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BENDING OF RECTANGULAR PLATES WITH LARGE DEFLECTIONS

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BENDING OF RECTANGULAR PLATES WITH LARGE DEFLECTIONS

By Chi-Teh Wang

SUMMARY

Von Kármán's equations for thin plates with large deflections are solved for the special cases of rectangular plates having ratios of length to width of 1.5 and 2 and loaded by uniform normal pressure. The boundary conditions are such as to approximate panels with riveted edges under normal pressure greater than that of the surrounding panels. Center deflections, membrane stresses, and extreme-fiber bending stresses are given as functions of the pressure for center deflections up to twice the thickness of the plate. For small deflections the results are consistent with those given by Timoshenko.

INTRODUCTION

A general numerical method for solving Von Kármán's equations for thin plates with large deflections was developed in reference 1. A square plate loaded by uniform normal pressure with simply supported edges was studied. The boundary conditions approximate the case when a riveted sheet-stringer panel is under normal pressure greater than that of the surrounding ones.

It was subsequently decided to extend the investigation to rectangular plates of various ratios of length to width. Two special cases are studied in this report; namely, rectangular plates for which the ratio of length to width has the value of 1.5 or 2. (For rectangular plates having a length equal to or greater than twice their width, experimental evidence (reference 2) indicates that they can practically be regarded as infinitely long.) Center deflections, membrane stresses, and extreme-fiber bending stresses are given as functions of the pressure for center deflections up to twice the thickness of the plate. For small deflections, the results are consistent with those obtained by Timoshenko (reference 3).

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SYMBOLS

a, b	length and width of plate; a, shorter side of rectangular plate
h	thickness of plate
x, y, z	coordinates of a point in plate
u, v	horizontal displacements in x- and y-directions of points in middle surface; nondimensional forms are ua/h^2 and va/h^2, respectively
w	deflection of middle surface out of its initial plane; nondimensional form is w/h
p	normal load on plate per unit area; nondimensional form is pa^4/Eh^4
E, μ	Young's modulus and Poisson's ratio, respectively
D	flexural rigidity of plate $\left(\frac{Eh^3}{12(1-\mu^2)}\right)$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4}$$

σ_x', σ_y', τ_{xy}'	membrane stresses in middle surface; nondimensional forms are $\sigma_x'a^2/Eh^2$, $\sigma_y'a^2/Eh^2$, and $\tau_{xy}'a^2/Eh^2$, respectively
σ_x'', σ_y'', τ_{xy}''	extreme-fiber bending and shearing stresses; nondimensional forms are $\sigma_x''a^2/Eh^2$, $\sigma_y''a^2/Eh^2$, and $\tau_{xy}''a^2/Eh^2$, respectively
ϵ_x', ϵ_y', γ_{xy}'	membrane strains in middle surface; nondimensional forms are $\epsilon_x'a^2/h^2$, $\epsilon_y'a^2/h^2$, and $\gamma_{xy}'a^2/h^2$, respectively
ϵ_x'', ϵ_y'', γ_{xy}''	extreme-fiber bending and shearing strains; nondimensional forms are $\epsilon_x''a^2/h^2$, $\epsilon_y''a^2/h^2$, $\gamma_{xy}''a^2/h^2$, respectively
F	stress function; nondimensional form is F/Eh^2

$\Delta, \Delta^2, \ldots \Delta^n$ first-, second-, . . . and nth-order differences, respectively

Δ_x, Δ_y first-order differences in x- and y-directions, respectively

FUNDAMENTAL EQUATIONS

The deformation of a thin plate, the deflections of which are large in comparison with its thickness but are still small as compared with the other dimensions, is governed by Von Kármán's equations:

$$\left.\begin{aligned}
&\frac{\partial^4 F}{\partial x^4} + 2\frac{\partial^4 F}{\partial x^2\,\partial y^2} + \frac{\partial^4 F}{\partial y^4} = E\left[\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2 - \frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2}\right] \\
&\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} + \frac{h}{D}\left(\frac{\partial^2 F}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 F}{\partial y^2}\,\frac{\partial^2 w}{\partial x^2} - 2\,\frac{\partial^2 F}{\partial x\,\partial y}\,\frac{\partial^2 w}{\partial x\,\partial y}\right)
\end{aligned}\right\} \quad (1)$$

where $D = \dfrac{Eh^3}{12(1 - \mu^2)}$. The median-fiber stresses are

$$\left.\begin{aligned}
\sigma_x' &= \frac{\partial^2 F}{\partial y^2} \\
\sigma_y' &= \frac{\partial^2 F}{\partial x^2} \\
\tau_{xy}' &= -\frac{\partial^2 F}{\partial x\,\partial y}
\end{aligned}\right\} \quad (2)$$

and the median-fiber strains are

$$\left.\begin{aligned}
\epsilon_x' &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial y^2} - \mu\,\frac{\partial^2 F}{\partial x^2}\right) \\
\epsilon_y' &= \frac{1}{E}\left(\frac{\partial^2 F}{\partial x^2} - \mu\,\frac{\partial^2 F}{\partial y^2}\right) \\
\gamma_{xy}' &= -\frac{2(1+\mu)}{E}\,\frac{\partial^2 F}{\partial x\,\partial y}
\end{aligned}\right\} \tag{3}$$

The extreme-fiber bending and shearing stresses are

$$\left.\begin{aligned}
\sigma_x'' &= -\frac{Eh}{2(1-\mu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \mu\,\frac{\partial^2 w}{\partial y^2}\right) \\
\sigma_y'' &= -\frac{Eh}{2(1-\mu^2)}\left(\frac{\partial^2 w}{\partial y^2} + \mu\,\frac{\partial^2 w}{\partial x^2}\right) \\
\tau_{xy}'' &= -\frac{Eh}{2(1+\mu)}\,\frac{\partial^2 w}{\partial x\,\partial y}
\end{aligned}\right\} \tag{4}$$

For a riveted panel under normal pressure greater than that of the surrounding panels, the boundary conditions are formulated in reference 1 and are as follows:

$$\left.\begin{aligned}
w &= 0 \\
\frac{\partial^2 w}{\partial y^2} &= 0 \\
\frac{\partial^2 F}{\partial y^2} - \mu\,\frac{\partial^2 F}{\partial x^2} &= 0 \\
\text{along } y = \pm\frac{b}{2} \text{ and} & \\
\int_{y=0}^{y=\frac{b}{2}} \left[\frac{\partial^2 F}{\partial x^2} - \mu\,\frac{\partial^2 F}{\partial y^2} - \frac{E}{2}\left(\frac{\partial w}{\partial y}\right)^2\right] dy &= 0
\end{aligned}\right\} \tag{5}$$

along any line x = Constant.

And,

$$\left.\begin{aligned} w &= 0 \\ \frac{\partial^2 w}{\partial x^2} &= 0 \\ \frac{\partial^2 F}{\partial x^2} - \mu \frac{\partial^2 F}{\partial y^2} &= 0 \\ \text{along } x = \pm\frac{a}{2} \text{ and} \\ \int_{x=0}^{x=\frac{a}{2}} \left[\frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} - \frac{E}{2}\left(\frac{\partial w}{\partial x}\right)^2\right] dx &= 0 \end{aligned}\right\} \qquad (6)$$

along any line y = Constant.

These expressions can be made nondimensional by writing $\frac{F}{h^2E}$, $\frac{w}{h}$, $\frac{pa^4}{Eh^4}$, $\frac{\sigma}{E}\left(\frac{a}{h}\right)^2$, $\frac{x}{a}$, $\frac{y}{a}$, and $\epsilon\left(\frac{a}{h}\right)^2$ in place of F, w, p, σ, x, y, and ϵ, respectively, where a is the smaller side of the rectangular plate. (These latter symbols are used to effect a reduction in the amount of writing involved in the equations.) The resulting differential equations can then be transformed into finite-difference equations. In terms of finite differences, the differential equations (1) are replaced by the following equations:

$$\left.\begin{aligned} \Delta_x^4 F + 2\Delta_{xy}^2 F + \Delta_y^4 F &= \left[\left(\Delta_{xy} w\right)^2 - \Delta_x^2 w \Delta_y^2 w\right] \\ \Delta_x^4 w + 2\Delta_{xy}^2 w + \Delta_y^4 w &= 10.8(\Delta l)^4 p + 10.8\Big(\Delta_x^2 F \Delta_y^2 w + \Delta_y^2 F\, \Delta_x^2 w \\ &\qquad - 2\Delta_{xy}^2 F \Delta_{xy}^2 w\Big) \end{aligned}\right\} \qquad (7)$$

where $\Delta l = \Delta x = \Delta y$ and μ^2 is taken to be 0.1, which value is characteristic for aluminum alloys.

In terms of finite differences, the boundary conditions are

$$\left.\begin{aligned} w_{m,k} &= 0 \\ \left(\Delta_y{}^2 w\right)_{m,k} &= 0 \\ \left(\Delta_y{}^2 F - \mu\Delta_x{}^2 F\right)_{m,k} &= 0 \\ \text{along } y = \pm\frac{b}{2}, \text{ and} \\ \sum_{n=0}^{k-1}\left[\Delta_x{}^2 F - \mu\Delta_y{}^2 F - \frac{1}{2}\left(\Delta_y w\right)^2\right]_{i,n} &= 0 \end{aligned}\right\} \tag{8}$$

where $n = k$ denotes points along the edges $y = \pm\frac{b}{2}$, $n = 0$ denotes points along the center lines $y = 0$, and $m = i$ denotes any point along the line $x = \text{Constant}$ in the plate. Similarly, along $x = \pm\frac{a}{2}$, the boundary conditions are

$$\left.\begin{aligned} w_{k,n} &= 0 \\ \left(\Delta_x{}^2 w\right)_{k,n} &= 0 \\ \left(\Delta_x{}^2 F - \mu\Delta_y{}^2 F\right)_{k,n} &= 0 \\ \text{and} \\ \sum_{m=0}^{k-1}\left[\Delta_y{}^2 F - \mu\Delta_x{}^2 F - \frac{1}{2}\left(\Delta_x w\right)^2\right]_{m,i} &= 0 \end{aligned}\right\} \tag{9}$$

where $m = k$ denotes points along the edges $x = \pm\frac{a}{2}$, $m = 0$ denotes points along the center line $x = 0$ and $n = i$ denotes any point along the line $y = \text{Constant}$ in the plate.

