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A THEORY FOR THE LATERAL RESPONSE OF AIRPLANES TO RANDOM ATMOSPHERIC TURBULENCE

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TABLE OF CONTENTS

Page

SUMMARY 1

INTRODUCTION 1

SYMBOLS 4

LATERAL EQUATIONS OF MOTION 7
- Systems of Axes and Description of Turbulence 7
- System of Equations 8
- Power Spectral Response 10
- Correlation Between Airplane Components 14
- Transfer Functions 18
- Complex Moment and Force Coefficients 21
  - Rolling moments 21
  - Yawing moments 23
  - Side force 24

POWER SPECTRA OF GUST VELOCITY 24

APPLICATION OF METHOD 26
- Lateral Response of Three Airplanes 26
- Analysis 30
  - Characteristics of the method 30
  - Effect of trim angle of attack 31
  - Effect of horizontal gusts 32
  - Response characteristics of example airplanes 32
- Simplified Equations 33

COMPARISON OF METHODS 34
- Comparison of Gust Power Spectra 34
  - Side gusts 35
  - Rolling gusts 35
- Comparison of Lateral Response Power Spectra 37

CONCLUDING REMARKS 38

APPENDIX A - PROOF OF POWER SPECTRAL AND CROSS-POWER SPECTRAL RELATIONSHIPS 40

APPENDIX B - RELATIONSHIP BETWEEN POWER SPECTRA OF AIRPLANE FORCES AND MOMENTS AND POWER SPECTRA OF GUST VELOCITIES 44

REFERENCES 47

TABLES 49

FIGURES 51

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SUMMARY

The lateral motions of an airplane flying through continuous random isotropic turbulence have been derived in terms of (1) the transfer functions relating the motion in the various degrees of freedom to the yawing moment, rolling moment, and side force, (2) the statistical forces and moments at the center of gravity due to gust velocities acting on the lifting surfaces of the airplane, and (3) the power spectra of the three orthogonal components of gust velocity acting on the airplane along the flight path. The method takes into account the random variations of gust velocity across the span and along the fuselage. Solutions are given in the form of equations relating the power spectra of the angular motions of the airplane to the power spectra of the gust velocities.

Three airplanes of different size are used to demonstrate the method, illustrate characteristic trends, and exhibit some simplifications possible in the calculations. For these airplanes the effects of horizontal gusts (that is, gusts parallel to the flight path) and side forces due to gusts on the airplanes were found to be negligible.

By using one of the example airplanes, a comparison is drawn between the present theory and several less comprehensive theories for calculating the effect of gusts on wings of finite span and the effect of gusts on the motions of the complete airplane.

INTRODUCTION

The classical theory of stability and control of airplanes has been applied to the calculation of response to controls and response to gusts. In the calculation of response to gusts, however, the angle-of-attack distributions which produced the forces and moments applied to the airplane by the gust have customarily been assumed to be equivalent to those resulting from a rigid-body motion of the airplane. This method was applied in NACA Report 1 and other early reports (ref. 1) in the study

of the longitudinal response of airplanes to gusts and in reference 2 to the calculation of the lateral response to gusts. Because of the method of accounting for the effect of the gusts, these analyses were restricted to the consideration of the effects of isolated gusts of some specified shape. Most early developments in the theory of response to gusts were attempts to account for factors which were important in the calculation of loads. For example, the calculation of loads due to vertical gusts was extended in reference 3 to include the effects of unsteady lift and of flexibility.

In the early methods of calculating gust response, the analysis did not include such factors as the penetration effect (a result of the penetration of a given gust at different times by different parts of the airplane) and the variation of gust velocities along the fuselage and spanwise directions. In recent years, some attempts have been made to include these effects without changing the basic method of approach. In reference 4 an approximate method was introduced for estimating the penetration effect in the calculation of longitudinal response to gusts by taking into account first-order effects of the time lag between the penetration of the gust by the wing and by the tail. The method used was similar to that introduced in reference 5 for calculating the effect of lag of downwash. A similar application of this concept to the calculation of lateral response to gusts was presented in reference 6.

A more realistic treatment of random disturbances in the atmosphere was made possible with the introduction of a method of calculating the statistical response of an airplane in terms of the statistical properties of atmospheric gust velocities. This method, known as statistical dynamics, was first applied in reference 7 to the measurement of the longitudinal response of an airplane to random vertical gusts and has since been used extensively in longitudinal gust loads studies.

A theoretical analysis for the statistical lateral response of an airplane to gusts was given in reference 8 where, in order to approximate the effects of the span, the turbulence was considered to be represented by a combination of rolling gusts and side gusts having arbitrary distributions along the flight path. With this representation of turbulence, however, the question arose as to the relative magnitudes of the rolling and side gusts which would satisfactorily represent atmospheric turbulence. In reference 8 the assumption was made that an arbitrary gust distribution across the span could be represented by a constant antisymmetric gradient which was assumed to be equal to the gradients existing in the corresponding distribution of side gusts. Experimental flight measurements of reference 9 indicated, however, that the measured gradients of vertical gusts across the span did not agree with the theoretical calculations of reference 8, and the assumption of nonisotropic atmospheric turbulence was employed in order to fit the theory to the flight-test results.

A more realistic method of utilizing statistical dynamics to take into account random gusts both along the flight path and across the span was reported in reference 10 in outlining a means of calculating the longitudinal response of an airplane. The analysis was consistent with isotropic turbulence and required only the statistical gust distributions measured at one point in the turbulence.

Therefore, with the advent of statistical dynamics and more realistic means of analyzing the random distributions over the airplane as well as along the flight path, the original methods of treating gusts no longer appear to be justified. Furthermore, although it would be desirable to continue to treat the effects of gusts as equivalent to certain rigid-body motions of the airplane, such treatment is not consistent with statistical gust inputs. The forces and moments on the airplane rather than the resulting motions are statistically related to the gust input; that is, a random distribution of gusts across the span and along the fuselage of an airplane will produce changes in forces and moments which are related to the gust velocities along the flight path only in a statistical manner. The purpose of this paper is to establish the statistical relations between gust velocities and the forces and moments affecting the lateral motions of the airplane and to use these forces and moments to compute the lateral motion.

In preparation for the calculations of the present report, two preceding papers have been published. One of these papers (ref. 11) contains a method for calculating the side force and yawing moment on the fuselage and tail of an airplane in continuous gusts. The other (ref. 12) presents the lateral forces and moments on a wing due to three-dimensional random turbulence for a number of wing load distributions and for wings of various spans. This work (ref. 12) is an extension of the theory given in reference 10 for defining the lift on a wing due to random turbulence.

This paper presents a method of utilizing the statistical forces and moments derived in references 11 and 12 for the calculation of the lateral motion of an airplane in any lateral degree of freedom due to atmospheric turbulence. First, the governing equations are derived and the power spectral solutions of the motions are given in terms of linear airplane transfer functions, statistical force and moment coefficients, and power spectra of the three orthogonal components of atmospheric gust velocity. General expressions are given or referenced for each of these independent functions. Next, solutions for three airplanes are shown with trends and possible simplifications noted. Finally, a comparison is made between the results of the present theory and those of earlier theories, both by comparing the gust distributions required to produce equivalent airplane motion and by comparing the airplane motions for the same assumed gust characteristics.

SYMBOLS

A	constant used in references 6, 8, and 9 to indicate relative amplitudes of power spectra; aspect ratio (table II)
B	matrix of equations of motion of airplane
b	wing span
D	nondimensional operator, $\frac{b}{U}\frac{d}{dt}$
$E(w_g)$	plot of Φ_{c_l} of reference 12
f	arbitrary function of time
F	Fourier transform of a quantity
h_p	pressure altitude
h_z	height of center of pressure of vertical tail above X stability axis of airplane
$i = \sqrt{-1}$	
$k_n = \omega x_n/U$	
$k' = \omega L/U$	
K	arbitrary constant
K_X	nondimensional radius of gyration about X stability axis
K_Z	nondimensional radius of gyration about Z stability axis
K_{XZ}	nondimensional product of inertia
l_t	tail length measured from center of gravity to center of pressure of vertical tail
L	integral scale of turbulence
m,n	arbitrary terms which may take on values 1, 2, 3 . . .
p	rolling velocity, $\dot{\phi}$
q	dynamic pressure

r	yawing velocity, $\dot{\psi}$
R	real part of complex quantity
S	wing area
s	profile height (see eq. (44))
t	time
T	particular value of time
U	relative velocity between airplane and general air mass
u	velocity along X stability axis
v	velocity along Y stability axis
w	velocity along Z stability axis
W	weight of airplane
X,Y,Z	three orthogonal stability axes (see fig. 1)
x,y,z	coordinate position with reference to stability axes
C_L	lift coefficient, $\frac{\text{Lift}}{qS}$
C_l	rolling-moment coefficient, $\frac{\text{Rolling moment}}{qSb}$
C_n	yawing-moment coefficient, $\frac{\text{Yawing moment}}{qSb}$
C_Y	side-force coefficient, $\frac{\text{Side force}}{qS}$
C_{D_o}	coefficient of drag at zero lift
α	angle of attack
β	angle of sideslip
$\beta' = b/L$	
Γ	effective dihedral angle

γ flight-path angle

θ angle of pitch

λ taper ratio

μ relative density factor, $\frac{\text{Mass}}{\rho S b}$

ρ density of atmosphere

σ sidewash angle

ϕ angle of roll

ψ angle of yaw

ω circular frequency

ω' reduced frequency, $\omega b/U$

ω'_n natural reduced frequency

Φ_f power spectral density of a function f

$\Phi_{f_1 f_2}$ cross-power spectral density of functions f_1 and f_2

C.P. cross power

Stability derivatives of airplane are indicated by subscript notation; for example,

$$C_{l_r} = \frac{\partial C_l}{\partial\left(\frac{rb}{2U}\right)} \qquad C_{n_p} = \frac{\partial C_n}{\partial\left(\frac{pb}{2U}\right)} \qquad C_{Y_\beta} = \frac{\partial C_Y}{\partial\left(\frac{v}{U}\right)}$$

Subscripts:

W wing

F fuselage

T vertical tail

FT fuselage and vertical tail

g gust

0 trim condition

j,k,m generalized matrix subscripts

Superscripts:

* conjugate of a complex number

Bar over quantity denotes mean value.

Dot over quantity denotes derivative with respect to time.

Matrix notation:

$|\ \ |$ absolute value of a quantity (determinant of a matrix)

$[\ \ ]$ rectangular matrix

$\{\ \ \}$ column matrix

LATERAL EQUATIONS OF MOTION

Systems of Axes and Description of Turbulence

The motions of an airplane flying through continuous atmospheric turbulence are defined as angular displacements, rates, or accelerations about a set of stability axes (with respect to the general air mass, the X stability axis is always alined with the projection of the resultant mean velocity on the plane of symmetry) whose origin is fixed at the center of gravity. These stability axes have some mean orientation with respect to the body axes of the airplane denoted by the angle α_o in the XZ-plane. The general orientation of the stability axes of the airplane with respect to the gust velocities and with respect to the body and earth axes is shown in figure 1.

At each position along the flight path of the airplane the instantaneous velocities of the local air mass, hereinafter designated as gusts, are instantaneously defined by three orthogonal components of velocity alined with the stability axes of the airplane at that position. The gusts are assumed to be essentially isotropic in nature, and their statistical characteristics remain unchanged regardless of the motion or orientation of the airplane. This assumption is implicit throughout this paper.

Furthermore, the assumption is made that, during the small interval of time in which each gust acts on the airplane, the gust velocity does not

change appreciably. On this basis the three components of gust velocity - alined with, but of opposite sign to, displacements along the stability axes of the airplane - are functions only of their space position along the flight path. With reference to the center of gravity of the airplane, these gusts are denoted by

$$u_g = u_g(x,y,z)$$

$$v_g = v_g(x,y,z)$$

$$w_g = w_g(x,y,z)$$

The orientation of the gust axes is shown in figure 1(a).

The assumption of isotropy further relates these velocities only to the extent of

$$\overline{u_g^2} = \overline{v_g^2} = \overline{w_g^2}$$

where the bar denotes the mean value.

System of Equations

The lateral equations of motion of an airplane defined in the stability system of axes are generally well known. Based on these equations of motion, the frequency response of an airplane in its three lateral degrees of freedom is defined by the matrix

$$\begin{Bmatrix} \phi(\omega) \\ \psi(\omega) \\ \beta(\omega) \end{Bmatrix} = \begin{bmatrix} \frac{\phi}{C_l}(\omega) & \frac{\phi}{C_n}(\omega) & \frac{\phi}{C_Y}(\omega) \\ \frac{\psi}{C_l}(\omega) & \frac{\psi}{C_n}(\omega) & \frac{\psi}{C_Y}(\omega) \\ \frac{\beta}{C_l}(\omega) & \frac{\beta}{C_n}(\omega) & \frac{\beta}{C_Y}(\omega) \end{bmatrix} \begin{Bmatrix} C_l(\omega) \\ C_n(\omega) \\ C_Y(\omega) \end{Bmatrix} \tag{1}$$

In equation (1) the ratios ϕ/C_l, ψ/C_n, and so forth, are complex transfer functions of the airplane and relate the response of the airplane in ϕ, ψ, and β to a sinusoidal rolling moment, yawing moment, or side force of unit amplitude. These moments and side force are functions of the gust

velocities referenced to a point (the center of gravity of the airplane). The necessary phase relations between these forces and moments and the gust velocities at the center of gravity are taken into account by expressing C_l, C_n, and C_Y as complex quantities. In terms of the gust velocities at the center of gravity, the lateral-force and moment coefficients may be expressed as follows:

$$\begin{Bmatrix} C_l(\omega) \\ C_n(\omega) \\ C_Y(\omega) \end{Bmatrix} = \begin{bmatrix} \frac{C_l}{u_g}(\omega) & \frac{C_l}{v_g}(\omega) & \frac{C_l}{w_g}(\omega) \\ \frac{C_n}{u_g}(\omega) & \frac{C_n}{v_g}(\omega) & \frac{C_n}{w_g}(\omega) \\ \frac{C_Y}{u_g}(\omega) & \frac{C_Y}{v_g}(\omega) & \frac{C_Y}{w_g}(\omega) \end{bmatrix} \begin{Bmatrix} u_g(\omega) \\ v_g(\omega) \\ w_g(\omega) \end{Bmatrix} \tag{2}$$

In equation (2) the elements of the rectangular matrix are transfer functions relating the forces and moments to the sinusoidal components of gust velocities of unit amplitude measured at the center of gravity. Inasmuch as some of these forces and moments arise as a result of gust velocities distributed over the airplane, which are, in turn, related to the gust components at the center of gravity only in a statistical sense, these transfer functions are likewise statistical in nature and are later expressed in the form of power spectra.

With the foregoing restrictions the complete relationship between the frequency response of the airplane in the three lateral motions and the gust velocities is given by the matrix equation

$$\begin{Bmatrix} \phi(\omega) \\ \psi(\omega) \\ \beta(\omega) \end{Bmatrix} = \begin{bmatrix} \frac{\phi}{C_l} & \frac{\phi}{C_n} & \frac{\phi}{C_Y} \\ \frac{\psi}{C_l} & \frac{\psi}{C_n} & \frac{\psi}{C_Y} \\ \frac{\beta}{C_l} & \frac{\beta}{C_n} & \frac{\beta}{C_Y} \end{bmatrix} \begin{bmatrix} \frac{C_l}{u_g} & \frac{C_l}{v_g} & \frac{C_l}{w_g} \\ \frac{C_n}{u_g} & \frac{C_n}{v_g} & \frac{C_n}{w_g} \\ \frac{C_Y}{u_g} & \frac{C_Y}{v_g} & \frac{C_Y}{w_g} \end{bmatrix} \begin{Bmatrix} u_g(\omega) \\ v_g(\omega) \\ w_g(\omega) \end{Bmatrix} \tag{3}$$

Thus, the moments due to the gusts and those due to the resulting motions are superposed. The equations of motion of the airplane are assumed to be linear within the range of this resulting motion.

Power Spectral Response

Inasmuch as turbulence in the atmosphere appears to be random, the statistical response of an airplane may, in general, be predicted in terms of the power spectral densities and mean-square values of the gusts in the atmosphere. An excellent summary of this field of dynamics is presented in reference 13. From the relationship (eq. (3)) between the gust velocities and the motions of the airplane in the frequency domain, the relationship between these quantities in terms of power spectra and cross-power spectra may be readily defined.

A quantity (herein denoted, in general, by the symbol $f(t)$) which has a random variation with time in the range $-T \leqq t \leqq T$ and is zero elsewhere is related to its Fourier transform $F(\omega)$ by

$$F(\omega) = \frac{1}{\pi} \int_{-T}^{T} f(t)\ e^{-i\omega t}\ dt \tag{4}$$

where, in turn,

$$f(t) = \frac{1}{2} \int_{-\infty}^{\infty} F(\omega)\ e^{i\omega t}\ d\omega \tag{5}$$

The power spectral density of the quantity is defined (see, for example, ch. 6 of ref. 14 or appendix of ref. 9) by

$$\begin{aligned} \Phi_f(\omega) &= \lim_{T\to\infty} \frac{1}{T} F^*(\omega)\ F(\omega) \\ &= \lim_{T\to\infty} \frac{1}{T} \left|F(\omega)\right|^2 \end{aligned} \tag{6}$$

where asterisks are used to denote the complex conjugate of a quantity and the term $\frac{1}{T}$ indicates that the spectra are symmetrical and only positive values of ω are considered when taking the average value.

When two related quantities are measured concurrently (for example, when the yawing and rolling moments due to each gust component are

measured at the same values of time), a cross-power spectral density exists between these two quantities. Where the frequency content of these two quantities is defined by

$$\left.\begin{aligned} F_1(\omega) &= \frac{1}{\pi}\int_{-T}^{T} f_1(t)e^{-i\omega t}\,dt \\ F_2(\omega) &= \frac{1}{\pi}\int_{-T}^{T} f_2(t)e^{-i\omega t}\,dt \end{aligned}\right\} \tag{7}$$

the cross-power spectral density is defined by

$$\Phi_{f_1f_2} = \lim_{T\to\infty}\frac{1}{T}F_1^*(\omega)F_2(\omega)$$

or

$$\begin{aligned} \Phi_{f_2f_1} &= \lim_{T\to\infty}\frac{1}{T}F_2^*(\omega)F_1(\omega) \\ &= \Phi_{f_1f_2}^* \end{aligned} \tag{8}$$

The significance of this cross power lies in the fact that, if the quantities are related, a certain interchange of energy takes place between them and the amount of this interchange is defined by the cross-power spectrum.

In the special case where F_1 and F_2 are the same function and are not removed in time or space (that is, $f_1(t) = f_2(t)$), equations (6) and (8) are identical and the cross power of f with itself is, in fact, the power spectrum.

These definitions may be found in most texts and in a number of papers on power spectral analysis, among them references 9, 13, and 14. On the basis of the preceding discussion, it may be shown that if each complex equation of the system defined by equation (3) is multiplied by its conjugate equation, divided through by the quantity T, and the limit as $T\to\infty$ taken on both sides of each equation, then the system of equations, by definition, is in the power spectral form. The power

spectra and cross-power spectra of the variables may be recognized and so denoted, the spectra and cross spectra of the gust velocities being unknown variables. An example of this operation is shown in appendix A.

By using this or any other suitable method of power spectral analysis, the system of equation (1) may be defined in terms of power and cross-power spectral densities such that

$$\begin{Bmatrix} \Phi_{\phi} \\ \Phi_{\psi} \\ \Phi_{\beta} \end{Bmatrix} = \begin{bmatrix} \left|\frac{\phi}{C_l}\right|^2 & \left|\frac{\phi}{C_n}\right|^2 & \left|\frac{\phi}{C_Y}\right|^2 \\ \left|\frac{\psi}{C_l}\right|^2 & \left|\frac{\psi}{C_n}\right|^2 & \left|\frac{\psi}{C_Y}\right|^2 \\ \left|\frac{\beta}{C_l}\right|^2 & \left|\frac{\beta}{C_n}\right|^2 & \left|\frac{\beta}{C_Y}\right|^2 \end{bmatrix} \begin{Bmatrix} \Phi_{C_l} \\ \Phi_{C_n} \\ \Phi_{C_Y} \end{Bmatrix} + 2R \begin{bmatrix} \left(\frac{\phi}{C_l}\right)^* \frac{\phi}{C_n} & \left(\frac{\phi}{C_n}\right)^* \frac{\phi}{C_Y} & \left(\frac{\phi}{C_Y}\right)^* \frac{\phi}{C_l} \\ \left(\frac{\psi}{C_l}\right)^* \frac{\psi}{C_n} & \left(\frac{\psi}{C_n}\right)^* \frac{\psi}{C_Y} & \left(\frac{\psi}{C_Y}\right)^* \frac{\psi}{C_l} \\ \left(\frac{\beta}{C_l}\right)^* \frac{\beta}{C_n} & \left(\frac{\beta}{C_n}\right)^* \frac{\beta}{C_Y} & \left(\frac{\beta}{C_Y}\right)^* \frac{\beta}{C_l} \end{bmatrix} \begin{Bmatrix} \Phi_{C_l C_n} \\ \Phi_{C_n C_Y} \\ \Phi_{C_Y C_l} \end{Bmatrix} \tag{9}$$

where 2R means twice the real part of the product of the two matrices.

On the right-hand side of equation (9), the product of the first two matrices is the contribution of the power spectra of the forces and moments, and the product of the second two matrices is the contribution of the cross-power spectra between the moments and forces produced by any one gust component and also the cross powers between the forces and moments produced by any two gust components. The powers and cross powers of the forces and moments may, in turn, be related to the powers and cross powers of the gust velocities. In so doing, a great simplification is obtained when atmospheric turbulence is considered to be isotropic within the frequency range affecting the response of the airplane.

By virtue of the assumption of isotropic homogeneous turbulence, the phasing between the u_g and w_g components and the v_g and w_g components is unrelated and the cross power of the components is zero. The phasing between the two gust components lying in the XY-plane of the airplane, that is, u_g and v_g, is unrelated when these velocities are measured at a given spanwise station (for example, along the airplane center line) but is related when u_g at one spanwise station is compared with v_g at another spanwise station. However, the effects

of the cross power between these two components are assumed to be insignificant, and justification for this assumption is made in a subsequent section. Thus,

$$\Phi_{u_g v_g}(\omega) = \Phi_{v_g w_g}(\omega) = \Phi_{w_g u_g}(\omega) = 0 \tag{10}$$

By the method of multiplying by the conjugate and taking the limits, the matrix relationship of equation (2) is defined in the form of power spectra by

$$\begin{Bmatrix} \Phi_{C_l} \\ \Phi_{C_n} \\ \Phi_{C_Y} \end{Bmatrix} = \begin{bmatrix} \left|\frac{C_l}{u_g}\right|^2 & \left|\frac{C_l}{v_g}\right|^2 & \left|\frac{C_l}{w_g}\right|^2 \\ \left|\frac{C_n}{u_g}\right|^2 & \left|\frac{C_n}{v_g}\right|^2 & \left|\frac{C_n}{w_g}\right|^2 \\ \left|\frac{C_Y}{u_g}\right|^2 & \left|\frac{C_Y}{v_g}\right|^2 & \left|\frac{C_Y}{w_g}\right|^2 \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \tag{11}$$

Likewise by cross-multiplying the equations in the matrix of equation (2) to conform with the definition of a cross-power spectrum, the cross spectra between the forces and moments are given by the matrix

$$\begin{Bmatrix} \Phi_{C_l C_n} \\ \Phi_{C_n C_Y} \\ \Phi_{C_Y C_l} \end{Bmatrix} = \begin{bmatrix} \left(\frac{C_l}{u_g}\right)^* \frac{C_n}{u_g} & \left(\frac{C_l}{v_g}\right)^* \frac{C_n}{v_g} & \left(\frac{C_l}{w_g}\right)^* \frac{C_n}{w_g} \\ \left(\frac{C_n}{u_g}\right)^* \frac{C_Y}{u_g} & \left(\frac{C_n}{v_g}\right)^* \frac{C_Y}{v_g} & \left(\frac{C_n}{w_g}\right)^* \frac{C_Y}{w_g} \\ \left(\frac{C_Y}{u_g}\right)^* \frac{C_l}{u_g} & \left(\frac{C_Y}{v_g}\right)^* \frac{C_l}{v_g} & \left(\frac{C_Y}{w_g}\right)^* \frac{C_l}{w_g} \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \tag{12}$$

A proof of the relationships of equations (11) and (12) is given in appendix B.

