REPORT No. 144

THE DECAY OF A SIMPLE EDDY

BY

H. BATEMAN
California Institute of Technology

REPORT No. 144.

THE DECAY OF A SIMPLE EDDY.

By H. Bateman.

INTRODUCTION.

This subject, which has been studied recently by G. I. Taylor[1] and H. A. Webb[2] after some initial investigations by Lees, is of considerable mathematical interest, as the theory depends upon an exact solution of the equations of motion of an incompressible viscous fluid. Since very few exact solutions of these equations are known,[3] it seems worth while to study in detail the one which describes the behavior of a simple eddy and to find, if possible, some further solutions of the equations. The results of this study are herein set forth for publication by the National Advisory Committee for Aeronautics.

SUMMARY.

The principal result obtained in this paper is a generalization of Taylor's formula for a simple eddy. The discussion of the properties of the eddy indicates that there is a slight analogy between the theory of eddies in a viscous fluid and the quantum theory of radiation. Another exact solution of the equations of motion of a viscous fluid yields a result which reminds one of the well-known condition for instability in the case of a horizontally stratified atmosphere.

SOLUTION OF EQUATIONS OF TWO-DIMENSIONAL MOTION.

The equations of two-dimensional motion of an incompressible viscous fluid may be written in the form

$$\frac{\partial U}{\partial t}+U\frac{\partial U}{\partial r}+\frac{V}{r}\frac{\partial U}{\partial \theta}-\frac{V^2}{r}=R-\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left\{\frac{\partial^2 U}{\partial r^2}+\frac{1}{r}\frac{\partial U}{\partial r}+\frac{1}{r^2}\left(\frac{\partial^2 U}{\partial \theta^2}-2\frac{\partial V}{\partial \theta}-U\right)\right\} \tag{1}$$

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial r}+\frac{V}{r}\frac{\partial V}{\partial \theta}+\frac{UV}{r}=\Theta-\frac{1}{\rho}\frac{\partial p}{r\partial \theta}+\nu\left\{\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}+\frac{1}{r^2}\left(\frac{\partial V}{\partial \theta^2}+2\frac{\partial U}{\partial \theta}-V\right)\right\} \tag{2}$$

$$\frac{\partial}{\partial r}(rU)+\frac{\partial V}{\partial \theta}=0 \tag{3}$$

where U is the radial velocity, V the transverse velocity, p the pressure, ρ the density, and ν the kinematic viscosity. The variables (r, θ) are cylindrical polar coordinates, t is the time, and (R, Θ) are the radial and transverse components of the external force on unit mass of the fluid.

If $\Theta=0$ and p, U, V are independent of θ, equations (2) and (3) give

$$\frac{\partial V}{\partial t}+U\frac{\partial V}{\partial r}+\frac{UV}{r}=\nu\left\{\frac{\partial^2 V}{\partial r^2}+\frac{1}{r}\frac{\partial V}{\partial r}-\frac{1}{r^2}V\right\} \tag{4}$$

$$\frac{\partial}{\partial r}(rU)=0 \tag{5}$$

[1] British Advisory Committee for Aeronautics, R. & M. No. 596, December, 1918.

[2] R. & M. No. 602, May, 1919.

[3] For some cases of two-dimensional steady motion see G. B. Jeffery, Phil. Mag. t. 29 (1915), p. 455; G. I. Taylor, Proc. Roy. Soc. A. (1917), p. 99; and Lamb's Hydrodynamics.

The last equation gives $rU=K$, where K is a function of t only. Equation (4) now becomes

$$\frac{\partial V}{\partial t}=\left(\nu\frac{\partial}{\partial r}-\frac{K}{r}\right)\left(\frac{\partial V}{\partial r}+\frac{V}{r}\right) \tag{6}$$

When K is constant an interesting solution of the partial differential equation is

$$V=B\frac{r^{2s+1}}{t^{s+2}}e^{-\frac{r^2}{4\nu t}} \tag{7}$$

where $s=\frac{K}{2\nu}$ and B is a constant. When $s=0$ there is no radial velocity and we obtain Taylor's solution.

In the general case our solution represents an eddy with a source or sink at the center, i. e., along the axis; it thus gives a very simple representation of the type of motion considered by Webb and used as a model of a case in which the decay of an eddy arises from both dynamic and viscous causes.

The same solution may also be used to represent an eddy whose center moves with uniform velocity, provided that U and V are interpreted as radial and transverse velocities relative to the moving center.

It should be noticed that V is a maximum when

$$r^2=4\nu t\left(s+\frac{1}{2}\right)=2t\,(K+\nu). \tag{8}$$

If, following Taylor, we define the radius of the eddy at time t as the radius of the ring of maximum velocity, it appears that this radius is proportional to the square root of $K+\nu$. The constants K and ν are thus of equal importance as far as the rate of increase in size of the eddy is concerned.

