REPORT No. 213

A RÉSUMÉ OF THE ADVANCES IN THEORETICAL AERONAUTICS MADE BY MAX M. MUNK

By JOSEPH S. AMES
National Advisory Committee for Aeronautics

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INTRODUCTION

In order to apply profitably the mathematical methods of hydrodynamics to aeronautical problems, it is necessary to make certain simplifications in the physical conditions of the latter. To begin with, it is allowable in many problems, as Prandtl has so successfully shown, to treat the air as having constant density and as free of viscosity. But this is not sufficient. It is also necessary to specify certain shapes for the solid bodies whose motion through the air is discussed, shapes suggested by the actual solids—airships or airfoils—it is true, but so chosen that they lead to solvable problems.

In a valuable paper presented by Dr. Max M. Munk, of the National Advisory Committee for Aeronautics, Washington, before the Delft Conference in April, 1924, these necessary simplifying assumptions are discussed in detail. It is the purpose of the present paper to present in as simple a manner as possible some of the interesting results obtained by Dr. Munk's methods. For fuller details and a discussion of many practical questions reference should be made to Munk's original papers:

1. The Aerodynamic Forces on Airship Hulls. N. A. C. A. Report No. 184, 1924.
2. Elements of the Wing Section Theory and of the Wing Theory. N. A. C. A. Report No. 191, 1924.
3. Remarks on the Pressure Distribution Over the Surface of an Ellipsoid, Moving Translationally Through a Perfect Fluid. N. A. C. A. Technical Note No. 196, 1924.
4. The Minimum Induced Drag of Aerofoils. N. A. C. A. Report No. 121, 1921.
5. General Theory of Thin Wing Sections. N. A. C. A. Report No. 142, 1922.
6. Determination of the Angles of Attack of Zero Lift and of Zero Moment, Based on Munk's Integrals. N. A. C. A. Technical Note No. 122, 1923.
7. General Biplane Theory. N. A. C. A. Report No. 151, 1922.

GENERAL PRINCIPLES OF HYDRODYNAMICS

In all the practical problems to be discussed, only the most general principles of hydrodynamics are used and in practically all cases the problems are reduced to questions involving only energy and momentum. It may be worth while to deduce the few equations necessary, although they are given in every textbook.

Since air is a fluid, the pressure is everywhere perpendicular to any surface through which it is transferred. If u, v, w are components of the velocity of flow at any point,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

since the density is considered to be constant. The entire energy of the flow is kinetic, and therefore

$$T=\frac{1}{2}\rho\int(u^2+v^2+w^2)\,d\tau$$

where $d\tau$ is an element of volume of the fluid. By Newton's law of motion

$$\rho \frac{du}{dt} dx\, dy\, dz = \left[p - \left(p + \frac{\partial p}{\partial x} dx\right)\right] dy\, dz = -\frac{\partial p}{\partial x} dx\, dy\, dz$$

or

$$\rho \frac{du}{dt} = -\frac{\partial p}{\partial x}$$

This may be written

$$\rho\, du = -\frac{\partial (pdt)}{\partial x}$$

The impulse per unit area in the time dt is, by definition, pdt. So the infinitesimal change in velocity du can be considered as produced by the infinitesimal impulse pdt, and a finite velocity u may be considered as produced from a state of rest by the finite impulse $P \equiv \int pdt$. where, then

$$\rho u = -\frac{\partial P}{\partial x}$$

or

$$u = \frac{\partial}{\partial x}\left(-\frac{P}{\rho}\right)$$

Similarly, the other two components of the velocity of flow at any point will be defined by

$$v = \frac{\partial}{\partial y}\left(-\frac{P}{\rho}\right), w = \frac{\partial}{\partial z}\left(-\frac{P}{\rho}\right)$$

Flows such as this, where

$$u = \frac{\partial \varphi}{\partial x}, v = \frac{\partial \varphi}{\partial y}, w = \frac{\partial \varphi}{\partial z}$$

are called "potential flows," and φ is called the "velocity potential." In this case, when the flow is considered as produced by an impulse, P,

$$\varphi = -\frac{P}{\rho} \qquad (1)$$

or, the impulse per unit area, equals $-\rho\varphi$.

There are cases of potential flow in which φ is not a single-valued function, and in such

$$P = -\rho\,(\varphi_1 - \varphi_2) \qquad (1a)$$

where φ_1 and φ_2 are the values of φ at the same point. Since $\varphi_2 - \varphi_1 = \int_1^2 (udx + vdy + wdz)$, if $_1$ and $_2$ refer to the same point, the integral is called the "circulation," and, if its value is μ, the equation may be written $P = +\rho\mu$, where P is in the direction of the flow.

As an illustration, consider the flow discussed later, equation (41), in which, for any point on the axis of x,

$$\varphi = A_0 \sin^{-1} x$$

The flow is a two-dimensional one, as shown in the figure. Consider an imaginary surface at x, having a minute length along the axis of x and unit length perpendicular to the plane of the paper. Let the point *1* be on the lower side of the surface and the point *2* on the upper.

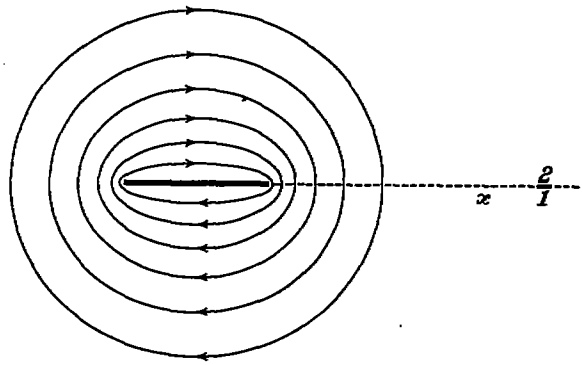

FIG. 1

$$\varphi_2-\varphi_1=\int_1^2 d\varphi=2\pi A_0$$

when the points approach each other indefinitely. The impulse per unit area at the point *1* is $-\rho\varphi_1$ and its action is downward, being perpendicular to the fluid surface below the imaginary surface; at the point *2* the impulse per unit area is $-\rho\varphi_2$, acting upward, since it is perpendicular to the fluid surface above the imaginary surface. Therefore the total impulse per unit area acting downward on the fluid is

$$P=\rho\ (\varphi_2-\varphi_1)=\rho 2\pi A_0$$

Again, since

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

φ must satisfy everywhere in the fluid the equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2}=0$$

Making use again of Newton's equation, and taking into account the fact that, in general, u, v, and w are functions of (t, x, y, z), the general equations of motion are

$$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

and two similar ones for $\frac{dv}{dt}$ and $\frac{dw}{dt}$.

But

$$\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x} \text{ and } \frac{\partial u}{\partial z}=\frac{\partial w}{\partial x}$$

since

$$u=\frac{\partial\varphi}{\partial x},\ v=\frac{\partial\varphi}{\partial y},\ w=\frac{\partial\varphi}{\partial z};$$

therefore

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial x}+w\frac{\partial w}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or

$$\frac{\partial}{\partial t}\left(\frac{\partial\varphi}{\partial x}\right)+\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)=-\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

with two similar equations for y and z. On integration, these three equations give

$$\frac{\partial\varphi}{\partial t}+\frac{1}{2}(u^2+v^2+w^2)=-\frac{p}{\rho}+\text{a constant.}$$

Written differently,

$$p=-\rho\frac{\partial\varphi}{\partial t}+C-\frac{\rho}{2}(u^2+v^2+w^2)$$

In the case of a steady state $\frac{\partial\varphi}{\partial t}=O$, and

$$p=C-\frac{\rho}{2}(u^2+v^2+w^2) \qquad (2)$$

which is Bernouilli's famous theorem. If there is a portion of space in which the fluid is at rest, the pressure there equals C.

The work done by an impulse is proved in mechanics to be the product of the impulse by the average of the initial and final velocities in the direction of the impulse. If a solid is moving through a fluid otherwise at rest, and if the existing fluid motion is considered as having been produced from rest by impulses applied by the surface of the body, the velocity normal to any element of surface is $\frac{d\varphi}{dn}$ where dn is drawn from the body into the fluid, and the mean value of this and the initial zero velocity is $\frac{1}{2}\frac{d\varphi}{dn}$; further, the impulse, normal to the surface dS, acting on the fluid is $-\rho\varphi\cdot dS$. Therefore, the kinetic energy of the fluid is

$$T=-\frac{\rho}{2}\int\varphi\frac{d\varphi}{dn}dS, \text{ taken over the surface of the solid body} \qquad (3)$$

Other general principles will be discussed as the occasion arises.

PROBLEMS MORE SPECIALLY CONCERNING AIRSHIPS

INTRODUCTION

The fundamental problems concerning airships are: (1) the determination of the moments acting on them under varying conditions of flight; (2) the determination of the distribution of transverse forces; (3) the distribution of pressure over the envelope.

These problems can be solved, at least approximately, by the application of certain general theorems.

When a body moves through a fluid otherwise at rest, there is a certain amount of kinetic energy of the fluid caused by the motion of the body. If the latter is moving with a velocity V in a definite direction, if T is the kinetic energy of the fluid due to the motion of the body, and if ρ is the density of the fluid, by definition $\frac{T}{\frac{1}{2}V^2}$ is called the "apparent additional mass" of the body for motion in that particular direction, and is written $K\rho$.

As an illustration, consider an infinitely long circular cylinder moving transversely in a definite direction with a velocity V. Choose this direction as the axis for a set of polar coordinates whose origin is on the axis of the moving cylinder. The velocity of any particle of the fluid will be in a plane perpendicular to the axis of the cylinder, so the flow is called two-dimensional, or uniplanar. A particle in contact with the cylinder must have the same component of velocity normal to the cylinder as the wall of the cylinder at that point. So, if r and θ are the polar coordinates of any point of the fluid in a particular transverse plane, and if R is the radius of the cylinder, this condition may be expressed by writing

$$\left(\frac{\partial\varphi}{\partial r}\right)_{r=R}=V\cos\theta_0,$$

if θ_0 denotes the point on the cylinder. This leads at once to the value of φ for any point in the fluid, r, θ, viz

$$\varphi=-\frac{VR^2\cos\theta}{r}$$

for it may be proved that this satisfies both

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0.$$

and the condition just expressed for the surface of the cylinder. Hence the kinetic energy of flow

$$T=-\frac{\rho}{2}\int\varphi\frac{d\varphi}{dn}dS$$

becomes, since at $r=R$, $\varphi=-VR\cos\theta_o$, $\frac{d\varphi}{dn}=V\cos\theta_o$ and $dS=Rd\theta_o h$, where h is any length desired of the cylinder,

$$T=\frac{\rho}{2}V^2R^2h\int_0^{2\pi}\cos^2\theta_o\,d\theta_o=\frac{\rho}{2}V^2\pi R^2h$$

Consequently the apparent additional mass is

$$\frac{T}{\frac{1}{2}V^2}=\rho\pi R^2h$$

i. e., is the mass of the fluid displaced by the cylinder. This is sometimes expressed, with reference to the two-dimensional flow, by saying that the "apparent mass of a circle is $\rho\pi R^2$."

