCULATED BY THE APPROXIMATE METHOD

c/b	k
3.7	0.16399
3.8	.16381
3.9	.16389

The results agree remarkably well with those given in Tables VI and VII as the result of more exact analysis.

Flange of isotropic material.

After substituting the assumed deflection (61) in the integral (59) for the energy of deformation of the flange in the case of isotropic material and equating T and V as given by (59) and (60) it is found that

$$p=\frac{1}{12\pi^2(1-\sigma^2)\rho}\left(\frac{d_0+d_1z+d_2z^2}{c_0+c_1z+c_2z^2}\right)E\frac{h^2}{b^2} \tag{70}$$

where

$$p=\frac{P}{h}$$

and expressions for d_0, d_1, d_2 are found from (64) by writing

$$E_x=E_y=E,\ \ \sigma_{xy}=\sigma_{yx}=\sigma,\ \text{and}\ \mu_{xy}=\mu=\frac{E}{2(1+\sigma)}.$$

The quantities c_0, c_1, and c_2 have the values given by (67).

If equation (70) is written in the form

$$p=kE\frac{h^2}{b^2} \tag{71}$$

the value of the minimum k for a given value of the ratio c/b can be calculated as with nonisotropic material. A few values in the vicinity of the half wave length for which the critical stress is a minimum are given in Table XII; Poisson's ratio σ was taken as 0.25.

TABLE XII

THEORETICAL CONSTANTS FOR FLANGES OF ISOTROPIC MATERIAL UNDER LONGITUDINAL COMPRESSION, CALCULATED BY THE APPROXIMATE METHOD AND WITH POISSON'S RATIO TAKEN AS 0.25

c/b	k
1.65	1.16434
1.66	1.16390
1.67	1.16407

The minimum values of k in Table XII differ from those of Table VIII by a small fraction of 1 per cent. The half wave lengths at which the minimum critical stress occurs differ by about 1.5 per cent. Plotting the curve connecting critical stress and half wave length in the vicinity of the minimum critical stress will show that this difference has little significance. For steel, with Poisson's ratio taken as 0.3, a similar calculation gives a minimum k of 1.1592 corresponding to a value of c/b of 1.60.

BASE OF FLANGE IMPERFECTLY FIXED

Discussion.

The condition of perfect fixity assumed in the preceding sections of this report for the edge of the flange $y=0$ (fig. 18) is probably never realized. This is due to two circumstances, which will be considered separately. Both result from the moment induced at the edge $y=0$ by the deformation of the outstanding flange bounded by this edge. This moment causes twisting of the whole cross section of the column and it also causes elastic giving of the material along the junction of the base of the flange and the body of the column. Both of these phenomena, twisting of the section and elastic giving at the base of the flange, are accompanied by a change in the inclination of the flange at its base from the value zero required by the condition of perfect fixity. The twisting phenomenon is easily expressed in terms of the torsional rigidity of the section. The elastic giving appears to involve factors that are best determined experimentally.

Effect of twisting of column.

We proceed to calculate the effect of the twisting of the column induced by the moments acting along the edge $y=0$. (See Timoshenko. Reference 21, p. 400.)

Let ϕ denote the angle of rotation of a cross section the abscissa of which is x. If elastic giving of the material is neglected for the present,

$$\phi=\left(\frac{\partial w}{\partial y}\right)_{y=0}. \qquad (72)$$

The torsional couple in any section is then

$$M=GK\frac{\partial\phi}{\partial x}=GK\left(\frac{\partial^2 w}{\partial x\partial y}\right)_{y=0}$$

where G is the modulus of rigidity of the material and K is the torsion constant of the section. (Reference 22, p. 11, and 1929 annual report, p. 681.) The couple applied per unit length is then

$$m=\frac{\partial M}{\partial x}=GK\left(\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=0}.$$

In applying the approximate method, the strain energy resulting from the twisting of the column (in whole or in segments) should be added to the strain energy of deformation of the corresponding portion of the outstanding flange. The strain energy per half wave length c, resulting from twisting, is

$$V_1=\int_0^c\frac{GK}{2}\left(\frac{\partial\phi}{\partial x}\right)^2dx=\frac{GK}{2}\int_0^c\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2_{y=0}dx.$$

If

$$w=f(y)\sin\frac{\pi x}{c}$$

$$V_1=\frac{GK\pi^2}{4c}\Big[f'(y)\Big]^2_{y=0} \qquad (73)$$

To apply the approximate method let

$$w=\left[Ay+A_1\left(1-\cos\frac{\pi y}{b}\right)\right]\sin\frac{\pi x}{c}. \qquad (74)$$

If $A_1=0$ the edge $y=0$ is simply supported. If $A=0$ the edge $y=0$ is fixed. Hence, by allowing A_1/A to vary from zero to infinity, all conditions on the edge $y=0$ intermediate between those for an edge simply supported and those for one perfectly fixed can be satisfied by a deflection in the form given by (74).