The distribution of pressure may be inferred from equation (1). Putting $R=0$, we have

$$\frac{1}{\rho}\frac{\partial p}{\partial r}=\frac{K^2}{r^3}+B^2\frac{r^{4s+1}}{t^{2s+4}}e^{-\frac{r^2}{2\nu t}}. \tag{9}$$

The pressure thus increases from the center outward.

The kinetic energy of the fluid may be regarded as made up of energy of radial motion and energy of transverse motion. The former is infinite when there is a source or sink at the origin, but the latter is always finite. Its value is in fact

$$\begin{aligned}T&=\int_0^\infty 2\pi r\,dr\left(\frac{1}{2}\rho V^2\right)\\&=\pi\rho B^2\int_0^\infty\frac{r^{4s+3}}{t^{2s+4}}e^{-\frac{r^2}{2\nu t}}dr\\&=\frac{1}{2}\pi\rho\left(\frac{B}{t}\right)^2(2\nu)^{2s+2}\int_0^\infty e^{-\xi}\,\xi^{2s+1}\,d\xi.\end{aligned} \tag{10}$$

The total energy of transverse motion at time t is thus represented by an expression of type $\frac{A}{t^2}$, whatever be the real value of s, and so the law of decay is always the same.

If we regard the circle of maximum velocity as a boundary separating the inside of the vortex from the outside, we find that the energy of transverse motion inside the vortex is always a constant fraction of the total amount. This fraction F is given by

$$F\int_0^\infty e^{-\xi}\,\xi^{2s+2}\,d\xi=\int_0^{2s+1}e^{-\xi}\,\xi^{2s+2}\,d\xi. \tag{11}$$

When $s=0$, $F=1-\frac{2}{e}$ and is less than $\frac{1}{3}$. In this case we have also

$$\frac{p}{\rho}=C-\frac{B^2\nu}{t^3}e^{-\frac{r^2}{2\nu t}} \tag{12}$$

where C is a function of t. It should be noticed that the pressure does not become a minimum when $\frac{1}{2}\rho V^2$ is a maximum but is a minimum when $r=0$.

The angular momentum of the eddy is

$$M=\int_0^\infty 2\pi r dr x \rho r V$$

$$=2\pi\rho B\int_0^\infty \frac{r^{2s+3}}{t^{s+2}}e^{-\frac{r^2}{4\nu t}}dr$$

$$=\pi\rho B\,(4\nu)^{s+2}\int_0^\infty e^{-\zeta}\zeta^{s+1}d\zeta. \tag{13}$$

The angular momentum remains constant, as we should expect from dynamical considerations.

The angular momentum within the eddy is a constant fraction f of the whole, where

$$f\int_0^\infty e^{-\zeta}\zeta^{s+1}d\zeta-\int_0^{s+\frac{1}{2}}e^{-\zeta}\zeta^{s+1}d\zeta. \tag{14}$$

When $s=0, f=1-\frac{3}{2}e^{-\frac{1}{2}}$ and is quite small; thus most of the angular momentum is outside the vortex.

The circulation in a ring is $2\pi r V$, and this is a maximum when

$$\frac{r^2}{4\nu t}=s+1. \tag{15}$$

It is easily seen that the maximum circulation decays according to a law of type $\frac{C}{t}$, while the maximum velocity decays according to a law of type $\frac{D}{t^{\frac{3}{2}}}$.

The angular velocity

$$\Omega=\frac{V}{r}=B\frac{r^{2s}}{t^{s+2}}e^{-\frac{r^2}{4\nu t}} \tag{16}$$

is a maximum when

$$\frac{r^2}{4\nu t}=s. \tag{17}$$

The maximum angular velocity decays according to a law of type

$$\Omega_{\max.}=\frac{E}{t^2} \tag{18}$$

where E is a constant.

VISCOUS FLUID MOTION AND THE QUANTUM THEORY OF RADIATION.

It follows from the last result that the total energy of transverse motion is proportional to the maximum angular velocity. This result is slightly analogous to the quantum law in the theory of radiation. Pursuing the analogy we shall attempt to compare critical values of the angular momentum in Bohr's atomic theory with critical values of the Reynolds number $\frac{VL}{\nu}$ in the theory of viscous fluid motion. It should be noticed in the first place that angular momentum has the same dimensions as (a Reynolds number) multiplied by (a mass and a kinematic viscosity). The mass required for our analogy may be the mass of an electron and the viscosity the viscosity of the ether.

It should be noticed also that we can actually find instances of viscous fluid motion which exist only for certain particular values of a constant.