The normal median-fiber stresses are

$$\left.\begin{aligned} \sigma'_x &= \frac{\Delta_y{}^2F}{(\Delta l)^2} \\ \sigma'_y &= \frac{\Delta_x{}^2F}{(\Delta l)^2} \end{aligned}\right\} \tag{10}$$

and the extreme-fiber bending stresses are

$$\left.\begin{aligned} \sigma_x'' &= -\frac{1}{2(1-\mu^2)(\Delta l)^2}\left(\Delta_x{}^2w + \mu\Delta_y{}^2w\right) \\ \sigma_y'' &= -\frac{1}{2(1-\mu^2)(\Delta l)^2}\left(\Delta_y{}^2w + \mu\Delta_x{}^2w\right) \end{aligned}\right\} \tag{11}$$

RECTANGULAR PLATE WITH LENGTH-WIDTH RATIO OF 1.5 UNDER UNIFORM NORMAL PRESSURE

The plate is divided into 24 square meshes. (See fig. 1.) The points 2', 5', 8', 9', 10', 11', and 11" are fictitious points outside the plate in order to make possible a better approximation to the boundary conditions. Since the plate is symmetrical with respect to the center lines, only one-quarter of the plate needs to be considered. In order to get satisfactory results in the subsequent computations, it is convenient to use a number of figures beyond those normally considered justifiable in view of the precision of the basic data.

With $\mu^2 = 0.1$ or $\mu = 0.316228$, the compatibility equations become

$$\left.\begin{aligned}
&20F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 = K_0\\
&-8F_0 + 21F_1 - 8F_2 + 4F_3 - 16F_4 + 4F_5 + 2F_7 + F_{2'} = K_1\\
&-8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 = K_3\\
&2F_0 - 8F_1 + 2F_2 - 8F_3 + 22F_4 - 8F_5 + 2F_6 - 8F_7 + 2F_8 + F_{10} + F_{5'} = K_4\\
&F_0 - 8F_3 + 4F_4 + 20F_6 - 16F_7 + 2F_8 - 8F_9 + 4F_{10} + F_{9'} = K_6\\
&F_1 + 2F_3 - 8F_4 + 2F_5 - 8F_6 + 21F_7 - 8F_8 + 2F_9 - 8F_{10} + 2F_{11} + F_{8'} + F_{10'} = K_7
\end{aligned}\right\} \tag{12}$$

where K_0, K_1, K_3, K_4, K_6, and K_7 are equal to $\left[\left(\Delta_{xy}w\right)^2 - \Delta_x{}^2w\Delta y^2w\right]$ at points 0, 1, 3, 4, 6, and 7, respectively.

The equilibrium equations are

$$\left.\begin{aligned}
&20w_0 - 16w_1 - 16w_3 + 2w_2 + 8w_4 + 2w_6 = p' + 10.8\Big[\alpha'_0\left(2w_3 - 2w_0\right)\\
&\quad + \beta'_0\left(2w_1 - 2w_0\right) - 2\gamma'_0\left(w_4 + w_0 - w_1 - w_3\right)\Big]\\
&21w_1 - 8w_0 - 8w_2 - 16w_4 + 4w_3 + 4w_5 + 2w_7 + w_{2'} = p' + 10.8\Big[\alpha'_1\left(2w_4 - 2w_1\right)\\
&\quad + \beta'_1\left(w_2 - 2w_1 + w_0\right) - 2\gamma'_1\left(w_5 + w_1 - w_2 - w_4\right)\Big]\\
&21w_3 - 8w_0 - 16w_4 - 8w_6 + 4w_1 + 2w_5 + 4w_7 + w_9 = p' + 10.8\Big[\alpha'_3\left(w_6 - 2w_3 + w_0\right)\\
&\quad + \beta'_3\left(2w_4 - 2w_3\right) - 2\gamma'_3\left(w_7 + w_3 - w_4 - w_6\right)\Big]\\
&22w_4 + 2w_0 - 8w_1 + 2w_2 - 8w_3 - 8w_5 - 8w_7 + 2w_6 + 2w_8 + w_{10} + w_{5'} = p'\\
&\quad + 10.8\Big[\alpha'_4\left(w_7 - 2w_4 + w_1\right) + \beta'_4\left(w_5 - 2w_4 + w_3\right) - 2\gamma'_4\left(w_8 + w_4 - w_5 - w_7\right)\Big]\\
&20w_6 + w_0 - 8w_3 + 4w_4 - 16w_7 - 8w_9 + 2w_8 + 4w_{10} + w_{9'} = p'\\
&\quad + 10.8\Big[\alpha'_6\left(w_3 - 2w_6 + w_9\right) + \beta'_6\left(2w_7 - 2w_6\right) - 2\gamma'_6\left(w_{10} + w_6 - w_7 - w_9\right)\Big]\\
&21w_7 + w_1 + 2w_3 - 8w_4 + 2w_5 - 8w_6 - 8w_8 + 2w_9 - 8w_{10} + 2w_{11} + w_{8'} + w_{10'} = p'\\
&\quad + 10.8\Big[\alpha'_7\left(w_{10} - 2w_7 + w_4\right) + \beta'_7\left(w_8 - 2w_7 + w_6\right) - 2\gamma'_7\left(w_{11} + w_7 - w_8 - w_{10}\right)\Big]
\end{aligned}\right\} \tag{13}$$

where $p' = 12(1 - \mu^2)(\Delta l)^4 p = 0.0421875p$ since $\Delta l = \frac{1}{4}$, and α', β', and γ' are $\Delta_x{}^2F$, $\Delta_y{}^2F$, and $\Delta_{xy}F$ at the points indicated by the subscripts, respectively.

The conditions for zero edge displacements are

$$\left.\begin{aligned}
&\left(2F_3 - 2F_0\right) - \mu\left(2F_1 - 2F_0\right) + \left(4F_4 - 4F_1\right) - \mu\left(2F_2 - 4F_1 + 2F_0\right) + \left(2F_5 - 2F_2\right) \\
&\qquad - \mu\left(F_1 - 2F_2 + F_{2'}\right) = S_1 \\
&\left(F_0 - 2F_3 + F_6\right) - \mu\left(2F_4 - 2F_3\right) + \left(2F_7 - 4F_4 + 2F_1\right) - \mu\left(2F_5 - 4F_4 + 2F_3\right) \\
&\qquad + \left(F_2 - 2F_5 + F_8\right) - \mu\left(F_4 - 2F_5 + F_{5'}\right) = S_2 \\
&\left(F_3 - 2F_6 + F_9\right) - \mu\left(2F_7 - 2F_6\right) + \left(2F_4 - 4F_7 + 2F_{10}\right) - \mu\left(2F_6 - 4F_7 + 2F_8\right) \\
&\qquad + \left(F_5 - 2F_8 + F_{11}\right) - \mu\left(F_{8'} - 2F_8 + F_7\right) = S_3 \\
&\left(2F_1 - 2F_0\right) - \mu\left(2F_3 - 2F_0\right) + \left(4F_4 - 4F_3\right) - \mu\left(2F_0 - 4F_3 + 2F_6\right) + \left(4F_7 - 4F_6\right) \\
&\qquad - \mu\left(2F_3 - 4F_6 + 2F_9\right) + \left(2F_{10} - 2F_9\right) - \mu\left(F_6 - 2F_9 + F_{9'}\right) = S_4 \\
&\left(F_2 - 2F_1 + F_0\right) - \mu\left(2F_4 - 2F_1\right) + \left(2F_5 - 4F_4 + 2F_3\right) - \mu\left(2F_1 - 4F_4 + 2F_7\right) \\
&\qquad + \left(2F_8 - 4F_7 + 2F_6\right) - \mu\left(2F_4 - 4F_7 + 2F_{10}\right) + \left(F_{11} - 2F_{10} + F_9\right) \\
&\qquad - \mu\left(F_7 - 2F_{10} + F_{10'}\right) = S_5
\end{aligned}\right\} \tag{14}$$

where

$$\begin{aligned}
S_1 &= \left(w_2 - w_1\right)^2 + \left(w_1 - w_0\right)^2 \\
S_2 &= \left(w_5 - w_4\right)^2 + \left(w_4 - w_3\right)^2 \\
S_3 &= \left(w_8 - w_7\right)^2 + \left(w_7 - w_6\right)^2 \\
S_4 &= \left(w_9 - w_6\right)^2 + \left(w_6 - w_3\right)^2 + \left(w_3 - w_0\right)^2 \\
S_5 &= \left(w_{10} - w_7\right)^2 + \left(w_7 - w_4\right)^2 + \left(w_4 - w_1\right)^2
\end{aligned}$$

The boundary conditions are

$$w_2 = 0,\ w_5 = 0,\ w_8 = 0,\ w_9 = 0,\ w_{10} = 0,\ w_{11} = 0$$

$$w_2' - 2w_2 + w_1 = 0$$
$$w_5' - 2w_5 + w_4 = 0$$
$$w_8' - 2w_8 + w_7 = 0$$
$$w_9' - 2w_9 + w_6 = 0$$
$$w_{10'} - 2w_{10} + w_7 = 0$$

$$F_2' - 2F_2 + F_1 - \mu\left(2F_5 - 2F_2\right) = 0$$
$$F_5' - 2F_5 + F_4 - \mu\left(F_8 - 2F_5 + F_2\right) = 0$$
$$F_8' - 2F_8 + F_7 - \mu\left(F_{11} - 2F_8 + F_5\right) = 0$$
$$F_9' - 2F_9 + F_6 - \mu\left(2F_{10} - 2F_9\right) = 0$$
$$F_{10'} - 2F_{10'} + F_7 - \mu\left(F_{11} - 2F_{10} + F_9\right) = 0$$

Now the boundary-value problem determines the values of w uniquely and the values of F to within an unknown constant. Since the actual value of the constant is irrelevant, it may be defined by letting $F_{11} = 0$.