Correlation Between Airplane Components

Generally, a force or moment acting on or about the center of gravity of the airplane is due to the penetration of the gust by each component of the airplane, and the correlation or phasing between the development of these forces and moments must be included. Complete generalization to all possible combinations of components would lead to lengthy and unnecessary expansions; however, all lifting surfaces which might reasonably be expected to contribute significantly to the lateral motion of the airplane as well as the components of the gust velocity that will significantly influence the forces and moments on these lifting surfaces are incorporated. For some unusual configuration where other lifting surfaces are thought to be important, the method of including these factors is clearly indicated.

The force and moment coefficients which include only those functions deemed pertinent to the lateral motion of the airplane are defined by

$$\left.\begin{aligned} C_l(u_g,v_g,w_g) &= (C_l)_W(u_g,v_g,w_g) + (C_l)_T(v_g) \\ C_n(u_g,v_g,w_g) &= (C_n)_W(u_g,w_g) + (C_n)_{FT}(v_g) \\ C_Y(u_g,v_g,w_g) &= (C_Y)_{FT}(v_g) \end{aligned}\right\} \tag{13}$$

In terms of the complex force and moment derivatives, the force and moment matrix is thus defined by

$$\begin{Bmatrix} (C_l)_W \\ (C_l)_T \\ (C_n)_W \\ (C_n)_{FT} \\ (C_Y)_{FT} \end{Bmatrix} = \begin{bmatrix} \left(\frac{C_l}{u_g}\right)_W & \left(\frac{C_l}{v_g}\right)_W & \left(\frac{C_l}{w_g}\right)_W \\ 0 & \left(\frac{C_l}{v_g}\right)_T & 0 \\ \left(\frac{C_n}{u_g}\right)_W & 0 & \left(\frac{C_n}{w_g}\right)_W \\ 0 & \left(\frac{C_n}{v_g}\right)_{FT} & 0 \\ 0 & \left(\frac{C_Y}{v_g}\right)_{FT} & 0 \end{bmatrix} \begin{Bmatrix} u_g \\ v_g \\ w_g \end{Bmatrix} \tag{14}$$

By the same method as was applied to equation (2) to obtain equations (11) and (12) (multiplying by the conjugate and taking the limits), the power spectra and cross-power spectra of the force and moment relationships of equation (14) are immediately definable as

$$\begin{Bmatrix} \Phi(C_l)_W \\ \Phi(C_l)_T \\ \Phi(C_n)_W \\ \Phi(C_n)_{FT} \\ \Phi(C_Y)_{FT} \end{Bmatrix} = \begin{bmatrix} \left|\frac{C_l}{u_g}\right|_W^2 & \left|\frac{C_l}{v_g}\right|_W^2 & \left|\frac{C_l}{w_g}\right|_W^2 \\ 0 & \left|\frac{C_l}{v_g}\right|_T^2 & 0 \\ \left|\frac{C_n}{u_g}\right|_W^2 & 0 & \left|\frac{C_n}{w_g}\right|_W^2 \\ 0 & \left|\frac{C_n}{v_g}\right|_{FT}^2 & 0 \\ 0 & \left|\frac{C_Y}{v_g}\right|_{FT}^2 & 0 \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \tag{15}$$

$$\begin{Bmatrix} \Phi(C_l)_W(C_l)_T \\ \Phi(C_l)_W(C_n)_W \\ \Phi(C_l)_W(C_n)_{FT} \\ \Phi(C_l)_W(C_Y)_{FT} \\ \Phi(C_l)_T(C_n)_W \\ \Phi(C_l)_T(C_n)_{FT} \\ \Phi(C_l)_T(C_Y)_{FT} \\ \Phi(C_n)_W(C_n)_{FT} \\ \Phi(C_n)_W(C_Y)_{FT} \\ \Phi(C_n)_{FT}(C_Y)_{FT} \end{Bmatrix} = \begin{bmatrix} 0 & \left(\frac{C_l}{v_g}\right)_W^*\left(\frac{C_l}{v_g}\right)_T & 0 \\ \left(\frac{C_l}{u_g}\right)_W^*\left(\frac{C_n}{u_g}\right)_W & 0 & \left(\frac{C_l}{w_g}\right)_W^*\left(\frac{C_n}{w_g}\right)_W \\ 0 & \left(\frac{C_l}{v_g}\right)_W^*\left(\frac{C_n}{v_g}\right)_{FT} & 0 \\ 0 & \left(\frac{C_l}{v_g}\right)_W^*\left(\frac{C_Y}{v_g}\right)_{FT} & 0 \\ 0 & 0 & 0 \\ 0 & \left(\frac{C_l}{v_g}\right)_T^*\left(\frac{C_n}{v_g}\right)_{FT} & 0 \\ 0 & \left(\frac{C_l}{v_g}\right)_T^*\left(\frac{C_Y}{v_g}\right)_{FT} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \left(\frac{C_n}{v_g}\right)_{FT}^*\left(\frac{C_Y}{v_g}\right)_{FT} & 0 \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \tag{16}$$

The complete power-spectral-density relationships including the lateral moments and forces due to the various lifting surfaces of the airplane are similar to those given in equation (9) with the quantities defined in equations (15) and (16) inserted. When the left-hand side of equation (15) is denoted by $\left\{\Phi_{C_j}\right\}$ and the left-hand side of equation (16), by $\left\{\Phi_{C_jC_k}\right\}_{j\neq k}$, the complete relationship is defined by

$$\left\{\Phi_{\phi_i}\right\} = \left[\left|\frac{\phi_i}{C_j}\right|^2\right]\left\{\Phi_{C_j}\right\} + 2R\left[\left(\frac{\phi_i}{C_j}\right)^*\frac{\phi_i}{C_k}\right]\left\{\Phi_{C_jC_k}\right\}_{j\neq k}$$

where ϕ_1 is ϕ, ϕ_2 is ψ, and so forth. The complex force and moment derivatives contained in equation (14) were derived in such a manner that the phase relationships between the moments and the gust disturbances are given with reference to gusts acting at the center of gravity. For this reason, the transfer functions $\frac{\phi_i}{C_j}$ are those given in equation (1). As an illustration, the total response of the airplane in roll is obtained in the following form:

$$\begin{aligned}\Phi_\phi = {} & \left|\frac{\phi}{C_l}\right|^2\left\{\Phi(C_l)_W + \Phi(C_l)_T\right\} + \left|\frac{\phi}{C_Y}\right|^2 \Phi(C_Y)_{FT} + \left|\frac{\phi}{C_n}\right|^2\left\{\Phi(C_n)_W + \Phi(C_n)_{FT}\right\} + \\ & 2R\left[\left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_l}\,\Phi(C_l)_W(C_l)_T + \left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n}\,\Phi(C_l)_W(C_n)_W + \right. \\ & \left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n}\,\Phi(C_l)_W(C_n)_{FT} + \left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_Y}\,\Phi(C_l)_W(C_Y)_{FT} + \\ & \left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n}\,\Phi(C_l)_T(C_n)_{FT} + \left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_Y}\,\Phi(C_l)_T(C_Y)_{FT} + \\ & \left.\left(\frac{\phi}{C_n}\right)^*\frac{\phi}{C_Y}\,\Phi(C_n)_{FT}(C_Y)_{FT} + 3 \text{ zero terms}\right]\end{aligned} \tag{17}$$

Other degrees of freedom may be obtained directly by replacing ϕ with the appropriate symbol, such as ψ or β.

By substituting in equation (17) the expressions for the power and cross-power terms of the force and moments from equations (15) and (16), the motion is defined in terms of the gust velocities:

$$\Phi_{\phi} = \left|\frac{\phi}{u_g}\right|^2 \Phi_{u_g} + \left|\frac{\phi}{v_g}\right|^2 \Phi_{v_g} + \left|\frac{\phi}{w_g}\right|^2 \Phi_{w_g} \tag{18}$$

where

$$\left|\frac{\phi}{u_g}\right|^2 = \left|\frac{\phi}{C_l}\right|^2 \left|\frac{\partial C_l}{\partial u_g}\right|_W^2 + \left|\frac{\phi}{C_n}\right|^2 \left|\frac{\partial C_n}{\partial u_g}\right|_W^2 + 2R\left[\left(\frac{\partial C_l}{\partial u_g}\right)_W^* \left(\frac{\partial C_n}{\partial u_g}\right)_W \left(\frac{\phi}{C_l}\right)^* \frac{\phi}{C_n}\right] \tag{19}$$

$$\begin{aligned}\left|\frac{\phi}{v_g}\right|^2 = {} & \left|\frac{\phi}{C_l}\right|^2 \left|\frac{\partial C_l}{\partial v_g}\right|_W^2 + \left|\frac{\phi}{C_l}\right|^2 \left|\frac{\partial C_l}{\partial v_g}\right|_T^2 + \left|\frac{\phi}{C_n}\right|^2 \left|\frac{\partial C_n}{\partial v_g}\right|_{FT}^2 + \left|\frac{\phi}{C_Y}\right|^2 \left|\frac{\partial C_Y}{\partial v_g}\right|_{FT}^2 + \\ & 2R\left\{\left(\frac{\partial C_l}{\partial v_g}\right)_W^* \left(\frac{\partial C_l}{\partial v_g}\right)_T \left|\frac{\phi}{C_l}\right|^2 + \left(\frac{\partial C_n}{\partial v_g}\right)_{FT}^* \left(\frac{\partial C_Y}{\partial v_g}\right)_{FT} \left(\frac{\phi}{C_n}\right)^* \frac{\phi}{C_Y} + \right. \\ & \left[\left(\frac{\partial C_l}{\partial v_g}\right)_W^* \left(\frac{\partial C_n}{\partial v_g}\right)_{FT} + \left(\frac{\partial C_l}{\partial v_g}\right)_T^* \left(\frac{\partial C_n}{\partial v_g}\right)_{FT}\right] \left(\frac{\phi}{C_l}\right)^* \frac{\phi}{C_n} + \\ & \left.\left[\left(\frac{\partial C_l}{\partial v_g}\right)_W^* \left(\frac{\partial C_Y}{\partial v_g}\right)_{FT} + \left(\frac{\partial C_l}{\partial v_g}\right)_T^* \left(\frac{\partial C_Y}{\partial v_g}\right)_{FT}\right] \left(\frac{\phi}{C_l}\right)^* \frac{\phi}{C_Y}\right\}\end{aligned} \tag{20}$$

$$\left|\frac{\phi}{w_g}\right|^2 = \left|\frac{\phi}{C_l}\right|^2 \left|\frac{\partial C_l}{\partial w_g}\right|_W^2 + \left|\frac{\phi}{C_n}\right|^2 \left|\frac{\partial C_n}{\partial w_g}\right|_W^2 + 2R\left[\left(\frac{\partial C_l}{\partial w_g}\right)_W^* \left(\frac{\partial C_n}{\partial w_g}\right)_W \left(\frac{\phi}{C_l}\right)^* \frac{\phi}{C_n}\right] \tag{21}$$

Thus, the response of the airplane in any degree of freedom depends on the power spectra of the three gust components, the eight complex coefficients of equation (14) which relate the gust velocities to the forces and moments, and, for each degree of freedom, the three complex transfer functions relating the motion to a sinusoidal rolling moment, yawing moment, and side force of unit amplitude. These coefficients, transfer functions, and power spectra are defined in subsequent sections of this paper.

Transfer Functions

The lateral equations of motion usually accepted for the rigid dynamics of an airframe for which noncoupling with the longitudinal mode and all assumptions associated with small perturbations about a trim condition are valid are given by the matrix

$$\left[B\right]\begin{Bmatrix}\phi\\ \psi\\ \beta\end{Bmatrix}=\begin{Bmatrix}C_l(u_g,v_g,w_g)\\ C_n(u_g,v_g,w_g)\\ C_Y(u_g,v_g,w_g)\end{Bmatrix} \tag{22}$$

where

$$\left[B\right]=\begin{bmatrix}2\mu K_X{}^2D^2-\frac{1}{2}C_{l_p}D & -2\mu K_{XZ}D^2-\frac{1}{2}C_{l_r}D & -C_{l_\beta}\\ -2\mu K_{XZ}D^2-\frac{1}{2}C_{n_p}D & 2\mu K_Z{}^2D^2-\frac{1}{2}C_{n_r}D & -C_{n_\beta}\\ -\frac{1}{2}C_{Y_p}D-C_L & \left(2\mu-\frac{1}{2}C_{Y_r}\right)D-C_L\tan\gamma & 2\mu D-C_{Y_\beta}\end{bmatrix} \tag{23}$$

The solution, in terms of the rolling moment, yawing moment, and side force may be found by methods of operational calculus. One method is used by Mokrzycki in reference 15 where, except for the sign of K_{XZ}, the identical equations of motion are used but with moments from aerodynamic-control surfaces as forcing functions. Another means of solution is through matrix algebra where the solution as given by equation (1) is of the form

$$\begin{Bmatrix}\phi\\ \psi\\ \beta\end{Bmatrix}=\left[B\right]^{-1}\begin{Bmatrix}C_l\\ C_n\\ C_Y\end{Bmatrix} \tag{24}$$

where

$$\left[B\right]^{-1}=\frac{\text{Adjoint }\left[B'\right]}{\left[B\right]} \tag{25}$$

The prime denotes the transpose of the matrix B. Such a method is found in reference 16 and other texts on matrix algebra. The solution by either method gives the complex transfer functions with real constant coefficients of the powers of D. With the common denominator denoted by $|B|$ (equivalent to $D\bar{\Delta}$ of ref. 15), the transfer functions are given by

$$\frac{\phi}{C_l} = \left[4\mu^2 K_Z^2 D^3 - \mu\left(2K_Z^2 C_{Y_\beta} + C_{n_r}\right)D^2 + \left(\frac{1}{2} C_{n_r} C_{Y_\beta} + 2\mu C_{n_\beta} - \frac{1}{2} C_{n_\beta} C_{Y_r}\right)D - C_{n_\beta} C_L \tan\gamma\right]\frac{1}{|B|} \tag{26}$$

$$\frac{\psi}{C_l} = \left[4\mu^2 K_{XZ} D^3 + \mu\left(C_{n_p} - 2K_{XZ} C_{Y_\beta}\right)D^2 + \frac{1}{2}\left(C_{n_\beta} C_{Y_p} - C_{Y_\beta} C_{n_p}\right)D + C_L C_{n_\beta}\right]\frac{1}{|B|} \tag{27}$$

$$\frac{\beta}{C_l} = \left\{\mu\left[-2K_{XZ}\left(2\mu - \frac{1}{2} C_{Y_r}\right) + K_Z^2 C_{Y_p}\right]D^3 - \left[-2\mu C_L\left(K_{XZ}\tan\gamma + K_Z^2\right) + \frac{1}{2} C_{n_p}\left(2\mu - \frac{1}{2} C_{Y_r}\right) + \frac{1}{4} C_{Y_p} C_{n_r}\right]D^2 + \frac{C_L}{2}\left(C_{n_p}\tan\gamma - C_{n_r}\right)D\right\}\frac{1}{|B|} \tag{28}$$

$$\frac{\phi}{C_n} = \left[4\mu^2 K_{XZ} D^3 + \mu\left(C_{l_r} - 2K_{XZ} C_{Y_\beta}\right)D^2 - \left(\frac{1}{2} C_{l_r} C_{Y_\beta} + 2\mu C_{l_\beta} - \frac{1}{2} C_{Y_r} C_{l_\beta}\right)D + C_{l_\beta} C_L \tan\gamma\right]\frac{1}{|B|} \tag{29}$$

$$\frac{\psi}{C_n} = \left[4\mu^2 K_X^2 D^3 - \mu\left(C_{l_p} + 2K_X^2 C_{Y_\beta}\right)D^2 + \frac{1}{2}\left(C_{l_p} C_{Y_\beta} - C_{l_\beta} C_{Y_p}\right)D - C_{l_\beta} C_L\right]\frac{1}{|B|} \tag{30}$$

$$\frac{\beta}{C_n} = \left\{\mu\left[K_{XZ}C_{Y_p} - 2K_X^2\left(2\mu - \frac{1}{2}C_{Y_r}\right)\right]D^3 + \left(\frac{1}{4}C_{l_r}C_{Y_p} + 2\mu K_{XZ}C_L + \mu C_{l_p} - \frac{1}{4}C_{l_p}C_{Y_r} + 2\mu K_X^2 C_L \tan\gamma\right)D^2 + \frac{C_L}{2}\left(C_{l_r} - C_{l_p}\tan\gamma\right)D\right\}\frac{1}{|B|} \tag{31}$$

$$\frac{\phi}{C_Y} = \left[2\mu\left(K_Z^2 C_{l_\beta} + K_{XZ}C_{n_\beta}\right)D^2 + \frac{1}{2}\left(C_{l_r}C_{n_\beta} - C_{n_r}C_{l_\beta}\right)D\right]\frac{1}{|B|} \tag{32}$$

$$\frac{\psi}{C_Y} = \left[2\mu\left(K_X^2 C_{n_\beta} + K_{XZ}C_{l_\beta}\right)D^2 + \frac{1}{2}\left(C_{l_\beta}C_{n_p} - C_{n_\beta}C_{l_p}\right)D\right]\frac{1}{|B|} \tag{33}$$

$$\frac{\beta}{C_Y} = \left\{4\mu^2\left(K_X^2K_Z^2 - K_{XZ}^2\right)D^4 - \mu\left[K_X^2C_{n_r} + K_{XZ}\left(C_{l_r} + C_{n_p}\right) + K_Z^2C_{l_p}\right]D^3 + \frac{1}{4}\left(C_{l_p}C_{n_r} - C_{l_r}C_{n_p}\right)D^2\right\}\frac{1}{|B|} \tag{34}$$

$$\begin{aligned}|B| = {} & 8\mu^3\left(K_X^2K_Z^2 - K_{XZ}^2\right)D^5 - 2\mu^2\left[K_X^2\left(2K_Z^2C_{Y_\beta} + C_{n_r}\right) + K_Z^2C_{l_p} + K_{XZ}\left(C_{n_p} + C_{l_r} - 2K_{XZ}C_{Y_\beta}\right)\right]D^4 \\ & + \mu\left[K_Z^2\left(C_{Y_\beta}C_{l_p} - C_{Y_p}C_{l_\beta}\right) + K_X^2\left(C_{Y_\beta}C_{n_r} + 4\mu C_{n_\beta} - C_{Y_r}C_{n_\beta}\right) + \frac{1}{2}\left(C_{l_p}C_{n_r} - C_{n_p}C_{l_r}\right) + K_{XZ}\left(C_{Y_\beta}C_{l_r} + C_{Y_\beta}C_{n_p} + 4\mu C_{l_\beta} - C_{Y_r}C_{l_\beta} - C_{Y_p}C_{n_\beta}\right)\right]D^3 \\ & + \left\{-2\mu C_L\left[\tan\gamma\left(K_X^2C_{n_\beta} + K_{XZ}C_{l_\beta}\right) + K_Z^2C_{l_\beta} + K_{XZ}C_{n_\beta}\right] + \frac{1}{4}\left[C_{Y_\beta}\left(C_{n_p}C_{l_r} - C_{l_p}C_{n_r}\right) + C_{l_\beta}\left(C_{n_r}C_{Y_p} - C_{Y_r}C_{n_p}\right) + C_{n_\beta}\left(C_{Y_r}C_{l_p} - C_{Y_p}C_{l_r}\right)\right] + \mu\left(C_{l_\beta}C_{n_p} - C_{n_\beta}C_{l_p}\right)\right\}D^2 + \frac{C_L}{2}\left[\tan\gamma\left(C_{n_\beta}C_{l_p} - C_{l_\beta}C_{n_p}\right) + C_{l_\beta}C_{n_r} - C_{n_\beta}C_{l_r}\right]D\end{aligned} \tag{35}$$

In addition to the angular-displacement transfer functions given here, the response in terms of the derivatives of the angular displacements may be obtained by multiplying both sides of equations (26) to (34) by the operator D. For example,

$$\frac{\dot{\psi}}{C_Y} = \frac{U}{b} D \frac{\psi}{C_Y} = \frac{U}{b}\left[2\mu\left(K_X{}^2 C_{n_\beta} + K_{XZ} C_{l_\beta}\right)D^3 + \frac{1}{2}\left(C_{l_\beta} C_{n_p} - C_{n_\beta} C_{l_p}\right)D^2\right]\frac{1}{|B|} \tag{36}$$

For the frequency response of these transfer functions, the operator

$$D = \frac{b}{U}\frac{d}{dt} = \frac{b}{U} i\omega = i\omega' \tag{37}$$

where ω' is the reduced frequency in terms of radians per span length. Thus, the transfer functions relating the dynamics of the airframe to the lateral forces and moments are obtained from the inertial characteristics and stability derivatives of the airframe under steady flow conditions (that is, nonturbulent flow conditions).

Complex Moment and Force Coefficients

The moment and force coefficients which relate the rolling moment, yawing moment, and side force to a sinusoidal gust velocity of unit maximum amplitude acting in planes of (but of opposite sign to) the reference axes of the airplane are, in general, frequency-dependent, and only their statistical average properties are definable. Their derivation is based on generalized harmonic analysis and basic aerodynamic theory. Where the derivation is lengthy, only the results of referenced papers are given or indicated. Furthermore, only those eight coefficients necessary to equation (14) are defined. All others are considered as having small or negligible influence on the results.

Rolling moments.- Four contributions to the rolling moment on the airplane are considered. They are the rolling moment of the wing due to all three components of gust velocity and the rolling moment due to the side gust acting on the vertical tail.

In reference 12 are derived the power spectral densities of the rolling moments of a wing in isotropic turbulence due to horizontal, vertical, and side gusts (that is, the three components of gusts considered herein). The theory considers random turbulence across the span as well as along the flight path, and the results are given for several span-loading distributions and for a range of values of the ratio of

span length to the integral scale of atmospheric turbulence. The power spectra of rolling moment given in reference 12 are not repeated herein but are simply related to the notation of this paper:

$$\left|\frac{C_l}{w_g}\right|_W^2 \Phi_{u_g}(\omega) = \Phi_{C_l} \text{ (due to horizontal gusts, ref. 12)} \tag{38}$$

$$\left|\frac{C_l}{u_g}\right|_W^2 \Phi_{w_g}(\omega) = \Phi_{C_l} \text{ (due to vertical gusts, ref. 12)} \tag{39}$$

For the side gust v_g, the random variation along the flight path was assumed to be uniform across the span, an assumption that is particularly valid for airplanes having wing loading due to sideslip concentrated over a small region of the span near the plane of symmetry. Such an assumption is likewise made herein, so that

$$\left(\frac{C_l}{v_g}\right)_W = \frac{1}{U}\left(C_{l_\beta}\right)_W \tag{40}$$

is assumed to be independent of frequency.

The vertical-tail contribution to the rolling moment is defined by

$$(C_l)_T(\omega) = \left(\frac{\partial C_Y}{\partial \beta}\right)_T \beta_T \frac{h_z}{qSb} e^{-i(l_t\omega/U)} = \left(C_{Y_\beta}\right)_T \frac{v_g}{U} \frac{h_z}{b} \left(1 + \frac{\partial \sigma}{\partial \beta}\right) e^{-i(l_t\omega/U)}$$

and, hence,

$$\left(\frac{C_l}{v_g}\right)_T(\omega) = \left[\left(C_{Y_\beta}\right)_T \frac{h_z}{bU}\left(1 + \frac{\partial \sigma}{\partial \beta}\right)\right] e^{-i(l_t\omega/U)} \tag{41}$$

Thus, the rolling moment due to side gusts may be directly calculated from the known characteristics of the airframe in still air, and the rolling moment due to horizontal and vertical gusts may be obtained in power spectral form from the plots of reference 12. The fact that the real and imaginary parts may not be ascertained for the rolling moment due to u_g and w_g components is shown to be irrelevant; that is, the real and imaginary parts, as such, are not necessary to the calculation of the cross-power terms involved in calculating the response of the airplane.