It will be proved later that if a plane lamina infinite in length and of width b is moving transversely with a velocity V, the flow being again two dimensional, the apparent additional mass of a length h of the lamina is $\rho\pi\left(\frac{b}{2}\right)^2h$ (the same as for a circular cylinder whose diameter is b.) So the apparent transverse mass of a straight line of length b is $\rho\pi\left(\frac{b}{2}\right)^2$ in a two-dimensional flow, this really being the apparent mass of a portion of length unity of an infinitely long lamina whose width is b.

If a body is moving in a definite direction with a constant velocity, the flow accompanies the body, so that the kinetic energy does not change, therefore there is no drag, which would absorb energy. Further, if the flow gives rise to a single-valued velocity potential, there is no lift. (See a later section.) But although, therefore, the resultant force is zero, there may be a moment acting on the body.

This may best be seen by a consideration of the momentum of the flow. When the body is moving with the velocity V in a definite direction, let there be a component of momentum of flow perpendicular to this direction and let its amount be $A\rho V$. Then, with reference to any axis perpendicular to the plane including the line of velocity and the direction of the component of momentum, there is a certain moment of momentum; and, as the body moves a distance V in a unit time, this moment of momentum increases in that time by an amount $V\cdot A\rho V$. An equal but opposite moment around the specified axis must, therefore, be acting on the body. Hence, moment = velocity of body × component of momentum of flow perpendicular to the axis of the moment and to the direction of the velocity. The "sense" of the moment is easily seen.

Conversely, if the body does not experience any moment, the momentum of the flow must be entirely in the line of motion of the body.

If a solid body is held stationary in a uniform flow, the kinetic energy of the entire infinite flow is of course infinite, but less than it would be if the body were absent, owing to two reasons: (1) The solid displaces an equal volume of the fluid, which otherwise would be in motion; (2) the velocity of the flow is reduced in front of and behind the body. This decrease in kinetic energy for a definite velocity of flow equals the kinetic energy of the total flow if the solid is moving in a stationary fluid with the same velocity as the velocity of flow in the first case.

When the solid is at rest in a uniform flow, let it be turned slightly through an angle $d\alpha$ about a definite axis; if there is a moment about this axis acting on the body in such a sense that it opposes the rotation $d\alpha$, work will be required to turn the body, and the kinetic energy of the fluid, T', will increase by an amount equal to the product of this moment and the angular displacement. Similarly, if the moment is in the same sense as $d\alpha$, $-dT'=Md\alpha$. Therefore, if now the body is moving through a stationary fluid, $Md\alpha=dT$, since $dT=-dT'$. Hence

$$M=\frac{dT}{d\alpha} \qquad (4)$$

where M is the moment acting on the body around a definite axis, in the same "sense" as $d\alpha$, the angular displacement around this axis.

If, therefore, for a given direction of motion, T is a maximum, a slight change $d\alpha$ would result in a decrease of T, and M would be negative, indicating a moment acting on the body in such a sense as to oppose the change $d\alpha$. Such a direction of motion would therefore be one of stable equilibrium. Similarly, if T is a minimum for a given line of motion, there is unstable equilibrium.

In general, if any motion is generated from rest by an impulse, the work done equals the product of the impulse by half the component of the velocity in the direction of the impulse. The impulse equals the momentum; therefore, the kinetic energy equals one-half the scalar product of the momentum and the velocity. This theorem may be applied to the fluid motions produced by the motion of solids through them.

A body gives rise to a definite kinetic energy of flow if it has a constant velocity in a specified direction; and, if its motion is reversed, it will give rise to the same amount, because the flow at each point of the fluid is reversed. (This is evident because the effect of the presence of the solid body when at rest in a stream of fluid may be duplicated by a certain distribution of sources and sinks, giving rise to the same field of velocity potential as before, outside the space previously occupied by the solid; then, if the stream is reversed and each source is made into a sink of an equal strength and vice versa, the potential field is exactly reversed, so that the velocity at each point is reversed.) The kinetic energy is different for directions of motion other than as specified, but it is always a positive number. Therefore, as the orientation of the line of motion of the body is changed from some definite one to its opposite, there must be two lines of motion—somewhere between—for one of which the kinetic energy is a maximum and for the other of which it is a minimum. For motion in either of these directions, therefore, $\delta T=0$; that is, for any small angular displacement $\delta\alpha$ around an axis perpendicular to the direction of motion $\delta T=0$. Consequently there is no moment acting on the body if it is moving in either of these two directions; and the body may therefore be said to be in equilibrium, stable if T is a maximum and unstable if T is a minimum.

Let a body be moving with a velocity V in such a direction that it is in equilibrium; call the direction $\overline{A}$. The momentum of the flow must be in the same direction, otherwise there

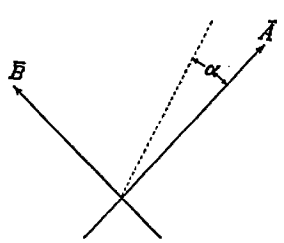

FIG. 2

would be a moment; call its value $K_1\rho V$. Keeping the orientation of the body unchanged, make the line of motion with the velocity V perpendicular to $\overline{A}$, i. e. along $\overline{B}$; let the component along $\overline{B}$ of the momentum of the flow be called $C\rho V$, its component along $\overline{A}$ be called

$D\rho V$, and its component perpendicular to the plane of $\overline{A}$ and $\overline{B}$ be called $E\rho V$. Again, let its line of motion with velocity V be in the plane of $\overline{A}$ and $\overline{B}$, making an angle α with $\overline{A}$. Then the momentum along $\overline{A}$, which may be written G_A, has the value

$$G_A = K_1\rho V \cos\alpha + D\rho V \sin\alpha. \text{ Similarly, along } \overline{B}$$
$$G_B = C\rho V \sin\alpha; \text{ and, perpendicular to the plane of } \overline{AB},$$
$$G_{AB} = E\rho V \sin\alpha$$

also

$$V_A = V\cos\alpha$$
$$V_B = V\sin\alpha$$
$$V_{AB} = 0$$

Consequently the kinetic energy of the flow, which equals one-half the scalar product of the momentum and the velocity, is given by the equation

$$T = \frac{\rho}{2} V^2 (K_1 \cos^2\alpha + C\sin^2\alpha + D\sin\alpha\cos\alpha)$$

Under these circumstances the moment acting on the body around an axis perpendicular to the plane of $\overline{A}$ and $\overline{B}$ is

$$M = \frac{dT}{d\alpha} = \frac{\rho}{2} V^2 ((C - K_1)\sin 2\alpha + D\cos 2\alpha)$$

But if $\alpha = 0$, $M = 0$, since $\overline{A}$ is a line of equilibrium, therefore $D = 0$. Consequently, when the body is moving in a direction $\overline{B}$ at right angles to $\overline{A}$—a line of equilibrium, there is a component of momentum $C\rho V$ along $\overline{B}$ and a component $E\rho V$ perpendicular to the plane of $\overline{A}$ and $\overline{B}$, but none parallel to $\overline{A}$. If the body is now rotated about the line $\overline{A}$, through 180°, and again set moving along $\overline{B}$ with a velocity V, the momentum will have a component of momentum $C\rho V$ along $\overline{B}$, and a component—$E\rho V$ perpendicular to the plane $\overline{A}$ and $\overline{B}$. Therefore, as the body is rotated about $\overline{A}$ as an axis, there must be some definite orientation such that, for a velocity along $\overline{B}$, the component of momentum perpendicular to the plane of $\overline{A}$ and $\overline{B}$ is zero. For this orientation, then, the momentum is entirely along $\overline{B}$. Therefore, the present location of $\overline{A}$ and $\overline{B}$ with reference to the body are what may be called "axes of equilibrium." They are at right angles to each other. Similarly, it will be possible to find a third axis of equilibrium which is perpendicular to the other two. Every body possesses, therefore, three mutually perpendicular axes of equilibrium, and, in general, no more. Let the apparent additional masses with reference to these three axes be called $K_1\rho$, $K_2\rho$, $K_3\rho$; that is, if V_1, V_2, V_3 are the components of V with reference to these same axes, the flow momenta parallel to these axes are $K_1\rho V_1$, $K_2\rho V_2$, $K_3\rho V_3$. Consequently the kinetic energy of the flow is

$$T = \frac{1}{2}\rho (K_1V_1^2 + K_2V_2^2 + K_3V_3^2) \qquad (5)$$

The moment acting upon the body is determined by the equations previously given. If the line of velocity is in a plane including two axes of equilibrium, the equations are specially simple. Let the velocity make the angle α with the axis 1; then

$$G_1 = K_1\rho V\cos\alpha;\ G_2 = K_2\rho V\sin\alpha;\ G_3 = 0.$$

The component perpendicular to V is

$$K_2\rho V\sin\alpha\cos\alpha - K_1\rho V\cos\alpha\sin\alpha = \frac{\rho}{2} V (K_2 - K_1)\sin 2\alpha$$

and therefore the moment $=\frac{\rho}{2} V^2 (K_2 - K_1) \sin 2\alpha$. This is about an axis perpendicular to the plane of *1* and *2* and is clockwise. (Of course, if $K_2 < K_1$, it is actual counterclockwise.)

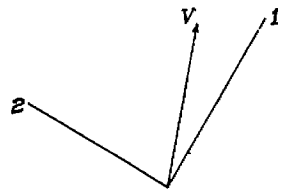

FIG. 3

As stated above, the three "principal" momenta of the flow are $K_1\rho V_1$, $K_2\rho V_2$, $K_3\rho V_3$, where V_1, V_2, V_3 are the components of the velocity $\overline{V}$. But if the localized vector is formed which represents $K_1\rho V_1$, i. e., the resultant of the parallel vectors representing the components of the momentum along this axis of each individual particle of the fluid; and similarly the localized vectors representing $K_2\rho V_2$ and $K_3\rho V_3$, it will be found that, in general, these three localized vectors do not pass through a common point. Therefore they can not be compounded to form a single localized vector, and we can not in general speak of "the momentum" of the flow. If, however, the moving body is one of revolution, or if it has three mutually perpendicular planes of symmetry, then there is a point common to the three lines of action of the principal momenta, and it is called the "aerodynamic center." In this case we may speak of "the flow-momentum" G, and our previous formulas for moments and kinetic energy may be written

$$\overline{M} = [\overline{G} \cdot \overline{V}] \qquad (6)$$

$$T = \frac{1}{2} (\overline{G} \cdot \overline{V}) \qquad (7)$$

MOMENTS AND FORCES ACTING ON AIRSHIPS

Airships may often be considered as having surfaces of revolution described by rotation about the longitudinal axis. The central portion of an airship may be considered as a circular cylinder, and therefore, from what has been proved for circular cylinders, the transverse apparent mass of the airship equals the mass of the fluid displaced, approximately. The longitudinal mass is small, because in longitudinal motion of the airship the air displaced by the bow escapes transversely on the whole and the air flowing in at the stern also flows in transversely, so that the momentum of the air in the direction of motion is small. On the other hand, when the airship moves transversely, the air in a transverse layer perpendicular to the longitudinal axis remains in the layer, so that the flow is a two-dimensional one about a circle. This is true near the central portion of the airship and approximately so elsewhere. Call the longitudinal apparent mass $K_1\rho$, the transverse apparent mass $K_2\rho$.