It follows from (73) and (74) that

$$V_1=\gamma\frac{A^2c}{4} \qquad (75)$$

where

$$\gamma=\frac{GK\pi^2}{c^2}. \qquad (76)$$

In calculating V_1 for a column of spruce the modulus of rigidity G may be taken as the mean modulus that would be given by a torsion test on a cylinder of circular section. This value may be conveniently taken as Young's modulus in the longitudinal direction divided by 15.6. (Reference 22, pp. 21 and 24, and 1929 annual report, pp. 691 and 694.)

Flange of nonisotropic material.

For nonisotropic material, such as wood, with three mutually perpendicular planes of elastic symmetry it follows from (58) and (74) that

$$V=\frac{E_xh^3c\pi^4}{48(1-\sigma_{xy}\sigma_{yx})}\left\{A^2\left[\frac{b^3}{3c^4}-\frac{4\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}\frac{b}{\pi^2c^2}\right]\right.$$
$$+A_1^2\left[\frac{b}{c^4}\left(\frac{3}{2}-\frac{4}{\pi}\right)+\frac{E_y}{E_x}\frac{1}{32b^3}+\frac{\sigma_{yx}}{c^2b}\left(\frac{1}{4}-\frac{1}{\pi}\right)\right.$$
$$\left.+\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}\frac{1}{2c^2b}\right]+AA_1\left[\frac{b^2}{c^4}\left(1-\frac{4}{\pi}+\frac{8}{\pi^3}\right)\right.$$
$$\left.\left.-\frac{\sigma_{yx}}{\pi c^2}\left(1-\frac{2}{\pi}\right)+\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}\frac{8}{\pi^2c^2}\right]\right\}.$$

From (60) and (74)

$$T=\frac{P\pi^2}{4c}\left[A^2\frac{b^3}{3}+A_1^2b\left(\frac{3}{2}-\frac{4}{\pi}\right)+AA_1b^2\left(1-\frac{4}{\pi}+\frac{8}{\pi^2}\right)\right].$$

From

$$T=V+V_1$$

using

$$\frac{A_1}{Ab}=z,\ \rho=\frac{c^2}{b^2},\ \text{and}\ p=\frac{P}{h} \tag{77}$$

it follows that

$$p=\left[\frac{\pi^2}{12(1-\sigma_{xy}\sigma_{yx})}\right]\frac{1}{\rho}\left[\frac{d_0+d_1z+d_2z^2}{c_0+c_1z+c_2z^2}\right]E_x\frac{h^2}{b^2} \tag{78}$$

or

$$p=kE_x\frac{h^2}{b^2} \tag{78a}$$

where for convenience the following notation has been used:

$$\epsilon=\frac{12(1-\sigma_{xy}\sigma_{yx})\gamma b}{E_xh^3}=12(1-\sigma_{xy}\sigma_{yx})\pi^2\frac{GK}{E_x}\frac{b}{c^2h^3} \tag{79}$$

$$d_0=\frac{1}{3}+\frac{4}{\cdot}\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}\frac{\rho}{\pi^2}+\frac{\epsilon\rho^2}{\pi^4}$$

$$d_1=1-\frac{4}{\pi}+\frac{8}{\pi^2}-\frac{\sigma_{yx}}{\pi}\left(1-\frac{2}{\pi}\right)\rho+\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}\frac{8}{\pi^2}\rho \tag{80}$$

$$d_2=\frac{3}{2}-\frac{4}{\pi}+\sigma_{yx}\left(\frac{1}{4}-\frac{1}{\pi}\right)\rho+\frac{\mu_{xy}(1-\sigma_{xy}\sigma_{yx})}{E_x}\frac{\rho}{2}+\frac{E_y}{E_x}\frac{\rho^2}{32}$$

$$c_0=\frac{1}{3},\ c_1=1-\frac{4}{\pi}+\frac{8}{\pi^2},\ \text{and}\ c_2=\frac{3}{2}-\frac{4}{\pi}. \tag{81}$$

Flange of isotropic material.

For isotropic material equation (78) with appropriate values of d_0, d_1, d_2, c_0, c_1, and c_2 becomes that given by Timoshenko, to whom the choice of the form (74) for the deflection w is due. (Reference 21, p. 401.)