Yawing moments.- The yawing moments due to the u_g and w_g components of the gust are primarily due to the unsymmetrical distribution of these gust velocities across the span of the wing. Since this unsymmetrical distribution is the same characteristic of turbulence that produces the rolling moment on the wing, the yawing moment may be fairly accurately related to the rolling moment through aerodynamic relations. As given in reference 12, the expressions for the yawing moment of the wing due to small horizontal- and vertical-gust disturbances are

$$\left(\frac{C_n}{u_g}\right)_W(\omega) = \left(\frac{C_{n_r}}{C_{l_r}}\right)_W\left(\frac{C_l}{u_g}\right)_W(\omega) \tag{42}$$

$$\left(\frac{C_n}{w_g}\right)_W(\omega) = \left(\frac{C_{n_p}}{C_{l_p}}\right)_W\left(\frac{C_l}{w_g}\right)_W(\omega) \tag{43}$$

where C_{n_r}, C_{l_p}, and so forth, are stability derivatives of the wing only at the trim (mean) angle of attack of the airplane and, as such, are not functions of frequency. Any yawing moments of the wing due to side gusts are neglected.

The yawing moment of the fuselage and vertical fin of an airplane due to penetration into side gusts v_g has been derived in reference 11 by using a simple profile shape to define the lift distribution over the fuselage and vertical tail due to sinusoidal side-gust components. As given in reference 11,

$$\left(\frac{C_n}{v_g}\right)_{FT}(\omega) = \frac{2\pi}{bSU}\left\{\frac{-2x_0s_0^2}{k_0^3}\left[\left(2k_0 + ik_0^2 + i2\right)e^{-ik_0} - i2\right] + \frac{(x_2 - x_1)(s_1 - s_0)^2}{(k_2 - k_1)^3}\left[\left(2k_2 - k_1 - i2 - ik_1k_2 + ik_2^2\right)e^{-ik_2} - (k_1 - i2)e^{-ik_1}\right]\right\} \tag{44}$$

where x_0, x_1, x_2, s_0, and s_1 are geometric dimensions of the profile shape and frequency appears in the form

$$k_n = \frac{\omega x_n}{U} \qquad (n = 0, 1, 2)$$

For any given flight condition for which the exact value of C_{n_β} is known, it is recommended in reference 11 that small adjustments in the profile shape be made so that at zero frequency

$$\left(\frac{C_n}{v_g}\right)_{FT}(\omega=0) = \frac{1}{U}\left(C_{n_\beta}\right)_{FT}$$

Side force.- The side force on an airplane in gusts is considered to be primarily due to the effect of side gusts on the fuselage and vertical fin (the profile shape) of the airplane. By using a simple profile shape and slender-body theory to define the lift distribution over the fuselage and vertical tail due to sinusoidal side gusts, the side force per unit side gust as derived in reference 11 is

$$\left(\frac{C_Y}{v_g}\right)_{FT}(\omega) = \frac{2\pi}{SU}\left\{\frac{2s_0^2}{k_0^2}\left[1 - (1 - ik_0)e^{ik_0}\right] + \left(\frac{s_1 - s_0}{k_2 - k_1}\right)^2\left[e^{-ik_1} - (1 - ik_1 + ik_2)e^{-ik_2}\right]\right\} \tag{45}$$

where the notation is defined in the preceding section. Again it is recommended that small adjustments be made in the representative profile shape so that at zero frequency

$$\left(\frac{C_Y}{v_g}\right)_{FT}(\omega=0) = \frac{1}{U}\left(C_{Y_\beta}\right)_{FT}$$

where $\left(C_{Y_\beta}\right)_{FT}$ is known from flight or wind-tunnel tests for the desired flight conditions.

POWER SPECTRA OF GUST VELOCITY

As discussed in reference 17, the relationship (correlation) between gust velocities measured at two points in isotropic turbulence may be completely defined in terms of their lateral and longitudinal components (components measured perpendicular and parallel, respectively, to the vector distance between two points). As treated herein, the airplane senses only the lateral components of the v_g and w_g gusts and both the lateral and longitudinal components of the u_g gusts. (See preceding section and ref. 12.)

Both theoretical methods (ref. 17) and flight measurements (ref. 18) have shown that the following expression approximates the spectrum of lateral components of turbulence in the atmosphere over the frequency range affecting the dynamic response of an airplane:

$$\frac{L}{\pi U} \frac{1 + 3(k')^2}{\left[1 + (k')^2\right]^2}$$

Its horizontal counterpart has the spectrum

$$\frac{L}{\pi U} \frac{2}{1 + (k')^2}$$

where $k' = \frac{\omega L}{U}$ and L is the integral scale of turbulence. The value $2\pi L$ may be considered an approximate measure of the average eddy size in the turbulence. The data of reference 18 further indicate that L is on the order of 1,000 to 2,000 feet, and probably closer to 1,000 feet.

The foregoing expressions, having asymptotic logarithmic slopes of -2 for values of $k' \rightarrow \infty$, were first suggested in reference 19 on the basis of wind-tunnel data and have since become generally accepted as reasonable expressions for defining the power spectra of lateral and longitudinal components of isotropic turbulence. Therefore, these expressions normalized by the mean-square values are used herein as the basis for calculations involving the components of atmospheric gusts as sensed at the center of gravity (point spectra):

$$\frac{\Phi_{v_g}}{\overline{v_g^2}} = \frac{\Phi_{w_g}}{\overline{w_g^2}} = \frac{L}{\pi U} \frac{1 + 3(k')^2}{\left[1 + (k')^2\right]^2} \tag{46}$$

$$\frac{\Phi_{u_g}}{\overline{u_g^2}} = \frac{L}{\pi U} \frac{2}{1 + (k')^2} \tag{47}$$

where

$$\overline{u_g^2} = \overline{v_g^2} = \overline{w_g^2}$$

$$k' = \frac{\omega L}{U}$$

$$L \approx 1{,}000 \text{ feet}$$

In the calculations of reference 12, which involved the gusts sensed by a wing having finite span, equations (46) and (47) were likewise used so that the data of reference 12 and this paper are in accord as to the spectra of gusts. However, equation (47), as such, is not used directly in this paper and for this reason only equation (46) is plotted herein. The variation of equation (46) with k' is shown in figure 2.

APPLICATION OF METHOD

Lateral Response of Three Airplanes

The equations which define the power spectral response of an airplane to atmospheric turbulence (eqs. (18) to (21)) have been applied to three airplanes of varying size and wing span. The flight conditions and stability derivatives of the example airplanes are given in table I, and some physical dimensions necessary to the calculation of frequency-dependent force and moment coefficients are given in table II. Some of these physical dimensions require reference 11 for their concise definition (s_0, s_1, x_0, x_1, and x_2).

By using the data presented in table I, the transfer functions of equations (26) to (35) were calculated. For discretely chosen values of reduced frequency ω', the in-phase and out-of-phase (real and imaginary parts of the transfer function) components were evaluated with a relay computer. Care was taken to choose as one point the natural frequency of the Dutch roll mode.

By using the data of table II the complex moment and force coefficients (eqs. (38) to (45)) were evaluated at the same values of frequency as were used for evaluating the transfer functions. The relationship

$$\omega' = \frac{\omega b}{U}$$

was used to relate circular frequency in radians per second to reduced frequency in radians per span. The evaluation of certain of these coefficients requires further explanation.

The variations of $\left(\frac{C_n}{v_g}\right)_{FT}$ (eq. (44)) and $\left(\frac{C_Y}{v_g}\right)_{FT}$ (eq. (45)) with frequency are plotted in the figures of reference 11 for the example airplanes of this paper with one adjustment:

$$C_n \text{ per unit side gust (ref. 11)} = U\left(\frac{C_n}{v_g}\right)_{FT}$$

$$C_Y \text{ per unit side gust (ref. 11)} = U\left(\frac{C_Y}{v_g}\right)_{FT}$$

The designation of example airplanes for this paper and reference 11 is consistent.

The two rolling- and two yawing-moment coefficients due to u_g and w_g components acting on the wing were taken directly from the figures of reference 12. Since these four complex moment coefficients represent the spanwise integration of the power spectra of the horizontal and vertical gust components, they appear in reference 12 only in the form of the absolute value squared, whereas equations (19) and (21) appear to require their real and imaginary components. However, by substituting equation (42) into equation (19), the expression for the motion due to horizontal gusts becomes

$$\left|\frac{\phi}{u_g}\right|^2 = \left|\frac{\phi}{C_l}\right|^2\left|\frac{C_l}{u_g}\right|_W^2 + \left|\frac{\phi}{C_n}\right|^2\left(\frac{C_{n_r}}{C_{l_r}}\right)_W^2\left|\frac{C_l}{u_g}\right|_W^2 + 2R\left[\left(\frac{C_l}{u_g}\right)_W^*\left(\frac{C_{n_r}}{C_{l_r}}\right)_W\left(\frac{C_l}{u_g}\right)_W\left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n}\right]$$

and since

$$z^*z = |z|^2$$

the equation becomes

$$\left|\frac{\phi}{u_g}\right|^2 = \left\{\left|\frac{\phi}{C_l}\right|^2 + \left|\frac{\phi}{C_n}\right|^2\left(\frac{C_{n_r}}{C_{l_r}}\right)_W^2 + 2R\left[\left(\frac{C_{n_r}}{C_{l_r}}\right)_W\left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n}\right]\right\}\left|\frac{C_l}{u_g}\right|_W^2 \tag{48}$$

where only the square of the absolute value of $\left(\frac{C_l}{u_g}\right)_W$ appears and the product

$$\left|\frac{C_l}{u_g}\right|_W^2\Phi_{u_g} \equiv \Phi_{C_l} \text{ (due to horizontal gusts)}$$

is plotted in figure 9 of reference 12.

In the parallel evaluation of the motion due to vertical gusts, the substitution of equation (43) into equation (21) yields

$$\left|\frac{\phi}{w_g}\right|^2 = \left\{\left|\frac{\phi}{C_l}\right|^2 + \left|\frac{\phi}{C_n}\right|^2\left(\frac{C_{n_p}}{C_{l_p}}\right)_W^2 + 2R\left[\left(\frac{C_{n_p}}{C_{l_p}}\right)_W\left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n}\right]\right\}\left|\frac{C_l}{w_g}\right|_W^2 \tag{49}$$

where the product

$$\left|\frac{C_l}{w_g}\right|_W^2 \Phi_{w_g} \equiv \Phi_{C_l} \text{ (due to vertical gusts)}$$

is plotted in figure 7 of reference 12.

Inasmuch as the figures of reference 12 are plotted against k', the relationship

$$k' = \frac{\omega L}{U} = \frac{\omega b}{\beta' U} = \frac{\omega'}{\beta'} \tag{50}$$

where

$$\beta' = \frac{b}{L}$$

was used in the compatible choice of frequencies. Since the ratio of wing spans between the three example airplanes is on the order of

$$b_A : b_B : b_C \approx 1:2:3.3$$

the values of β' were taken as

$$\beta'_A = 0.03125$$

$$\beta'_B = 0.0625$$

$$\beta'_C = 0.103$$

which correspond to a value of L on the order of 1,100 feet. A rectangular span load distribution was used for airplane A and elliptical span load distributions were used for airplanes B and C. This choice was a matter of convenience since the plots of reference 12 indicate only a minor effect due to span loading distribution.

The power spectral responses of the airplanes were completely calculated in three degrees of freedom, ϕ, ψ, and β, by using the power

spectra of the gust components given by equation (47). Angular displacements of the airplanes are shown in figures 3 to 11 as follows:

Airplane	Angular displacement	Figure number
A	ϕ	3
	ψ	4
	β	5
B	ϕ	6
	ψ	7
	β	8
C	ϕ	9
	ψ	10
	β	11

In the (a) parts of figures 3 to 11 are plotted the three components and the sum for that particular angular displacement as expressed by equation (18). As may be seen from these figures, in none of the cases considered does the u component of gust velocity contribute perceptibly to the resulting motion. Therefore, in the (b) parts of figures 3 to 11 are plotted the relative contributions to the motion of the terms due only to the v and w components of the gusts. The (b) part of each figure then is a plot of equations (20) and (49) (where eq. (49) is a modified form of eq. (21)) and, as such, does not include any power-spectrum terms. For instance, the three components due to vertical gusts w_g shown in each (b) plot have been summed, multiplied by the power spectrum defined by equation (39), and plotted in the (a) part of each figure. The u_g and v_g contributions were treated likewise with the exception that the components due to u_g are not plotted in the (b) part of each figure.

Figures 3 to 11 are plotted in this manner to permit observation of the end results without undue complication. When the relative magnitudes of the contributing forces and moments are of particular interest, the (b) parts of the figures may be studied more closely.

Since in order to obtain the (a) parts of these figures, the w_g components are multiplied by $\left|\frac{C_l}{w_g}\right|_W^2 \Phi_{w_g}$ and the v_g components multiplied by Φ_{v_g}, it should be noted that in the (b) parts of these figures the units of the w_g components are dimensionless while the v_g components have the units $\left(\frac{\text{radians}}{\text{ft/sec}}\right)^2$.

The β response plotted in figures 5, 8, and 11 is the angular displacement with respect to the general air mass (that is, still air or fixed axes) and as such does not include the displacement with respect to the local air mass (gusts).

From these example cases and figures, conclusions may be drawn on characteristics of method, trends, and further simplification of the calculations and a comparison of results from other methods of analysis may be made.

Analysis

Characteristics of the method.- In applying the method of this paper, several characteristics appear to be worthy of special attention. The first of these is the importance of calculating the response of the airplane at the exact frequency where the Dutch roll mode has a maximum amplitude. Since the Dutch roll mode is generally lightly damped and the square of the amplitude is required in the calculations, the power spectral response of the angular displacements of the airplane will generally exhibit a sharp peak in this frequency range. Enough datum points should be calculated in this region to insure an accurate representation of the response.

A second characteristic of the method is the inaccuracy of equation (20) as $\omega \rightarrow 0$ for the ϕ and ψ modes of motion. The reason for this inaccuracy becomes apparent when the equation relating ϕ (or ψ) to v_g

$$\frac{\phi}{v_g}(\omega) = \frac{\phi}{C_l}\frac{C_l}{v_g}(\omega) + \frac{\phi}{C_n}\frac{C_n}{v_g}(\omega) + \frac{\phi}{C_Y}\frac{C_Y}{v_g}(\omega) \tag{51}$$

is written in the form

$$\frac{\phi}{v_g}(\omega) = \frac{1}{U}\left\{\frac{\phi}{C_l}\left[C_{l_\beta} + \frac{C_l}{\beta_g}(\omega)\right] + \frac{\phi}{C_n}\left[C_{n_\beta} + \frac{C_n}{\beta_g}(\omega)\right] + \frac{\phi}{C_Y}\left[C_{Y_\beta} + \frac{C_Y}{\beta_g}(\omega)\right]\right\}$$

$$\frac{\phi}{v_g}(\omega) = \frac{1}{U}\left(\frac{\phi}{C_l} C_{l_\beta} + \frac{\phi}{C_n} C_{n_\beta} + \frac{\phi}{C_Y} C_{Y_\beta}\right) + \frac{1}{U}\left[\frac{\phi}{C_l}\frac{C_l}{\beta_g}(\omega) + \frac{\phi}{C_n}\frac{C_n}{\beta_g}(\omega) + \frac{\phi}{C_Y}\frac{C_Y}{\beta_g}(\omega)\right] \tag{52}$$

The first half of equation (52) now represents the steady-state response to a uniform change in side gust over the airplane. The second half represents the additional response due to the nonuniform distribution of the gusts over the airplane. If the frequency-dependent expressions for the transfer functions $\frac{\phi}{C_l}$, $\frac{\phi}{C_n}$, and $\frac{\phi}{C_Y}$ as given by equations (26), (29), and (32), respectively, are substituted into equation (52) and the coefficients having the same powers of ω are grouped $\left(\frac{C_l}{\beta_g}(\omega)\right.$ may be expanded in the form $\left.\sum_{m=1}^{\infty} a_m\omega^m\right)$, the coefficient of the lowest order power of ω in the numerator may be shown to be identically equal to zero. In the numerical evaluations at low frequencies where the lowest order terms of equation (52) (or the square of the absolute value of eq. (52)) are predominant, small differences between large numbers appear unless these terms are canceled beforehand. Since in the present method this cancellation is not convenient, a scattering of calculated values for $\frac{\phi}{v_g}$ and $\frac{\psi}{v_g}$ begins to occur at some frequency below that of the Dutch roll mode. However, since this scattering occurs only as $\frac{\phi}{v_g}$ and $\frac{\psi}{v_g}$ approach a constant (at $\omega = 0$), this constant may be evaluated and the curve faired to that value. In figures 3, 4, 6, 7, 9, and 10 this fairing is denoted with dashed lines. In the range of frequencies where the response in ϕ and ψ due to v_g is faired, the total response of the three airplanes is due almost entirely to the w_g component of the gusts.

Effect of trim angle of attack.- In analyzing the motions of the three example airplanes, the effect of small differences in trim-lift coefficients or trim angles of attack appears to be important. Although the trim angles of attack for all three airplanes are low, their relative magnitudes are

$$B:A:C = 1:1.36:2.50$$

Thus, airplane C was at an angle of attack from two to two and a half times as great as airplanes A and B. The stability derivatives for the wing alone in incompressible flow are functions of angle of attack (and, hence, lift coefficient) of the order (see ref. 20):

$$C_{l_p} = \text{Constant}$$

$$C_{n_p} \propto \alpha$$

$$C_{l_r} \propto \alpha$$

$$C_{n_r} \propto \alpha^2 \quad \text{and} \quad C_{D_o}$$

The effect of increased angle of attack on the ratio of yawing moment to rolling moment may be seen to increase by

$$\frac{C_{n_p}}{C_{l_p}} \propto \alpha \qquad \frac{C_{n_r}}{C_{l_r}} \propto \alpha \quad \text{and} \quad \frac{C_{D_o}}{\alpha}$$

These ratios may explain trends such as that shown in the case of airplane C for which the response in ψ and β to w_g at the higher frequencies is primarily due to the ability of w_g to produce yawing moment, whereas for airplanes A and B the response is primarily due to the ability of w_g to produce rolling moment.

Effect of horizontal gusts.- While there is, in general, some cross power (or cross correlation) between any two velocity components measured at any two points in the turbulence, only those components lying in the XY-plane of the airplane are considered herein since the Z-gradients of the gusts are unimportant to the airplane. The airplane will, however, sense the correlation between the u_g component acting at one point on the wing (at any instant) and the v_g component acting at some other point on the wing or vertical tail (a vortex effect). It is the additional effect of this correlation that has been neglected in the method of this paper. While this correlation is probably small, the fact that u_g has a negligible effect on the airplanes considered is reason enough for neglecting any cross-power effects involving u_g. In cases where the u_g component of gusts does have a pronounced effect on the airplane, the contribution of the cross power must be determined or further justification be made for neglecting it.

Response characteristics of example airplanes.- The calculated lateral responses of all three example airplanes exhibited a number of

characteristics in common. Most important of these was the total motion in each of the three degrees of freedom.

In each case the response in roll was due primarily to w_g at low frequencies, v_g near the Dutch roll frequency, and w_g at the higher frequencies. The response of the example airplanes in yaw was due primarily to w_g at very low frequencies and due primarily to v_g at the Dutch roll and higher frequencies. The response in β of the three example airplanes was due primarily to v_g throughout the frequency spectrum.

Simplified Equations

Based on the three conventional example airplanes, certain simplifications to the response equations (18) to (21) appear to be justified. For the low trim lift coefficients (or angles of attack) investigated, the effect of the horizontal gust (u_g) was not discernible in any of the three modes of motion ((a) parts of figs. 3 to 11) and could have been neglected. Furthermore, in every case investigated, the effect of side force due to side gusts $\left(\frac{C_Y}{v_g}\right)$ was negligible ((b) parts of figs. 3 to 11) apparently because of the magnitude of C_{Y_β} and the magnitude of the $\frac{\phi}{C_Y}$, $\frac{\psi}{C_Y}$, and $\frac{\beta}{C_Y}$ transfer functions for the example airplanes; however, these orders of magnitude are typical of most present-day airplanes. Other parameters appear to be negligible in some cases. When computing the response of the airplane in sideslip, neglecting the effect of w_g as well as that of u_g appears to be justified. Although for some cases, for example, airplanes A and B, the effect of $\frac{C_n}{w_g}$ appears to be of small order in all modes, this effect is not generally true, as is shown by airplane C. As pointed out in a preceding section, the response in sideslip was due primarily to v_g.

In general, then, the preceding discussion leads to the simplification of equations (18) to (21) as follows, where ϕ may again be replaced by any of the lateral degrees of freedom in the form of angular displacements, velocities, or accelerations:

$$\Phi_\phi = \left|\frac{\phi}{v_g}\right|^2 \Phi_{v_g} + \left|\frac{\phi}{w_g}\right|^2 \Phi_{w_g} \tag{53}$$

and

$$\left|\frac{\phi}{v_g}\right|^2 = \left|\frac{\phi}{C_l}\right|^2\left|\frac{C_l}{v_g}\right|_W^2 + \left|\frac{\phi}{C_l}\right|^2\left|\frac{C_l}{v_g}\right|_T^2 + \left|\frac{\phi}{C_n}\right|^2\left|\frac{C_n}{v_g}\right|_{FT}^2 + 2R\left\{\left[\left(\frac{C_l}{v_g}\right)_W^*\left(\frac{C_n}{v_g}\right)_{FT} + \left(\frac{C_l}{v_g}\right)_T^*\left(\frac{C_n}{v_g}\right)_{FT}\right]\left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n} + \left(\frac{C_l}{v_g}\right)_W^*\left(\frac{C_l}{v_g}\right)_T\left|\frac{\phi}{C_l}\right|^2\right\} \tag{54}$$

$$\left|\frac{\phi}{w_g}\right|^2 = \left\{\left|\frac{\phi}{C_l}\right|^2 + \left|\frac{\phi}{C_n}\right|^2\left(\frac{C_{n_p}}{C_{l_p}}\right)_W^2 + 2R\left[\left(\frac{C_{n_p}}{C_{l_p}}\right)_W\left(\frac{\phi}{C_l}\right)^*\frac{\phi}{C_n}\right]\right\}\left|\frac{C_l}{w_g}\right|_W^2 \tag{55}$$

COMPARISON OF METHODS

Both airplanes A and B have been analyzed in other papers (refs. 6 to 9) at the same flight condition as that used herein for their responses to atmospheric turbulence by using methods which treat gusts as equivalent motions of the airplane in still air. These gusts are denoted as side gusts β_g, rolling gusts $D\phi_g$ or $\frac{\partial w_g}{\partial y}$, and yawing gusts $D\psi_g$ or $\frac{\partial u_g}{\partial y}$. The analysis of airplane A given in reference 6 is based on the power spectra derived from the analysis of reference 9. Although the analysis of reference 6 also incorporates a refinement by considering the lag of the vertical tail penetrating the gust, the effect proved to be negligible on such a small fast airplane. Since in this analysis (ref. 6) the rolling gust affected only the wing, a comparison with the analysis of this paper (where the vertical gust likewise affects only the wing) may be made. Therefore, a comparison is made herein by using airplane A to illustrate any differences in the results of the two methods.

Comparison of Gust Power Spectra

Since the power spectrum of the v_g component of atmospheric turbulence as used in references 9 and 6 and that used herein have in common an asymptotic value of k^{-2} at large values (relatively speaking) of k, the arbitrary constant A, given in references 9 and 6, is assigned a value such that the referenced v_g spectrum and the v_g spectrum of this report are asymptotically congruent. With this same constant, the

$D\phi_g$ and $\frac{\partial w_g}{\partial y}$ spectra are compared, and all comparisons of the calculated motions of airplane A are likewise based on this same constant.

Side gusts.- The power spectrum of side gusts expressed by equation (47) written in the form

$$\frac{\Phi_{v_g}}{\overline{v_g^2}} = \frac{\Phi_{\beta_g}}{\overline{v_g^2}/U^2} = \frac{L}{\pi U} \frac{1 + 3\left(\frac{\omega L}{U}\right)^2}{\left[1 + \left(\frac{\omega L}{U}\right)^2\right]^2} \tag{56}$$

is comparable to that shown in figure 4 of reference 6 except for a constant. That is,

$$\frac{\Phi_{v_g}}{\overline{v_g^2}} \approx \frac{\Phi_{\beta_g}}{A} \tag{57}$$

where

$$A = K \frac{\overline{v_g^2}}{U^2} \tag{58}$$

Plotted in figure 12 are the side-gust spectrum given by equation (56) and that of reference 6 with the constant K of equation (58) given the value

$$K = \frac{72L}{\pi} \tag{59}$$

in order to get the agreement shown. Since both spectra are multiplied by the factor L, this agreement is independent of the choice of magnitude for L.