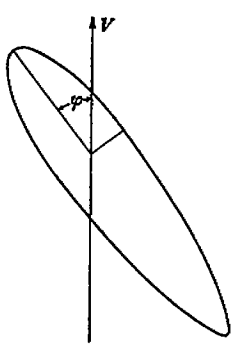

FIG. 4

Let the airship move in a straight line with a velocity V having an angle of yaw (or pitch) φ. The longitudinal momentum $= V \cos \varphi \cdot K_1\rho$; the transverse momentum $= V \sin \varphi \cdot K_2\rho$; hence the component perpendicular to the line of V is $\frac{\rho}{2} V (K_2 - K_1) \sin 2\varphi$, and the moment acting on the airship is

$$M = \frac{\rho}{2} V^2 (K_2 - K_1) \sin 2\varphi \qquad (8)$$

about an axis perpendicular to the plane including V and the longitudinal axis and is of such a "sense" as to increase φ. It is therefore called the "unstable" moment.

Let the airship move in a horizontal circle of radius r, with a velocity V and at an angle of yaw φ. Call the "apparent moment of inertia" about a transverse axis through the aerodynamic center $K'\rho$. The longitudinal velocity is $V \cos \varphi$; the transverse velocity is $V \sin \varphi$; and the angular velocity is $\frac{V}{r}$. Therefore, the longitudinal momentum is $K_1\rho V \cos \varphi$; the transverse momentum is $K_2\rho V \sin \varphi$, and the angular momentum, which remains constant, is $K'\rho \frac{V}{r}$. Since the aerodynamic center moves in a circle, the resultant force acting on the fluid must always pass through the center of this circle. During the motion the two components of momentum remain constant in amount but their directions rotate with the angular velocity $\frac{V}{r}$. If a vector representing momentum G rotates with an angular velocity ω, a force $G\omega$ must be

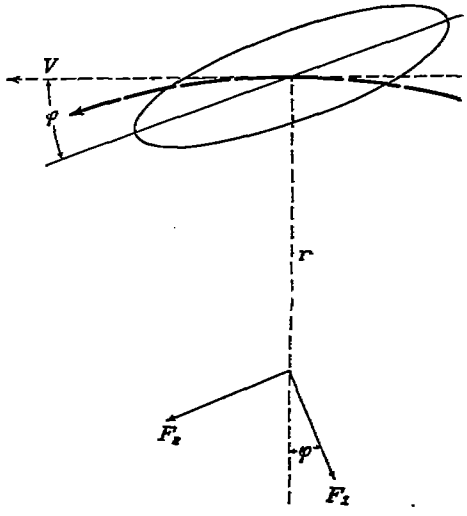

FIG. 5

acting perpendicular to the line of G. Therefore there must be acting on the fluid (1) a transverse force F_1, opposite in direction to the transverse momentum, equal to $K_1\rho V \cos \varphi \cdot \frac{V}{r}$, (2) a longitudinal force F_2, in the same direction as the longitudinal momentum, equal to $K_2\rho V \sin \varphi \cdot \frac{V}{r}$. The moment of these forces about an axis through the aerodynamic center, perpendicular to the plane of the motion is $(K_2 - K_1) \frac{\rho}{2} V^2 \sin 2\varphi$. This moment, acting on the fluid, is clockwise (in the drawing); therefore the moment acting on the airship is counterclockwise, tending to increase φ. (There is also a "negative drag.")

This moment is the same in amount as that found for the airship in straight flight with the same angle of yaw; but the distribution of forces along the airship is different in the two cases, as will now be shown by making a closer analysis of the two flows.

Consider an airship flying in a straight line with velocity V, and with an angle of pitch φ downward. In a stationary transverse plane perpendicular to the axis, and therefore approximately vertical, the flow may be regarded as two-dimensional, as explained before. The airship displaces a circle, which changes its size as the ship advances and also its position, owing to the pitch. The apparent mass of the two-dimensional flow in a layer of thickness dx, if S is the area of the circle, is $\rho S\, dx$, since the apparent transverse mass of a circular cylinder, if the flow is two-dimensional, is known to be equal to the mass of the fluid displaced. The transverse velocity is $V \sin \varphi$, and therefore the transverse momentum upward (in the drawing)

in the layer is $\rho S V \sin \varphi \, dx$. The rate of change of this is $\rho V \sin \varphi \cdot dx \cdot \frac{dS}{dt}$. But

$$\frac{dS}{dt}=\frac{dS}{dx}\frac{dx}{dt}=+V \cos \varphi \frac{dS}{dx}.$$

Hence at any element of length dx there is a transverse force downward on the airship, given by

$$\frac{\rho}{2} V^2 \sin 2\varphi \frac{dS}{dx} dx$$

This force is in opposite directions at the two ends, and produces the unstable moment.

Now consider the airship flying with constant velocity V, and angle of yaw φ, in a circle of radius r. The transverse momentum of a layer of thickness dx, outward, away from the circle, is, as before, $\rho S v \, dx$, where v is the transverse velocity. This now varies with the time. So the rate of change of this outward momentum is

$$\rho dx\left(S \frac{dv}{dt}+v \frac{dS}{dt}\right).$$

v is made up of two terms $V \sin \varphi$, due to the translation, and $\frac{V}{r} x$, due to the rotation, where x is measured along the axis from the aerodynamic center. Hence

$$\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=\frac{V}{r} V \cos \varphi; \frac{dS}{dt}=\frac{dS}{dx} V \cos \varphi.$$

Thus the rate of change of the transverse fluid momentum outward is

$$\rho dx\left(S \frac{V^2}{r} \cos \varphi+V^2 \frac{dS}{dx} \sin \varphi \cos \varphi+\frac{V^2}{r} x \cos \varphi \frac{dS}{dx}\right)$$

$$=\left(V^2 \frac{\rho}{2} \sin 2\varphi \frac{dS}{dx}+V^2 \frac{\rho}{r} S \cos \varphi+V^2 \frac{\rho}{r} \cos \varphi . x \frac{dS}{dx}\right) dx \qquad (9)$$

Therefore this gives the transverse force inward, toward the inside of the circle, on an element of length dx of the airship.

The first of the three terms is the same as found for the case of straight flight. The last two terms combine to form $V^2 \frac{\rho}{r} \cos \varphi \frac{d(xS)}{dx} dx$, and the resultant moment due to this force vanishes. The distribution of these three forces is shown in the accompanying figure.

In discussions of apparent masses it is customary to introduce three constants, defined as follows:

$K_1 \equiv k_1$ volume; $K_2 \equiv k_2$ volume; $K' \equiv k' J$ where J is the moment of inertia of the volume when occupied by matter of density one.

In deducing the transverse forces on an actual airship, it is not correct to assume that the transverse flow is two-dimensional, especially near the ends. A fairly satisfactory formula may be obtained by multiplying each of the three terms in the approximate formula by a definite factor, depending upon the shape of the airship. Munk adduces reasons for multiplying the first term by k_2-k_1 and the other two by k'. (In this discussion there is omitted the transverse component of the centrifugal force produced by the air which is flowing longitudinally and gives rise to the longitudinal mass. It is very small.)

What has been said above applies to airships without fins. One function of the fins is to counterbalance the unstable moment. If S is the effective area of a pair of fins and b the total span, the lift exerted on them, as proved in a later section, is

$$L=2\pi \frac{\rho}{2} V^2 \frac{S}{1+\frac{2S}{b^2}} \cdot \varphi$$

where φ is the angle of attack measured in radians. If the mean distance of the fins from the center of the airship is written a, then, for the lift to balance the unstable moment,

$$La = (k_2 - k_1) \cdot \text{volume} \cdot \frac{\rho}{2} V^2 2\varphi, \text{ since } \varphi \text{ is small.}$$

Hence the area of the fins

$$S = \frac{(k_2 - k_1) \cdot \text{volume}}{a} \frac{1 + \frac{2S}{b^2}}{\pi}$$

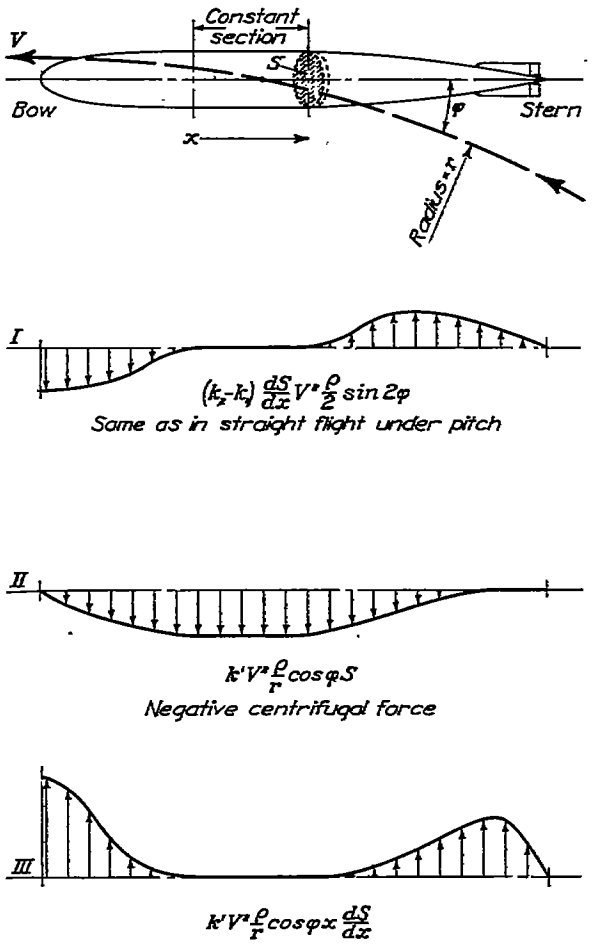

FIG. 6.—Diagram showing the direction of the transverse air forces acting on an airship flying in a turn. The three terms are to be added together

If the ship is flying in a circle of radius r, not simply must the air force on the fins balance the unstable moment, but it must produce the force required to make the airship move in a circle, i. e., $\rho \cdot \text{volume} \cdot \frac{V^2}{r}$. This can therefore be equated to the unstable moment divided by a and hence

$$\rho \cdot \text{volume} \cdot \frac{V^2}{r} = \frac{(k_2 - k_1) \cdot \text{volume} \cdot \frac{\rho}{2} V^2 2\varphi}{a}$$

or

$$\varphi = \frac{a}{r(k_2 - k_1)}$$

and this value may be substituted in the formulas giving the distribution of the transverse forces.