On combining the boundary conditions with equations (12), (13), and (14), the final equations are

$$
\left.
\begin{aligned}
&20F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 = K_0 \\
&-8F_0 + 20F_1 - 6.632456F_2 + 4F_3 - 16F_4 + 4.632456F_5 + 2F_7 = K_1 \\
&-8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 = K_3 \\
&2F_0 - 8F_1 + 2.316228F_2 - 8F_3 + 21F_4 - 6.632456F_5 + 2F_6 - 8F_7 + 2.316228F_8 \\
&\qquad\qquad + F_{10} = K_4 \\
&F_0 - 8F_3 + 4F_4 + 19F_6 - 16F_7 + 2F_8 - 6.632456F_9 + 4.632456F_{10} = K_6 \\
&F_1 + 2F_3 - 8F_4 + 2.316228F_5 - 8F_6 + 19F_7 - 6.632456F_8 + 2.316228F_9 \\
&\qquad\qquad - 6.632456F_{10} = K_7 \\
&-2F_0 - 3.367544F_1 - 2.432456F_2 + 2F_3 + 4F_4 + 1.8F_5 = S_1 \\
&F_0 + 2F_1 + 0.9F_2 - 2F_3 - 3.367544F_4 - 2.432456F_5 + F_6 + 2F_7 + 0.9F_8 = S_2 \\
&F_3 + 2F_4 + 0.9F_5 - 2F_6 - 3.367544F_7 - 2.432456F_8 + F_9 + 2F_{10} + F_{11} = S_3 \\
&-2F_0 + 2F_1 - 4F_3 + 4F_4 - 3.367544F_6 + 4F_7 - 2.432456F_9 + 1.8F_{10} = S_4 \\
&F_0 - 2F_1 + F_2 + 2F_3 - 4F_4 + 2F_5 + 2F_6 - 3.367544F_7 + 2F_8 + 0.9F_9 \\
&\qquad\qquad - 2.432456F_{10} = S_5
\end{aligned}
\right\} \quad (15)
$$

and

$$
\left.
\begin{aligned}
&\left[20 + 21.6\left(\alpha'_0 + \beta'_0 + \gamma'_0\right)\right] w_0 - \left[16 + 21.6\left(\beta'_0 + \gamma'_0\right)\right] w_1 \\
&\quad - \left[16 + 21.6\left(\alpha'_0 + \gamma'_0\right)\right] w_3 + \left(8 + 21.6\gamma'_0\right) w_4 + 2w_6 = p' \\
&- \left(8 + 10.8\beta'_1\right) w_0 + \left[20 + 21.6\left(\alpha'_1 + \beta'_1 + \gamma'_1\right)\right] w_1 + 4w_3 \\
&\quad - \left[16 + 21.6\left(\alpha'_0 + \gamma'_0\right)\right] w_4 + 2w_7 = p' \\
&- \left(8 + 10.8\alpha'_3\right) w_0 + 4w_1 + \left[21 + 21.6\left(\alpha'_3 + \beta'_3 + \gamma'_3\right)\right] w_3 \\
&\quad - \left[16 + 21.6\left(\beta'_3 + \gamma'_3\right)\right] w_4 - \left[8 + 10.8\left(\alpha'_3 + 2\gamma'_3\right)\right] w_6 \\
&\quad + \left(4 + 21.6\gamma'_3\right) w_7 + w_9 = p' \\
&2w_0 - \left(8 + 10.8\alpha'_4\right) w_1 - \left(8 + 10.8\beta'_4\right) w_3 + \left[21 + 21.6\left(\alpha'_4 + \beta'_4 + \gamma'_4\right)\right] w_4 \\
&\qquad + 2w_6 - \left[8 + 10.8\left(\alpha'_4 + 2\gamma'_4\right)\right] w_7 = p' \\
&w_0 - \left(8 + 10.8\alpha'_6\right) w_3 + 4w_4 + \left[19 + 21.6\left(\alpha'_6 + \beta'_6 + \gamma'_6\right)\right] w_6 \\
&\qquad - \left[16 + 21.6\left(\beta'_6 + \gamma'_6\right)\right] w_7 = p' \\
&w_1 + 2w_3 - \left(8 + 10.8\alpha'_7\right) w_4 - \left(8 + 10.8\beta'_7\right) w_6 \\
&\qquad + \left[19 + 21.6\left(\alpha'_7 + \beta'_7 + \gamma'_7\right)\right] w_7 = p'
\end{aligned}
\right\} \quad (16)
$$

By following the procedures outlined in reference 1, equations (15) and (16) can be solved simultaneously for w and F by the method of successive approximations.

RECTANGULAR PLATE WITH LENGTH-WIDTH RATIO OF 2 UNDER UNIFORM NORMAL PRESSURE

The plate is first divided into 8 square meshes and then into 32 square meshes. (See figs. 2 and 3, respectively.) On referring first to figure 2, points 1', 3', 4', 5', and 5" are fictitious points outside the plate in order to give a better approximation to the boundary conditions. Consider one-quarter of the plate; the compatibility equations are

$$
\left.
\begin{aligned}
&20F_0 - 16F_1 - 16F_2 + 8F_3 + 2F_4 + 2F_1' = K_0 \\
&20F_2 - 8F_0 - 16F_3 - 8F_4 + 4F_1 + 4F_5 + F_2 + 2F_3 + F_4' = K_1
\end{aligned}
\right\} \quad (17)
$$

where K_0 and K_1 are equal to $\left[\left(\Delta_{xy}w\right)^2 - \Delta_x{}^2w\Delta_y{}^2w\right]$ at points 0 and 1, respectively.

The equilibrium equations are

$$\left.\begin{aligned}
&20w_0 - 16w_1 - 16w_2 + 8w_3 + 2w_{11} + 2w_4 = p' + 10.8\left[\alpha'_0\left(2w_2 - 2w_0\right)\right.\\
&\qquad\left. + \beta'_0\left(2w_1 - 2w_0\right) - 2\gamma'_0\left(w_3 + w_0 - w_1 - w_2\right)\right]\\
&20w_2 - 8w_0 - 16w_3 - 8w_4 + 4w_1 + 4w_5 + w_2 + 2w_{3'} + w_{4'}\\
&\quad = p' + 10.8\left[\alpha'_2\left(w_0 - 2w_2 + w_4\right) + \beta'_2\left(2w_3 - 2w_2\right)\right.\\
&\qquad\qquad\left. - 2\gamma'_2\left(w_5 + w_2 - w_3 - w_4\right)\right]
\end{aligned}\right\} \quad (18)$$

where $p' = 12(1 - \mu^2)(\Delta l)^4 = 0.675p$ since $\Delta l = \frac{1}{2}$, and α', β', and γ' are $\Delta_x{}^2F$, $\Delta_y{}^2F$, and $\Delta_{xy}F$ at the points indicated by the subscripts, respectively.

The conditions for zero edge displacements are

$$\left.\begin{aligned}
&\left(2F_2 - 2F_0\right) - \mu\left(2F_1 - 2F_0\right) + \left(2F_3 - 2F_1\right) - \mu\left(F_0 - 2F_1 + F_{1'}\right) = w_0{}^2\\
&\left(F_0 - 2F_2 + F_4\right) - \mu\left(2F_3 - 2F_2\right) + \left(F_1 - 2F_3 + F_5\right) - \mu\left(F_2 - 2F_3 + F_{3'}\right) = w_2{}^2\\
&\left(2F_1 - 2F_0\right) - \mu\left(2F_2 - 2F_0\right) - 2\mu\left(F_0 - 2F_2 + F_4\right) + 2\left(2F_3 - 2F_2\right)\\
&\qquad - \mu\left(F_2 - 2F_4 + F_{4'}\right) + \left(2F_5 - 2F_4\right) = \left(w_2 - w_0\right)^2 + w_2{}^2
\end{aligned}\right\} \quad (19$$

The boundary conditions are

$$w_1 = 0,\ w_3 = 0,\ w_4 = 0,\ w_5 = 0$$

$$w_1' + w_0 = 0$$

$$w_3' + w_2 = 0$$

$$w_4' + w_2 = 0$$

$$F_1' - 2F_1 + F_0 - \mu\left(2F_3 - 2F_1\right) = 0$$

$$F_3' - 2F_3 + F_2 - \mu\left(F_1 - 2F_3 + F_5\right) = 0$$

$$F_4' - 2F_4 + F_2 - \mu\left(2F_5 - 2F_4\right) = 0$$

The problem can now be solved uniquely for the values of w and the values of F to within an unknown constant. As the actual value of the constant is irrelevant, it may be defined by letting $F_5 = 0$.