Rolling gusts.- Similarly, the power spectrum of rolling gusts $D\phi_g$ used in reference 6 may be compared with the power spectrum of rolling moment due to vertical gusts as given in reference 12 and used herein. Since

$$C_l = -C_{l_p}\frac{pb}{2U} = -\frac{C_{l_p}}{2} D\phi \tag{60}$$

then

$$\Phi_{D\phi_g} = \frac{4}{C_{l_p}^2} \Phi_{C_l} \tag{61}$$

and for airplane A, under the conditions considered herein,

$$\Phi_{D\phi_g} = \frac{4}{C_{l_p}^2} \frac{\overline{w_g^2} L C_{l_p}^2}{\pi U^3} E(w_g)$$

where $E(w_g)$ is the curve of $\dfrac{\Phi C_l(k')}{\overline{w_g^2} L C_{l_p}^2 / U^3 \pi}$ for $\beta' = 0.03125$ from reference 12, figure 7(a). If $\Phi_{D\phi_g}$ is expressed with the same normalization used for the side gust (see eq. (56)), then

$$\frac{\Phi_{D\phi_g}}{\overline{w_g^2}/U^2} = \frac{4L}{\pi U} E(w_g) \tag{62}$$

Equation (62) is plotted in figure 12 together with the power spectrum of $D\phi_g$ given in reference 6 again with

$$A = \frac{72L}{\pi} \frac{\overline{v_g^2}}{U^2} \tag{63}$$

Since the spectra of reference 6 are the same as those given in reference 9 except for a slight difference in frequency range, no difference is shown in the plots except that the range of data of reference 9 is denoted with ticks.

Likewise shown in figure 12 is the spectrum for rolling gusts as predicted by Decaulne in reference 8. The theory is based on the assumption that the variation of vertical gusts across the span $(\partial w_g/\partial y)$ can be approximated with a constant gradient (a linear variation) and has as a spectrum

$$\Phi_{\frac{\partial w_g}{\partial y}}(\omega) = \Phi_{w_g}(\omega) \frac{36}{b^2} \left[\frac{\sin \frac{\omega b}{2U}}{\left(\frac{\omega b}{2U}\right)^2} - \frac{\cos \frac{\omega b}{2U}}{\frac{\omega b}{2U}} \right]^2 \tag{64}$$

which for small values of ω reduces to

$$\Phi_{\frac{\partial w_g}{\partial y}}(\omega) = \Phi_{w_g}(\omega) \left(\frac{\omega}{U}\right)^2 \tag{65}$$

Based on this theory, when ω is less than the frequency corresponding to a wavelength of L (a value of L of 1,128 feet used in fig. 12), the vertical-gust spectrum will become a constant and the $\partial w_g/\partial y$ spectrum will fall off to zero at the rate of ω^2 as ω approaches zero. This result is denoted by dashed lines in figure 12.

At the higher frequencies shown in figure 12, the calculated spectrum of reference 8 predicts almost no attenuation of rolling power of the wing with frequency. The spectrum of the same quantity according to references 9 and 6 would predict an attenuation for the wing of airplane A, the attenuation beginning at about 8 radians/sec with the power falling off rapidly thereafter. The more comprehensive theory of reference 12 shows a more pronounced attenuation of power over the plotted frequency range although all theories have approximately the same value at the frequency corresponding to the value of L.

On the basis of this comparison, the concept of constant gust gradients across the span would appear to be insufficient, despite the agreement in the region of the break frequency $\left(\omega = \frac{U}{L}\right)$. The more comprehensive theory of random gusts across the span predicts no attenuation below the break frequency such as is predicted by the theories of references 9 and 6. At frequencies greater than the break frequency, the attenuation predicted by the two theories again becomes inconsistent.

Comparison of Lateral Response Power Spectra

With the assigned value for the constant A given by equation (63), the responses of airplane A in ϕ, ψ, and β_o due to β_g and $D\phi_g$ as calculated in reference 6 (fig. 10(a)) are plotted in figures 13(a), (b), and (c), respectively. Also replotted in figure 13 are the responses in ϕ, ψ, and β of airplane A due to v_g and w_g as obtained by the more comprehensive method of this paper (β_o of ref. 6 is equivalent to β of this paper). Since reference 6 did not consider the u_g component of gust velocity, no comparison is made of the contribution of this component. Flight conditions are the same in both cases.

In reproducing the frequency-response calculations of reference 6, the calculated transfer functions of $\psi/D\phi_g$ were found to be in error; however, this error did not affect the conclusions made therein. The power spectral response of $(\psi)_{D\phi_g}$ shown in figure 13(b) has therefore been corrected and plotted for a somewhat larger frequency range than is given in reference 6. In the extended frequency range the $D\phi_g$ spectrum which would be predicted by equation (65) is used.

At the lower frequencies the differences in the response of the airplane due to side gusts as calculated by the two methods are due principally to the differences in the spectra of side gusts used by the two methods, as shown in figure 12. In the intermediate frequency range where the Dutch roll mode is predominant, a large difference in peak amplitudes is indicated by the two methods. This difference appears to be due to failure to evaluate the transfer functions at the exact frequency at which the Dutch roll mode has maximum frequency content (ω_n = 3.16 radians/sec for ref. 6 as compared with ω_n = 3.08 radians/sec for this paper). In power spectral analyses where frequency amplitudes are squared, this factor can result in appreciable differences, as demonstrated in figure 13.

In the higher frequency range the differences in the theories are more pronounced since in this range the calculated effectiveness of the vertical gusts acting on the wing to produce lateral moments differs with the two theories. As pointed out in reference 11, the distribution of gust velocities along the fuselage begins to become an important factor in calculating the yawing moment and side force due to side gusts in this frequency range.

CONCLUDING REMARKS

By resolving the gust velocities of random isotropic atmospheric turbulence into three components alined with the stability axes of an airplane, the lateral forces and moments on the airplane and the resulting lateral motions of the airplane have been derived. The method involves the computation of the statistical forces and moments experienced by the airplane due to isotropic turbulence having random variations along the fuselage and across the wing span. On the basis of these gust-induced forces and moments referenced to the center of gravity of the airplane, the power spectral response of the airplane in any lateral degree of freedom (be it displacement, velocity, or acceleration) may be computed by using the appropriate transfer functions of the airplane. Since the gust velocities, the forces, and the moments are defined along the flight path in terms of their statistical average, the resulting motions are defined in terms of their power spectra, the mean-square value of the gust velocities, and the scale of atmospheric turbulence.

By using this method, calculations have been made on three airplanes, which differed in size, experiencing continuous atmospheric turbulence while in a trimmed condition of low angle of attack. On the basis of these examples, certain possible simplifications to the equations have been brought out. Primary among these are the negligible effect of horizontal gusts u_g and the minor contribution of side force in computing

the motions. In general, then, the lateral motions at low trim angles of attack are due primarily to the ability of the side gusts v_g and vertical gusts w_g to produce yawing and rolling moments on the airplane.

With regard to lateral motions the present method of considering random variations of the gust velocities along the fuselage and across the span is a concept hitherto only approximated with constant gradients of gust velocity or completely ignored. Such approximations do not appear to be justified for gusts having wavelengths shorter than twice the span or fuselage length of the airplane. Although these less comprehensive methods correctly predict the trends in the vicinity of the Dutch roll mode of the airplane, they lead to the erroneous conclusion that the power spectra of the lateral motion due to the unsymmetrical components of the gust across the span ($\partial w_g/\partial y$ and $\partial u_g/\partial y$) fall off to zero at wavelengths greater than the integral scale of the turbulence. The magnitude of difference between the methods increases with frequency above the Dutch roll mode since in this general region of frequencies the short-period (with respect to the size of the airplane) random variations of gust velocity along the fuselage and across the span have a predominant effect on the moments and motions of the airplane.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., January 18, 1957.

APPENDIX A

PROOF OF POWER SPECTRAL AND CROSS-POWER SPECTRAL RELATIONSHIPS

The physical and theoretical aspects of statistical methods of analyzing dynamic systems (generalized harmonic analysis) have been treated in a number of excellent papers. (Among these are refs. 13, 14, and 21.) The following definitions and proofs are purely a mathematical means of utilizing the statistical concept and should be treated as useful supplements to the referenced statistical theory.

Consider the simple equation

$$\theta = \frac{\theta}{\delta}\delta + \frac{\theta}{\gamma}\gamma \tag{A1}$$

where θ, δ, and γ are complex variables and θ/δ and θ/γ are complex transfer functions such that

$$\left.\begin{aligned} \theta &= R_\theta + iI_\theta & \frac{\theta}{\delta} &= \left|\frac{\theta}{\delta}\right| e^{i\phi(\theta/\delta)} \\ \delta &= R_\delta + iI_\delta & & \\ \gamma &= R_\gamma + iI_\gamma & \frac{\theta}{\gamma} &= \left|\frac{\theta}{\gamma}\right| e^{i\phi(\theta/\gamma)} \end{aligned}\right\} \tag{A2}$$

With the use of an asterisk to denote the conjugate of a complex quantity, it is also true that

$$\theta^* = \left(\frac{\theta}{\delta}\right)^* \delta^* + \left(\frac{\theta}{\gamma}\right)^* \gamma^* \tag{A3}$$

Thus

$$\begin{aligned} \theta\theta^* &= \frac{\theta}{\delta}\left(\frac{\theta}{\delta}\right)^* \delta\delta^* + \frac{\theta}{\gamma}\left(\frac{\theta}{\gamma}\right)^* \gamma\gamma^* + \frac{\theta}{\delta}\left(\frac{\theta}{\gamma}\right)^* \delta\gamma^* + \frac{\theta}{\gamma}\left(\frac{\theta}{\delta}\right)^* \delta^*\gamma \\ &= \left|\frac{\theta}{\delta}\right|^2 \left|\delta\right|^2 + \left|\frac{\theta}{\gamma}\right|^2 \left|\gamma\right|^2 + \frac{\theta}{\delta}\left(\frac{\theta}{\gamma}\right)^* \delta\gamma^* + \frac{\theta}{\gamma}\left(\frac{\theta}{\delta}\right)^* \delta^*\gamma \end{aligned} \tag{A4}$$

By expansion in terms of equations (A2) it may be seen that

$$\delta\gamma^* = (\delta^*\gamma)^*$$

$$\frac{\theta}{\delta}\left(\frac{\theta}{\gamma}\right)^* = \left[\left(\frac{\theta}{\delta}\right)^*\frac{\theta}{\gamma}\right]^*$$

and, in fact,

$$\frac{\theta}{\delta}\left(\frac{\theta}{\gamma}\right)^*\delta\gamma^* = \left[\left(\frac{\theta}{\delta}\right)^*\frac{\theta}{\gamma}\,\delta^*\gamma\right]^* \tag{A5}$$

Since the sum of a complex number and its conjugate is equal to twice the real part, that is,

$$(R + iI) + (R - iI) = 2R$$

then the substitution of equation (A5) into equation (A4) yields

$$\left|\theta\right|^2 = \left|\frac{\theta}{\delta}\right|^2\left|\delta\right|^2 + \left|\frac{\theta}{\gamma}\right|^2\left|\gamma\right|^2 + 2R\left[\left(\frac{\theta}{\delta}\right)^*\frac{\theta}{\gamma}\,\delta^*\gamma\right] \tag{A6}$$

It is equally true to write

$$2R\left[\left(\frac{\theta}{\delta}\right)^*\frac{\theta}{\gamma}\,\delta^*\gamma\right] = 2R\left[\frac{\theta}{\delta}\left(\frac{\theta}{\gamma}\right)^*\delta\gamma^*\right] \tag{A7}$$

so that either form may be used.

Dividing both sides of equation (A6) by T and taking the limit of the statistical variables as T approaches infinity give the form

$$\lim_{T\to\infty}\frac{1}{T}\left|\theta\right|^2 = \left|\frac{\theta}{\delta}\right|^2\lim_{T\to\infty}\frac{1}{T}\left|\delta\right|^2 + \left|\frac{\theta}{\gamma}\right|^2\lim_{T\to\infty}\frac{1}{T}\left|\gamma\right|^2 + 2R\left[\left(\frac{\theta}{\delta}\right)^*\frac{\theta}{\gamma}\lim_{T\to\infty}\frac{1}{T}\,\delta^*\gamma\right] \tag{A8}$$

where, by the definitions of equations (5) and (6) of the text, equation (A8) is equivalent to

$$\Phi_\theta = \left|\frac{\theta}{\delta}\right|^2\Phi_\delta + \left|\frac{\theta}{\gamma}\right|^2\Phi_\gamma + 2R\left[\left(\frac{\theta}{\delta}\right)^*\frac{\theta}{\gamma}\,\Phi_{\delta\gamma}\right] \tag{A9}$$

If the two variables γ and δ are statistically independent, their cross power is zero. For equations with more than two variables the process is identical and, of course, yields more terms. For an equation whose solution is a function of m variables, there will result m power-spectrum terms and $\sum_{n=1}^{m}(n-1)$ cross-power terms in the form of equation (A9), or in the equivalent form

$$\Phi_\theta = \left|\frac{\theta}{\delta}\right|^2 \Phi_\delta + \left|\frac{\theta}{\gamma}\right|^2 \Phi_\gamma + 2R\left[\frac{\theta}{\delta}\left(\frac{\theta}{\gamma}\right)^* \Phi_{\gamma\delta}\right] \tag{A10}$$

based on the identity of equation (A7).

For the cross spectrum between two statistical variables not appearing in the same equation, consider the equations

$$\theta = \frac{\theta}{\delta}\,\delta + \frac{\theta}{\gamma}\,\gamma$$

and

$$\epsilon = \frac{\epsilon}{\delta}\,\delta + \frac{\epsilon}{\gamma}\,\gamma \tag{A11}$$

where the complex variable ϵ, in addition to the definitions of equation (A2), is defined by

$$\epsilon = R_\epsilon + iI_\epsilon \tag{A12}$$

Multiplying equation (A3) by equation (A11) gives

$$\theta^*\epsilon = \left(\frac{\theta}{\delta}\right)^*\frac{\epsilon}{\delta}\,\delta^*\delta + \left(\frac{\theta}{\gamma}\right)^*\frac{\epsilon}{\gamma}\,\gamma^*\gamma + \left(\frac{\theta}{\delta}\right)^*\frac{\epsilon}{\gamma}\,\delta^*\gamma + \left(\frac{\theta}{\gamma}\right)^*\frac{\epsilon}{\delta}\,\gamma^*\delta \tag{A13}$$

where the identity

$$\left(\frac{\theta}{\delta}\right)^*\frac{\epsilon}{\gamma}\,\delta^*\gamma \equiv \left[\frac{\theta}{\delta}\left(\frac{\epsilon}{\gamma}\right)^*\delta\gamma^*\right]^* \tag{A14}$$

may be proved by the use of equations (A2) and (A11), so that

$$\theta^*\epsilon = \left(\frac{\theta}{\delta}\right)^*\frac{\epsilon}{\delta}\left|\delta\right|^2 + \left(\frac{\theta}{\gamma}\right)^*\frac{\epsilon}{\gamma}\left|\gamma\right|^2 + 2R\left[\left(\frac{\theta}{\gamma}\right)^*\frac{\epsilon}{\delta}\,\gamma^*\delta\right] \tag{A15}$$

Dividing both sides by T and taking the limit of the statistically dependent variables as T approaches infinity yield

$$\lim_{T\to\infty} \frac{1}{T}\,\theta^{*}\epsilon = \left(\frac{\theta}{\delta}\right)^{*}\frac{\epsilon}{\delta}\lim_{T\to\infty}\frac{1}{T}\left|\delta\right|^{2} + \left(\frac{\theta}{\gamma}\right)^{*}\frac{\epsilon}{\gamma}\lim_{T\to\infty}\frac{1}{T}\left|\gamma\right|^{2} + 2R\left[\left(\frac{\theta}{\gamma}\right)^{*}\frac{\epsilon}{\delta}\lim_{T\to\infty}\gamma^{*}\delta\right] \tag{A16}$$

By the definition of equations (5) and (6) of the text, equation (A16) becomes

$$\Phi_{\theta\epsilon} = \left(\frac{\theta}{\delta}\right)^{*}\frac{\epsilon}{\delta}\,\Phi_{\delta} + \left(\frac{\theta}{\gamma}\right)^{*}\frac{\epsilon}{\gamma}\,\Phi_{\gamma} + 2R\left[\left(\frac{\theta}{\gamma}\right)^{*}\frac{\epsilon}{\delta}\,\Phi_{\gamma\delta}\right] \tag{A17}$$

where again the identity

$$2R\left[\left(\frac{\theta}{\gamma}\right)^{*}\frac{\epsilon}{\delta}\,\Phi_{\gamma\delta}\right] \equiv 2R\left[\frac{\theta}{\gamma}\left(\frac{\epsilon}{\delta}\right)^{*}\Phi_{\delta\gamma}\right]$$

may be used in equation (A17) when desired. As before, $\Phi_{\gamma\delta} = \Phi_{\delta\gamma} = 0$ where γ and δ are statistically independent; but, in general, where the two equations are both functions of m variables, there will result m power-density terms and $\sum_{n=1}^{m}(n-1)$ cross-power terms.

APPENDIX B

RELATIONSHIP BETWEEN POWER SPECTRA OF AIRPLANE FORCES AND MOMENTS AND POWER SPECTRA OF GUST VELOCITIES

Consider the system defined by equation (2) of the text, which is rewritten here for convenience in the matrix form

$$\left\{C_j\right\} = \left[a_{jm}\right]\left\{u_m\right\} \tag{B1}$$

where C_j is a force or moment coefficient (C_l, C_n, or C_Y) and a_{jm} is a transfer function relating C_j to u_m. The gust velocities u_1, u_2, and u_3 (u_m where $m = 1, 2,$ or 3) are the three orthogonal velocity components of isotropic turbulence, and the cross power between any two components is zero; that is,

$$\Phi_{u_1u_2} = \Phi_{u_2u_3} = \Phi_{u_3u_1} = 0 \tag{B2}$$

In the power-spectral domain (see appendix A) the first equation of the system is defined by

$$\Phi_{C_1} = \left|a_{11}\right|^2\Phi_{u_1} + \left|a_{12}\right|^2\Phi_{u_2} + \left|a_{13}\right|^2\Phi_{u_3} + 2R\left(a_{11}^{*}a_{12}\Phi_{u_1u_2} + a_{12}^{*}a_{13}\Phi_{u_2u_3} + a_{13}^{*}a_{11}\Phi_{u_3u_1}\right) \tag{B3}$$

All the cross-power terms of equation (B3) are zero. The other equations yield similar results; therefore, the system is defined by

$$\left\{\Phi_{C_j}\right\} = \left[\left|a_{jm}\right|^2\right]\left\{\Phi_{u_m}\right\} \tag{B4}$$

which is the result given by equation (9) of the text.

Now consider the cross-power terms of the C_j matrix, where, for instance,

$$\Phi_{C_1C_2} = \lim_{T\to\infty} \frac{1}{T} C_1^* C_2$$

$$= \lim_{T\to\infty} \frac{1}{T}\left[\sum_{m=1}^{3} a_{1m}u_m\right]^*\left[\sum_{m=1}^{3} a_{2m}u_m\right]$$

$$= \lim_{T\to\infty} \frac{1}{T}\sum_{m=1}^{3} a_{1m}^* u_m^* \sum_{m=1}^{3} a_{2m}u_m$$

$$= \lim_{T\to\infty} \frac{1}{T}\left(\sum_{m=1}^{3} a_{1m}^* a_{2m} u_m^* u_m + 6 \text{ cross-power terms}\right.$$

$$\left.\text{involving } u_1^* u_2,\ u_2^* u_3, \text{ and so forth}\right) \tag{B5}$$

In the limit, then

$$\Phi_{C_1C_2} = \sum_{m=1}^{3} a_{1m}^* a_{2m} \lim_{T\to\infty} \frac{1}{T} u_m^* u_m + 0$$

$$= \sum_{m=1}^{3} a_{1m}^* a_{2m}\, \Phi_{u_m} \tag{B6}$$

where by definition

$$\Phi_{u_m} = \lim_{T\to\infty} \frac{1}{T} u_m^* u_m = \lim_{T\to\infty} \frac{1}{T} \left|u_m\right|^2 \tag{B7}$$

A similar derivation for the other two cross-power combinations yields similar results; therefore, in general,

$$\left\{\Phi_{C_jC_k}\right\} = \left[a_{jm}^{*}a_{km}\right]\left\{\Phi_{u_m}\right\} \qquad (j\neq k) \qquad \text{(B8a)}$$

where C_k is also a force or moment coefficient and a_{km} is a transfer function relating C_k to u_m, and

$$\left\{\Phi_{C_jC_k}\right\} = \left[\left|a_{jm}\right|^2\right]\left\{\Phi_{u_m}\right\} \qquad (j=k) \qquad \text{(B8b)}$$

Equation (B8b) is seen to be identical to equation (B4). In expanded form, for the case considered

$$\begin{Bmatrix} \Phi_{C_1C_2} \\ \Phi_{C_2C_3} \\ \Phi_{C_3C_1} \end{Bmatrix} = \begin{bmatrix} a_{11}^{*}a_{21} & a_{12}^{*}a_{22} & a_{13}^{*}a_{23} \\ a_{21}^{*}a_{31} & a_{22}^{*}a_{32} & a_{23}^{*}a_{33} \\ a_{31}^{*}a_{11} & a_{32}^{*}a_{12} & a_{33}^{*}a_{13} \end{bmatrix} \begin{Bmatrix} \Phi_{u_g} \\ \Phi_{v_g} \\ \Phi_{w_g} \end{Bmatrix} \qquad \text{(B9)}$$

In the terms of equation (2) of the text, this result is equivalent to equation (10) of the text. Furthermore, other equally correct arrangements are possible, since

$$\left\{\Phi_{C_kC_j}\right\} = \left\{\Phi_{C_j}^{*}C_k\right\} \qquad \text{(B10a)}$$

$$\left\{\Phi_{C_kC_j}\right\} = \left[a_{jm}a_{km}^{*}\right]\left\{\Phi_{u_m}\right\} \qquad (j\neq k) \qquad \text{(B10b)}$$

$$\left\{\Phi_{C_kC_j}\right\} = \left[\left|a_{jm}\right|^2\right]\left\{\Phi_{u_m}\right\} \qquad (j=k) \qquad \text{(B10c)}$$

Equation (B10c) is identical to equations (B4) and (B8b).