DISTRIBUTION OF PRESSURE OVER THE ENVELOPE OF AN AIRSHIP

It is proved in Lamb's Hydrodynamics, Chapter V, that, if an ellipsoid is moving through a fluid with constant velocity U, parallel to a principal axis, which may be called the x-axis, the velocity potential of the flow at any point of its surface is

$$\varphi = A\,U x$$

where A is a constant for a given ellipsoid.

This constant A may be expressed in terms of the apparent mass of the ellipsoid for motion parallel to the x-axis. The kinetic energy of the flow is

$$T = -\frac{1}{2}\rho \int \varphi \frac{d\varphi}{dn}\, dS \text{ over the ellipsoid.}$$

$$\frac{d\varphi}{dn}\, dS \text{ may obviously be replaced by } U\, dy\, dz.$$

Therefore

$$T = -\frac{1}{2}\rho \int A\, U^2 x\, dy\, dz = -\frac{\rho}{2} U^2 A \int x\, dy\, dz = -\frac{\rho}{2} U^2 A \cdot \text{volume of ellipsoid.}$$

But by the definition of apparent mass

$$T = \frac{\rho}{2} U^2 k_1 \cdot \text{volume} \cdot$$

Hence

$$A = -k_1, \text{ and } \varphi = -k_1 U x$$

Similarly, if the motion of the ellipsoid is oblique, so that its velocity has the components U, V, W with reference to its principal axes, the velocity potential at any point of the surface is

$$\varphi = -k_1 U x - k_2 V y - k_3 W z \qquad (10)$$

the origin of coordinates being at the center of the ellipsoid.

The values of the k's are given by certain definite integrals. If a, b, c are the semiaxes of the ellipsoid,

$$k_1 = \frac{\alpha}{2-\alpha} \text{ where } \alpha = abc \int_0^\infty \frac{dp}{(a^2+p)\sqrt{(a^2+p)(b^2+p)(c^2+p)}}$$

$$k_2 = \frac{\beta}{2-\beta} \text{ where } \beta = abc \int_0^\infty \frac{dp}{(b^2+p)\sqrt{(a^2+p)(b^2+p)(c^2+p)}}$$

$$k_3 = \frac{\gamma}{2-\gamma}, \text{ etc.}$$

For an ellipsoid of revolution

$$b = c,\ k_1 = \frac{1-k_2}{2k_2},\ k_3 = k_2$$

The following table gives values of k_1 and k_2 for different elongation ratios of an ellipsoid of revolution.

Length (diameters)	k_1 (longitudinal)	k_2 (transverse)	k_2-k_1	k' (rotation)
1	0.500	0.500	0	0
1.50	.305	.621	.316	.094
2.00	.209	.702	.493	.240
2.51	.156	.763	.607	.367
2.99	.122	.803	.681	.465
3.99	.082	.860	.778	.608
4.99	.059	.895	.836	.701
6.01	.045	.918	.873	.764
6.97	.036	.933	.897	.805
8.01	.029	.945	.916	.840
9.02	.024	.954	.930	.865
9.97	.021	.960	.939	.883
∞	.000	1.000	1.000	1.000

Similarly, if the ellipsoid is stationary in a stream of air whose velocity has the components U, V, W, the velocity potential at any point on the surface is

$$\varphi = A'Ux + B'Vy + C'Wz \qquad (11)$$

where

$$A' = 1 - A = k_1 + 1,\ B' = k_2 + 1,\ C' = k_2 + 1,$$

and are therefore known quantities for a given ellipsoid. Further, the velocity of flow at any point on the surface is along the surface; and points of constant potential lie on parallel ellipses, intersections of the ellipsoid by the family of planes $\varphi = C$.

Consider the intersection of the surface of the ellipsoid by the plane $\varphi = A'Ux + B'Vy + C'Wz = 0$. At these points the gradient of φ is along the surface; hence the velocity of flow has the components $A'U$, $B'V$, $C'W$. At any other point on the surface, the direction of the gradient is not along the surface; and, if Δh is the constant perpendicular distance between any two planes whose potentials have a constant difference and if Δs is the shortest distance along the surface between the ellipses in which these planes cut the surface, $\Delta s = \Delta h$ at the first one of the points referred to above, while at any other point $\Delta h = \Delta s \cos \epsilon$, where ϵ is the angle between the normals to the surface at the two points. Consequently the velocity of flow has its maximum at the points first described; and, calling this v_{max}, the magnitude of the velocity at any other point of the surface is $v_{max} \cos \epsilon$.

For the case of an ellipsoid of revolution the velocity at any point on the surface may be found by simple geometry, as follows. Call the plane through the line of general flow and the axis of revolution of the ellipsoid the $X-Y$ plane—in order to have a simple mode of description. Then the transverse axis of the ellipsoid which lies in this plane is the only one which need be considered. The components of the velocity of the general flow are U, V, 0; hence the maximum velocity has the components $A'U$, $B'V$, 0. Let α, β, γ be the direction cosines of the normal to the surface at any point. At a point on the surface where the velocity of flow is a maximum draw a line parallel to this normal, and call the component of v_{max} along it v_1 and that perpendicular to it v_2. v_2 is, from what has been said before, equal to the velocity of the flow at the point where the normal was originally drawn. But

$$v^2_{max} = v_1^2 + v_2^2 = (A'U)^2 + (B'V)^2 = (k_1+1)^2 U^2 + (k_2+1) V^2$$

and

$$v_1 = A'U \cdot \alpha + B'V \cdot \beta = (k_1+1)\, U \cdot \alpha + (k_2+1)\, V \cdot \beta$$

Hence

$$v_2^2 = (k_1+1)^2 U^2 + (k_2+1)^2 V^2 - ((k_1+1)\, U\, \alpha + (k_2+1) V \cdot \beta)^2 \qquad (12)$$

Then, by Bernouilli's theorem, viz, $p + \frac{1}{2}\rho V_2^2 = \text{constant}$, the pressure may be calculated.

With a very elongated ellipsoid, k_1 is small and k_2 nearly equals 1. Hence $A' = 1$ and $B' = 2$, so the components of maximum velocity are U and $2V$. Consequently, while the angle of attack is defined by $\tan \alpha = \frac{V}{U}$, the direction of maximum flow makes an angle φ with the longitudinal axis, where $\tan \varphi = \frac{2V}{U}$. Therefore φ is about twice α.

CONCLUSION

Considering an airship as an ellipsoid of revolution of known volume and elongation ratio, so that k_1, k_2, and k' (and also K_1 and K_2) are known,

1. The unstable moment, for an angle of yaw φ, in straight or circling flight, is

$$M = \frac{\rho}{2} V^2 (K_2 - K_1) \sin 2\varphi$$

2. The transverse force per unit length is, where S is the cross section at a point x,

(a) For straight flight,

$$(k_2-k_1)\frac{\rho}{2}V^2\sin 2\varphi\cdot\frac{dS}{dx}$$

(b) For circling flight

$$(k_2-k_1)\frac{\rho}{2}V^2\cdot\sin 2\varphi\frac{dS}{dx}+k'V^2\frac{\rho}{r}S\cos\varphi+k'V^2\frac{\rho}{r}\cos\varphi x\frac{dS}{dx}$$

3. The pressure over the envelope is given by the formula

$$p=C-\frac{1}{2}\rho v^2$$

where, if U and V are the components of flight velocity with respect to the longitudinal and transverse axes,

$$v^2=(k_1+1)^2U^2+(k_2+1)^2V^2-((k_1+1)U\alpha+(k_2+1)V\cdot\beta)^2$$

α and β being the direction cosines of the normal to the surface at the point at which the pressure is to be calculated.

PROBLEMS MORE SPECIALLY CONCERNING AIRFOILS AND AIRPLANES

INTRODUCTION

In outlining a theory of an airplane wing it is necessary to show how, assuming certain constructional data, it is possible to calculate, among other things, the lift, the drag due to other causes than viscosity, the pitching moment, and the rolling moment. In the simplest type of wing, that whose chord section is a straight line, flying at a definite angle of attack, the values of the lift and the pitching moment can be calculated immediately. They are seen to depend upon the transverse velocity of the air flow perpendicular to the chord. Similarly, in discussing the properties of a wing whose section is a curved line, if the distribution of the transverse velocity at the points of the chord is known, the lift and pitching moment may be calculated, as will be shown. So the first essential step in the theory of the wing is to discuss mathematical methods of representing arbitrary distributions of transverse velocity over the chord, and to deduce the nature of the consequent flow. It will be shown how the distribution of velocity may be so expressed as to lead to formulas for the lift and pitching moment in terms of quantities known to the designer.

In all this discussion an essential element is the angle of attack; but it is evident that the geometrical angle of attack is not the effective one, owing to the fact that the direction of the relative wind is affected by the presence of the wing. Owing to this modification of the air flow, there is a drag introduced, known as the "induced drag," and the effective angle of attack is the geometrical one diminished by what is called the "induced" angle of attack. The problem is to calculate these and then their effect upon the lift. One method of approach to the problem is to assume as known the distribution of lift along the span, but another and better one is to assume as known the angle of attack at all points along the span and to apply the general method to wings having particular plane forms. It will be seen that all these methods lead back to the discussion of the distribution of the transverse down-wash velocity along the span.

If an airfoil has an infinite span, the flow around it when the air stream is perpendicular to its span may be regarded as two-dimensional. The air particles in a longitudinal plane, i. e., one including the line of flow of the air stream and perpendicular to the span, remain in the plane. Further, if an airfoil of finite span is advancing into a transverse vertical layer of air, it imparts to the air velocity in this plane, so that again one can consider this transverse flow as being two-dimensional about a plane figure which is the projection of the airfoil on this transverse plane. The simplest case of motion in the longitudinal plane is to consider the longitudinal section of the airfoil to be a straight line of a length equal to the chord, and the simplest case of motion

in the transverse plane is to consider the front aspect of an airfoil to be a straight line of a length equal to the span.