Combined with the boundary conditions, equations (17), (18), and (19) are then

$$\left.\begin{aligned}
&18F_0 - 13.264912F_1 - 16F_2 + 9.264912F_3 + 2F_4 = K_0 \\
&-8F_0 + 4.632456F_1 + 18F_2 - 13.264912F_3 - 6.632456F_4 = K_1 \\
&-1.367544F_0 - 2.532456F_1 + 2F_2 + 1.8F_3 = w_0^2 \\
&F_0 + 0.9F_1 - 1.367544F_2 - 2.532456F_3 + F_4 = w_2^2 \\
&-2F_0 + 2F_1 - 3.367544F_2 + 4F_3 - 2.532456F_4 = \left(w_2 - w_0\right)^2 + w_2^2
\end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned}
&\left[18 + 21.6\left(\alpha'_0 + \beta'_0 + \gamma'_0\right)\right]w_0 - \left[16 + 21.6\left(\alpha'_0 + \gamma'_0\right)\right]w_2 = p' \\
&-\left(8 + 10.8\alpha'_2\right)w_0 + \left[18 + 21.6\left(\alpha'_2 + \beta'_2 + \gamma'_2\right)\right]w_2 = p'
\end{aligned}\right\} \quad (21)$$

Equations (20) and (21) can now be solved simultaneously for w and F by the method of successive approximations.

With the values of w thus computed as a first approximation, the case wherein the plate is divided into 32 square meshes is now to be studied. On referring to figure 3, points 2', 5', 8', 11', 12', 13' 14', and 14" are again fictitious points outside the plate in order to give a better approximation to the boundary conditions. Consider one-quarter of the plate; the compatibility equations are

$$\left.\begin{aligned}
&20F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 = K_0 \\
&-8F_0 + 21F_1 - 8F_2 + 4F_3 - 16F_4 + 4F_5 + 2F_7 + F_{2'} = K_1 \\
&-8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 = K_3 \\
&2F_0 - 8F_1 + 2F_2 - 8F_3 + 22F_4 - 8F_5 + 2F_6 - 8F_7 + 2F_8 + F_{10} + F_{5'} = K_4 \\
&F_0 - 8F_3 + 4F_4 + 20F_6 - 16F_7 + 2F_8 - 8F_9 + 4F_{10} + F_{12} = K_6 \\
&F_1 + 2F_3 - 8F_4 + 2F_5 - 8F_6 + 21F_7 - 8F_8 + 2F_9 - 8F_{10} + 2F_{11} + F_{13} + F_{8'} = K_7 \\
&F_3 - 8F_6 + 4F_7 + 20F_9 - 16F_{10} + 2F_{11} - 8F_{12} + 4F_{13} + F_{12'} = K_9 \\
&F_4 + 2F_6 - 8F_7 + 2F_8 - 8F_9 + 21F_{10} - 8F_{11} + 2F_{12} - 8F_{13} + 2F_{14} + F_{11'} + F_{13'} = K_{10}
\end{aligned}\right\} \tag{22}$$

where $K = \left[\left(\Delta_{xy}w\right)^2 - \Delta_x{}^2 w \Delta_y{}^2 w\right]$ at the points indicated by the subscripts.

The equilibrium equations are

$$
\left.
\begin{aligned}
&20w_0 - 16w_1 - 16w_3 + 8w_4 + 2w_2 + 2w_6 = p' + 10.8\Big[\alpha'_0\left(2w_3 - 2w_0\right) \\
&\qquad + \beta'_0\left(2w_1 - 2w_0\right) - 2\gamma'_0\left(w_4 - w_1 - w_3 + w_0\right)\Big] \\
&-8w_0 + 21w_1 - 8w_2 + 4w_3 - 16w_4 + 4w_5 + 2w_7 + w_{2'} = p' + 10.8\Big[\alpha'_1\left(2w_4 - 2w_1\right) \\
&\qquad + \beta'_1\left(w_2 - 2w_1 + w_0\right) - 2\gamma'_1\left(w_5 - w_2 - w_4 + w_1\right)\Big] \\
&-8w_0 + 4w_1 + 21w_3 - 16w_4 + 2w_5 - 8w_6 + 4w_7 + w_9 = p' + 10.8\Big[\alpha'_3\left(w_0 - 2w_3 + w_6\right) \\
&\qquad + \beta'_3\left(2w_4 - 2w_3\right) - 2\gamma'_3\left(w_7 - w_4 - w_6 + w_3\right)\Big] \\
&2w_0 - 8w_1 + 2w_2 - 8w_3 + 22w_4 - 8w_5 + 2w_6 - 8w_7 + 2w_8 + w_{10} \\
&\qquad + w_{5'} = p' + 10.8\Big[\alpha'_4\left(w_7 - 2w_4 + w_1\right) + \beta'_4\left(w_5 - 2w_4 + w_3\right) \\
&\qquad\qquad -2\gamma'_4\left(w_8 - w_5 - w_7 + w_4\right)\Big] \\
&w_0 - 8w_3 + 4w_4 + 20w_6 - 16w_7 + 2w_8 - 8w_9 + 4w_{10} + w_{12} \\
&\qquad = p' + 10.8\Big[\alpha'_6\left(w_3 - 2w_6 + w_9\right) + \beta'_6\left(2w_7 - 2w_6\right) \\
&\qquad\qquad -2\gamma'_6\left(w_{10} - w_7 - w_9 + w_6\right)\Big] \\
&w_1 + 2w_3 - 8w_4 + 2w_5 - 8w_6 + 21w_7 - 8w_8 + 2w_9 - 8w_{10} \\
&\qquad + 2w_{11} + w_{13} + w_{8'} = p' + 10.8\Big[\alpha'_7\left(w_4 - 2w_7 + w_{10}\right) \\
&\qquad + \beta'_7\left(w_8 - 2w_7 + w_6\right) - 2\gamma'_7\left(w_{11} - w_8 - w_{10} + w_7\right)\Big] \\
&w_3 - 8w_6 + 4w_7 + 20w_9 - 16w_{10} + 2w_{11} - 8w_{12} + 4w_{13} + w_{12'} = p' \\
&\quad + 10.8\Big[\alpha'_9\left(w_6 - 2w_9 + w_{12}\right) - 2\gamma'_9\left(w_{13} - w_{10} - w_{12} + w_9\right)\Big] \\
&w_4 + 2w_6 - 8w_7 + 2w_8 - 8w_9 + 21w_{10} - 8w_{11} + 2w_{12} - 8w_{13} + 2w_{14} + w_{11'} + w_{13'} \\
&= p' + 10.8\Big[\alpha'_{10}\left(w_7 - 2w_{10} + w_{13}\right) + \beta'_{10}\left(w_{11} - 2w_{10} + w_9\right) \\
&\qquad\qquad - 2\gamma'_{10}\left(w_{14} - w_{11} - w_{13} + w_{10}\right)\Big]
\end{aligned}
\right\}
\tag{23}
$$

where $p' = 12\left(1 - \mu^2\right)(\Delta l)^4 p = 0.0421875p$ since $\Delta l = \frac{1}{4}$, and α', β', and γ' are $\Delta_x{}^2F$, $\Delta_y{}^2F$, and $\Delta_{xy}F$ at the points indicated by the subscripts, respectively.

The conditions for zero edge displacements are

$$
\left.
\begin{aligned}
&2F_3 - 2F_0 - \mu\left(2F_1 - 2F_0\right) + 4F_4 - 4F_1 - 2\mu\left(F_2 - 2F_1 + F_0\right) + 2F_5 - 2F_2 \\
&\qquad - \mu\left(F_{2'} - 2F_2 + F_1\right) = S_1 \\
&F_0 - 2F_3 + F_6 - \mu\left(2F_4 - 2F_3\right) + 2F_1 - 4F_4 + 2F_7 - 2\mu\left(F_3 - 2F_4 + F_5\right) \\
&\qquad + \left(F_2 - 2F_5 + F_8\right) - \mu\left(F_{5'} - 2F_5 + F_4\right) = S_2 \\
&F_3 - 2F_6 + F_9 - \mu\left(2F_7 - 2F_6\right) + 2F_4 - 4F_7 + 2F_{10} - 2\mu\left(F_6 - 2F_7 + F_8\right) \\
&\qquad + \left(F_5 - 2F_8 + F_{11}\right) - \mu\left(F_7 - 2F_8 + F_{8'}\right) = S_3 \\
&F_6 - 2F_9 + F_{12} - \mu\left(2F_{10} - 2F_9\right) + 2F_7 - 4F_{10} + 2F_{13} - 2\mu\left(F_9 - 2F_{10} + F_{11}\right) \\
&\qquad + F_8 - 2F_{11} + F_{14} - \mu\left(F_{10} - 2F_{11} + F_{11'}\right) = S_4 \\
&2F_1 - 2F_0 - \mu\left(2F_3 - 2F_0\right) + 4F_4 - 4F_3 - 2\mu\left(F_0 - 2F_3 + F_6\right) + 4F_7 - 4F_6 \\
&\quad - 2\mu\left(F_3 - 2F_6 + F_9\right) + 4F_{10} - 4F_9 - 2\mu\left(F_6 - 2F_9 + F_{12}\right) + 2F_{13} - 2F_{12} \\
&\qquad - \mu\left(F_9 - 2F_{12} + F_{12'}\right) = S_5 \\
&F_2 - 2F_1 + F_0 - \mu\left(2F_4 - 2F_1\right) + 2F_5 - 4F_4 + 2F_3 - 2\mu\left(F_1 - 2F_4 + F_7\right) + 2F_8 - 4F_7 \\
&\quad + 2F_6 - 2\mu\left(F_4 - 2F_7 + F_{10}\right) + 2F_{11} - 4F_{10} + 2F_9 - 2\mu\left(F_7 - 2F_{10} + F_{13}\right) \\
&\qquad + F_{14} - 2F_{13} + F_{12} - \mu\left(F_{10} - 2F_{13} + F_{13'}\right) = S_6
\end{aligned}
\right\} \tag{24}
$$

where $S_i = \sum_{m=0}^{k-1} \left(w_{m+1,n} - w_{m,n}\right)^2$ for $i = 1, 2, 3, 4$