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TABLE I.- FLIGHT CONDITIONS AND STABILITY DERIVATIVES OF EXAMPLE AIRPLANES

(a) Flight conditions

Flight condition	Airplane		
	A	B	C
h_p, ft	30,000	4,000	35,000
U, ft/sec	696	318	700
W, lb	12,600	28,000	125,000
μ	50	10.9	31.83
C_L	0.242	0.177	0.443
tan γ	0	0	0
α_o, radians	0.0586	0.0325	0.0823

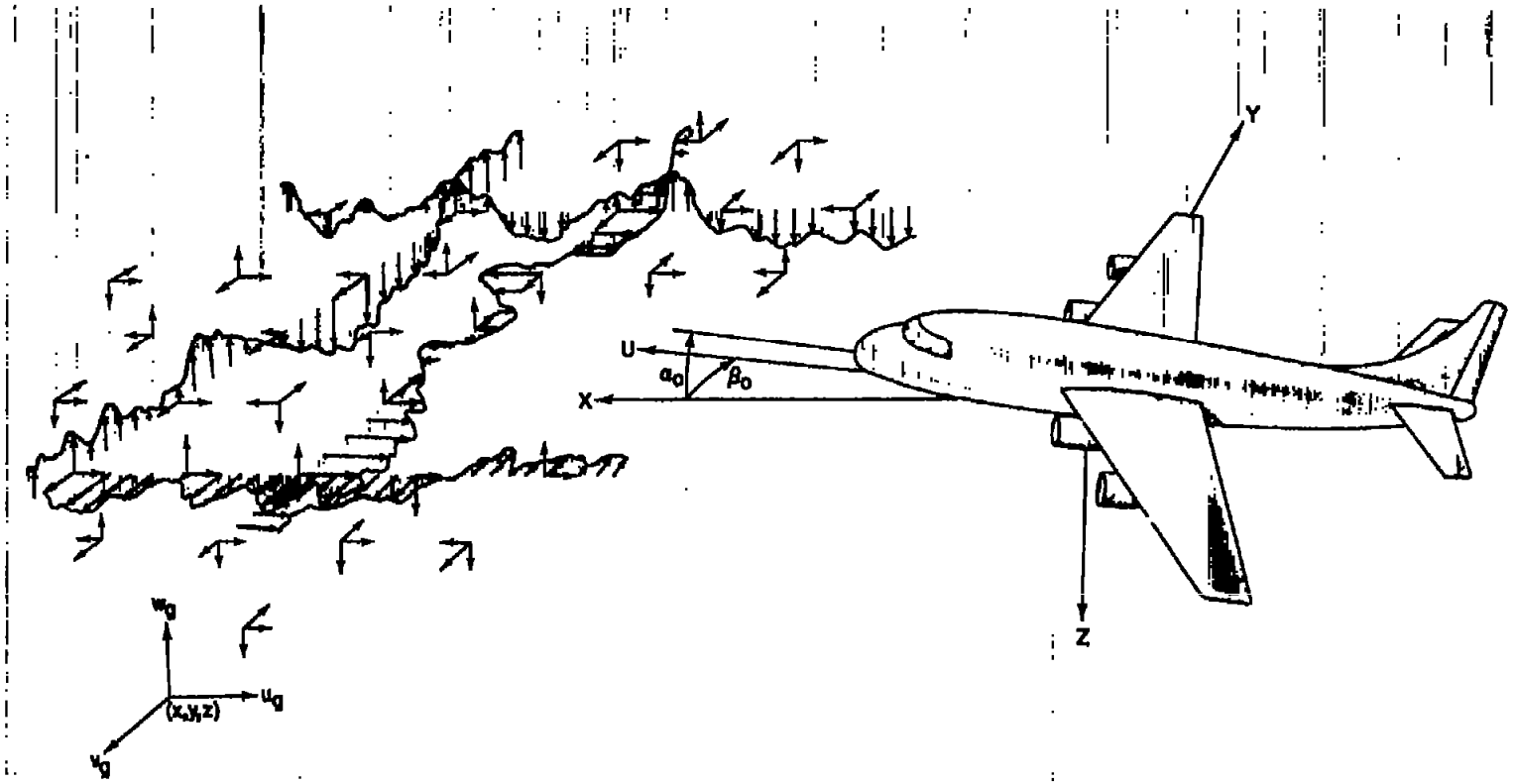
(b) Stability derivatives

Stability derivative	Airplane		
	A	B	C
Total airplane			
K_X^2	0.01485	0.0158	0.0311
K_Z^2	0.0504	0.0300	0.072
K_{XZ}	-0.00062	-0.00138	0
C_{l_p}	-0.45	-0.4875	-0.44
C_{l_r}	0.040	0.1053	0.149
C_{l_β}	-0.11	-0.1084	-0.14
C_{n_p}	-0.035	-0.0308	-0.0275
C_{n_r}	-0.15	-0.1041	-0.156
C_{n_β}	0.12	0.0831	0.12
C_{Y_p}	-0.013	0	0
C_{Y_r}	0.225	0	0
C_{Y_β}	-0.58	-0.763	-0.61
C_{L_α}	4.13	5.44	5.38
Wing			
C_{n_p}/C_{l_p}	0.025	0.0208	0.054
C_{n_r}/C_{l_r}	-0.075	-0.053	-0.052
C_{l_β}	-0.0475	-0.0666	-0.0876
C_{D_o}	0.01	0.006	0.01
Vertical tail			
C_{Y_β}	-0.42	-0.337	-0.384
$\partial\sigma/\partial\beta$	0.19	0.242	0.215
C_{n_β}	0.183	0.129	0.154
C_{l_β}	-0.0625	-0.0418	-0.0524

TABLE II.- PHYSICAL DIMENSIONS

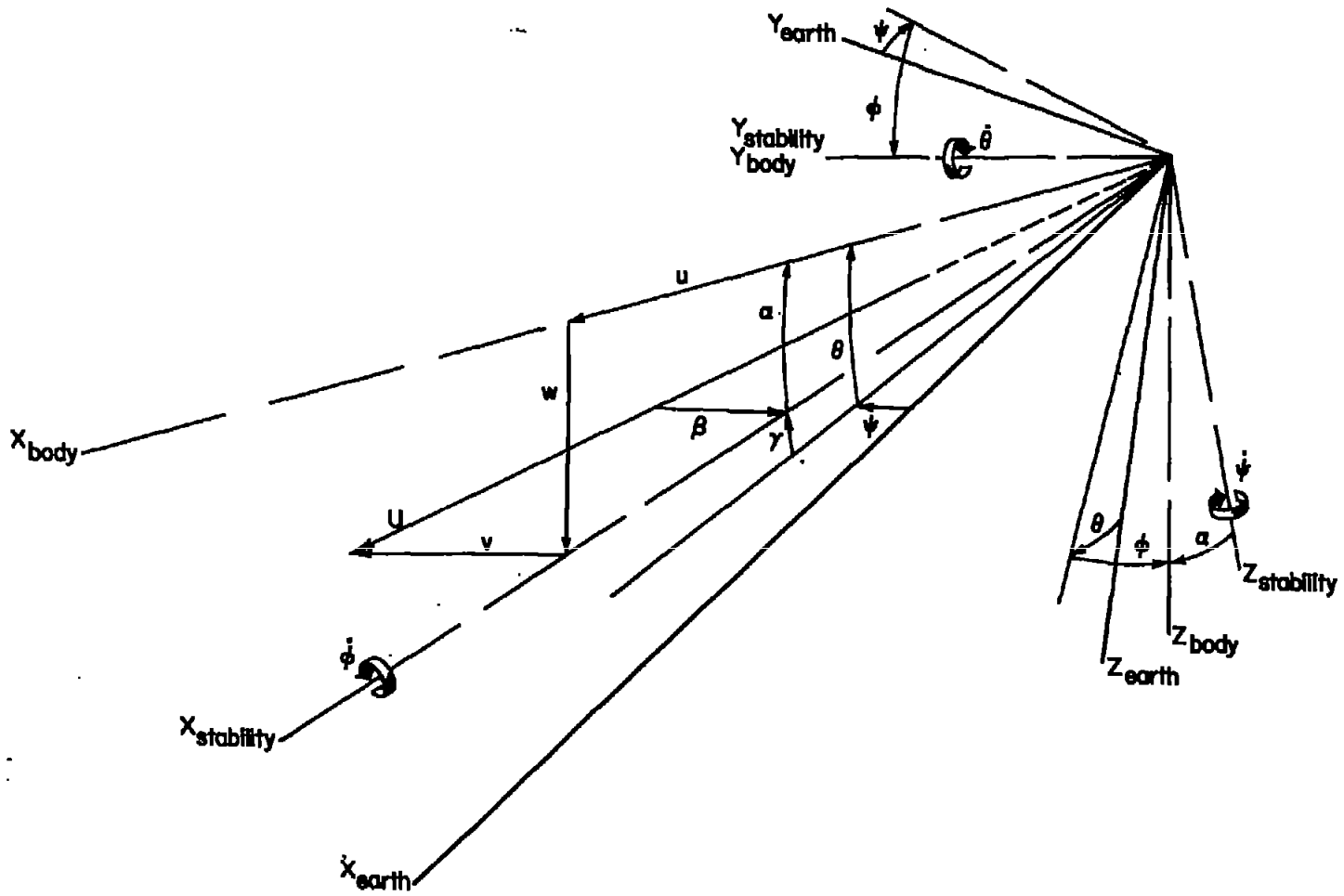
OF EXAMPLE AIRPLANES

Dimension	Airplane		
	A	B	C
b, ft	35.25	70	116
S, sq ft	250	540	1,428
S_T, sq ft	55	71.35	230
A	4.98	9.07	9.4
Γ, deg	4.0	4.50	3.6
λ	0.46	0.45	0.42
h_z, ft	4.4	7.0	13.0
l_t, ft	14.8	27.6	46.6
x_0, ft	18	15.25	48.5
x_1, ft	5.7	22.0	36.6
x_2, ft	20	29.4	51.6
s_0, ft	2.7	3.0	5.3
s_1, ft	8.5	10.6	18.5



(a) Turbulence structure and mean orientation of stability axes of airplane.

Figure 1.- Sign convention and stability axes of airplane flying through atmospheric turbulence.



(b) Instantaneous orientation of stability axes of airplane with respect to earth and body axes. Positive sense of axes, velocities, and angles is indicated by arrows.

Figure 1.- Concluded.

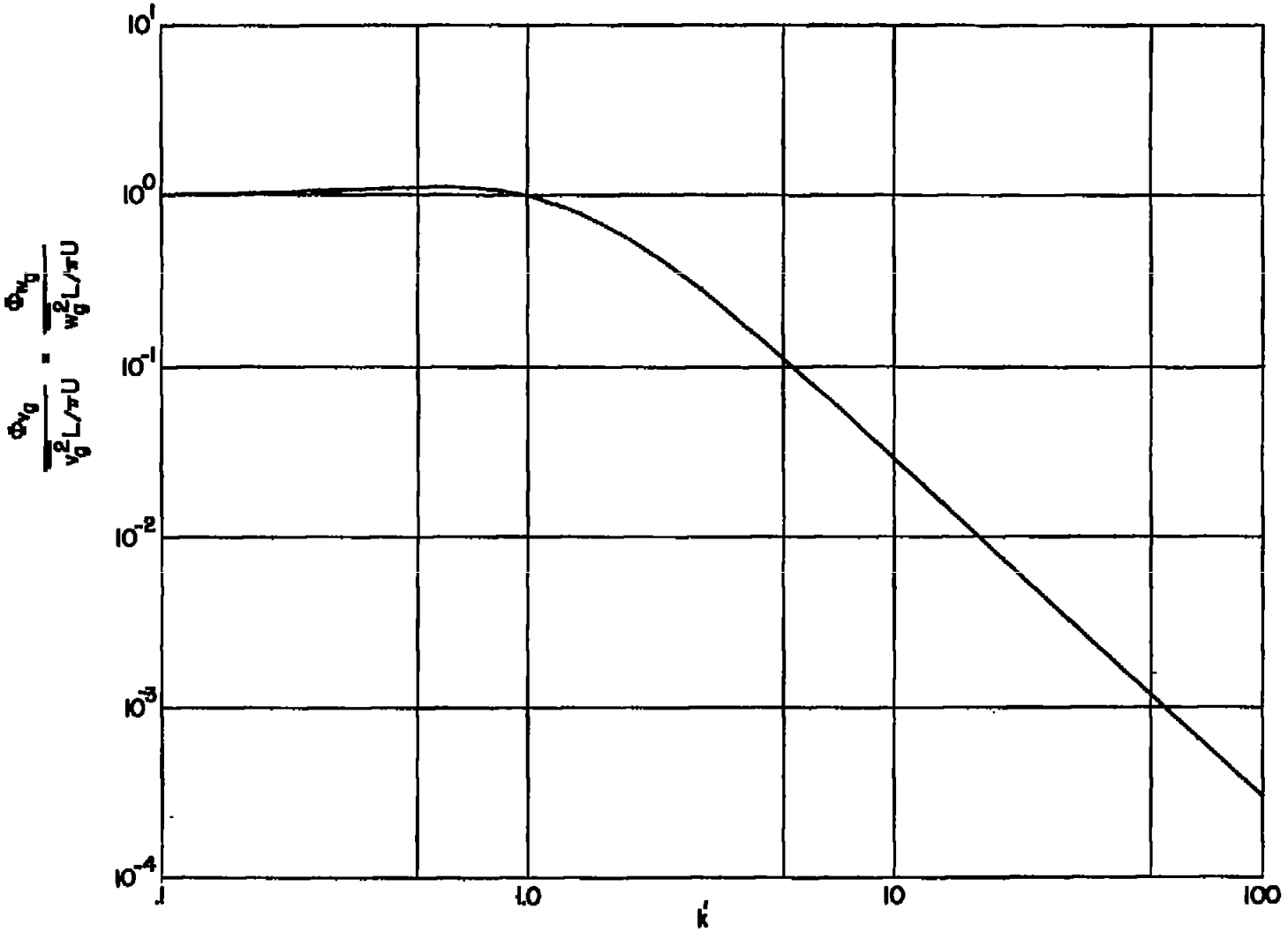
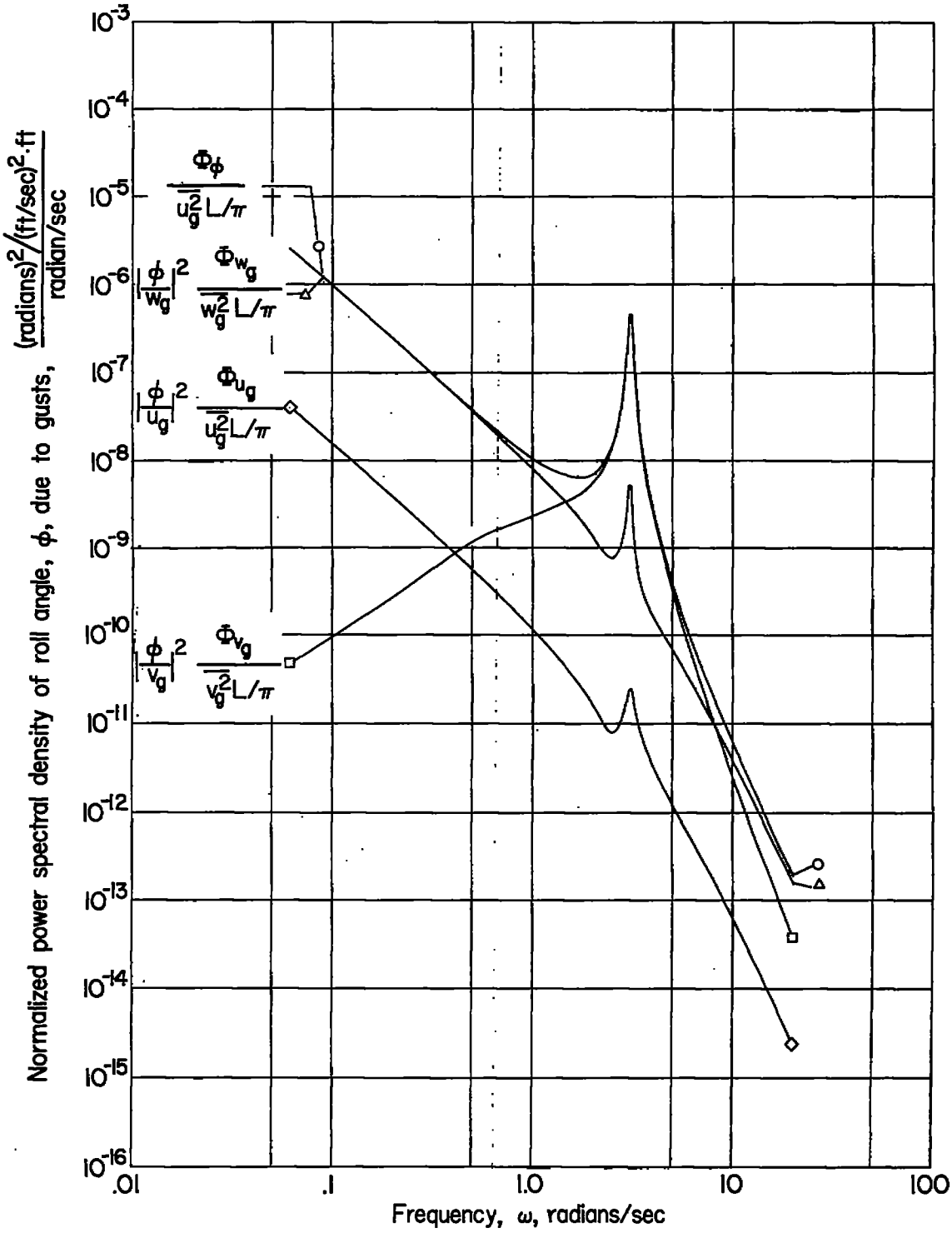
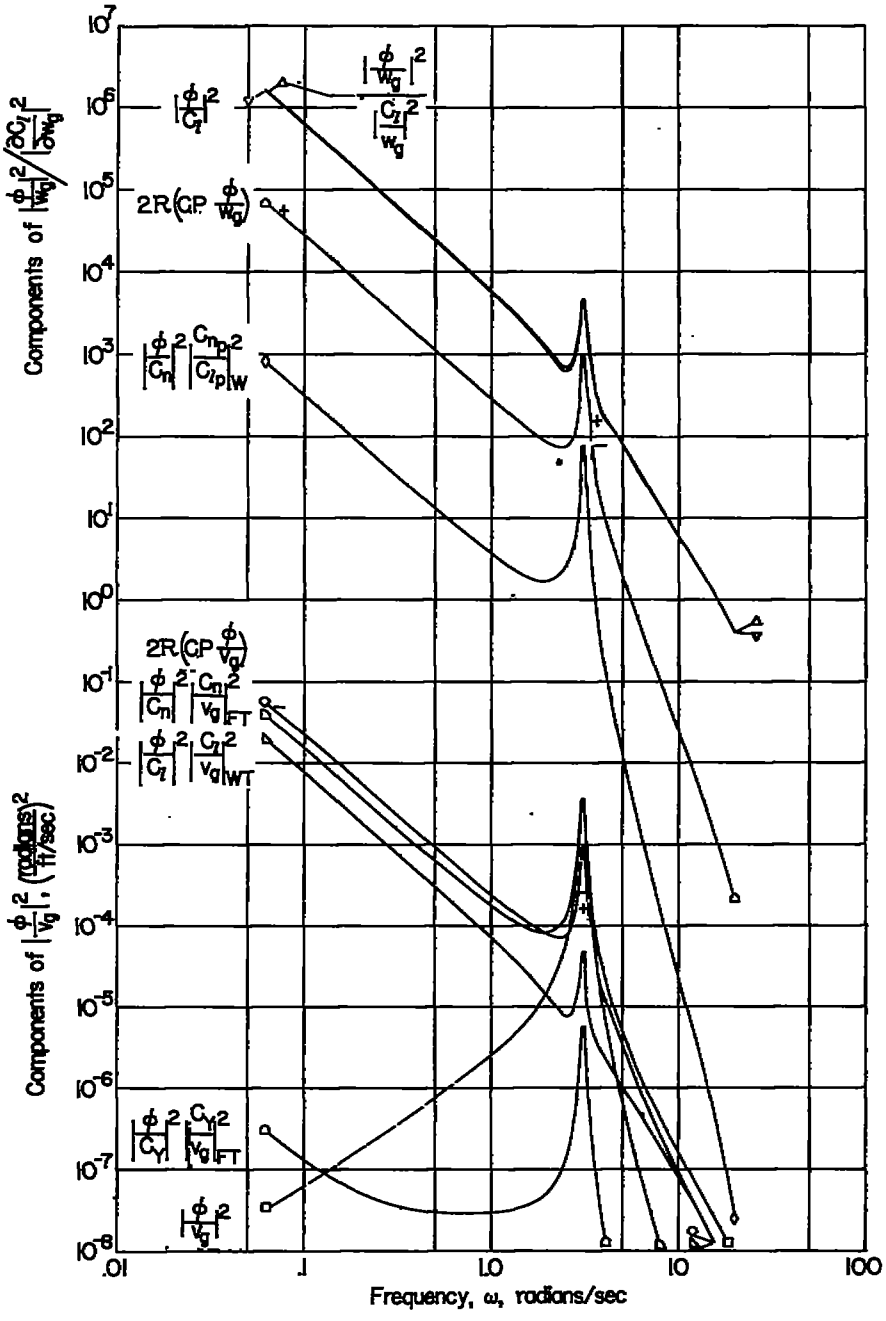


Figure 2.- Power spectral density of u, v, and w components of homogeneous isotropic turbulence as sensed at single point in turbulence.



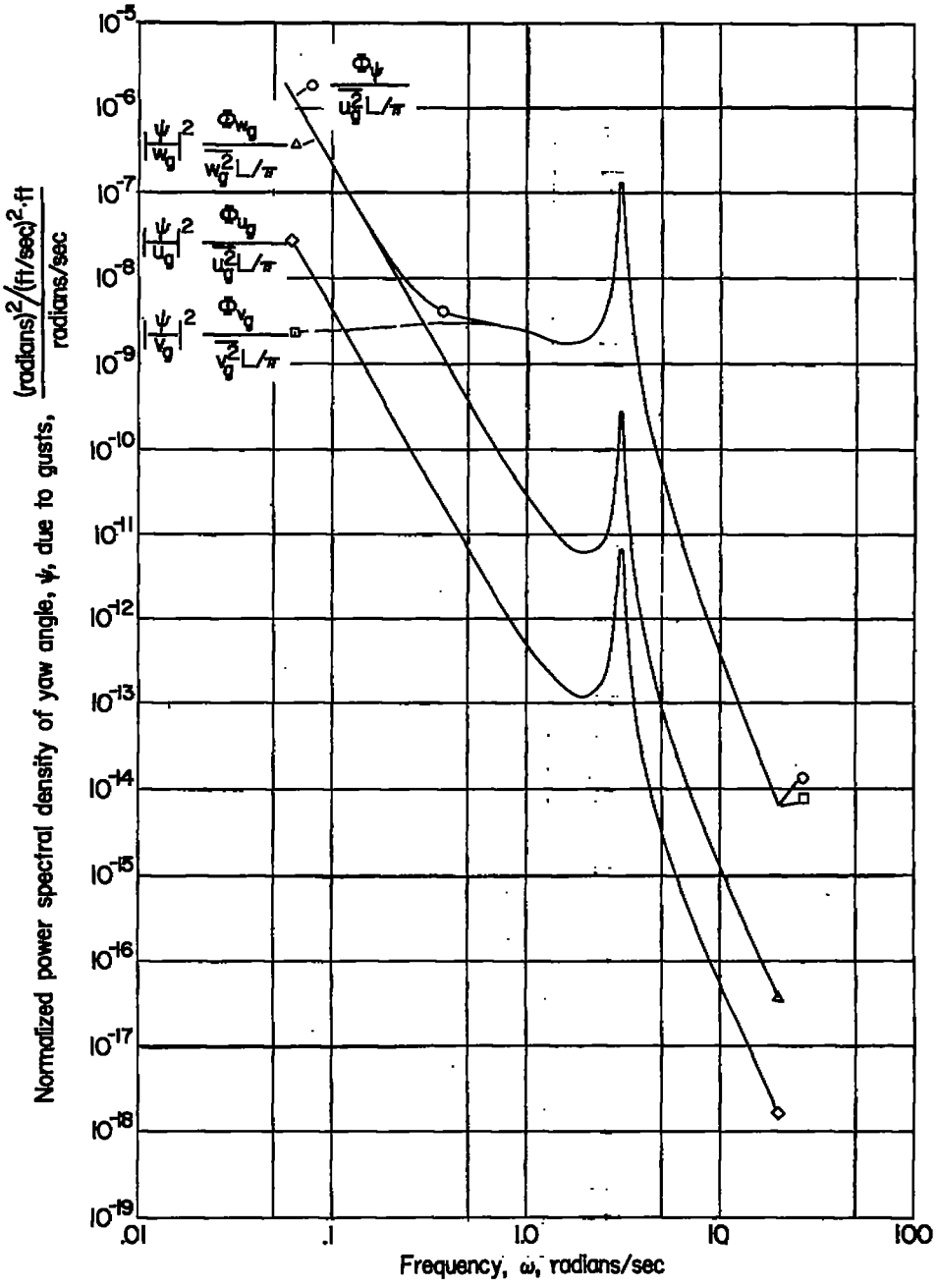
(a) Components and sum of components of power spectral density of roll angle due to three components of gust velocity.

Figure 3.- Response in roll angle of airplane A flying through continuous atmospheric turbulence.