The importance of two-dimensional flows requires a brief statement of the properties of conjugate functions which are so useful in all two-dimensional problems. All the cases of flow to be discussed will be those described by a velocity potential, which then satisfies the equation

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0$$

Writing $z=x+iy$, if

$$F=f(z)=P(x,y)+iQ(x,y),$$

P and Q are called conjugate functions, because if F is any analytic function

$$\frac{\partial P}{\partial x}=\frac{\partial Q}{\partial y},\ \frac{\partial P}{\partial y}=-\frac{\partial Q}{\partial x}.$$

It follows that

$$\frac{\partial^2 P}{\partial x^2}+\frac{\partial^2 P}{\partial y^2}=0.\quad \frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}=0$$

to that both P and Q satisfy the fundamental equation for the velocity potential, and, further shat

$$\frac{dF}{dz}=\frac{\partial P}{\partial x}-i\frac{\partial P}{\partial y}$$

Consequently, if P is chosen as the velocity potential φ, i. e., if φ is the real part of F, then the real part of $\frac{dF}{dz}$ gives the component of the velocity at any point in the direction of the x-axis, and the imaginary part of $\frac{dF}{dz}$ gives the component of the velocity, at any point, in the negative direction of the y-axis. Therefore the whole motion is defined by the knowledge of F as a unction of the single variable z.

ILLUSTRATIONS OF TWO-DIMENSIONAL FLOW

1. Let

$$F=iV(z-i\sqrt{1-z^2}) \qquad (13)$$

Then

$$\frac{dF}{dz}=iV\left(1+\frac{iz}{\sqrt{1-z^2}}\right)=-\frac{Vz}{\sqrt{1-z^2}}+iV \qquad (14)$$

There are evidently two singular points $z=\pm 1$, i. e., $x=1$, $y=0$ and $x=-1$, $y=0$; and at infinity $\frac{dF}{dz}=0$

Along the line joining the two singular points $(-1, 0)$ and $(+1, 0)$, $y=0$; hence

$$F_0=\pm V\sqrt{1-x_0^2}+iVx_0$$

$$F'_0\equiv\left(\frac{dF}{dz}\right)_0=\mp\frac{Vx_0}{\sqrt{1-x_0^2}}+iV$$

in which the upper signs apply to points on the positive side of the line (i. e., where y is positive), and conversely. Hence, for points on the line

$$\varphi_0=\pm V\sqrt{1-x_0^2} \qquad (15)$$

Further, the longitudinal velocity, i. e., the velocity along the x-axis, is

$$v_o = \mp \frac{V x_o}{\sqrt{1-x_o^2}} \qquad (16)$$

and the transverse velocity downward is

$$u_o = V \qquad (17)$$

The general function F, which was assumed to begin with, represents, therefore, the two-dimensional flow around a straight line of length 2 moving transversely downward with a velocity V. (Or, the general flow about a lamina, infinite in length and a width 2, moving transversely with velocity V.)

Near the positive end of the line, i. e., when $x_o = 1-\epsilon$, ϵ being a small quantity, $v = \mp \frac{V}{\sqrt{2\epsilon}}$, and therefore v becomes infinite on both sides of the line, but is "outward" on the lower side and inward on the upper.

It should be noted that in defining a function F which leads to a value $\varphi_o = \pm V\sqrt{1-x_o^2}$, the *difference of potential* between two points on opposite sides of the line is $2V\sqrt{1-x_o^2}$. This is at once evident if polar coordinates are used. In that case, writing $z = \cos\delta$, where δ is a complex number,

$$F = iVe^{-i\delta} = V(\sin\delta + i\cos\delta) \qquad (13a)$$

$$F' = \frac{dF}{dz} = \frac{dF}{d} \cdot \frac{d\delta}{dz} = -\frac{Ve^{-i\delta}}{\sin\delta} = -\frac{V}{\sin\delta}(\cos\delta - i\sin\delta) \qquad (14a)$$

For points on the line $x_o = \cos\delta_o$ where δ_o is real, and

$$\varphi_o = V\sin\delta_o \qquad (15a)$$

$$v_o = -\frac{V\cos\delta_o}{\sin\delta_o},\ u_o = V$$

hence at

$$\delta_o = O,\ v_{edge} = -\frac{V}{(\sin\delta_o)_o} \qquad (18)$$

As x_o goes from $+1$ to -1 on the upper side of the line δ increased from O to π; and, as the point returns on the lower side of the line, δ increases from π to 2π. (The flow is shown in the Figure 7.)

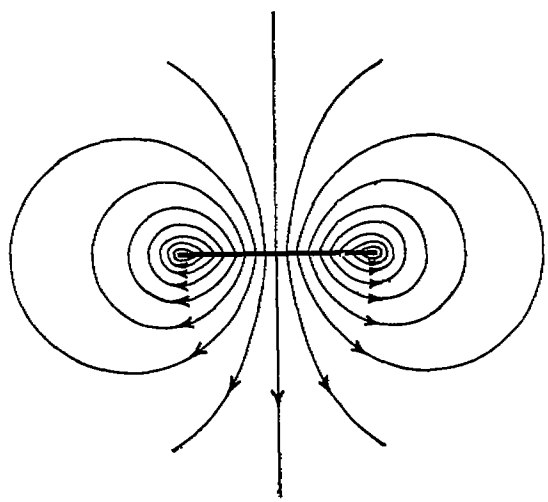

FIG. 7.—Transverse flow, produced by a moving straight line

If the line has the length b, stretching from $\left(-\frac{b}{2}, O\right)$ to $\left(+\frac{b}{2}, O\right)$, we may write

$$F=iV\frac{b}{2}\left(\frac{2}{b}z-i\sqrt{1-\left(\frac{2}{b}z\right)^2}\right) \qquad (13')$$

Then

$$F'=-\frac{V\frac{2}{b}z}{\sqrt{1-\left(\frac{2}{b}z\right)^2}}+iV \qquad (14')$$

Thus

$$\varphi_0=\pm V\frac{b}{2}\sqrt{1-\left(\frac{2}{b}x_0\right)^2} \qquad (15')$$

$$v_0=\mp\frac{V\frac{2}{b}x_0}{\sqrt{1-\left(\frac{2}{b}x_0\right)^2}},\ u_0=V \qquad (16')$$

Or, using polar coordinates, writing $\frac{2}{b}z=\cos\delta$,

$$F=iV\frac{b}{2}e^{-i\delta}=V\frac{b}{2}(\sin\delta+i\cos\delta) \qquad (13'a)$$

$$F'=-\frac{V}{\sin\delta}(\cos\delta-i\sin\delta) \qquad (14'a)$$

$$\varphi_0=V\frac{b}{2}\sin\delta_0 \qquad (15'a)$$

$$v_0=-\frac{V\cos\delta_0}{\sin\delta_0},\ u_0=V$$

and

$$v_{edge}=-\frac{V}{(\sin\delta_0)_0} \qquad (18')$$

It has been proved that the kinetic energy of the flow is

$$T=-\frac{\rho}{2}\int\varphi\frac{d\varphi}{dn}dS$$

integrated over the moving body, where dn is the normal away from the body. In the case o the line of length b, moving transversely with velocity V

$$\varphi_0=\pm V\frac{b}{2}\sqrt{1-\left(\frac{2}{b}x_0\right)^2};\ \left(\frac{d\varphi}{dn}\right)_0=\mp V;\ dS=dx_0$$

Hence, integrating over both sides,

$$T=\frac{\rho}{2}V^2\frac{b}{2}\cdot 2\int_{-b/2}^{+b/2}\sqrt{1-\left(\frac{2}{b}x_0\right)^2}dx_0$$

or writing

$$\cos\delta_0=\frac{2}{b}x_0$$

$$T=\frac{\rho}{2}V^2b\cdot\frac{b}{2}\int_0^\pi\sin^2\delta_0 d\delta_0=\frac{\rho}{2}V^2\pi\left(\frac{b}{2}\right)^2 \qquad (19)$$

But $\pi\left(\frac{b}{2}\right)^2$ is the area of a circle described about the line of length b as a diameter. The apparent transverse mass per unit length, then, of a lamina infinite in length and of width b is $\rho\pi\left(\frac{b}{2}\right)^2$, the same as for a circular cylinder of diameter b moving transversely.

2. A second function, suggested by the first, is

$$F=iA_n e^{-in\delta}=A_n(\sin n\delta+i\cos n\delta) \qquad (20)$$

where $z=\cos\delta$.

Therefore

$$F'=-A_n\frac{ne^{-in\delta}}{\sin\delta}=-A_n\frac{n\cos n\delta}{\sin\delta}+iA_n\frac{n\sin n\delta}{\sin\delta} \qquad (21)$$

Hence

$$\varphi_o=A_n\sin n\delta_o \text{ [for points on the line joining } (-1, O) \text{ and } (+1, O)] \qquad (22)$$

$$v_o=-A_n\frac{n\cos n\delta_o}{\sin\delta_o} \qquad (23)$$

$$u_o=A_n\frac{n\sin n\delta_o}{\sin\delta_o} \qquad (24)$$

Note that u_o is no longer the same for all points of the line.

For a line of length b, stretching from $\left(-\frac{b}{2}, O\right)$ to $\left(+\frac{b}{2}, O\right)$ we may write, if we wish the same expressions for v_o and u_o as just found,

$$F=A_n\frac{b}{2}(\sin n\delta+i\cos n\delta), \text{ where } \cos\delta=\frac{2}{b}z \qquad (20')$$

Therefore

$$F'=-A_n\frac{n\cos n\delta}{\sin\delta}+iA_n\frac{n\sin n\delta}{\sin\delta} \qquad (21')$$

$$\varphi_o=A_n\frac{b}{2}\sin n\delta_o \qquad (22')$$

$$v_o=-A_n\frac{n\cos n\delta_o}{\sin\delta_o} \qquad (23')$$

and

$$u_o=A_n\frac{n\sin n\delta_o}{\sin\delta_o}, \text{ where } \frac{2}{b}x_o=\cos\delta_o \qquad (24')$$

In the formula, therefore, for the kinetic energy of the flow

$$\varphi_o=A_n\frac{b}{2}\sin n\delta_o; \left(\frac{d\varphi}{dn}\right)_o=-A_n\frac{n\sin n\delta_o}{\sin\delta_o}$$

and

$$dS=dx_o=-\frac{b}{2}\sin\delta_o\,d\delta_o,$$

and

$$T=\frac{\rho}{2}A_n^2\left(\frac{b}{2}\right)^2 n\int_0^{2\pi}\frac{\sin^2 n\delta_o}{\sin\delta_o}\cdot\sin\delta_o d\delta_o=\frac{\rho}{2}A_n^2\pi\left(\frac{b}{2}\right)^2 n \qquad (25)$$

The impulse per unit length required to create the flow is $-\rho\varphi_0$; hence the total impulse is $-\rho\int\varphi_0\,dx$, taken over both sides of the line, i. e.,

$$P=\rho\int_0^{2\pi}A_n\frac{b}{2}\sin n\delta_0\cdot\frac{b}{2}\sin\delta_0 d\delta_0=\rho\pi\left(\frac{b}{2}\right)^2A_n$$

since $\int^{\pi}\sin n\theta\cdot\sin\theta\cdot d\theta=O$ unless $n=1$

3. A more general case would then be a superposition of flows like the one last considered, described by

$$F=i\{A_1(z-i\sqrt{1-z^2})+A_2(z-i\sqrt{1-z^2})^2+\text{etc.}\} \qquad (26)$$

where it is assumed that it is possible to so choose the coefficients as to make the series involved convergent. Obviously, then, for points on the line connecting $(-1, O)$ and $(+1, O)$,

$$\varphi_0'=A_1\sin\delta_0+A_2\sin 2\delta_0+\text{etc., where } \cos\delta_0=x_0 \qquad (27)$$

$$v_0=-\left\{\frac{A_1\cos\delta_0}{\sin\delta_0}+\frac{2A_2\cos 2\delta_0}{\sin\delta_0}+\text{etc.}\right\} \qquad (28)$$

$$u_0=\frac{A_1\sin\delta_0}{\sin\delta_0}+\frac{2A_2\sin 2\delta_0}{\sin\delta_0}+\text{etc} \qquad (29)$$

For a line of length b, i. e., making $\cos\delta_0=\frac{2}{b}x_0$

$$\varphi_0=\frac{b}{2}(A_1\sin\delta_0+A_2\sin 2\delta_0+\text{etc.}) \qquad (27')$$

if v_0 and u_0 are to have the same forms as before.