The values of the coefficients d_0, d_1, and d_2 are

$$d_0=\frac{1}{3}+\frac{2(1-\sigma)}{\pi^2}\rho+\frac{\epsilon\rho^2}{\pi^4}$$

$$d_1=1-\frac{4}{\pi}+\frac{8}{\pi^2}-\frac{\rho}{\pi}\left(1-\frac{2}{\pi}\right)+\frac{\rho}{\pi}(1-\sigma)\left(1+\frac{2}{\pi}\right) \tag{82}$$

$$d_2=\frac{3}{2}-\frac{4}{\pi}+\rho\left(\frac{1}{4}-\frac{\sigma}{\pi}\right)+\frac{\rho^2}{32}$$

where

$$\epsilon=12(1-\sigma^2)\pi^2\frac{GK}{E}\frac{b}{c^2h^3}. \tag{83}$$

The constants c_0, c_1, and c_2 are unchanged.

The critical stress p is given by

$$p=\frac{\pi^2}{12(1-\sigma^2)}\frac{1}{\rho}\left(\frac{d_0+d_1z+d_2z^2}{c_0+c_1z+c_2z^2}\right)E\frac{h^2}{b^2} \tag{84}$$

or

$$p=kE\frac{h^2}{b^2}. \tag{84a}$$

Application of formulas.

Equations (78) and (84) are of the same form as (68) and should be used in the same way. For a given ϵ and a series of values of the ratio c/b a series of critical stresses p are determined corresponding to a suitable value z. The ratio c/b associated with the minimum critical stress (if there is a minimum) determines the half wave length c that is ideal for the value of ϵ under consideration.

For the study of a given column it is more convenient to proceed in another way. The first step is to construct a table giving k in the formula for the critical stress p as a function of the fixity coefficient ϵ, for each of a series of values of the ratio c/b of the half wave length to the width of the outstanding flange. Table XIII was constructed in this way for flanges of spruce and Table XIV for flanges of isotropic material. The results in these tables are also shown in the curves of Figures 23 and 24.

The use of these curves in studying a given column is discussed in a later section of this appendix. In interpreting the curves, it must be borne in mind that the fixity coefficient ϵ depends upon the half wave length c and the outstanding width b as well as upon the thickness h and the torsion constant K.

TABLE XIII

THE COEFFICIENT k IN EQUATION (78a) FOR A FLANGE OF SPRUCE HAVING GROWTH RINGS AT AN ANGLE OF 45° WITH THE FACES

c/b	ϵ	k	c/b	ϵ	k
1	0.10	0.868	3	0.10	0.154
1	.05	.866	3	.05	.147
1	.03	.865	3	.03	.142
1	.01	.864	3	.01	.136
1	.00	.863	3	.00	.132
5	.10	.113	7	.10	.117
5	.05	.102	7	.05	.102
5	.03	.004	7	.03	.091
5	.01	.083	7	.01	.073
5	.00	.075	7	.00	.059
9	.10	.136	12	.10	.181
9	.05	.115	12	.05	.149
9	.03	.100	12	.03	.124
9	.01	.074	12	.01	.079
9	.00	.053	12	.00	.049
15	.10	.241	20	.10	.375
15	.05	.195	20	.05	.298
15	.03	.159	20	.03	.237
15	.01	.098	20	.01	.133
15	.00	.047	20	.00	.046