and $S_j = \sum_{n=0}^{k-1} \left(w_{m,n+1} - w_{m,n}\right)^2$ for $j = 5, 6$

The boundary conditions are

$$w_2 = 0,\ w_5 = 0,\ w_8 = 0,\ w_{11} = 0,\ w_{12} = 0,\ w_{13} = 0,\ w_{14} = 0$$

$$w_{2'} - 2w_2 + w_1 = 0$$

$$w_{5'} - 2w_5 + w_4 = 0$$

$$w_{8'} - 2w_8 + w_7 = 0$$

$$w_{11'} - 2w_{11} + w_{10} = 0$$

$$w_{12'} - 2w_{12} + w_9 = 0$$

$$w_{13'} - 2w_{13} + w_{10} = 0$$

$$F_1 - 2F_2 + F_{2'} - \mu\left(2F_5 - 2F_2\right) = 0$$

$$F_4 - 2F_5 + F_{5'} - \mu\left(F_2 - 2F_5 + F_8\right) = 0$$

$$F_7 - 2F_8 + F_{8'} - \mu\left(F_5 - 2F_8 + F_{11}\right) = 0$$

$$F_{10} - 2F_{11} + F_{11'} - \mu\left(F_8 - 2F_{11} + F_{14}\right) = 0$$

$$F_9 - 2F_{12} + F_{12'} - \mu\left(2F_{13} - 2F_{12}\right) = 0$$

$$F_{10} - 2F_{13} + F_{13'} - \mu\left(F_{14} - 2F_{13} + F_{12}\right) = 0$$

The problem can now be solved uniquely again for the values for w and the values of f to within an unknown constant. As the actual value of the constant is irrelevant, it may be defined by letting $F_{14} = 0$.

On combining with the boundary conditions, equations (22), (23), and (24) are then

$$
\left.
\begin{aligned}
&20F_0 - 16F_1 + 2F_2 - 16F_3 + 8F_4 + 2F_6 = K_0 \\
&-8F_0 + 20F_1 - 6.632456F_2 + 4F_3 - 16F_4 + 4.632456F_5 + 2F_7 = K_1 \\
&-8F_0 + 4F_1 + 21F_3 - 16F_4 + 2F_5 - 8F_6 + 4F_7 + F_9 = K_3 \\
&2F_0 - 8F_1 + 2.316228F_2 - 8F_3 + 21F_4 - 6.632456F_5 + 2F_6 - 8F_7 + 2.316228F_8 \\
&\qquad + F_{10} = K_4 \\
&F_0 - 8F_3 + 4F_4 + 20F_6 - 16F_7 + 2F_8 - 8F_9 + 4F_{10} + F_{12} = K_6 \\
&F_1 + 2F_3 - 8F_4 + 2.316228F_5 - 8F_6 + 20F_7 - 6.632456F_8 + 2F_9 - 8F_{10} \\
&\qquad + 2.316228F_{11} + F_{13} = K_7 \\
&F_3 - 8F_6 + 4F_7 + 19F_9 - 16F_{10} + 2F_{11} - 6.632456F_{12} + 4.632456F_{13} = K_9 \\
&F_4 + 2F_6 - 8F_7 + 2.316228F_8 - 8F_9 + 19F_{10} - 6.632456F_{11} + 2.316228F_{12} \\
&\qquad -6.632456F_{13} = K_{10} \\
&-2F_0 - 3.367544F_1 - 2.432456F_2 + 2F_3 + 4F_4 + 1.8F_5 = S_1 \\
&F_0 + 2F_1 + 0.9F_2 - 2F_3 - 3.367456F_4 - 2.432456F_5 + F_6 + 2F_7 + 0.9F_8 = S_2 \\
&F_3 + 2F_4 + 0.9F_5 - 2F_6 - 3.367456F_7 - 2.432456F_8 + F_9 + 2F_{10} + 0.9F_{11} = S_3 \\
&F_6 + 2F_7 + 0.9F_8 - 2F_9 - 3.367456F_{10} - 2.432456F_{11} + F_{12} + 2F_{13} = S_4 \\
&-2F_0 + 2F_1 - 4F_3 + 4F_4 - 4F_6 + 4F_7 - 3.367544F_9 + 4F_{10} - 2.432456F_{12} \\
&\qquad + 1.8F_{13} = S_5 \\
&F_0 - 2F_1 + F_2 + 2F_3 - 4F_4 + 2F_5 + 2F_6 - 4F_7 + 2F_8 + 2F_9 - 3.367544F_{10} \\
&\qquad + 2F_{11} + 0.9F_{12} - 2.432544F_{13} = S_6
\end{aligned}
\right\} \quad (25)
$$

and

$$
\left.
\begin{aligned}
&\left[20 + 21.6\left(\alpha'_0 + \beta'_0 + \gamma'_0\right)\right] w_0 - \left[16 + 21.6\left(\beta'_0 - \gamma'_0\right)\right] w_1 \\
&\quad - \left[16 + 21.6\left(\alpha'_0 + \gamma'_0\right)\right] w_3 + \left(8 + 10.8\gamma'_0\right) w_4 + 2w_6 = p' \\
&- \left(8 + 10.8\beta'_1\right) w_0 + \left[20 + 21.6\left(\alpha'_1 + \beta'_1 + \gamma'_1\right)\right] w_1 + 4w_3 \\
&\quad - \left[16 + 21.6\left(\alpha'_1 + \gamma'_1\right)\right] w_4 + 2w_7 = p' \\
&-\left(8 + 10.8\alpha'_3\right) w_0 + 4w_1 + \left[21 + 21.6\left(\alpha'_3 + \beta'_3 + \gamma'_3\right)\right] w_3 \\
&\quad - \left[16 + 21.6\left(\beta'_3 + \gamma'_3\right)\right] w_4 - \left[8 + 10.8\left(\alpha'_3 + 2\gamma'_3\right)\right] w_6 \\
&\qquad + \left(4 + 21.6\gamma'_3\right) w_7 + w_9 = p' \\
&2w_0 - \left(8 + 10.8\alpha'_4\right) w_1 - \left(8 + 10.8\beta'_4\right) w_3 + \left[21 + 21.6\left(\alpha'_4 + \beta'_4 + \gamma'_4\right)\right] w_4 \\
&\qquad + 2w_6 - \left[8 + 10.8\left(\alpha'_4 + 2\gamma'_4\right)\right] w_7 + w_{10} = p' \\
&w_0 - \left(8 + 10.8\alpha'_6\right) w_3 + 4w_4 + \left[20 + 21.6\left(\alpha'_4 + \beta'_4 + \gamma'_4\right)\right] w_6 \\
&\quad - \left[16 + 21.6\left(\beta'_6 + \gamma'_6\right)\right] w_7 - \left[8 + 10.8\left(\alpha'_6 + 2\gamma'_6\right)\right] w_9 \\
&\qquad + \left(4 + 21.6\gamma'_6\right) w_{10} = p' \\
&w_1 + 2w_3 - \left(8 + 10.8\alpha'_7\right) w_4 - \left(8 + 10.8\beta'_7\right) w_6 + \left[20 + 21.6\left(\alpha'_7 + \beta'_7 + \gamma'_7\right)\right] w_7 \\
&\qquad + 2w_9 - \left[8 + 10.8\left(\alpha'_7 + 2\gamma'_7\right)\right] w_{10} = p' \\
&w_3 - \left(8 + 10.8\alpha'_9\right) w_6 + 4w_7 + \left[19 + 21.6\left(\alpha'_9 + \beta'_9 + \gamma'_9\right)\right] w_9 \\
&\quad - \left[16 + 21.6\left(\beta'_0 + \gamma'_0\right)\right] w_{10} = p' \\
&w_4 + 2w_6 - \left(8 + 10.8\alpha'_{10}\right) w_7 - \left(8 + 10.8\beta'_{10}\right) w_9 \\
&\qquad + \left[19 + 21.6\left(\alpha'_{10} + \beta'_{10} + \gamma'_{10}\right)\right] w_{10} = p'
\end{aligned}
\right\} \quad (26)
$$

By the method of successive approximations, equations (25) and (26) can be solved simultaneously for values of w and F.

RESULTS AND DISCUSSION

The bending problem of rectangular plates under uniform normal pressure with simply supported edges is studied by finite-difference approximations. The difference equations are solved by the method of successive approximation. The case of a square plate is studied in reference 1. In the present report, the computations are extended to rectangular plates with ratios of length to width of 1.5 or 2. Experimental work (reference 2) indicates that a rectangular plate having a length equal to twice its width may be regarded practically as an infinitely long plate. The maximum normal pressure calculated is $\frac{pa^4}{Eh^4} = 250$, which gives the center deflections about twice the thickness of the plate.

The deflections at various points in the plate under different pressure ratios are tabulated in tables 1, 2, and 3. The center deflections are plotted against the normal pressure ratio in figures 4 and 5. The membrane stresses in the centers of the plates and at the centers of the edges are tabulated in tables 4, 5, and 6 and are plotted in figures 6 and 7. The bending and total stresses at the centers of the plates are tabulated in tables 7 and 8 and are plotted in figures 8 and 9.

As indicated in reference 1, the plate should be designed for strength on the basis of clamped edges and for maximum deflection or washboarding according to the boundary conditions such as formulated in the present report. The finite-difference equations give the values of deflections with good approximations, whereas the stresses, which are obtained by a second-order numerical differentiation, are always less accurate.

Brown University
Providence, R. I.
November 4, 1946

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2. Ramberg, Walter, McPherson, Albert E., and Levy, Samuel: Normal-Pressure Tests of Rectangular Plates. NACA TN No. 849, 1942.