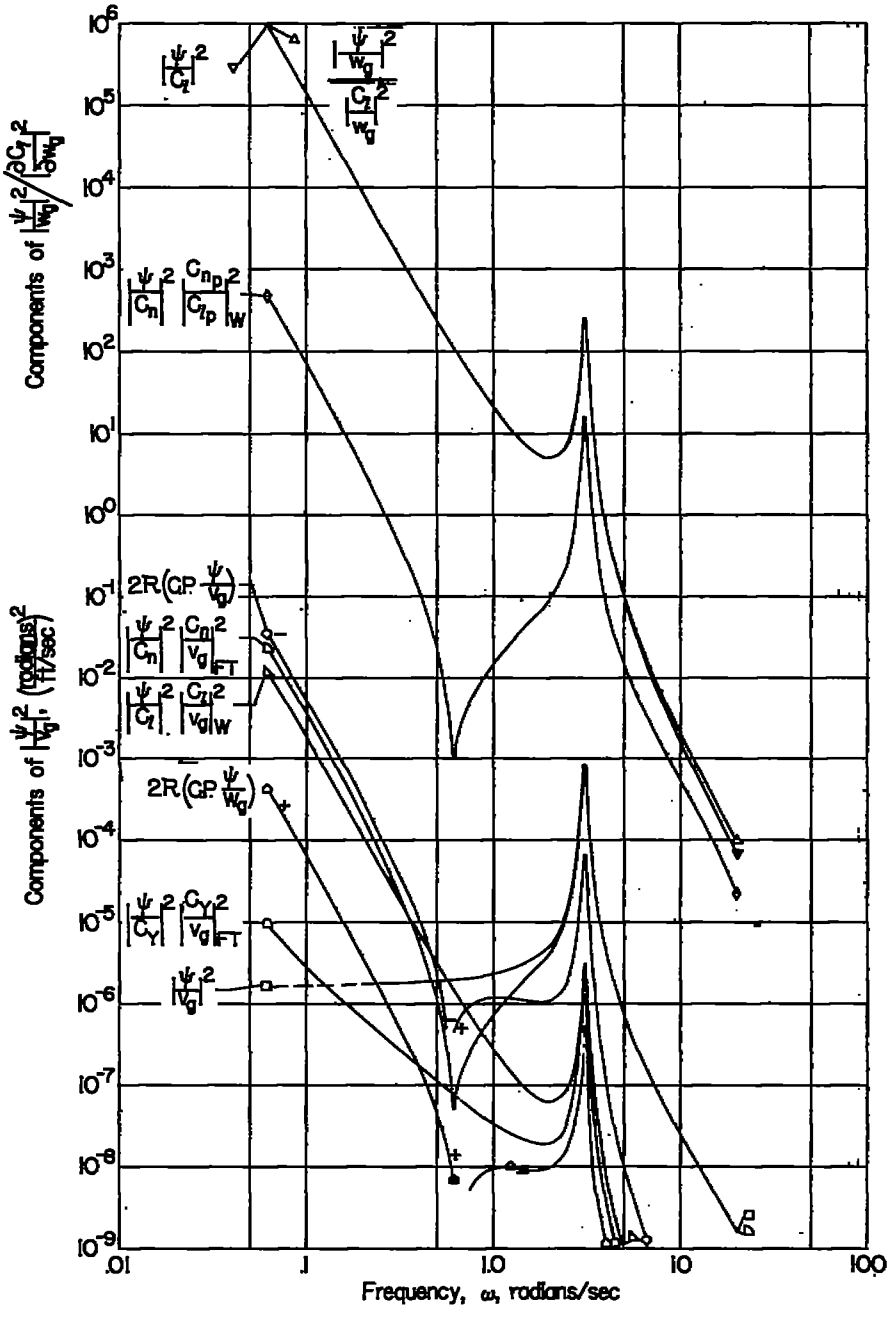
(b) Roll angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 3.- Concluded.



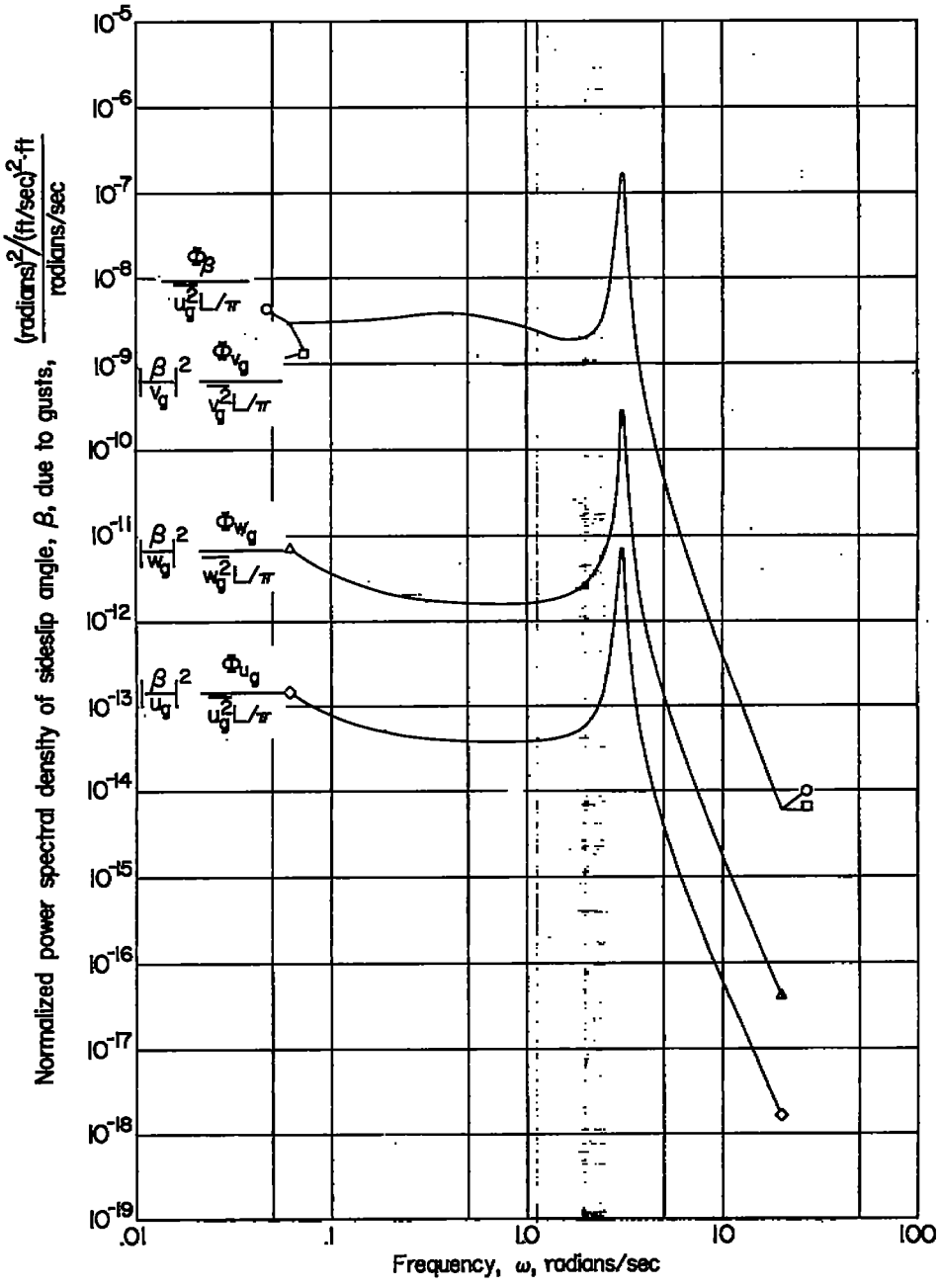
(a) Components and sum of components of power spectral density of yaw angle due to three components of gust velocity.

Figure 4.- Response in yaw angle of airplane A flying through continuous atmospheric turbulence.



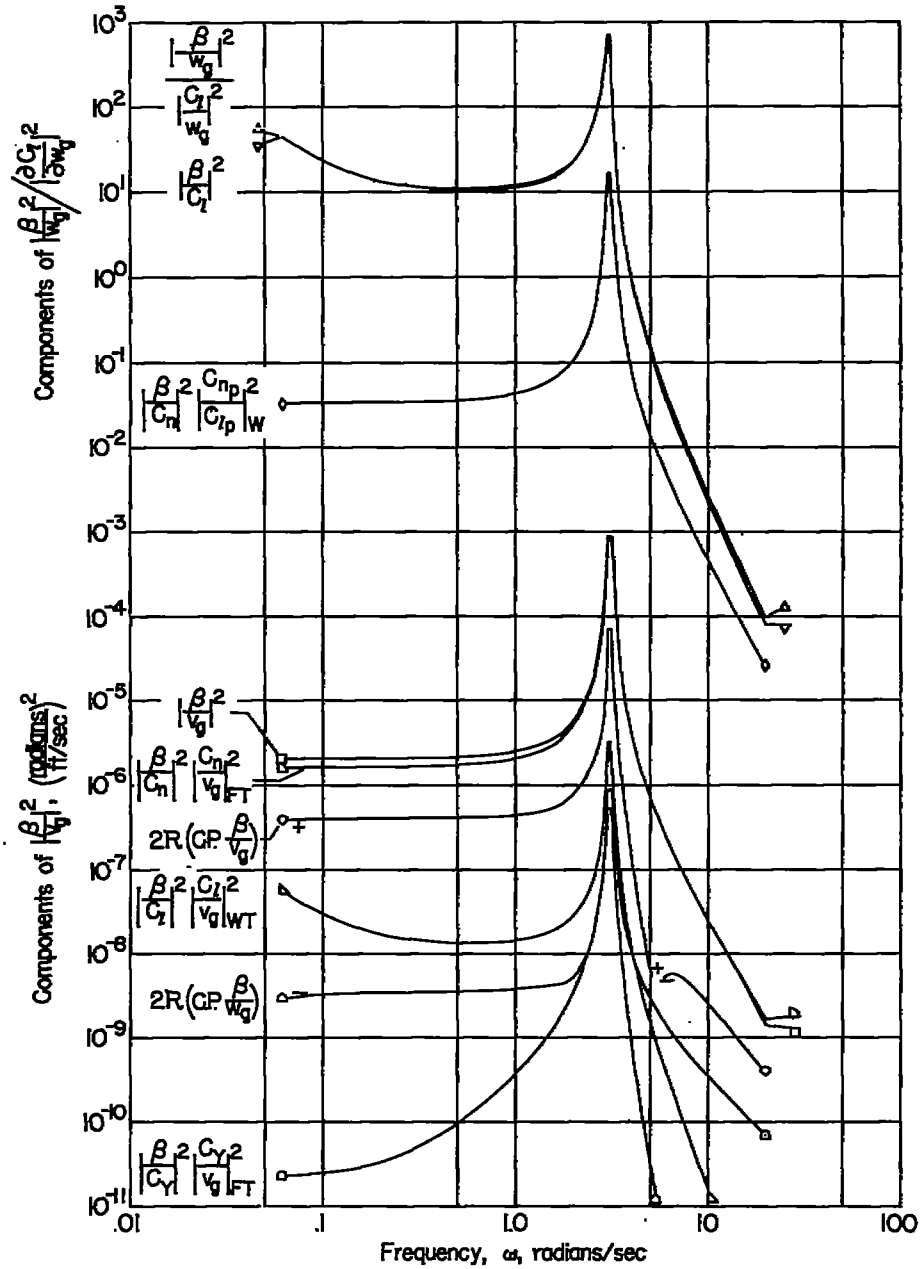
(b) Yaw angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 4.- Concluded.



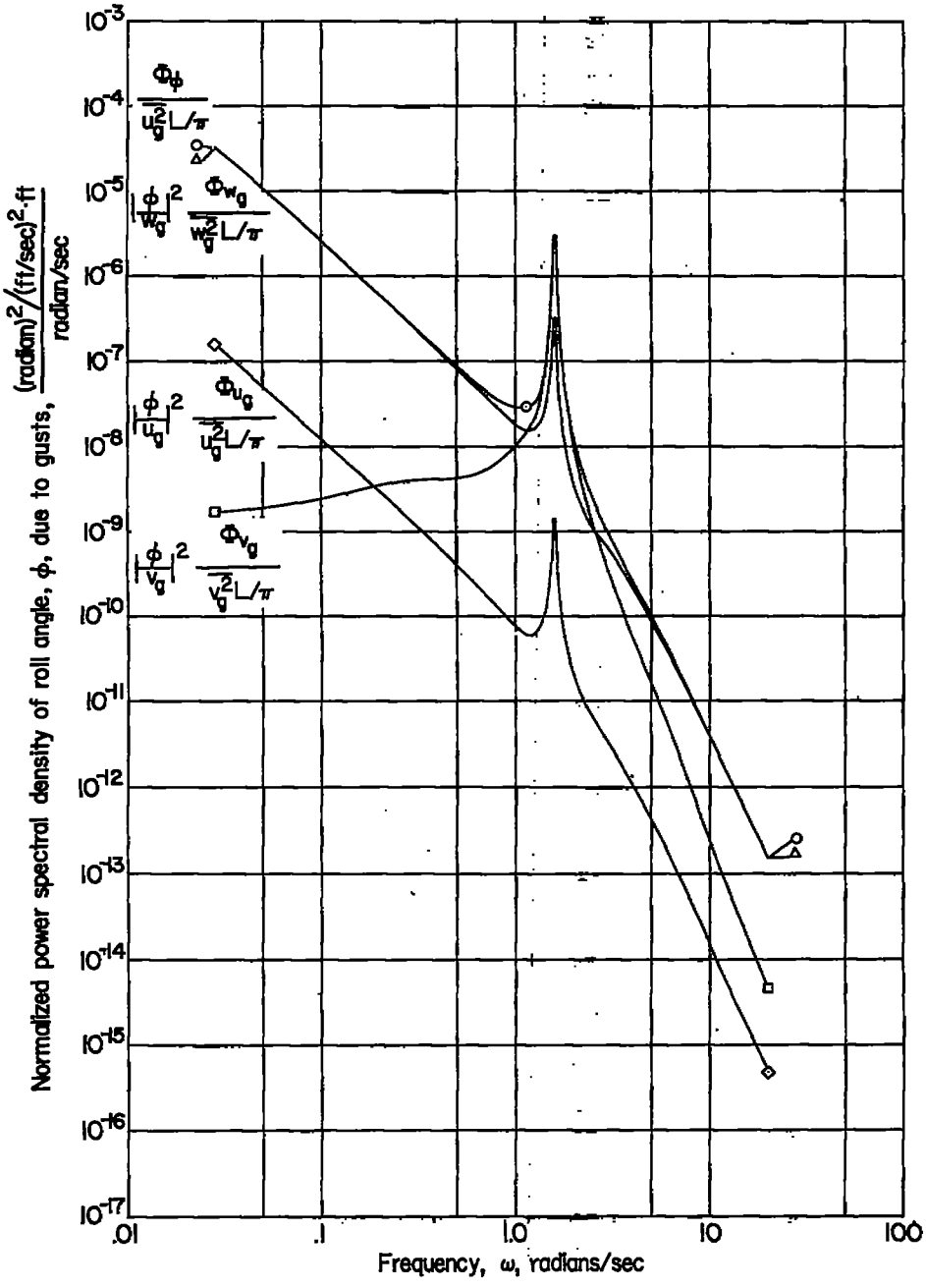
(a) Components and sum of components of power spectral density of sideslip angle due to three components of gust velocity.

Figure 5.- Response in sideslip angle of airplane A flying through continuous atmospheric turbulence.



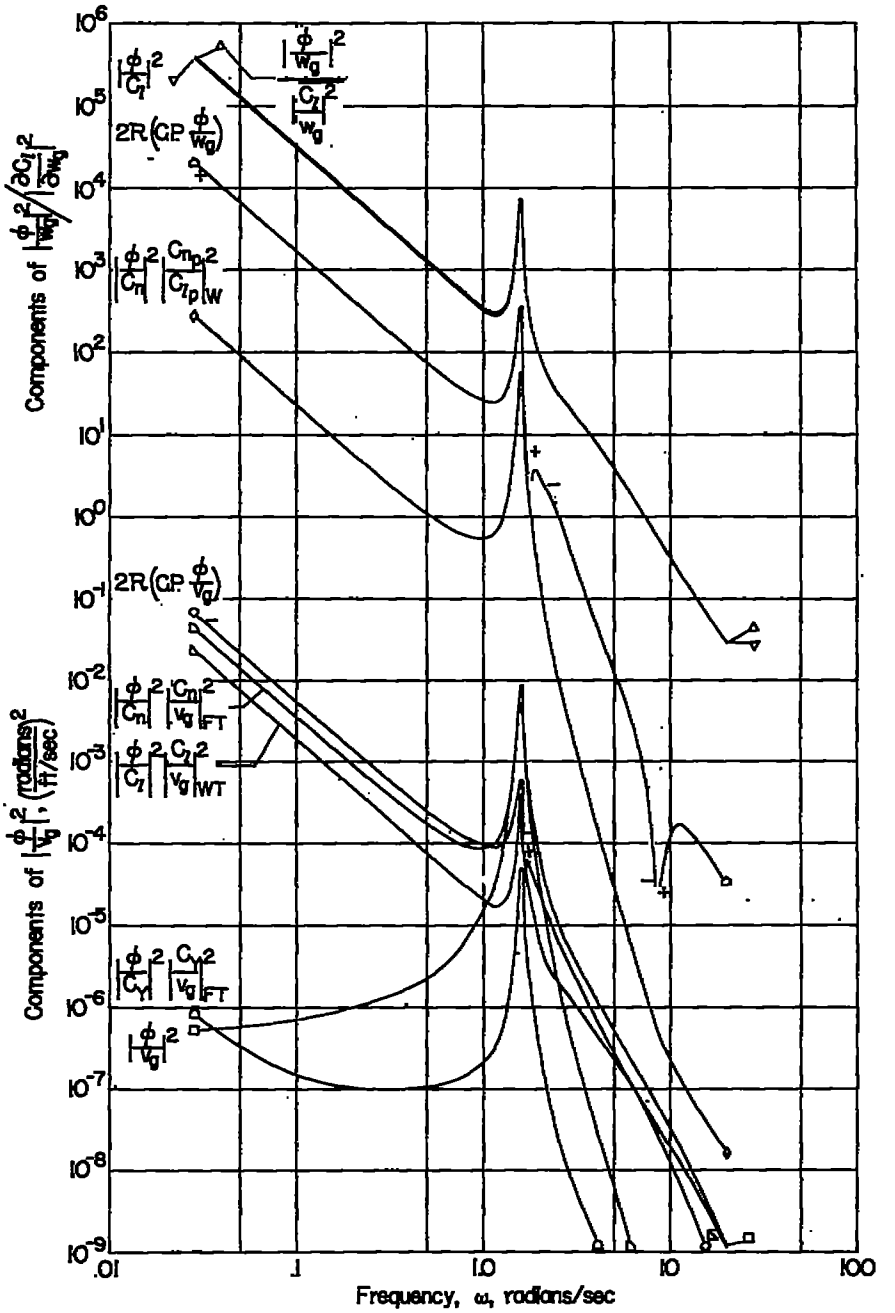
(b) Sideslip angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 5.- Concluded.



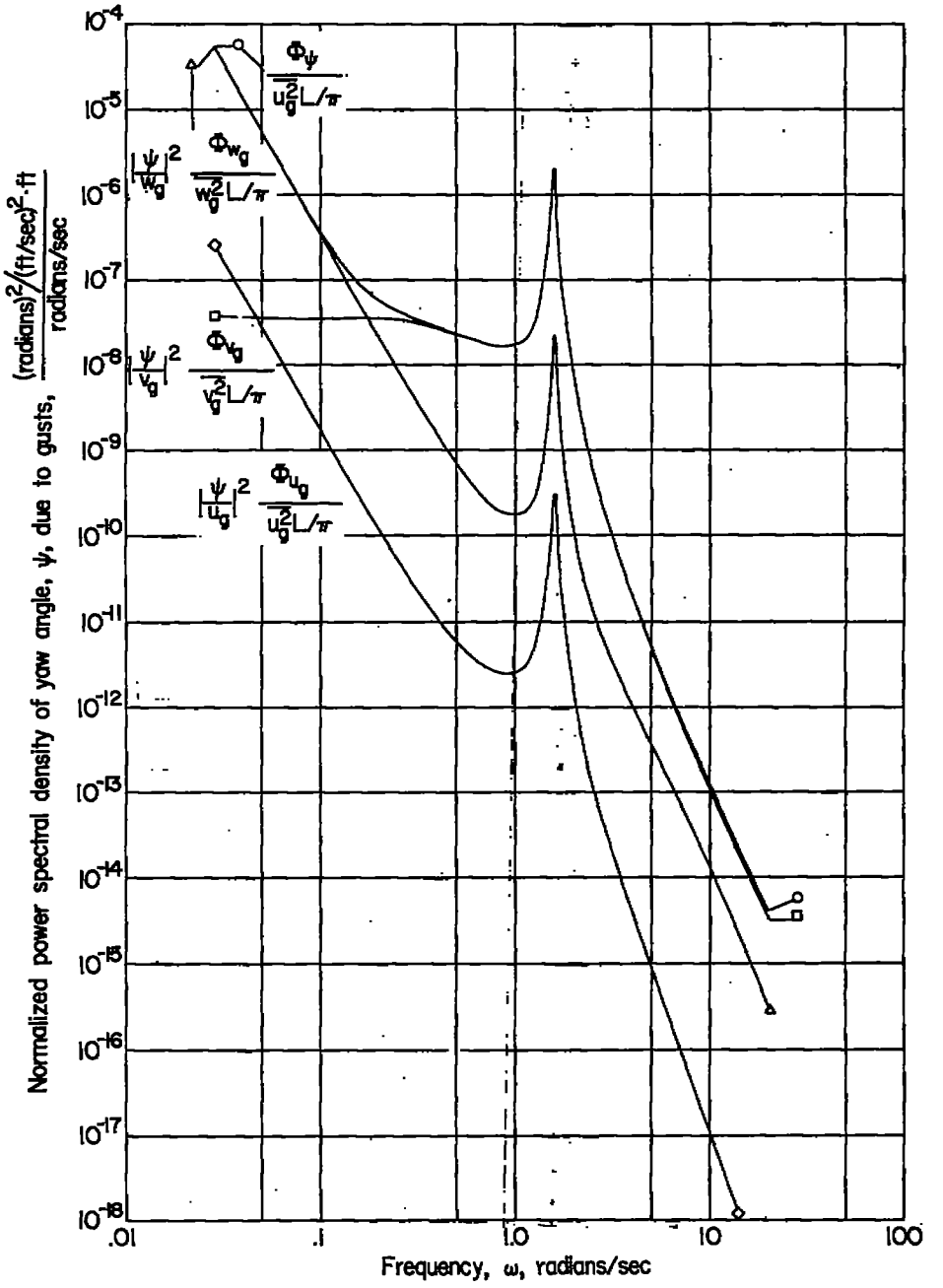
(a) Components and sum of components of power spectral density of roll angle due to three components of gust velocity.

Figure 6.- Response in roll angle of airplane B flying through continuous atmospheric turbulence.