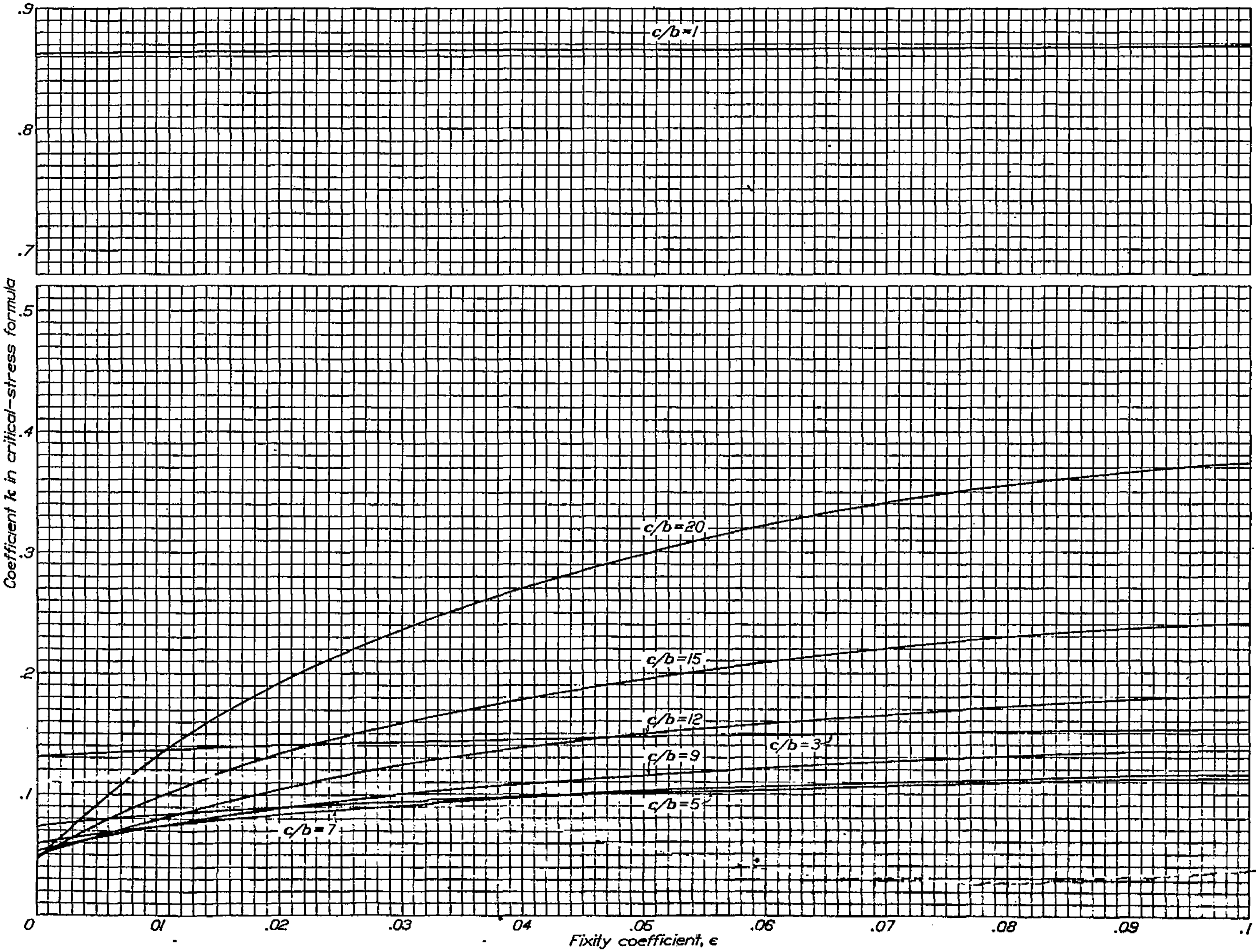

FIGURE 23.—The coefficient k in equation (78a) for a flange of spruce the growth rings of which make an angle of 45° with the faces