3. Timoshenko, S.: Theory of Plates and Shells. McGraw-Hill Book Co., Inc., 1940, pp. 113-133.

TABLE 1

CENTER DEFLECTIONS

$\frac{pa^4}{Eh^4}$	w_0/h		
	$\frac{b}{a} = 1.5;\ \Delta l = \frac{a}{4}$	$\frac{b}{a} = 2;\ \Delta l = \frac{a}{2}$	$\frac{b}{a} = 2;\ \Delta l = \frac{a}{4}$
0	0	0	0
12.5	.6249	.7322	.6958
25	.8790	1.002	.9463
50	1.175	1.323	1.241
75	1.374	1.542	1.440
100	1.528	1.713	1.596
150	1.769	1.982	1.840
200	1.957	2.194	2.033
250	2.115	2.372	2.196

TABLE 2

DEFLECTION AT VARIOUS POINTS

$$\left[\frac{b}{a} = 1.5; \quad \Delta l = \frac{a}{4}\right]$$

$\frac{pa^4}{Eh^4}$	$\frac{w_0}{h}$	$\frac{w_1}{h}$	$\frac{w_3}{h}$	$\frac{w_4}{h}$	$\frac{w_6}{h}$	$\frac{w_7}{h}$
0	0	0	0	0	0	0
12.5	.6249	.4526	.5627	.4077	.3563	.2582
25	.8790	.6398	.7993	.5818	.5180	.3750
50	1.175	.8606	1.080	.7902	.7168	.5183
75	1.374	1.009	1.269	.9314	.8537	.6167
100	1.528	1.125	1.416	1.042	.9613	.6942
150	1.769	1.306	1.646	1.215	1.130	.8140
200	1.957	1.449	1.827	1.350	1.263	.9091
250	2.115	1.569	1.977	1.465	1.375	.9880

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TABLE 3

DEFLECTIONS AT VARIOUS POINTS

$$\left[\frac{b}{a} = 2; \quad \Delta l = \frac{a}{4}\right]$$

$\frac{pa^4}{Eh^4}$	$\frac{w_0}{h}$	$\frac{w_1}{h}$	$\frac{w_3}{h}$	$\frac{w_4}{h}$	$\frac{w_6}{h}$	$\frac{w_7}{h}$	$\frac{w_9}{h}$	$\frac{w_{10}}{h}$
0	0	0	0	0	0	0	0	0
12.5	.6958	.5025	.6707	.4847	.5749	.4159	.3585	.2590
25	.9463	.6868	.9201	.6682	.8074	.5867	.5184	.3745
50	1.241	.9055	1.215	.8870	1.087	.7940	.7179	.5179
75	1.440	1.054	1.414	1.036	1.277	.9354	.8558	.6164
100	1.596	1.171	1.569	1.152	1.426	1.046	.9646	.6938
150	1.840	1.355	1.813	1.335	1.658	1.220	1.135	.8146
200	2.033	1.500	2.004	1.479	1.840	1.356	1.269	.9097
250	2.196	1.621	2.166	1.600	1.994	1.471	1.383	.9889

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TABLE 4[a]

MEMBRANE STRESSES

$\left[\frac{b}{a} = 1.5; \quad \Delta l = \frac{a}{4}\right]$

$\frac{pa^4}{Eh^4}$	$\frac{\sigma'_{x0}a^2}{Eh^2}$	$\frac{\sigma'_{y0}a^2}{Eh^2}$	$\frac{\sigma'_{x1}a^2}{Eh^2}$	$\frac{\sigma'_{y1}a^2}{Eh^2}$	$\frac{\sigma'_{x2}a^2}{Eh^2}$	$\frac{\sigma'_{y2}a^2}{Eh^2}$
0	0	0	0	0	0	0
12.5	1.064	.6050	1.072	.3389	.2316	.7325
25	2.109	1.213	2.150	.6799	.4775	1.510
50	3.783	2.202	3.903	1.234	.8930	2.824
75	5.176	3.039	5.381	1.702	1.251	3.955
100	6.410	3.784	6.694	2.117	1.574	4.977
150	8.605	5.110	9.037	2.858	2.154	6.811
200	10.55	6.292	11.13	3.518	2.673	8.452
250	12.34	7.375	13.04	4.125	3.153	9.972

[a]Subscript 0 denotes the center of the plate; subscript 1 denotes the center of the sides $x = \pm \frac{a}{2}$; and subscript 2 denotes the center of the sides $y = \pm \frac{b}{2}$.

TABLE 5[a]

MEMBRANE STRESSES

$\left[\frac{b}{a} = 2; \quad \Delta l = \frac{a}{2}\right]$

$\frac{pa^4}{Eh^4}$	$\frac{\sigma'_{x0}a^2}{Eh^2}$	$\frac{\sigma'_{y0}a^2}{Eh^2}$	$\frac{\sigma'_{x1}a^2}{Eh^2}$	$\frac{\sigma'_{y1}a^2}{Eh^2}$	$\frac{\sigma'_{x2}a^2}{Eh^2}$	$\frac{\sigma'_{y2}a^2}{Eh^2}$
0	0	0	0	0	0	0
12.5	1.149	.6330	1.115	.3525	.1897	.5998
25	2.132	1.201	2.103	.6652	.3751	1.186
50	3.692	2.119	3.690	1.167	.6821	2.157
75	4.995	2.893	5.024	1.589	.9437	2.984
100	6.156	3.585	6.215	1.965	1.178	3.726
150	8.218	4.816	8.332	2.635	1.597	5.049
200	10.06	5.916	10.22	3.233	1.971	6.234
250	11.75	6.928	11.96	3.782	2.316	7.324

[a]Subscript 0 denotes the center of the plate; subscript 1 denotes the center of the sides $x = \pm \frac{a}{2}$; and subscript 2 denotes the center of the sides $y = \pm \frac{b}{2}$.

TABLE 6[a]

MEMBRANE STRESSES

$$\left[\frac{b}{a} = 2; \quad \Delta l = \frac{a}{4}\right]$$

$\frac{pa^4}{Eh^4}$	$\frac{\sigma'_{x0}a^2}{Eh^2}$	$\frac{\sigma'_{y0}a^2}{Eh^2}$	$\frac{\sigma'_{x1}a^2}{Eh^2}$	$\frac{\sigma'_{y1}a^2}{Eh^2}$	$\frac{\sigma'_{x2}a^2}{Eh^2}$	$\frac{\sigma'_{y2}a^2}{Eh^2}$
0	0	0	0	0	0	0
12.5	1.293	.5421	1.298	.4106	.2372	.7500
25	2.403	1.035	2.441	.7718	.4718	1.492
50	4.155	1.827	4.264	1.349	.8730	2.761
75	5.612	2.496	5.788	1.830	1.220	3.859
100	6.911	3.094	7.149	2.261	1.534	4.852
150	9.220	4.160	9.570	3.026	2.100	6.640
200	11.28	5.111	11.72	3.707	2.606	8.240
250	13.16	5.989	13.70	4.334	3.076	9.728

[a]Subscript 0 denotes the center of the plate; subscript 1 denotes the center of the sides $x = \pm \frac{a}{2}$; and subscript 2 denotes the center of the sides $y = \pm \frac{b}{2}$.

TABLE 7[a]

EXTREME-FIBER BENDING AND TOTAL

STRESSES AT CENTER OF PLATE

$\left[\frac{b}{a} = 1.5; \quad \Delta l = \frac{a}{4}\right]$

$\frac{pa^4}{Eh^4}$	$\frac{\sigma''_{xO}a}{Eh^2}$	$\frac{\sigma''_{yO}a}{Eh^2}$	$\frac{\sigma''_{xO}a}{Eh^2}$	$\frac{\sigma''_{yO}a}{Eh^2}$
0	0	0	0	0
12.5	3.413	2.075	4.477	2.680
25	4.700	2.761	6.809	3.974
50	6.137	3.476	9.920	5.679
75	7.072	3.914	12.25	6.953
100	7.813	4.260	14.22	8.044
150	8.909	4.777	17.51	9.887
200	9.770	5.177	20.32	11.47
250	10.49	5.519	22.83	12.89

[a]Subscript O denotes the center of the plate.

TABLE 8[a]

EXTREME-FIBER BENDING AND TOTAL

STRESSES AT CENTER OF PLATE

$\left[\frac{b}{a} = 2\right]$

$\frac{pa^4}{Eh^4}$	$\frac{\sigma''_{x0}a^2}{Eh^2}$		$\frac{\sigma''_{y0}a^2}{Eh^2}$		$\frac{\sigma_{x0}a^2}{Eh^2}$		$\frac{\sigma_{y0}a^2}{Eh^2}$	
	$\Delta l = \frac{a}{2}$	$\Delta l = \frac{a}{4}$	$\Delta l = \frac{a}{2}$	$\Delta l = \frac{a}{4}$	$\Delta l = \frac{a}{2}$	$\Delta l = \frac{a}{4}$	$\Delta l = \frac{a}{2}$	$\Delta l = \frac{a}{4}$
0	0	0	0	0	0	0	0	0
12.5	3.431	3.577	1.588	1.532	4.580	4.871	2.221	2.074
25	4.662	4.761	2.069	1.925	6.795	7.164	3.271	2.960
50	6.122	6.108	2.625	2.349	9.814	10.26	4.745	4.176
75	7.116	7.005	3.003	2.637	12.11	12.62	5.896	5.133
100	7.895	7.703	3.300	2.864	14.05	14.61	6.885	5.957
150	9.118	8.792	3.767	3.224	17.34	18.01	8.583	7.385
200	10.08	9.651	4.138	3.519	20.14	20.93	10.05	8.630
250	10.90	10.39	4.452	3.766	22.64	23.55	11.38	9.755

[a]Subscript 0 denotes the center of the plate.