In the formula for the kinetic energy, then,

$$T=\frac{\rho}{2}\left(\frac{b}{2}\right)^2\int_0^{2\pi}(A_1\sin\delta_0+A_2\sin 2\delta_0+\cdots)(A_1\sin\delta_0+2A_2\sin 2\delta_0+\cdots)\,d\delta_0$$

$$=\frac{\rho}{2}\pi\left(\frac{b}{2}\right)^2(A_1^2+2A_2^2+\cdots\cdots) \qquad (30)$$

Therefore, not simply are the kinematic properties of the separate flows additive, but the energies also.

The constants in the formula for F may be determined by various physical specifications:

a. Let the distribution of potential be known at all points of the line between $(-1, O)$ and $(+1, O)$, the function having equal and opposite values at opposite points.

Then it is possible to so choose coefficients that φ_0 may be expressed as follows:

$$\varphi_0=A_1\sin\delta_0+A_2\sin 2\delta_0+\text{etc.}$$

for, on multiplying both sides of the equation by $\sin n\delta_0$ and integrating from O to π,

$$\int_0^{\pi}\varphi_0\cdot\sin n\delta_0 d\delta_0=\int_0^{\pi}(A_1\sin\delta_0+A_2\sin 2\delta_0+\cdots)\sin n\delta_0\cdot d\delta_0=\int_0^{\pi}A_n\sin^2 n\delta_0 d\delta_0=\frac{\pi}{2}A_n$$

Hence

$$A_n=\frac{2}{\pi}\int_0^{\pi}\varphi_0\sin n\delta_0 d\delta_0 \qquad (31)$$

and is therefore determined. These values of the A's may then be substituted in the general function F. (For a line of length b,

$$A_n=\frac{2}{b}\cdot\frac{2}{\pi}\int_0^{\pi}\varphi_0\sin n\delta_0 d\delta_0 \qquad (31')$$

in formula 27′, etc.)

(A better mode of expression would be to specify that the difference of potential between opposite points is known for each point of the line, i. e., $\Delta\varphi$ is specified. Then expand $\frac{1}{2}\Delta\varphi$ in a series $A_1 \sin \delta_0 + A_2 \sin 2\delta_0 + \text{etc.}$)

b. Let the distribution of longitudinal velocity be specified at each point of the line of length 2, being equal and opposite at opposite points.

We may then consider $v_0 \sin \delta_0$ as known at each point, and this can be expanded into a series

$$v_0 \sin \delta_0 = -(A_1 \cos \delta_0 + 2A_2 \cos 2\delta_0 + \text{etc.})$$

if the A's are given proper values, viz, since

$$\int_0^\pi v_0 \sin \delta_0 \cdot \cos n\delta_0 \, d\delta_0 = -\int_0^\pi nA_n \cos^2 n\delta_0 \, d\delta_0 = -\frac{\pi}{2} nA_n, \; A_n = -\frac{2}{n\pi}\int_0^\pi v_0 \sin \delta_0 \cdot \cos n\delta_0 \, d\delta_0 \quad (32)$$

and these values may be substituted in the function F in order to determine the flow at all points.

(These same values for the A's are to be used in the formulas for a line of length b.)

c. Let the transverse velocity be specified at each point of the line, having the same value at opposite points. Then $u_0 \sin \delta_0$ is known for each point, and this may be expanded into a series

$$u_0 \sin \delta_0 = A_1 \sin \delta_0 + 2A_2 \sin 2\delta_0 + \text{etc.}$$

by giving the coefficients proper values, viz, since

$$\int_0^\pi u_0 \sin \delta_0 \cdot \sin n\delta_0 \, d\delta_0 = \int_0^\pi nA_n \sin^2 n\delta_0 \, d\delta_0 = \frac{\pi}{2} nA_n, \; A_n = \frac{2}{\pi n}\int_0^\pi u_0 \sin \delta_0 \cdot \sin n\delta_0 \, d\delta_0 \quad (33)$$

These values may be substituted in F, etc. (These same values for the A's are to be used in the formulas for a line of length b.)

The essential thing is that, if specification a, b, or c is made, the flow at all points in space may be deduced.

4. If $f(z)$ is a flow function and contains a parameter x_0, then $f(z, x_0)\, u_0 \, dx_0$ is also a solution of the equations if u_0 is a function of x_0. Hence also

$$F = \int_{-1}^{+1} f(z, x_0)\, u_0 \, dx_0 \quad (34)$$

is a solution, and

$$F' = \int_{-1}^{+1} f'(z, x_0)\, u_0 \, dx_0, \text{ where } f' = \frac{df}{dz} \quad (35)$$

A value of $f(z, x_0)$ suggested by Munk is

$$f \equiv \frac{1}{\pi}\{\log(e^{i\delta} - e^{-i\delta_0}) - \log(e^{i\delta} - e^{i\delta_0})\} \quad (36)$$

where $\cos \delta = z$ and $\cos \delta_0 = x_0$.

This solution F may be interpreted physically by deducing the meaning of each elementary term.

$$f' = -\frac{1}{\pi}\frac{\sin \delta_0}{\sin \delta}\frac{1}{\cos \delta - \cos \delta_0} = \mp\frac{1}{\pi}\frac{1}{z - x_0}\sqrt{\frac{1 - x_0^2}{1 - z^2}} \quad (37)$$

where the negative sign is to be taken over the positive side of the line, and the positive sign on the other. There is evidently a singular point at $z = x_0$. For points close to this—not necessarily on the line—$f' = \mp\frac{1}{\pi}\frac{1}{z - x_0}$. Therefore an element of F', that is, $f' \, u_0 \, dx_0$, when applied to these points, has the value

$$\mp\frac{1}{\pi}\frac{1}{z - x_0} u_0 \, dx_0 = \mp\frac{1}{\pi} u_0 \, dx_0 \frac{x - x_0 - iy}{(x - x_0)^2 + y^2}$$

If a small circle of radius r is drawn around the point x_0, and the point x, y lies on it, $r^2=(x-x_0)^2+y^2$. The velocity along the x-axis for points on the positive side of the line is $-\frac{1}{\pi}u_0\,dx_0\,\frac{x-v_0}{r^2}$, and the velocity along the y-axis is $-\frac{1}{\pi}u_0\,dx_0\,\frac{y}{r^2}$; hence there is a radial velocity inward toward x_0, of the value $\frac{1}{\pi}u_0\,dx_0\cdot\frac{1}{r}$. Therefore the total flow per second in through the semicircle is $\rho u_0\,dx_0$. Similarly there is an equal outward flow through the semicircle on the negative side of the line.

This is equivalent, then, to there being a transverse velocity u_0 at all points of the element dx_0, toward it on the positive side of the line and away from it on the other. This gives the physical meaning of u_0.

The total function

$$F'=\mp\int_{-}^{+1}\frac{1}{\pi}\frac{u_0\,dx_0}{z-x_0}\sqrt{\frac{1-x_0^2}{1-z^2}} \qquad (38)$$

indicates the effect at a point z of a given distribution of transverse velocity, u_0 being the downward velocity at the point x_0, on both sides of the line. The longitudinal velocity, due to this distribution, at a point x on the line is

$$v=\mp\int_{-1}^{+1}\frac{1}{\pi}\frac{u_0\,dx_0}{x-x_0}\sqrt{\frac{1-x_0^2}{1-x^2}}$$

Interchanging symbols, the velocity at a point x_0 on the line is

$$v_0=\pm\int_{-1}^{+1}\frac{1}{\pi}\frac{u\,dx}{x-x_0}\sqrt{\frac{1-x^2}{1-x_0^2}} \qquad (39)$$

where u is the transverse velocity downward at the point x.

For a point near the positive edge, write $x_0=1-\epsilon$ where ϵ is small. Then since $\sin\delta_0=\sqrt{1-x_0^2}=\sqrt{2\epsilon}$,

$$v_{edge}=-\frac{1}{\pi\,(\sin\delta_0)_{\delta_0=0}}\int_{-1}^{+1}u\sqrt{\frac{1+x}{1-x}}\,dx \qquad (40)$$

The flow, due to a single element, is hown in Figure 8.

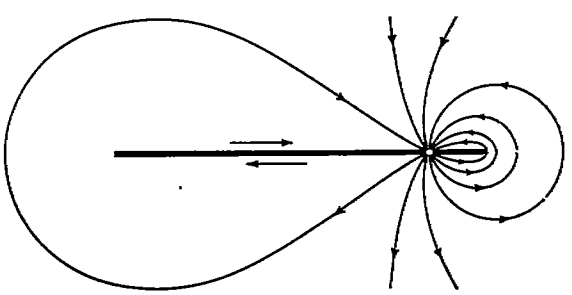

FIG. 8.—Flow around a straight line created by one element of the wing sect on

If the line has the length b stretching between

$$\left(-\frac{b}{2},0\right)\text{ and }\left(+\frac{b}{2},0\right)\text{ let}$$

$$F=\int_{-b/2}^{+b/2}f(z,\,x_0)\;u_0\,dx_0$$

where

$$f=\frac{1}{\pi}\left\{\log\left(e^{i\delta}-e^{-i\delta_0}\right)-\log\left(e^{i\delta}-e^{i\delta_0}\right)\right\}$$

and $\cos\delta=\frac{2}{b}z$ and $\cos\delta_0=\frac{2}{b}x_0$. This leads to a transverse downward velocity u_0 at x_0, etc.