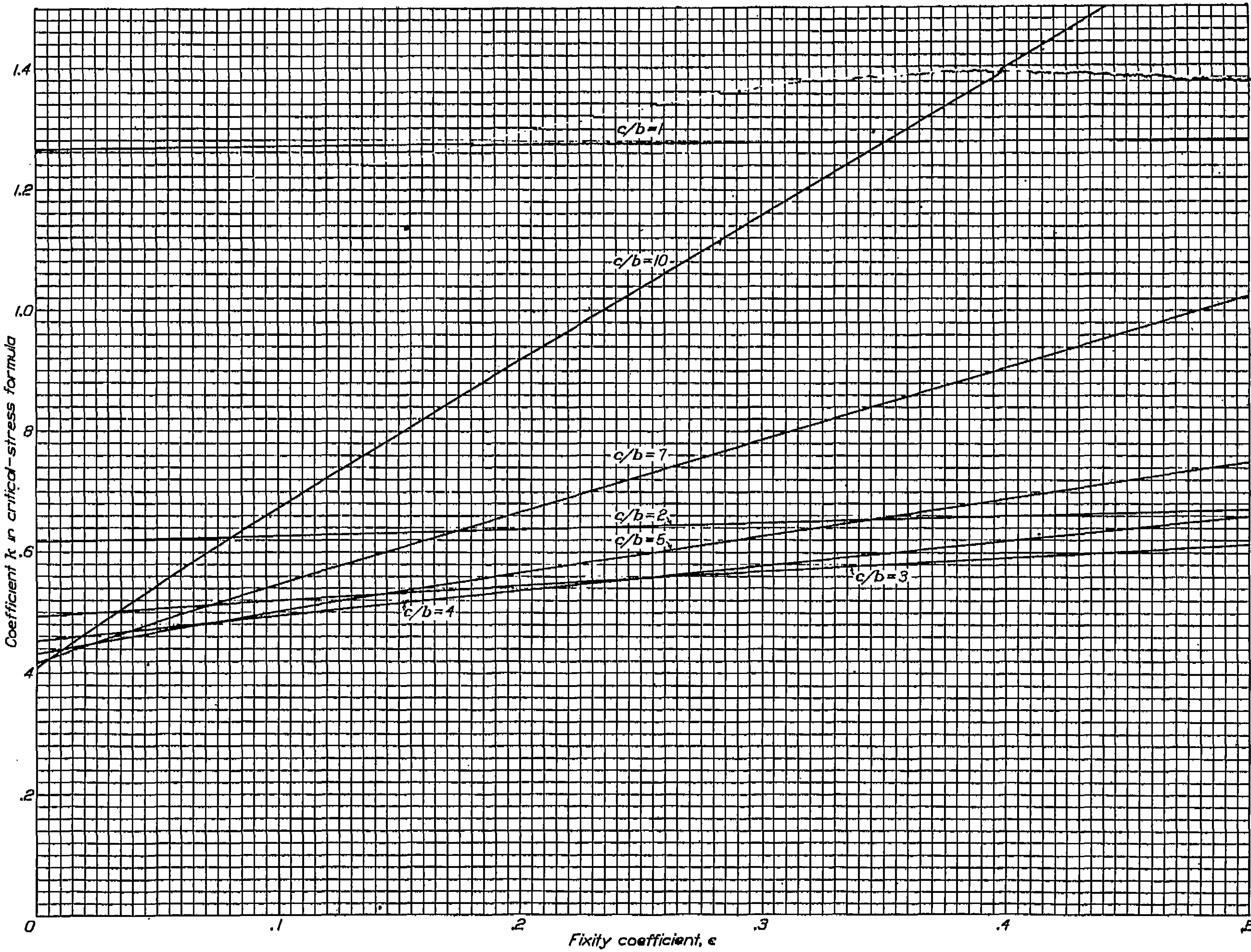

FIGURE 24.—The coefficient k in equation (84a) for a flange of isotropic material that has a Poisson's ratio of 0.25

TABLE XIV

THE COEFFICIENT k IN EQUATION (84a) FOR A FLANGE OF ISOTROPIC MATERIAL THAT HAS A POISSON'S RATIO OF 0.25

c/b	ϵ	k	c/b	ϵ	k
1	0.50	1.287	2	0.50	0.673
1	.10	1.278	2	.10	.628
1	.00	1.269	2	.05	.622
			2	.01	.617
			2	.00	.616
3	.50	.616	4	.50	.660
3	.10	.522	4	.10	.498
3	.05	.509	4	.05	.476
3	.01	.498	4	.03	.467
3	.00	.496	4	.01	.458
			4	.00	.454
5	.50	.750	7	.50	1.026
5	.10	.503	7	.10	.549
5	.05	.469	7	.05	.484
5	.02	.448	7	.03	.458
5	.00	.434	7	.01	.431
			7	.00	.418
10	.50	1.640			
10	.10	.675			
10	.05	.543			
10	.00	.409			

Flange with a simply supported edge, the limiting case as the fixity coefficient approaches zero.

As the fixity coefficient ϵ approaches zero in equations (78) and (84) it is found that the value of z corresponding to a minimum value of p approaches zero. This should be so for as ϵ approaches zero the edge $y=0$ becomes more and more nearly simply supported. The ratio of A_1 to A in (74) will then approach zero. By equation (77) this implies that z approaches zero, as just noted.

Accordingly the limiting critical stress as ϵ approaches zero is found to be

$$p=\left[\frac{\pi^2}{12(1-\sigma_{xy}\sigma_{yx})}\frac{1}{\rho}+\frac{\mu_{xy}}{E_x}\right]E_x\frac{h^2}{b^2} \tag{85}$$

by setting $\epsilon=0$ and $z=0$ in (78) and (80). The values of k given by this formula for a simply supported edge agree well with those of Table XIII for the fixity coefficient $\epsilon=0$. As ρ becomes large p decreases to the limiting value,

$$p=\left(\frac{\mu_{xy}}{E_x}\right)E_x\frac{h^2}{b^2}. \tag{86}$$

Using the elastic constants for spruce having the growth rings at an angle of 45° with the faces of the flange, (86) becomes

$$p=0.044\ E_x\frac{h^2}{b^2}. \tag{87}$$

If the growth rings are perpendicular to the faces of the flange

$$p=0.053\ E_x\frac{h^2}{b^2}, \tag{88}$$

while if they are parallel

$$p=0.037\ E_x\frac{h^2}{b^2}. \tag{89}$$

Thus for a flange with a simply supported edge the critical stress is less when the growth rings are parallel to the faces of the flange than when the rings are inclined to them at an angle of 45°. For a flange with a perfectly fixed edge, on the contrary, the critical stress was found to be less when the rings are inclined to the faces at an angle of 45° than when they are parallel to them. The relative variation of the critical stress with inclination of the rings is less for flanges with simply supported edges than for those with perfectly fixed edges.

In practice, the fixity at the bases of the flanges is small. Consequently the variation of the critical stress with the inclination of the growth rings may be expected to be similar to that for flanges with simply supported edges.

From this point on the discussion will be limited to flanges with growth rings at an angle of 45° to the faces. The results may be considered to be applicable to flanges with rings at any inclination except for the extreme cases of rings nearly parallel to the faces or nearly perpendicular to them. In the first case the calculated critical stress should be reduced somewhat, while in the second it should be increased somewhat. These formulas hold for long flanges. For short ones the effect of the first term of (85) must be included.