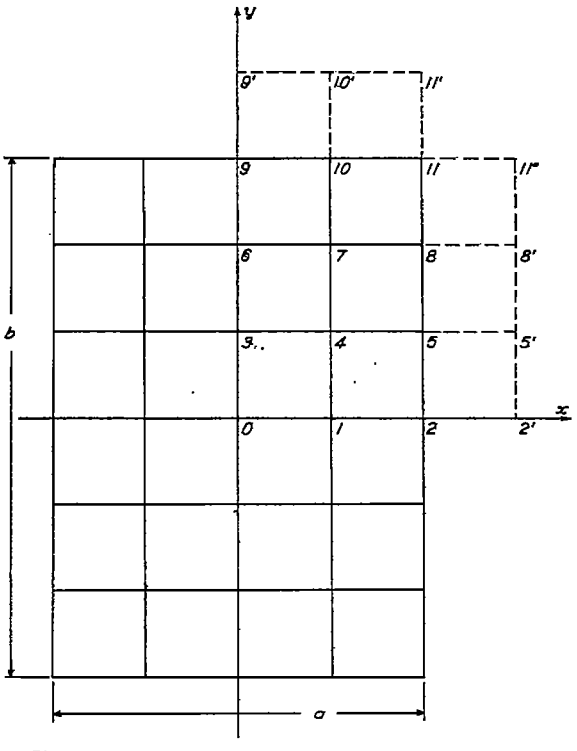

Figure 1.- Plate divided into 24 square meshes.

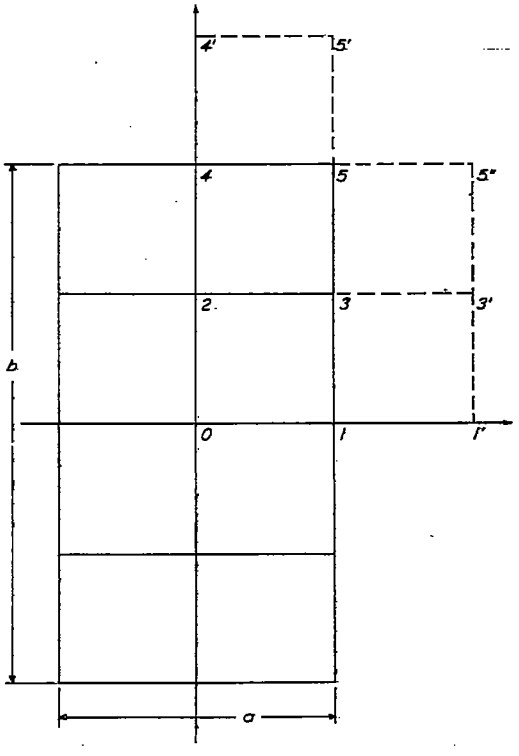

Figure 2.- Plate divided into 8 square meshes.

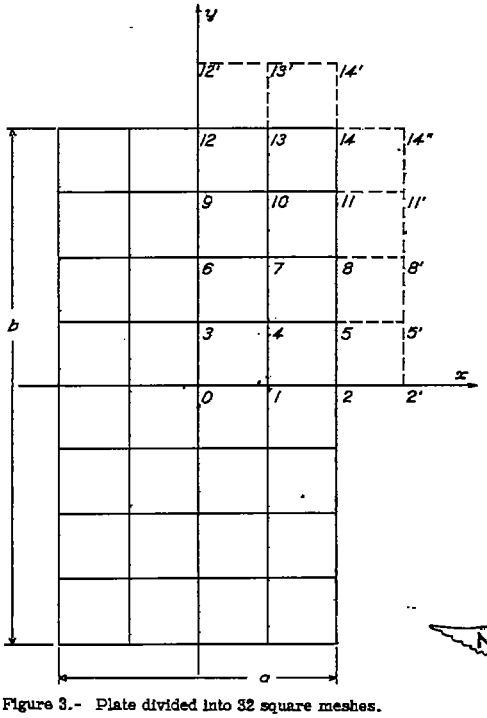

Figure 3.- Plate divided into 32 square meshes.

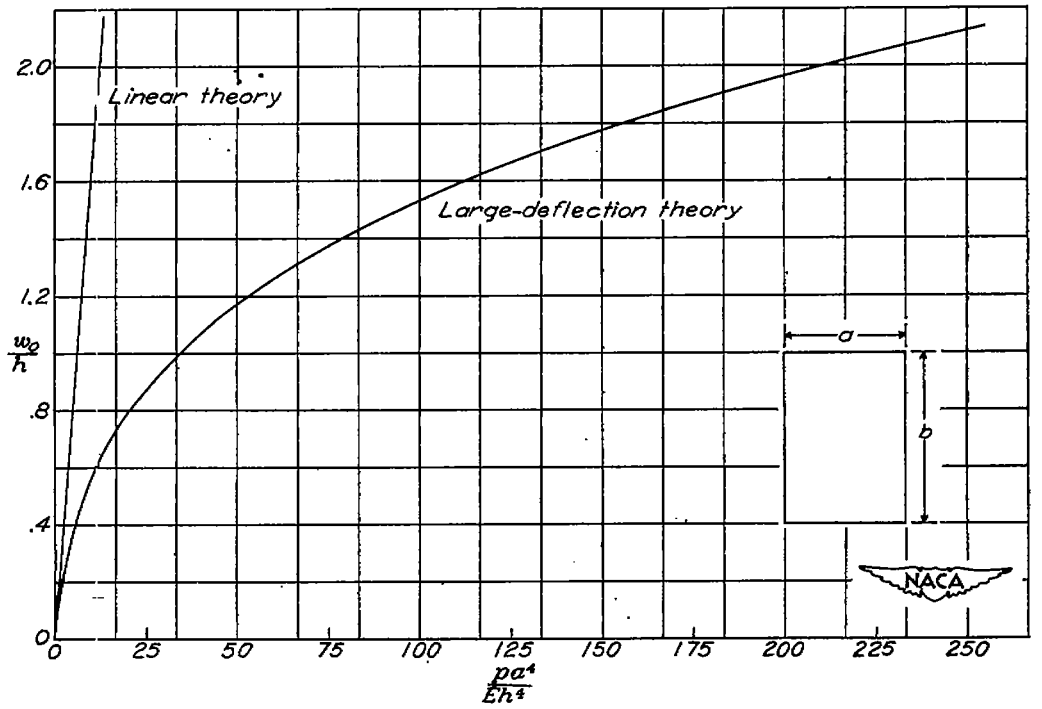

Figure 4.- Center deflections for a rectangular plate under normal pressure. $\frac{b}{a} = 1.5$.

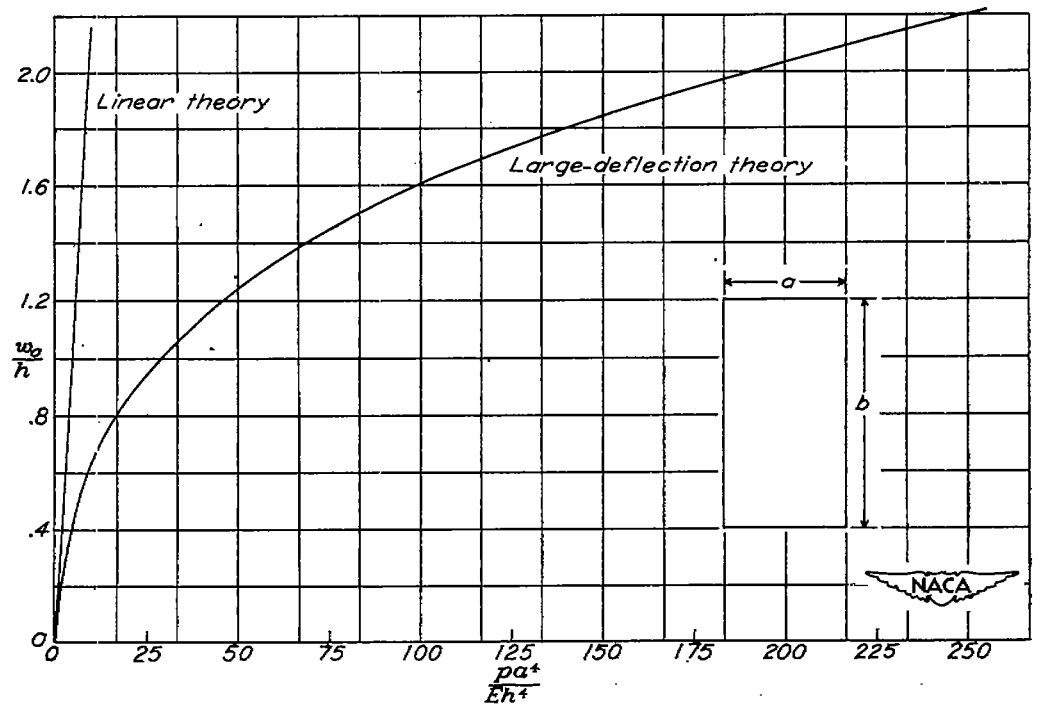

Figure 5.- Center deflections for a rectangular plate under normal pressure. $\frac{b}{a} = 2$.