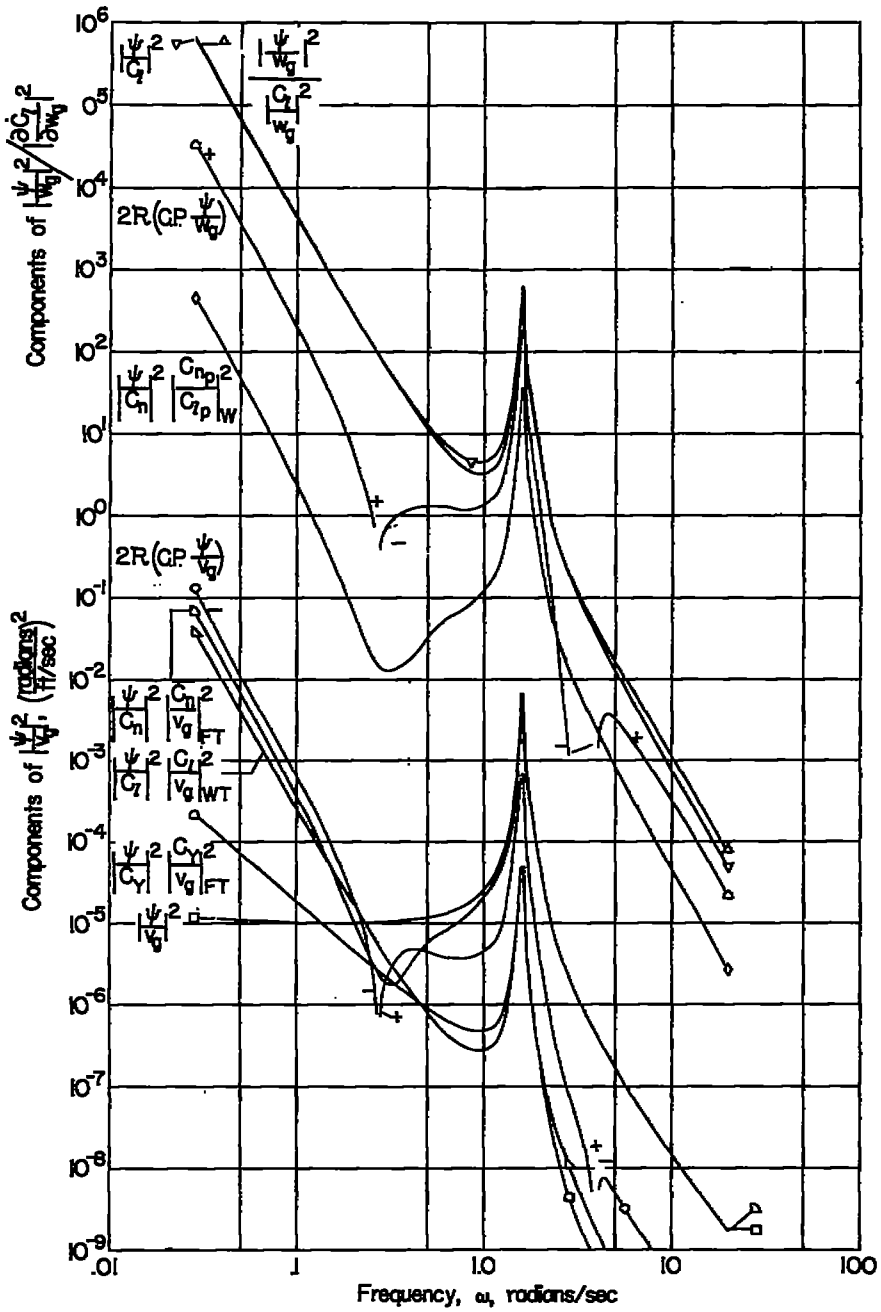
(b) Roll angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 6.- Concluded.



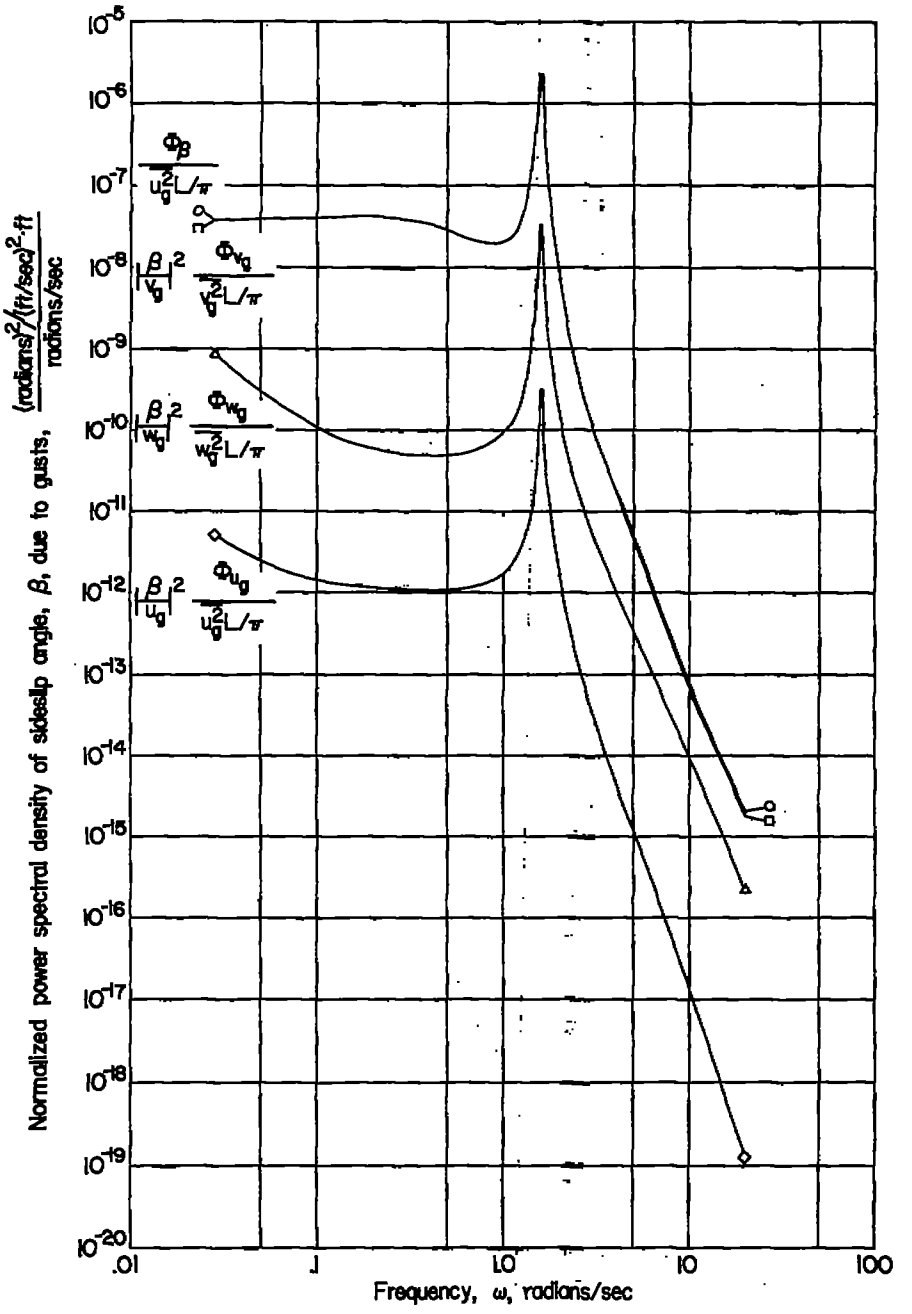
(a) Components and sum of components of power spectral density of yaw angle due to three components of gust velocity.

Figure 7.- Response in yaw angle of airplane B flying through continuous atmospheric turbulence.



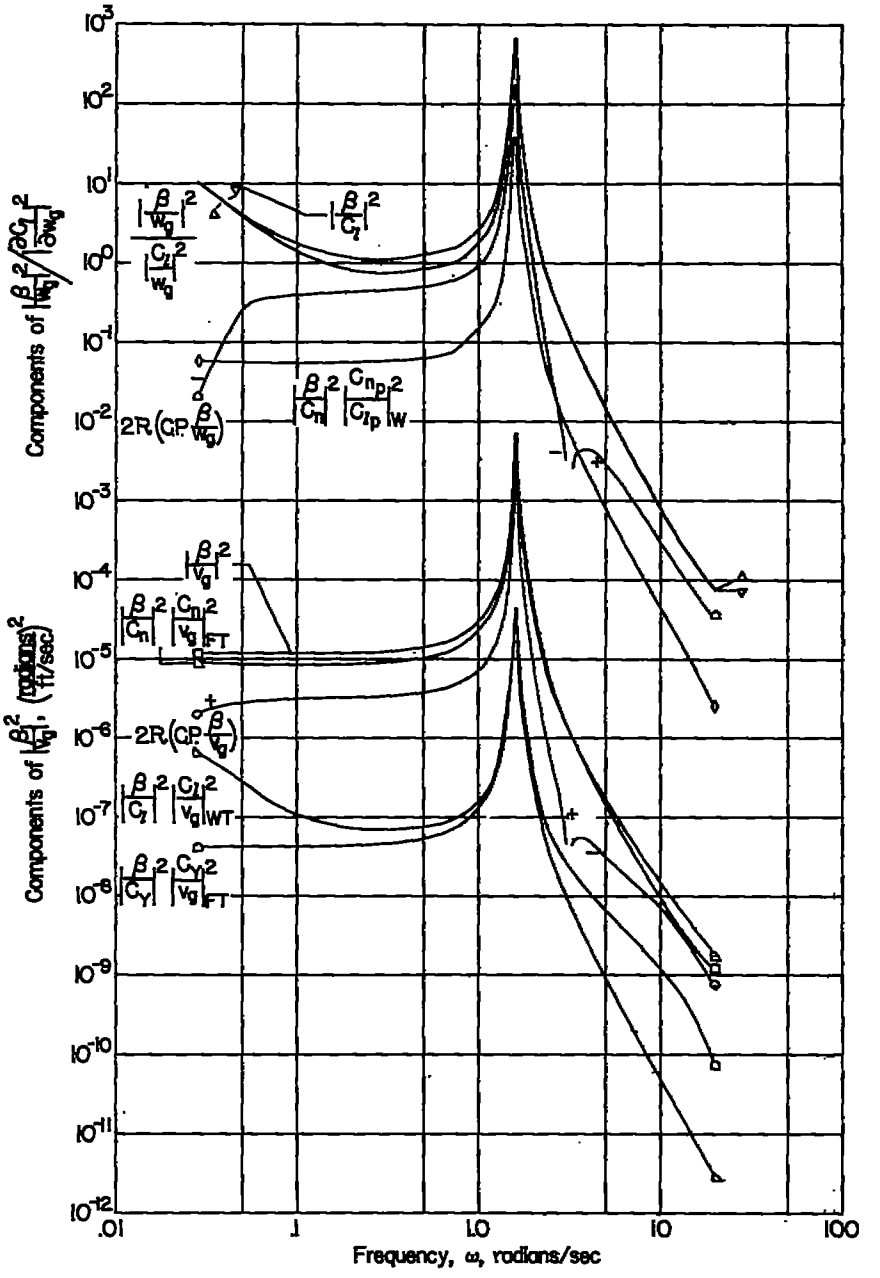
(b) Yaw angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 7.- Concluded.



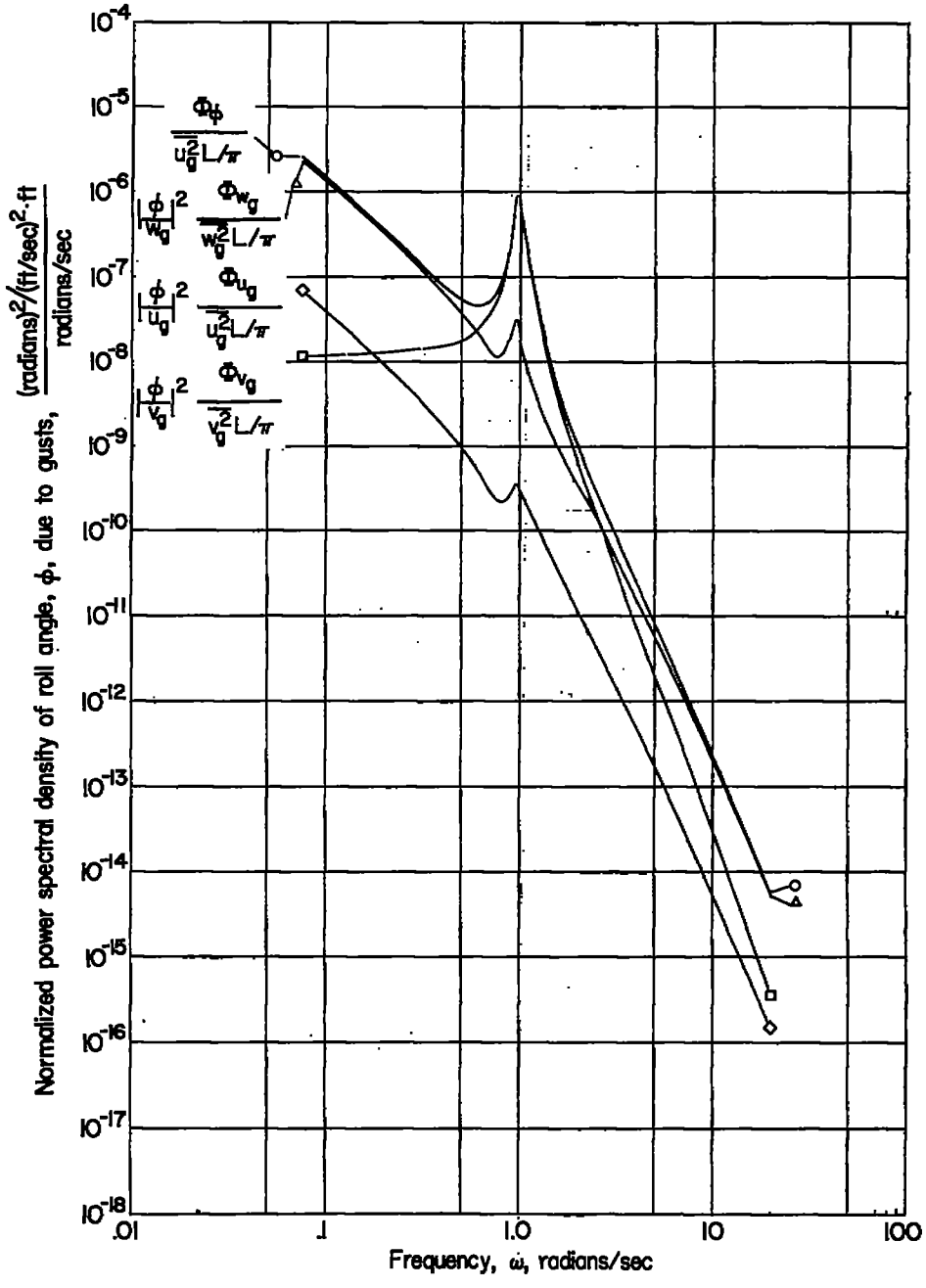
(a) Components and sum of components of power spectral density of sideslip angle due to three components of gust velocity.

Figure 8.- Response in sideslip angle of airplane B flying through continuous atmospheric turbulence.



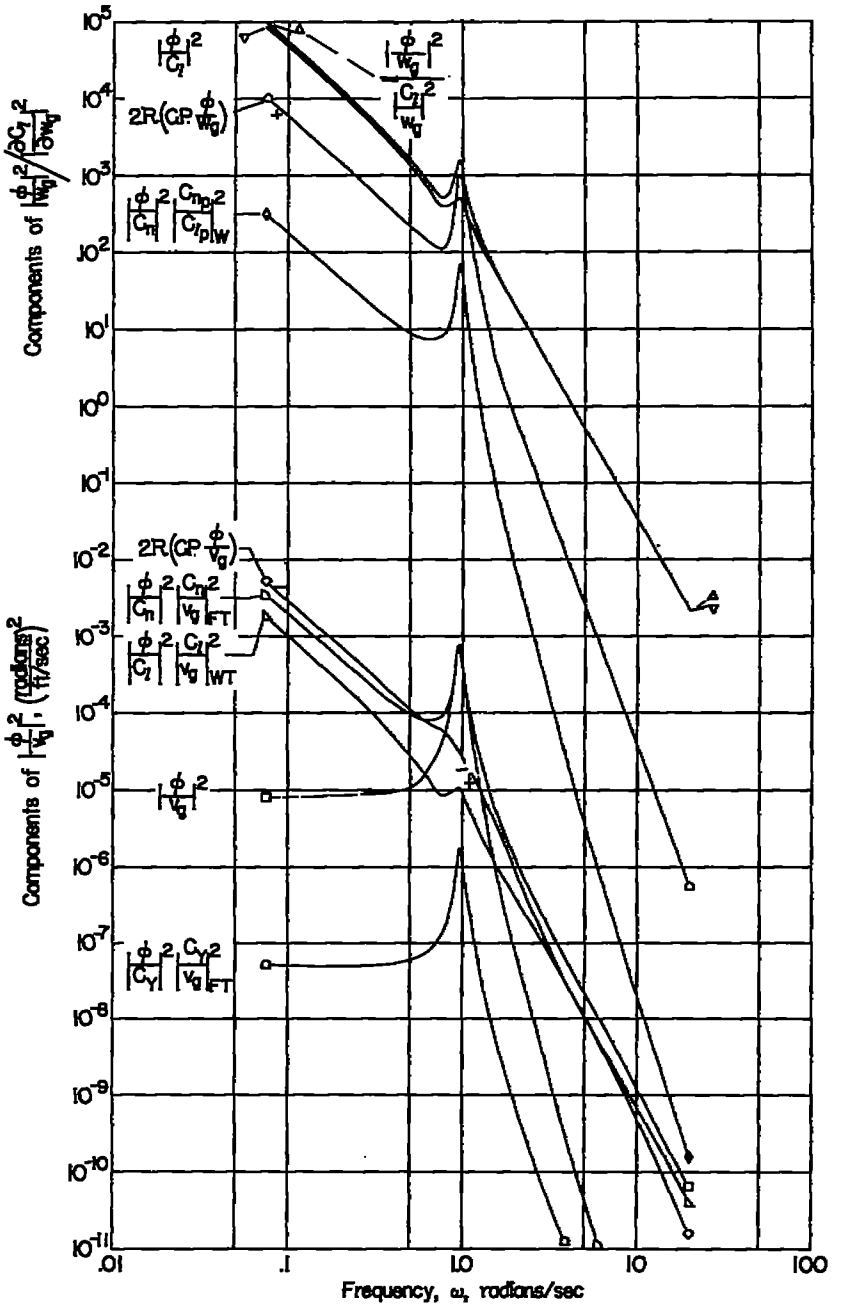
(b) Sideslip angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 8.- Concluded.



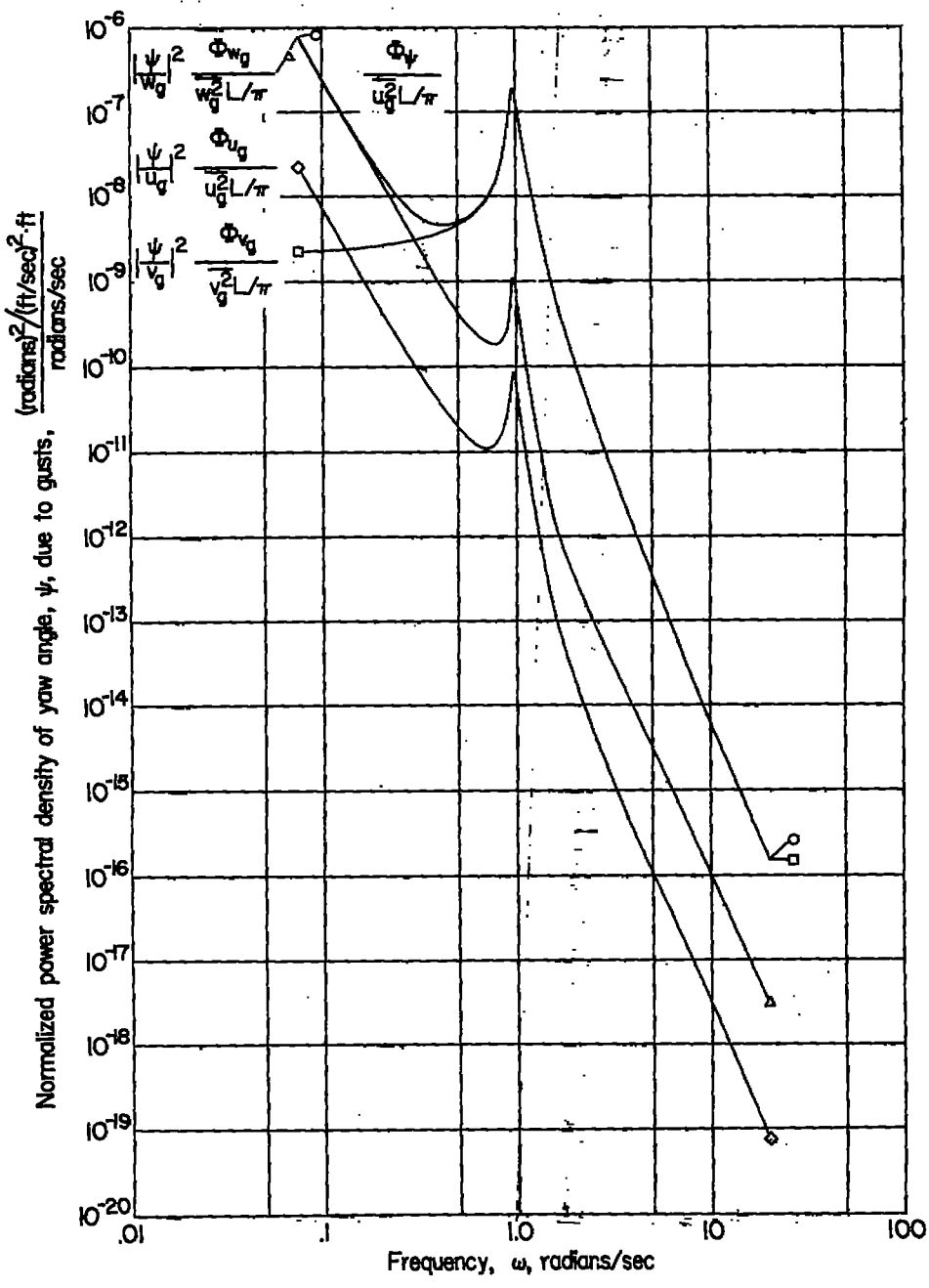
(a) Components and sum of components of power spectral density of roll angle due to three components of gust velocity.

Figure 9.- Response in roll angle of airplane C flying through continuous atmospheric turbulence.



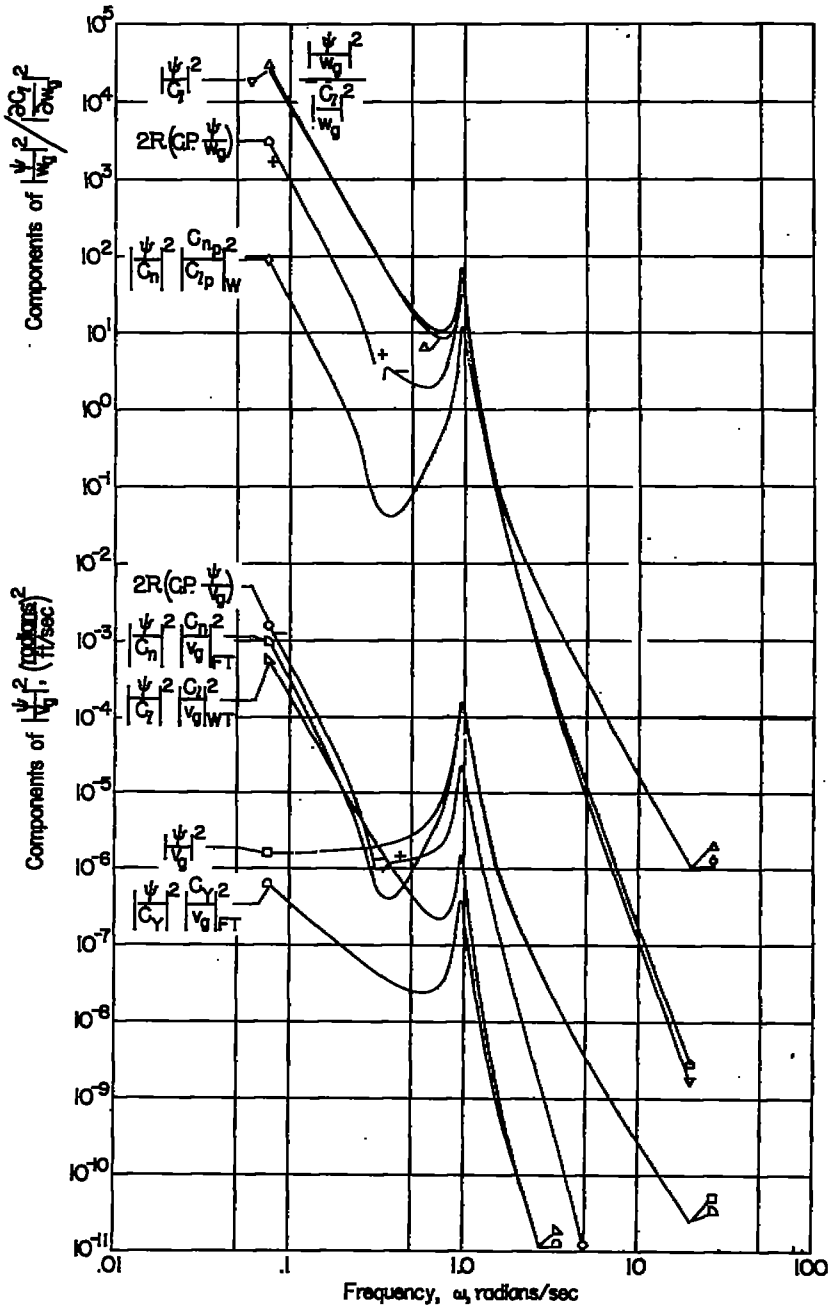
(b) Roll angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 9.- Concluded.