For isotropic material the equations corresponding to (85) and (86) are

$$p=\left[\frac{\pi^2}{12(1-\sigma^2)}\frac{1}{\rho}+\frac{1}{2(1+\sigma)}\right]E\frac{h^2}{b^2} \tag{90}$$

and

$$p=\frac{1}{2(1+\sigma)}E\frac{h^2}{b^2}. \tag{91}$$

With $\sigma=0.25$ equation (91) becomes

$$p=0.4E\frac{h^2}{b^2} \tag{92}$$

and with $\sigma=0.3$

$$p=0.385E\frac{h^2}{b^2}. \tag{93}$$

For short flanges the first term in (90) must be retained.

The results expressed by equations (85) to (93) for flanges with a simply supported edge at $y=0$ could have been obtained directly through the approximate method by assuming, for example, instead of (74) that

$$W=Ay.$$

This was done for isotropic flanges by Timoshenko. (Reference 21, p. 396.)

Effect of elastic giving of material at the base of the flange.

In obtaining the preceding results the lack of fixity of the edge $y=0$ was ascribed to the twisting of the column, either as a whole or in segments, in consequence of the moments applied at this edge by the deformation of the flange. Actually, however, the material at the base of the flange yields elastically under the action of these moments so that the angle of rotation of the section is less than $(\partial w/\partial y)_{y=0}$, the inclination of the flange at its base. Accordingly equation (72) should be replaced by

$$\phi=\eta\left(\frac{\partial w}{\partial y}\right)_{y=0}$$

where η is some proper fraction. The effect is to reduce the strain energy V_1 (equation (73)) resulting from the twisting of the column. To the reduced V_1 should be added the energy of deformation of the material at the base of the flange. This portion of the energy is relatively small. The result is that V_1, equation (73), which was added to V, equation (58), to express the whole energy of deformation of the flange and column in so far as it arises from the load on the flanges, should be reduced. This is equivalent to saying that ϵ as calculated by (79) from the torsional rigidity of the section should be reduced.

For flanges of wood in which the grain is longitudinal, such reduction in the fixity coefficient is very great. This is due to the extremely small relative value of the modulus of elasticity E in the direction parallel to the faces of the flange and perpendicular to its length, which ranges from 1/15 of the modulus in the longitudinal direction in quarter-sawn flanges of spruce to 1/100 of this modulus in flanges in which the growth rings make an angle of 45° with the faces. The tests show that, for calculated coefficients of fixity of the order of magnitude of 2 and above, the critical stress corresponds to an actual fixity of about 0.01. Corresponding reductions in the smaller calculated fixity coefficients are observed but the law that the reduction follows has not been determined.

The practical result of the reduction in fixity because of elastic giving is that the condition of a simply supported edge at the base of the flange is closely approximated when the calculated fixity coefficient is small. The material is unable to transmit the bending moment from the base of the flange to the body of the column, with the result that the flange itself is inclined nearly as if it were merely hinged or simply supported at its base and consequently a condition in which formula (87) is applicable is approached. This situation will be discussed further in connection with the study of two flanged columns with the aid of the curves of Figure 23.

A similar but probably not so great a reduction occurs in the calculated fixity coefficients of the flanges of structural steel columns in consequence of the elastic giving of the material at the bases of the flanges. Practically no data are available for use in determining the extent of this reduction.

Examples of the determination of the critical stresses, neglecting the effect of elastic giving at the bases of the flanges.

In the following paragraphs will be explained the procedure to follow in applying the results of the preceding mathematical analysis, using the fixity coefficient as calculated from the torsional rigidity of the section and the dimensions of the flange and neglecting the reduction in this coefficient that should be made to allow for the elastic giving of the material at the base of the flange. The method can then be applied when the reduced coefficients are known by substituting in each case for the fixity coefficient ϵ the reduced fixity coefficient ϵ'.

The method will be first applied to a column of spruce similar to many of those used in the tests. The dimensions are shown in Figure 25. The growth rings in the single outstanding flange will be assumed to make an angle of 45° with the faces of the flange. The fixity coefficient is given by

$$\epsilon=12(1-\sigma_{xy}\sigma_{yx})\pi^2\frac{G}{E_x}K\frac{b}{c^2h^3}.$$

If

$$\frac{G}{E_x}=\frac{1}{15.6}$$

it follows that

$$\epsilon=7.58\frac{K}{bh^3}\frac{b^2}{c^2}. \qquad (94)$$

With the given dimensions

where $\alpha=c/b$. $\qquad \epsilon=1066\frac{1}{\alpha^2}$.