Finally

$$v_{edge} = -\frac{2}{b}\frac{1}{\pi(\sin\delta_0)_{\delta_0=\delta}}\int_{-b/2}^{+b/2} u\sqrt{\frac{1+\frac{2}{b}x}{1-\frac{2}{b}x}}\,dx \qquad (40')$$

Two expressions have been deduced, therefore, for the flow due to an arbitrary distribution of transverse velocity over a line of length b:

1. $$\varphi = \frac{b}{2}(A_1 \sin\delta + A_2 \sin 2\delta + \text{etc.})$$

in which

$$A_n = \frac{2}{\pi n}\int_0^{\pi} u_0 \sin\delta_0 \cdot \sin n\delta_0 \cdot d\delta_0$$

where $\cos\delta = \frac{2}{b}z$ and u_0 is the transverse velocity downward at the point x_0.

2. φ is the real part of

$$F = \int_{-b/2}^{+b/2} f(z, x_0)\, u_0\, dx_0$$

in which

$$f(z, x_0) = \frac{1}{\pi}\{\log(e^{i\delta} - e^{-i\delta_0}) - \log(e^{i\delta} - e^{i\delta_0})\}$$

These are, of course, mathematically identical.

5. A flow of a different kind entirely is given by

$$F = A_0 \sin^{-1} z \qquad (41)$$

This makes

$$F' = \pm\frac{A_0}{\sqrt{1-z^2}} = \frac{A_0}{\sin\delta}, \text{ where } \cos\delta = z \qquad (42)$$

and

$$v_0 = \frac{A_0}{\sin\delta} \qquad (43)$$

Therefore for points on the line between $(-1, 0)$ and $(+1, 0)$, $u = 0$ on both sides of the line and v is positive on the upper side and negative on the other. The flow is as indicated.

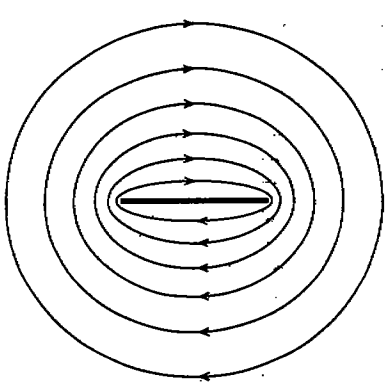

FIG. 9.—Circulation flow around a straight line

F is a multiple valued function, its modulus being $2\pi A_0$. For points on the line $y = 0$, beyond $x = -1$ and $x = +1$, $\varphi = A_0 \sin^{-1} x$; consequently there is a difference of potential $2\pi A_0$ between two points lying on opposite sides of the line, since each line of flow incloses the origin.

This flow can not be produced by impulsive pressures over the line between $x=-1$ and $x=+1$, because the flow is everywhere parallel to the surface. It can be imagined produced by impulses over all points of the line $y=0$, extending from one end of the line of length 2, out to infinity. At all points there is a potential difference $2\pi A_0$; hence the downward impulse per unit length of the line required to generate the motion is $2\pi A_0\rho$. But if the line of length 2 be considered an airplane wing, and if it moves with a velocity V longitudinally, it must deliver to the air per second a momentum downward equal to the lift on the wing, L. Therefore since this momentum is imparted in going a distance V, the momentum imparted per unit length, i. e., the impulse per unit length, is $\frac{L}{V}$. Hence

$$\frac{L}{V}=2\pi A_0\rho \cdots \text{(Kutta's theorem)} \qquad (44)$$

or

$$A_0=\frac{L}{2\pi\rho V}$$

and, from (43)

$$v_{edge}=\frac{L}{2\pi\rho V(\sin\delta_0)_{\delta_0=0}} \qquad (45)$$

For a line of length b, stretching from $\left(-\frac{b}{2}, 0\right)$ to $\left(+\frac{b}{2}, 0\right)$ write

$$F=A_0\sin^{-1}\left(\frac{2}{b}z\right) \qquad (41')$$

Hence

$$F'=\pm\frac{A_0\frac{2}{b}}{\sqrt{1-\left(\frac{2}{b}z\right)^2}}=\frac{A_0\frac{2}{b}}{\sin\delta} \qquad (42')$$

As before,

$$\frac{L}{V}=2\pi A_0\rho,$$

and therefore

$$v_{edge}=\frac{2}{b}\frac{L}{2\pi\rho V(\sin\delta_0)_{\delta_0=0}} \qquad (45')$$

In these formulas L is the lift per unit length along the infinite span, since the problem is treated as a two-dimensional one.

ANGLE OF ATTACK AND LIFT WING SECTION THEORY

In discussing suitable combinations of types of flow for application to airplane wings, it is essential to include a circulation flow so as to secure lift, and also so to choose the types that the total flow divides exactly at the trailing edge. The condition for the latter is that $v_{edge}=0$. (Kutta was the first to state this condition.)

A. STRAIGHT LINE, ANGLE OF ATTACK α

In order to introduce the angle of attack, consider the problem of the straight line of length 2 moving with a velocity V in a direction making the angle α with the line. The transverse velocity is $V\sin\alpha$, and hence the flow is given by (13a) as

$$F=V\sin\alpha\cdot i\cdot e^{-i\delta} \qquad (46)$$

and the longitudinal velocity at the trailing edge is, by (18),

$$-V \sin \alpha \frac{1}{(\sin \delta_0)_{\delta_0=0}} \qquad (47)$$

Since α is small, F can also be taken as the flow function for a line inclined to the axis of x by an angle α having a velocity V in the direction of the axis, v and u now referring to the line of flight. (This approximation was proposed and used by Munk.)

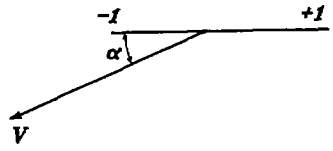

FIG. 10

Due to a circulation flow around the line of length 2, given by $F = A_0 \sin^{-1} z$, the longitudinal edge velocity is, from (45),

$$\frac{L}{2\pi\rho V} \frac{1}{(\sin \delta_0)_{\delta_0=0}}$$

Hence, if $v_{edge} = 0$ due to the two flows,

$$\frac{L}{2\pi\rho V} = V \sin \alpha$$

or

$$L = 2\pi\rho V^2 \sin \alpha \qquad (48)$$

Introducing the area, $S = 2$ since the span is one, and, writing α in place of $\sin \alpha$,

$$L = 2\pi \frac{\rho}{2} V^2 S\alpha \qquad (48a)$$

giving a lift coefficient

$$C_L = 2\pi\alpha$$

If the line has a length b, the two edge velocities are, by (18′) and (45′),

$$-V \sin \alpha \frac{1}{(\sin \delta_0)_{\delta_0=0}} \text{ and } \frac{2}{b} \frac{L}{2\pi\rho V} \frac{1}{(\sin \delta_0)_{\delta_0=0}}$$

Hence

$$L = 2\pi\rho V^2 \frac{b}{2} \sin \alpha \qquad (48')$$

But $S = b$, and therefore, as before,

$$L = 2\pi \frac{\rho}{2} V^2 S\alpha \qquad (48'a)$$

B. CURVED LINE, ZERO ANGLE OF ATTACK; "APPARENT" ANGLE OF ATTACK

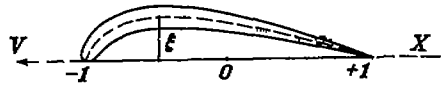

FIG. 11

If the wing is a thin cambered one, it is equivalent to a good approximation, to a curve which is the mean of the upper and lower curves of the wing section. Consider, then, the problem of the motion of such a curved line whose chord is the x-axis, having a velocity V in

the negative direction of the chord. Let ξ be the ordinate of the curve at the point x. Any element of the curve is then moving with the angle of attack whose tangent is $-\frac{d\xi}{dx}$. Therefore, at this element the component of V downward (i. e., as shown above, $V \sin \alpha$, or $V\alpha$) is $-V\frac{d\xi}{dx}$ if the curvature is small. This is to be substituted in the formula previously deduced for the case of a variable transverse velocity along the chord, viz, for a chord of length 2, from (40),

$$v_{edge}=\frac{V}{\pi\ (\sin \delta_0)_{\delta_0=0}}\int_{-1}^{+1}\frac{d\xi}{dx}\sqrt{\frac{1+x}{1-x}}\,dx \qquad (49)$$

This leads to a definition of the "mean apparent angle of attack," viz, the angle of attack which a straight line having a chord of equal length would have to possess in order to give this same value of edge velocity and therefore the same lift. Calling this angle α', the condition, then, is, from (47),

$$-V \sin \alpha' \frac{1}{(\sin \delta_0)_{\delta_0=0}}=\frac{V}{\pi\ (\sin \delta_0)_{\delta_0=0}}\int_{-1}^{+1}\frac{d\xi}{dx}\sqrt{\frac{1+x}{1-x}}\,dx$$

Hence

$$\alpha'=-\frac{1}{\pi}\int_{-1}^{+1}\frac{d\xi}{dx}\sqrt{\frac{1+x}{1-x}}\,dx=\frac{1}{\pi}\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}} \qquad (50)$$

since for $x=1$, $\xi=0$.

For a line of length b, the angle of attack of each element and the component of velocity downward are as before and, from (40′),

$$v_{edge}=\frac{2}{b}\frac{V}{\pi(\sin\delta_0)_{\delta_0=0}}\int_{-b/2}^{+b/2}\frac{d\xi}{dx}\sqrt{\frac{1+\frac{2}{b}x}{1-\frac{2}{b}x}}\,dx \qquad (49')$$

Hence

$$\alpha'=-\frac{2}{b}\frac{1}{\pi}\int_{-b/2}^{+b/2}\frac{d\xi}{dx}\sqrt{\frac{1+\frac{2}{b}x}{1-\frac{2}{b}x}}\,dx=\left(\frac{2}{b}\right)^2\frac{1}{\pi}\int_{-b/2}^{+b/2}\frac{\xi dx}{\left(1-\frac{2}{b}x\right)\sqrt{1-\left(\frac{2}{b}x\right)^2}} \qquad (50')$$

Since for any given wing section ξ is specified as a $f(x)$, these integrals may be evaluated and α' may be calculated.

CONCLUSION

Considering the wing as one of infinite span, the lift on a cambered wing of chord c and area S, when at zero angle of attack, is

$$L=2\pi\frac{\rho}{2}V^2S\alpha'$$

where

$$\alpha'=\left(\frac{2}{c}\right)^2\frac{1}{\pi}\int_{-c/2}^{+c/2}\frac{\xi dx}{\left(1-\frac{2}{c}x\right)\sqrt{1-\left(\frac{2}{c}x\right)^2}}$$

In this formula ξ is the ordinate from the chord to the mean curve of the upper and lower surfaces of the wing section.

(For simple methods of calculating α' from wing profiles, see N. A. C. A. Technical Note, 122.)