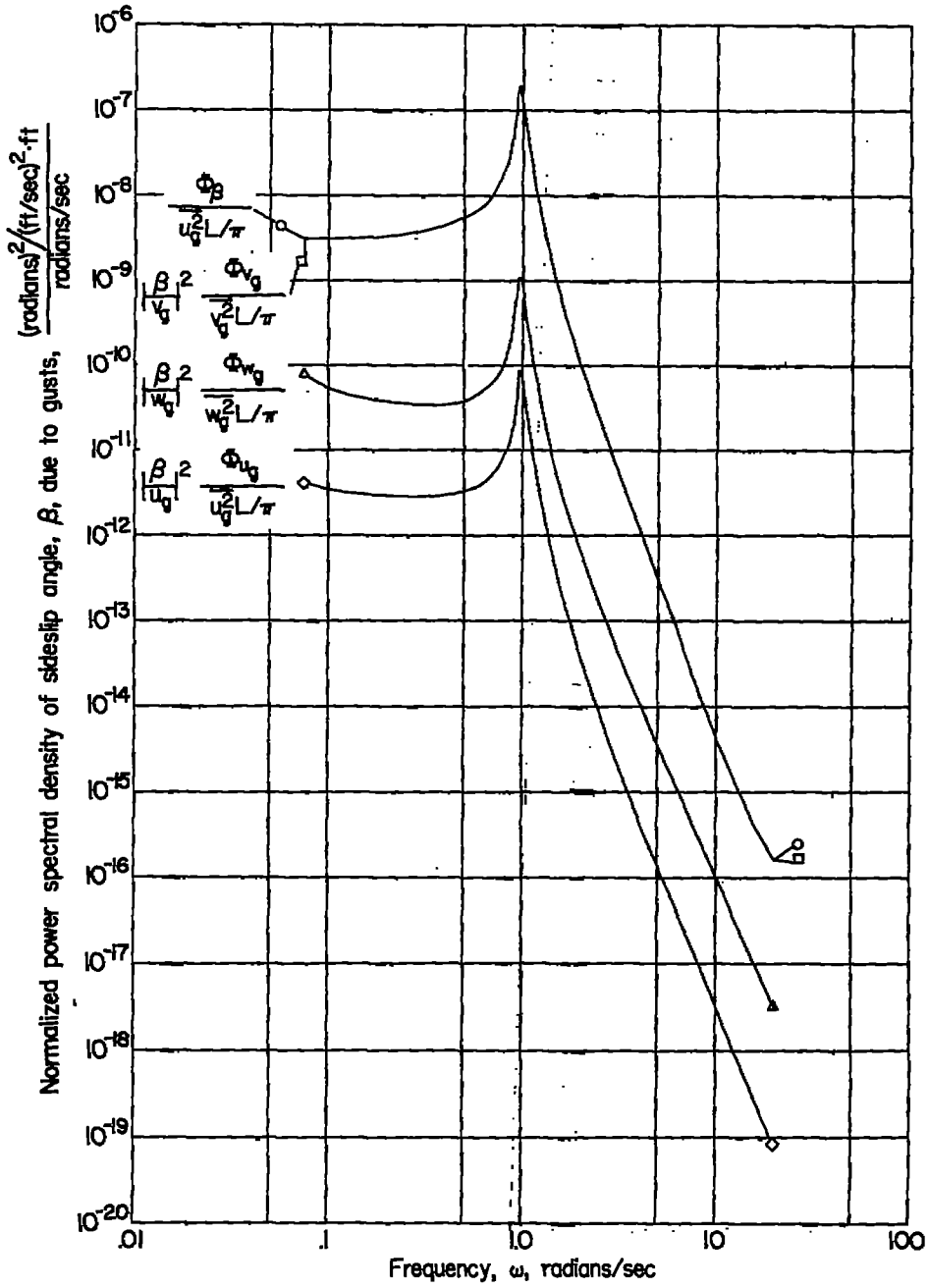
(a) Components and sum of components of power spectral density of yaw angle due to three components of gust velocity.

Figure 10.- Response in yaw angle of airplane C flying through continuous atmospheric turbulence.



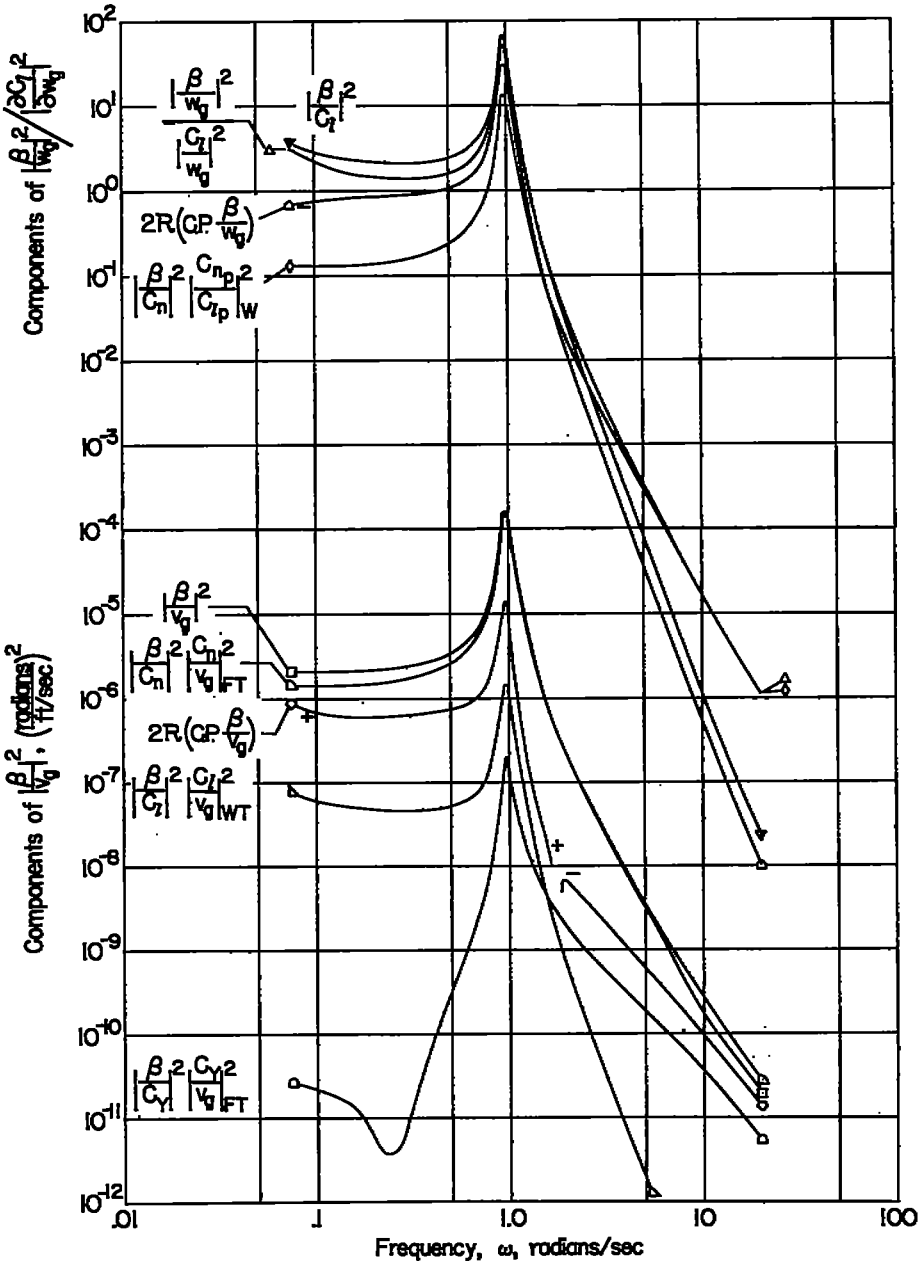
(b) Yaw angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 10.- Concluded.



(a) Components and sum of components of power spectral density of sideslip angle due to three components of gust velocity.

Figure 11.- Response in sideslip angle of airplane C flying through continuous atmospheric turbulence.



(b) Sideslip angle due to forces and moments induced by v_g and w_g components of gust velocity. Shown are sum and relative magnitudes of parts.

Figure 11.- Concluded.

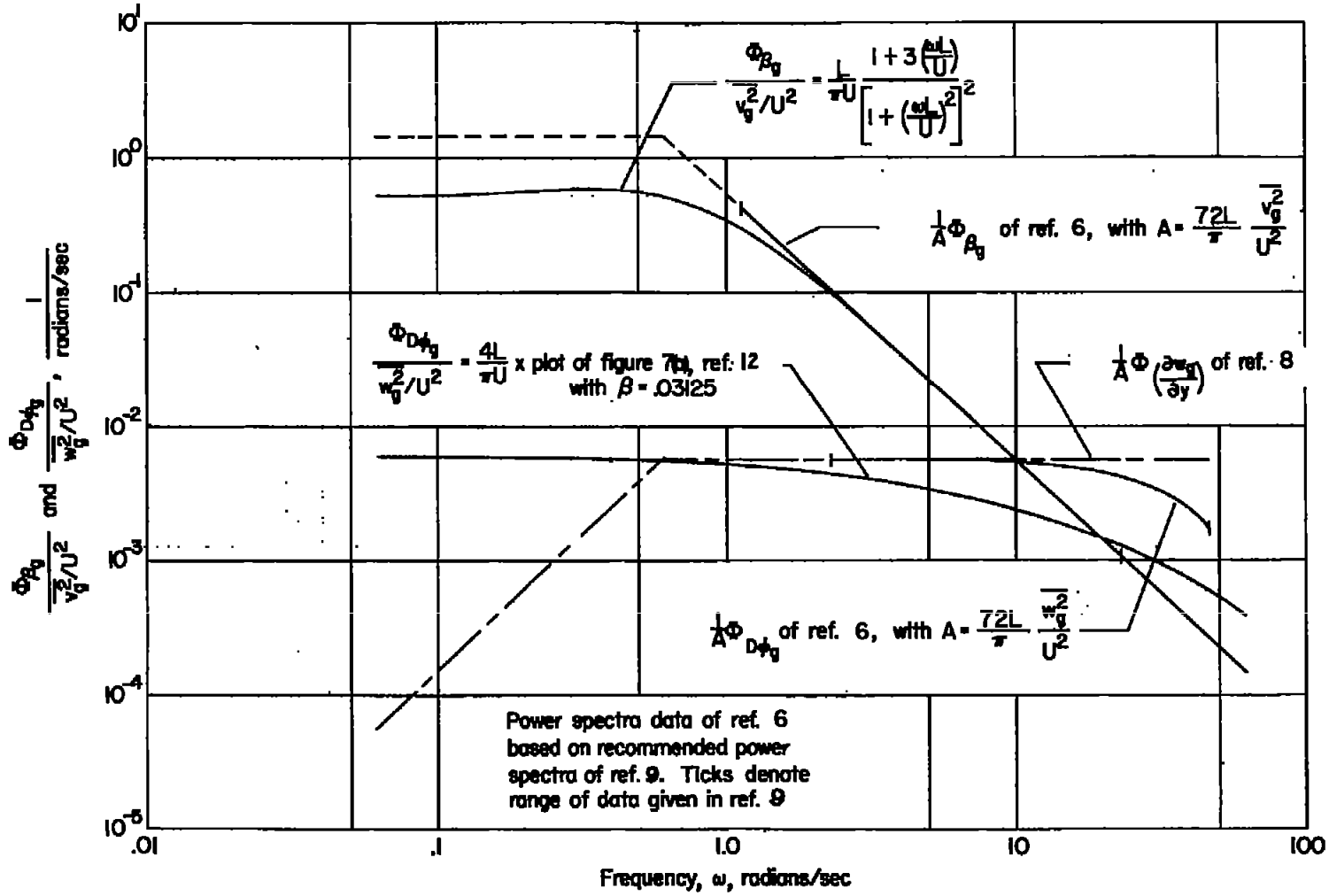
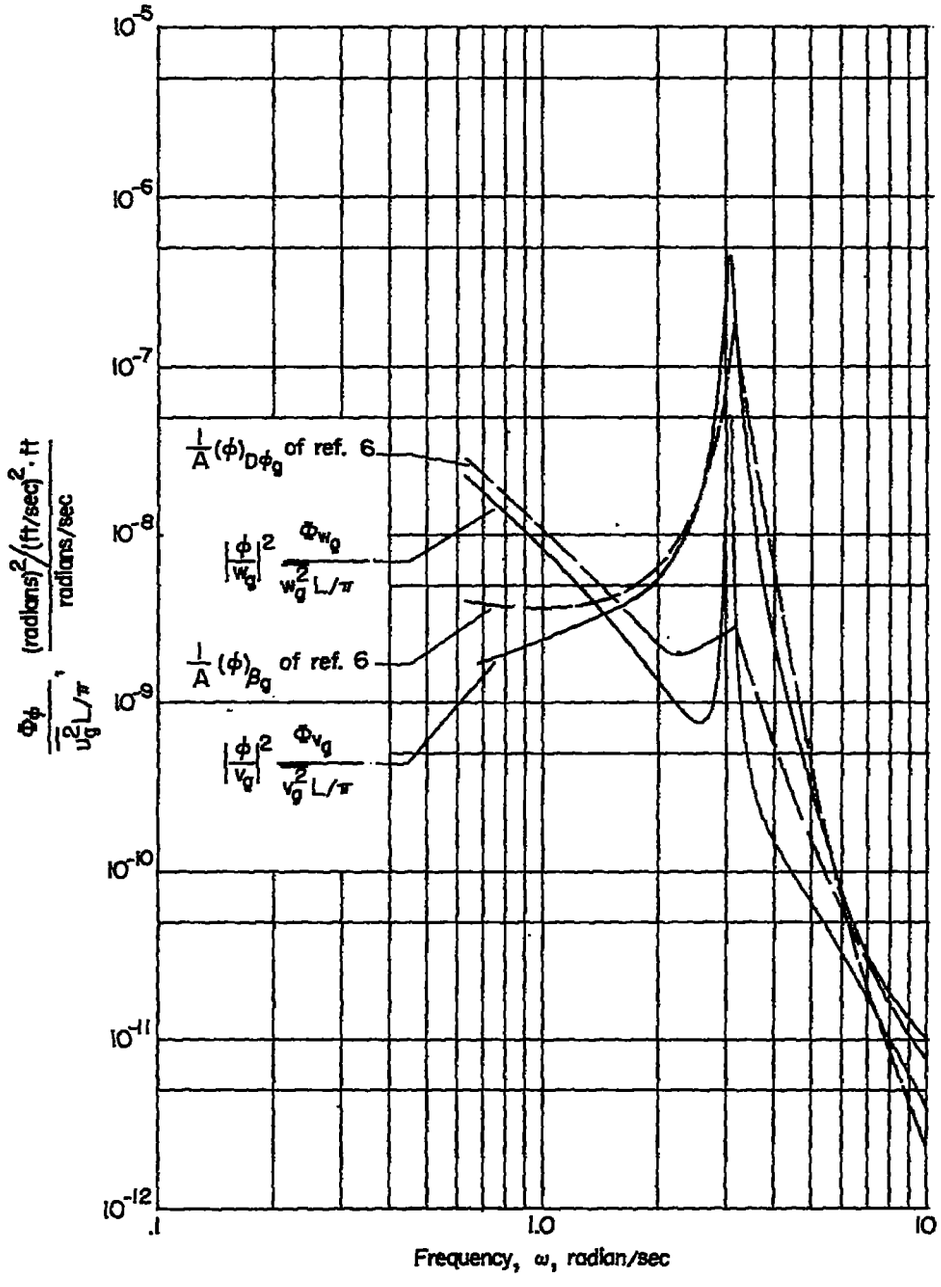
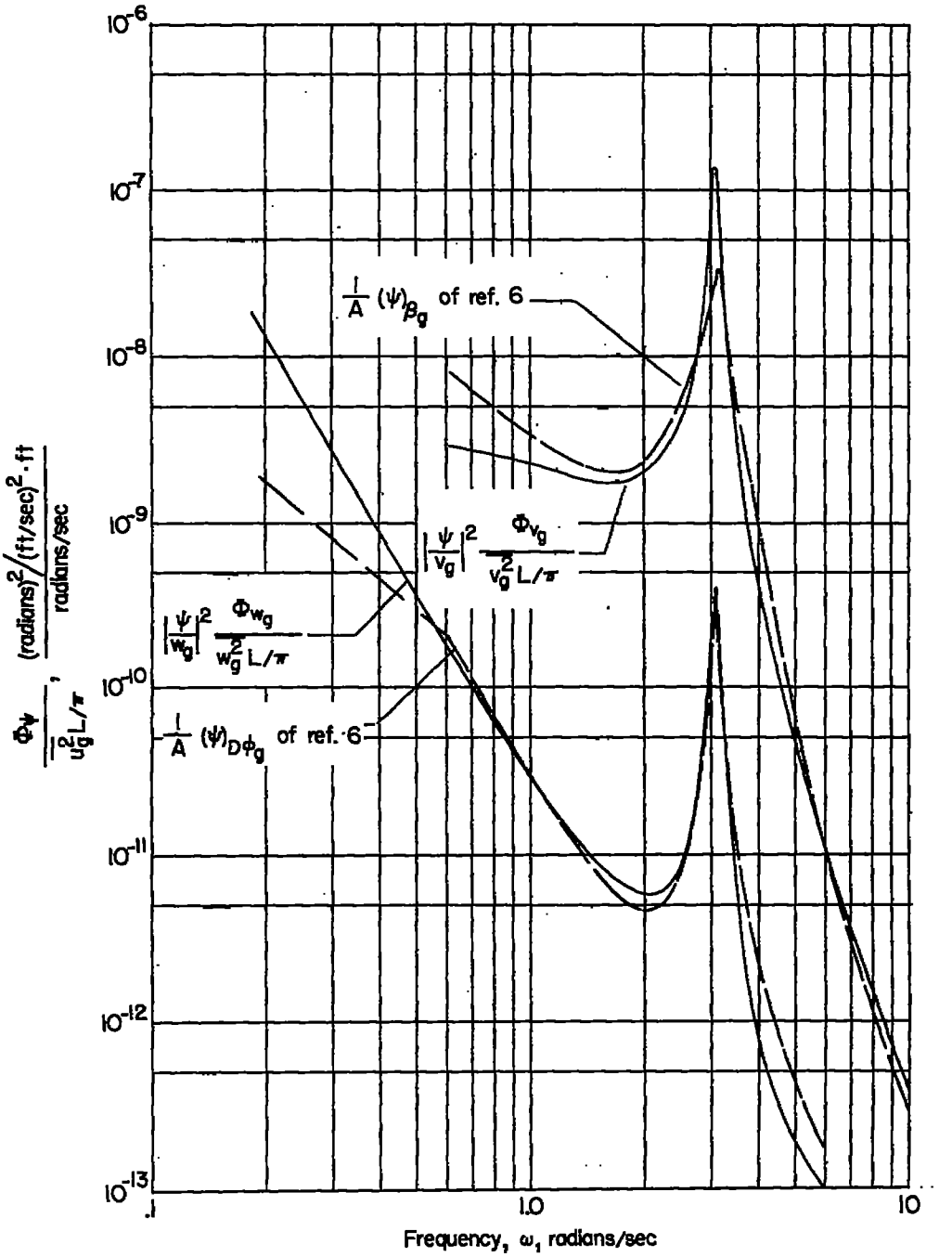


Figure 12.- Comparison of several theories for power spectra of side gusts and rolling gusts as experienced by airplane A. U = 696 ft/sec; b = 35.25 feet; L = 1,128 feet.



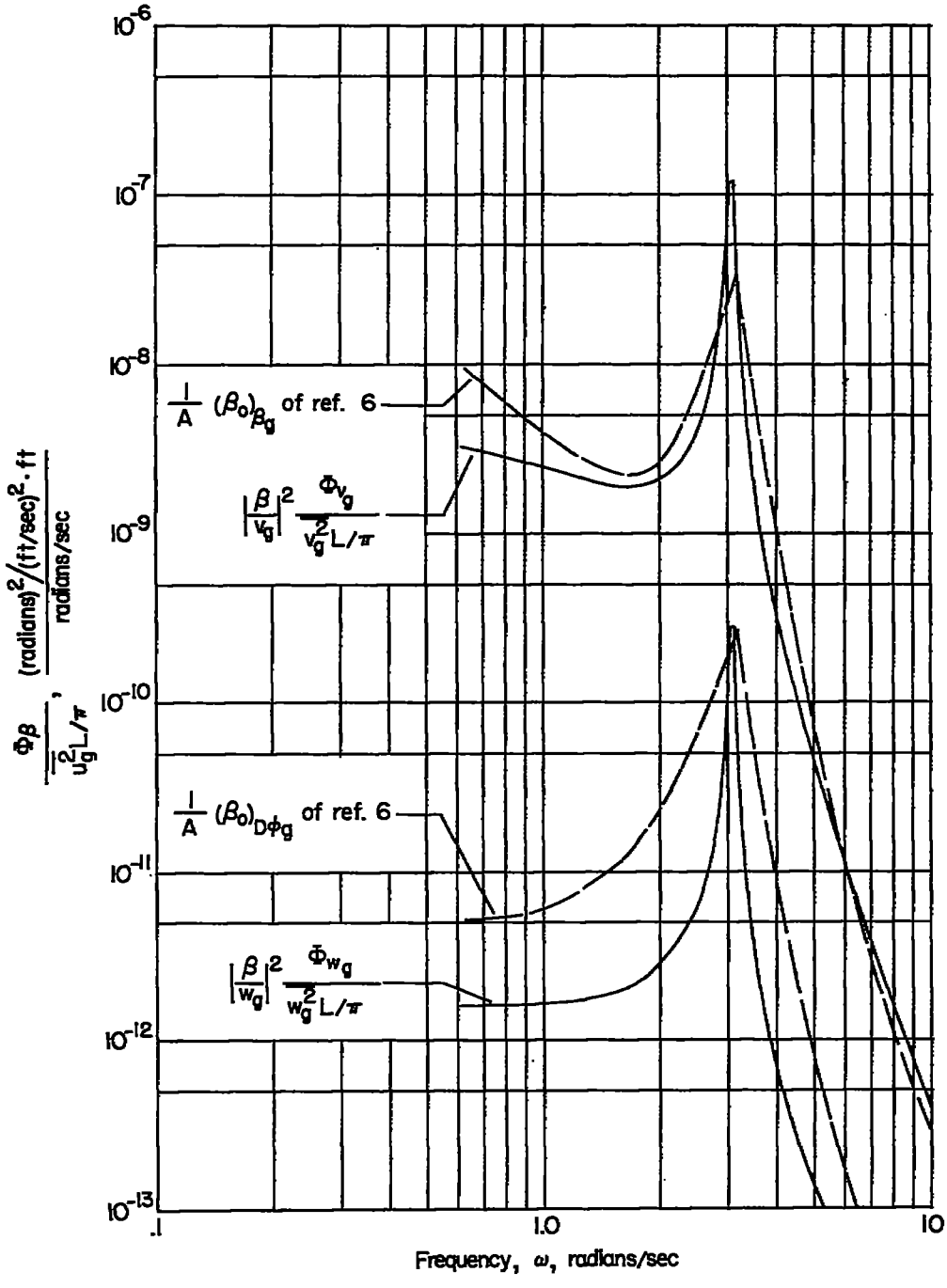
(a) Power spectral response in roll angle ϕ of airplane A.

Figure 13.- Comparison between the lateral responses of airplane A in atmospheric turbulence as determined by method of present paper and method of reference 6.



(b) Power spectral response in yaw angle ψ of airplane A.

Figure 13.- Continued.



(c) Power spectral response in sideslip angle β of airplane A.

Figure 13.- Concluded.