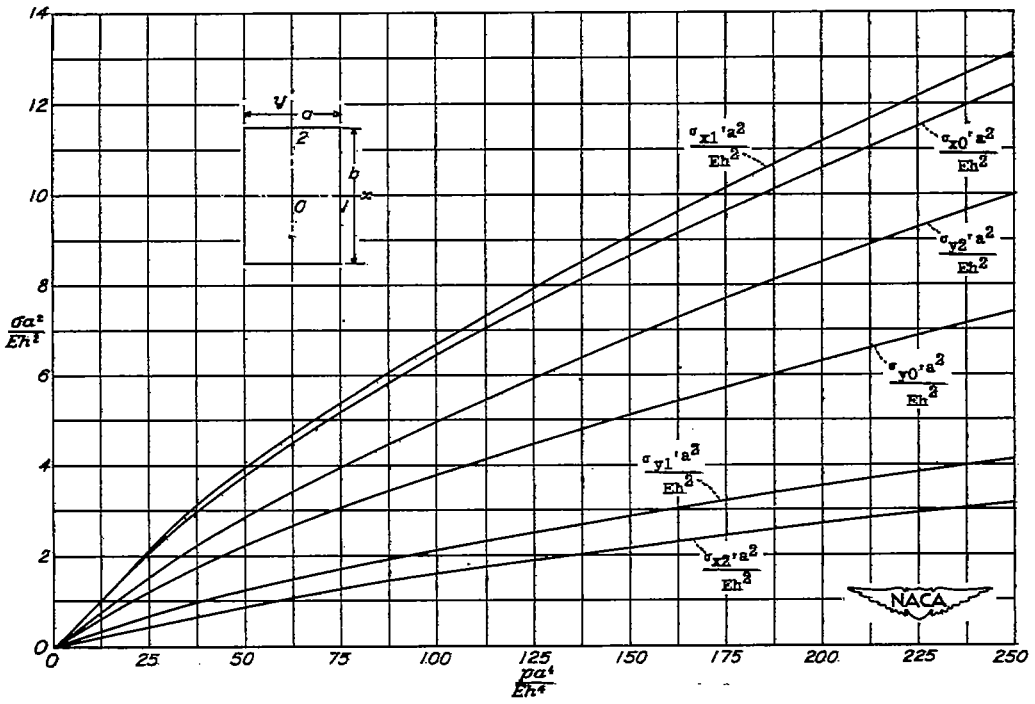

Figure 6.- Membrane stresses. $\frac{b}{a} = 1.5$; $\Delta l = \frac{a}{4}$.

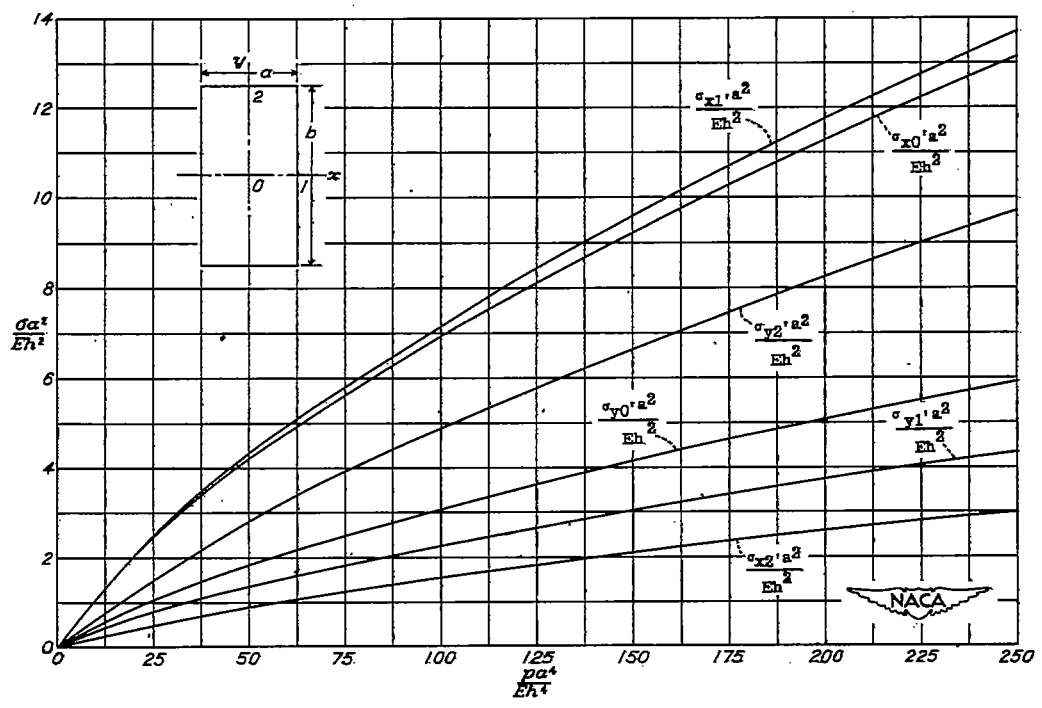

Figure 7.- Membrane stresses. $\frac{b}{a} = 2$; $\Delta l = \frac{a}{4}$.

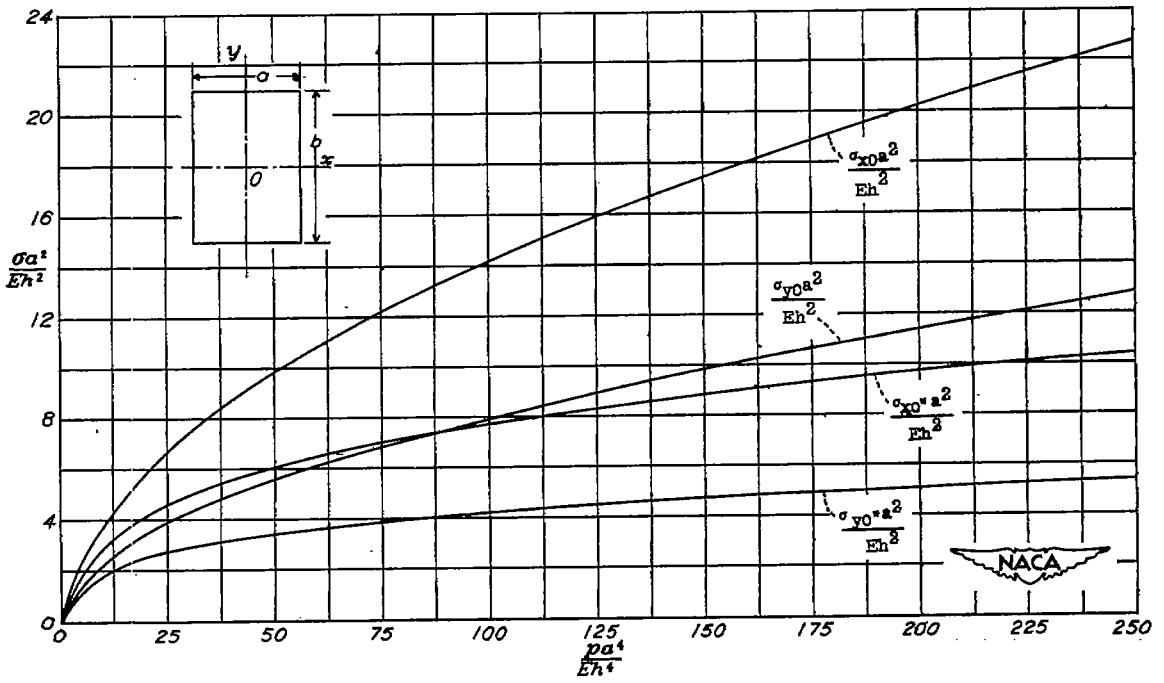

Figure 8.- Extreme-fiber total and bending stresses at center of plate. $\frac{b}{a} = 1.5$; $\Delta l = \frac{a}{4}$.

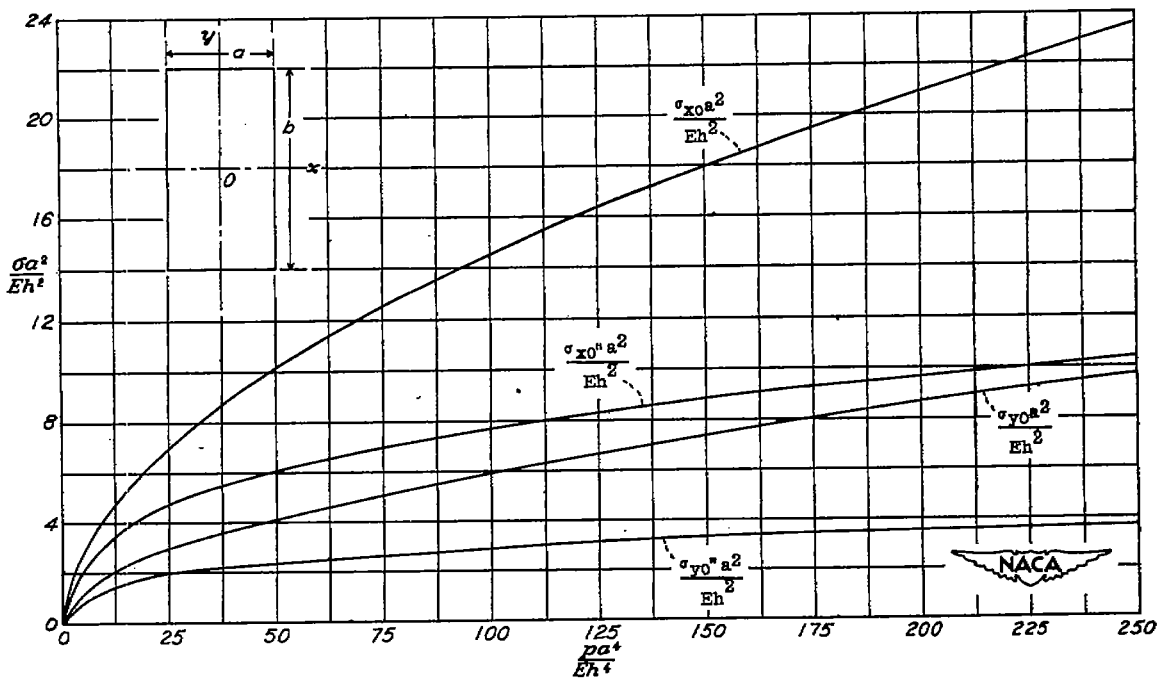

Figure 9.- Extreme-fiber total and bending stresses at center of plate. $\frac{b}{a} = 2$; $\Delta l = \frac{a}{4}$.