Let us consider, for instance, a type of eddy motion in which the transverse velocity V is zero for $r=O$ and $r=\infty$ while some derivative $\frac{\partial^n V}{\partial r^n}$ is finite and different from zero for $r=O$.

An expression of type (7) evidently satisfies the conditions when $2s+1=n$, hence a solution of the required type exists whenever $2s+1$ is a positive integer. Moreover, we get into difficulties when we try to satisfy the conditions by means of a solution of (6) of type

$$V=B\frac{r^n}{t^m}\left\{1-\frac{m}{(1+n-2s)\,(n+3)}\frac{r^2}{\nu t}+\frac{m\,(m+1)}{(1+n-2s)\,(3+n-2s)\,(n+3)\,(n+5)}\left(\frac{r^2}{\nu t}\right)^2\cdots\cdots\right\} \tag{19}$$

for this does not generally become zero when $r\rightarrow\infty$. When s is very large, for instance, V is practically equal to $B\frac{r^n}{t^m}$. To illustrate the way in which a solution of the required type can be derived from the infinite series, let us consider the case $n=1$, $m=2$. We may then write

$$\begin{aligned}V&=B\frac{r}{t^2}\left\{1-\frac{1}{1-s}\frac{r^2}{4\nu t}+\frac{1}{(1-s)\,(2-s)}\left(\frac{r^2}{4\nu t}\right)^2\cdots\cdots\right\}\\&=B\frac{r}{t^2}\,e^{-\frac{r^2}{4\nu t}}\left\{1+\frac{s}{s-1}\frac{r^2}{4\nu t}+\frac{s}{s-2}\times\frac{1}{1\times2}\left(\frac{r^2}{4\nu t}\right)^2+\frac{s}{s-3}\times\frac{1}{1\times2\times3}\left(\frac{r}{4\nu t}\right)^3+\cdots\cdots\right.\end{aligned} \tag{20}$$

Multiplying V by $s-k$, where k is a positive integer and making $s\rightarrow k$, we obtain a solution of the required type. Webb assumes that for different eddies of the same radius a, the radial velocity at $r=a$ is proportional to the strength. This means that

$$s=\frac{K}{2\nu}=\lambda\frac{a\,V}{\nu} \tag{21}$$

Where λ is a nondimensional constant, hence s is a Reynolds number and a solution of our problem exists only for certain critical values of a Reynolds number.

ANOTHER METHOD OF SOLUTION.

Another exact solution of the equations of variable motion of an incompressible viscous fluid may be obtained as follows:

Let us endeavor to satisfy the equation

$$\left.\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}-\nu\Delta^2u&=-\frac{1}{\rho}\frac{\partial p}{\partial x}\\\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}-\nu\Delta^2v&=-\frac{1}{\rho}\frac{\partial p}{\partial y}\end{aligned}\right\} \tag{22}$$

by expressions of type

$$u=ke^{\lambda t-ky}\cos qx,\ v=-qe^{\lambda t-ky}\sin qx \tag{23}$$

where k, q, and λ are constants. These expressions evidently satisfy the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=O \tag{24}$$

We find that equations (22) are satisfied if

$$\begin{aligned}-\frac{1}{\rho}\frac{\partial p}{\partial x}&=k\left\{\lambda-\nu\,(k^2-q^2)\right\}e^{\lambda t-ky}\cos qx\\-\frac{1}{\rho}\frac{\partial p}{\partial y}&=-q\left\{\lambda-\nu\,(k^2-q^2)\right\}e^{\lambda t-ky}\sin qx-kq^2\,e^{2(\lambda t-ky)}\end{aligned}$$

These equations can be satisfied either by putting $k=q$ or by putting $\lambda=\nu(k^2-q^2)$. The former case is of no interest, for it corresponds to a well-known case of irrotational motion.

In the latter case we have

$$\frac{p}{\rho}=C-\frac{1}{2}q^2\,e^{2(\lambda t-ky)} \tag{25}$$

where C is a constant, ρ being supposed to be also constant. When $k<q$ the motion is dying down while when $k>q$ the motion is becoming more violent. When $k=q$ the motion is steady and irrotational.

If at any instant we fix the value of p at $y=O$, we find that in the case of growing or unstable motion the pressure gradient $\left(\frac{\partial p}{\partial y}\right)_{y=O}$ is larger than in the steady case when $k=q$. In the case of a decaying motion the pressure gradient is less than in the steady case. This result may be compared with the well-known theorem that a horizontally stratified atmosphere is unstable when the lapse rate of temperature is higher than the adiabatic value.

Another point worth noticing is that

$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=(k^2-q^2)\,e^{\lambda t-ky}\cos qx.$$

Hence when the pressure over $y=O$ is fixed, the sign of the vorticity is different according as k is greater than or less than q, i. e., according as the motion is growing or decaying.