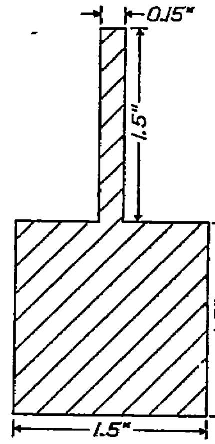

FIGURE 25.—The cross section of a wood test column with a single thin, outstanding flange the growth rings of which make an angle of 45° with the faces

It is important to observe that the coefficient ϵ depends upon the half wave length c. This coefficient was computed for a series of possible half wave lengths, the length of the column being 40 inches, and the quantity k, to which the corresponding critical stress is proportional, was then taken by extrapolation from the curves of Figure 23. The results are shown in Table XV. The numbers in the last column of the table are really estimated, since the values of ϵ concerned are far beyond the limits plotted on the curves of Figure 23. Through inspection of this column and the curves in Figure 23, however, it becomes clear that the flange will break into five half wave lengths, the critical stress being 0.12 $E_x h^2/b^2$, corresponding to the value 5.33 of the ratio c/b. These values agree well with those calculated for $\epsilon=\infty$. Indeed it is apparent from the behavior of the portions of the curves shown that the ordinates rapidly approach their limiting values as ϵ increases.

The approximate method used in calculating the curves of Figure 23 gives values of k that are slightly too large for the higher values of ϵ. High values of ϵ, however, do not occur in cases of practical interest, as will shortly be seen. The approximate method may therefore be considered entirely satisfactory.

TABLE XV

VARIATION OF CRITICAL STRESS WITH NUMBER OF HALF WAVE LENGTHS FOR THE 40-INCH COLUMN OF FIGURE 25

Number of half wave lengths.	c	$\alpha=c/b$	ϵ	k (estimated)
8	5.00	3.33	96.1	0.18
7	5.71	3.81	73.4	.13
6	6.67	4.45	53.8	.12
5	8.00	5.33	37.5	.12
4	10.00	6.67	24.0	.12
3	13.33	8.89	13.5	.14
2	20.00	13.33	6.0	.20
1	40.00	26.67	1.5	.40

Table XV was calculated on the assumption that the effects of the elastic giving of the material at the base of the flange could be neglected. This table indicated a minimum critical stress of 0.12 $E_x h^2/b^2$, corresponding to the value 5.33 of the ratio c/b. Actual tests, however, show that the flange wrinkles at a stress of 0.07 $E_x h^2/b^2$. (Part I, equation (5), p. 9.) This reduction in the critical stress should be attributed to the elastic giving of the material at the base of the flange. The curves show that this minimum critical stress should be attributed to a fixity coefficient in the vicinity of 0.01 and a ratio of c/b of about 7. This example is very informing, since it indicates a reduction in the fixity coefficient from a number of the order of 20.0 to one of the order of 0.01.

In the example just considered there was only one outstanding flange. If there are N flanges, the fixity coefficient as calculated should be divided by N.

Consider now the section of column T–25, Figure 5. The length of the column is taken as 120 inches. The growth rings of the wood will be assumed to make an angle of 45° with the faces of the flanges. In accordance with equation (79)

$$\epsilon=\frac{7.58}{4}\frac{K}{bh^3}\frac{b^2}{c^2}=2.736\frac{1}{\alpha^2} \tag{95}$$

where $\alpha=c/b$. Proceeding as before Table XVI was constructed with the aid of the curves of Figure 23.

TABLE XVI

VARIATION OF CRITICAL STRESS WITH NUMBER OF HALF WAVE LENGTHS FOR COLUMN T–25 OF FIGURE 5

Number of half wave lengths	c	$\alpha=c/b$	ϵ	k
8	15.00	4.67	0.1255	0.115
7	17.14	5.34	.0960	.112
6	20.00	6.23	.0705	.107
5	24.00	7.47	.0490	.105
4	30.00	9.34	.0314	.103
3	40.00	12.46	.0176	.102
2	60.00	18.68	.0078	.107
1	120.00	37.37	.0020	[1] .120

[1] Estimated.

The values of k in Table XVI indicate that at a critical stress of 0.102 $E_x h^2/b^2$ each flange will break into three half wave lengths corresponding to a fixity coefficient of 0.0176. The tests showed that each flange broke into a single half wave length and the column twisted at a critical stress of about 0.044 $E_x h^2/b^2$, the critical stress for a simply supported edge. This means that the calculated fixity coefficient has been reduced nearly to zero by the elastic giving of the material at the bases of the flanges.

Failure through twisting or wrinkling.

When, as in the example just given, the least critical stress is associated with a half wave length equal to the length of the column, the column fails by twisting about its axis. At the base of each flange, as a result of the beginning of failure, a torque that is in the same sense for the entire length of the column is applied to the column as a whole. If a flange breaks into several half wave lengths, however, the torques at its base are in opposite senses in adjoining half wave lengths and consequently oppose one another.

Practical rules for determining the critical stress, allowance being made for elastic giving of the material at the bases of the flanges.

In a cruciform section having equal arms and no fillets it appears from equation (95) that a change in the dimensions, b, the outstanding width, and h, the thickness of the flange, will not greatly alter the calculated fixity coefficient ϵ, since K, the torsion constant is nearly proportional to b and to h^3. (Part I, p. 7.)

Much the same situation exists in other sections, such as L, U, Z, and T, made up of component rectangles, all parts being of equal thickness and having no fillets. It appears from the data at our disposal that the flanges of such sections may be treated as having their bases simply supported. The critical stress for long columns of spruce of such sections may then be taken as 0.044 $E_x h^2/b^2$, provided that this stress is less than the one that would cause primary failure. If fillets are added to any of these sections or if the thickness of the back of a channel is increased, for example, the critical stress will increase. The exact amount of this increase can not be stated, since the law by which the calculated fixity coefficient is reduced through the giving of the material at the bases of the flanges is not known. Tests indicate, however, that the critical stress is increased approximately in the ratio of the torsional rigidity of the changed section to that of the original section. This relation may be taken to hold for spruce until the limiting critical stress 0.07 $E_x h^2/b^2$ is attained. From this point as the torsional rigidity increases the critical stress remains unchanged.

As the critical stress increases with increasing coefficient of fixity at the base of the flange, the type of failure changes from one through twisting to one through wrinkling. The distinction between these externally different types of failure does not appear to be important, since the one goes over gradually to the other.

For flanges on short columns the critical stresses will be higher than those for the long columns just considered.

As previously stated, the foregoing discussion applies to flanges of spruce in which the growth rings make an angle of 45° with the faces of the flange. Flanges of steel or other isotropic material can be treated in a similar way through the use of Table XIV and the curves of Figure 24. Sufficient experimental data for steel columns, to enable the authors to estimate the effect of the reduction in the calculated coefficient of fixity, have not been found in the literature.

CONCLUSIONS

1. Under a compressive load, the critical stress for a moderately long flange of spruce, perfectly fixed along its base and of thickness h and width b, is given by

$$p=0.228\ E_x\frac{h^2}{b^2}$$

when the growth rings are perpendicular to the faces of the flange (fig. 21), by

$$p=0.117\ E_x\frac{h^2}{b^2}$$

when the rings make an angle of 45° with the faces (fig. 22), and by

$$p=0.164\ E_x\frac{h^2}{b^2}$$

when the rings are parallel to the faces. In these formulas E_x is Young's modulus in the direction of the grain of the wood, which is taken as the direction of the length of the flange.

For a flange of steel the base of which is perfectly fixed the critical stress is given by

$$p=1.16\ E\frac{h^2}{b^2}$$

when Poisson's ratio is taken as 0.3.

2. If the base of the flange is simply supported the corresponding critical stresses are

$$p=0.053\ E_x\frac{h^2}{b^2}$$

and

$$p=0.044\ E_x\frac{h^2}{b^2}$$

and

$$p=0.037\ E_x\frac{h^2}{b^2}$$

for a flange of spruce and

$$p=0.385\ E\frac{h^2}{b^2}$$

for a flange of steel. Such a condition at the base of the flanges is found, for example, in the case of columns of L, U, Z, T, and + sections without fillets and having parts of the same thickness.

3. The condition of perfect fixity is not realized in practice because of the elastic giving of the material at the base of the flange. Tests indicate that the upper limit of the critical stress for moderately long flanges of spruce is given by

$$p=0.07E_x\frac{h^2}{b^2}.$$

This is an average value from tests of specimens in which the growth rings were at various inclinations to the faces of the flanges. For strictly quarter-sawn flanges the critical stress would be somewhat higher and for plain-sawn ones somewhat lower. The reduction from the values given for flanges with perfectly fixed edges should be attributed to the elastic giving of the material at the bases of the flanges.

Because of the same elastic giving the fixity of flanges with partially fixed bases is greatly reduced. For such flanges the critical stress ranges from

$$p=0.044E_x\frac{h^2}{b^2}$$

to the upper limit

$$p=0.07E_x\frac{h^2}{b^2}.$$

Both limiting stresses can be increased somewhat for strictly quarter-sawn flanges and should be reduced somewhat for plain-sawn ones.

Tests on steel flanges were not made. As a result of the elastic giving of the material at the base of the flange, however, it is probable that the upper limit of the critical stress will be found to be considerably less than that calculated for a flange with a perfectly fixed edge.

4. The critical stresses for short flanges are greater than those given by the preceding formulas.

5. The critical stresses obtained through use of these formulas will be of interest only if they are less than those that would cause a primary failure of the column under consideration.

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DEPARTMENT OF AGRICULTURE,
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