PITCHING MOMENT AND CENTER OF PRESSURE

In the case of a straight line of length 2 moving with a velocity V at an angle of attack α the moment acting on it due to the air forces may be calculated at once from the general theorem already proved, viz: The moment equals the product of the velocity and the component, perpendicular to the velocity, of the momentum of the air flow. Such a line has a transverse

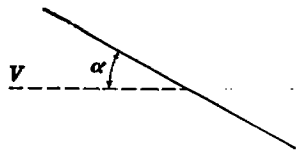

FIG. 12

mass $\pi\rho$, and hence a momentum, perpendicular to the line, of $\pi\rho \cdot V \sin \alpha$. Its component perpendicular to the line of V is then $\pi\rho\ V \sin \alpha \cdot \cos \alpha$; and therefore the pitching moment (clockwise), for unit span, is

$$M = V^2 \frac{\rho}{2}\ \pi \sin 2\alpha \qquad (51)$$

or

$$= 2\pi\ \frac{\rho}{2}\ V^2 \cdot \alpha$$

since α is small.

The lift was found, (48), to have the value, for a wing of unit span,

$$L = 2\pi\rho\ V^2 \cdot \alpha$$

Hence the distance of the "center of pressure" from the center of the line is

$$\frac{M}{L} = \frac{1}{2} \qquad (52)$$

It is therefore independent of α and is 25% of the length of the chord from the leading edge.

For a line of length b the transverse mass is $\pi \left(\frac{b}{2}\right)^2 \rho$, and hence

$$M = 2\pi \left(\frac{b}{2}\right)^2 \frac{\rho}{2}\ V^2 \alpha \qquad (51')$$

Further, from (48′),

$$L = 2\pi\ \frac{b}{2}\ \rho V^2 \alpha;$$

Hence

$$\frac{M}{L} = \frac{1}{2}\ \frac{b}{2} \qquad (52')$$

i. e., the center of pressure is "at 25%", and is independent of α.

In the case of a curved line, in order to deduce the center of pressure, it is necessary to calculate the distribution of pressure over the line. By Bernouilli's theorem the pressure at any point equals $C - \frac{\rho}{2}$ (velocity)^2 where C is a constant. The general formula for the longitudinal velocity is, (see (28))

$$v_o = -\left(A_1 \frac{\cos \delta_o}{\sin \delta_o} + 2A_2 \frac{\cos 2\delta_o}{\sin \delta_o} + \cdots\right)$$

This may be applied to any element of the curve, and is the velocity of the flow toward the right; but since the curved line itself has a velocity V toward the left, the relative velocity between the air and the wing is $V + v_o$ or

$$V - \left(A_1 \frac{\cos \delta_o}{\sin \delta_o} + 2A_2 \frac{\cos 2\delta_o}{\sin \delta_o} + \cdots\right)$$

In squaring this, the squares of the A's may be omitted, since in the integration given below the corresponding terms would disappear. Hence

$$p=C-\frac{\rho}{2}V^2+\rho V\left(A_1\frac{\cos\delta_0}{\sin\delta_0}+2A_2\frac{\cos 2\delta_0}{\sin\delta_0}+\cdots\right) \qquad (53)$$

The first two terms are the same for all points on both sides of the line and therefore produce no moment. The second term gives equal and opposite values of p at two opposite points on the line, i. e., if it is a pressure on one side it will be a suction on the other; therefore, the pitching moment (clockwise) for unit span,

$$M=2\int_{-1}^{+1}px\,dx=2\int_{-1}^{+1}p\cos\delta_0\sin\delta_0\,d\delta_0,$$

where $x=\cos\delta_0$

$$=2V\rho\int_0^{\pi}(A_1\cos\delta_0+2A_2\cos 2\delta_0+\cdots)\cos\delta_0\,d\delta_0=2V\rho\cdot A_1\frac{\pi}{2} \qquad (54)$$

But the value of A_1 in terms of the transverse velocity was found previously, (33), to be

$$A_1=\frac{2}{\pi}\int_0^{\pi}u_0\sin^2\delta_0\,d\delta_0$$

In the case of an element of the curved line

$$u_0=-V\frac{d\xi}{dx},$$

hence, for a wing of unit span,

$$\begin{aligned}M&=-2\rho V^2\int_0^{\pi}\frac{d\xi}{dx}\sin^2\delta_0\,d\delta_0\\&=-2\rho V^2\int_{-1}^{+1}\frac{d\xi}{dx}\sqrt{1-x^2}\,dx\\&=-2\rho V^2\int_{-1}^{+1}\frac{x\xi}{\sqrt{1-x^2}}\,dx \qquad (55)\end{aligned}$$

For a straight line having the same chord and the angle of attack α, the pitching moment was found to be

$$2\pi V^2\frac{\rho}{2}$$

so that, in order for the straight line to have the same moment as the curved line at zero angle of attack, the angle of attack of the former must be given by

$$2\pi V^2\frac{\rho}{2}\alpha''=-2\rho V^2\int_{-1}^{+1}\frac{x\xi}{\sqrt{1-x^2}}\,dx$$

or

$$\alpha''=-\frac{2}{\pi}\int_{-1}^{+1}\frac{x\xi}{\sqrt{1-x^2}}\,dx \qquad (56)$$

The lift of the straight line was found to be $2\pi\rho V^2\alpha$; hence the lift of the curved line at angle of attack zero is, for unit span,

$$L=2\pi\rho V^2\alpha', \text{ or on substitution from (50),}$$

$$=2\rho V^2\int_{-1}^{+1}\frac{\xi\,dx}{(1-x)\sqrt{1-x^2}} \qquad (57)$$

Hence the distance of the center of pressure from the center of the chord is

$$\frac{M}{L}=-\frac{\int_{-1}^{+1}\frac{x\xi}{\sqrt{1-x^2}}dx}{\int_{-1}^{+1}\frac{\xi dx}{(1-x)\sqrt{1-x^2}}} \qquad (58)$$

Writing this fraction equal to h, the position of the center of pressure is given by $\frac{1-h}{2}$ %.

If the length of the chord is b, the moment per unit length of the span is obviously

$$\begin{aligned} M&=2V\rho\left(\frac{b}{2}\right)^2 A_1\frac{\pi}{2}\\ &=-2\rho V^2\left(\frac{b}{2}\right)\int_{-b/2}^{+b/2}\frac{d\xi}{dx}\sqrt{1-\left(\frac{b}{2}x\right)^2}dx\\ &=-2\rho V^2\int_{-b/2}^{+b/2}\frac{\frac{2}{b}x\cdot\xi}{\sqrt{1-\left(\frac{b}{2}x\right)^2}}dx \qquad (55')\end{aligned}$$

For a straight line, by (51′),

$$M=2\pi V^2\frac{\rho}{2}\cdot\left(\frac{b}{2}\right)^2\alpha''$$

Therefore,

$$\alpha''=-\frac{2}{\pi}\left(\frac{2}{b}\right)^2\int_{-b/2}^{+b/2}\frac{\frac{2}{b}x\cdot\xi}{\sqrt{1-\left(\frac{2}{b}x\right)^2}}dx \qquad (56')$$

$L=2\pi\rho V^2\frac{b}{2}\alpha'$, per unit length of span, or, from (50′),

$$=2\rho V^2\left(\frac{2}{b}\right)\int_{-b/2}^{+b/2}\frac{\xi dx}{\left(1-\frac{2}{b}x\right)\sqrt{1-\left(\frac{2}{b}x\right)^2}} \qquad (57')$$

Hence

$$\frac{M}{L}=-\frac{b}{2}\frac{\int_{-b/2}^{+b/2}\frac{\frac{2}{b}x\cdot\xi}{\sqrt{1-\left(\frac{2}{b}x\right)^2}}dx}{\int_{-b/2}^{+b/2}\frac{\xi dx}{\left(1-\frac{2}{b}x\right)\sqrt{1-\left(\frac{2}{b}x\right)^2}}}=h\frac{b}{2} \qquad (58')$$

It follows at once that the position of the center of pressure is given by

$$\frac{1}{b}\frac{b}{2}(1-h)=\frac{1-h}{2}$$

The case of two or more wing sections, combined to form a biplane or multiplane, when surrounded by a two-dimensional flow in a longitudinal vertical plane may be treated in the same way as a single section. Each section determines by its slope at each point a distribution of vertical and horizontal velocity. This distribution being known, the resultant moment can be determined; from Kutta's condition for the two trailing edges the lift can be deduced; and finally the center of pressure may be calculated. The mathematical difficulties are, however, great.

CONCLUSION

Considering the wing as one of infinite span, the pitching moment acting on a cambered wing of chord c, per unit length of span, when at zero angle of attack, is

$$M = 2\pi \frac{\rho}{2} V^2 \left(\frac{c}{2}\right)^2 \alpha''$$

where

$$\alpha'' = -\frac{2}{\pi}\left(\frac{2}{c}\right)^2 \int_{-c/2}^{+c/2} \frac{\frac{2}{c} x \cdot \xi \cdot dx}{\sqrt{1-\left(\frac{2}{c} x\right)^2}};$$

and the ratio of the distance from the leading edge to the center of pressure to the length of the chord is $\frac{1-h}{2}$

where

$$h = -\frac{\int_{-c/2}^{+c/2} \frac{\frac{2}{c} x \cdot \xi \cdot dx}{\sqrt{1-\left(\frac{2}{c} x\right)^2}}}{\int_{-c/2}^{+c/2} \frac{\xi\, dx}{\left(1-\frac{2}{c} x\right)\sqrt{1-\left(\frac{2}{c} x\right)^2}}}$$

For simple means of calculation, see N. A. C. A. Technical Note No. 122.

INDUCED DRAG AND INDUCED ANGLE OF ATTACK

A. INDUCED DOWNWASH

In what has gone before we have considered only the two-dimensional flow in a vertical longitudinal plane; but this is only part of the motion, for it presupposes a wing of infinite span. If one views a finite wing from the front it is evident that, for many purposes, one may consider again the problem as that of a two-dimensional flow about a straight line, this time in a vertical transverse plane. The wing enters a stationary vertical layer of air and imparts to it a certain energy and momentum, this last giving rise to the lift. While it is passing through the layer it imparts to the air a certain velocity downward, and so is itself moving through air whose relative velocity is not in the direction of flight. This velocity downward, which modifies the direction of the relative velocity of the flow, is called the "induced downwash" u'. Its

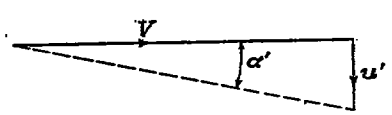

FIG. 13

effect is twofold. It evidently decreases the geometrical angle of attack α_g by an angle whose tangent is $\frac{u'}{V}$, or since u' is small compared with V, by an angle $\frac{u'}{V}$. This is called the "induced" angle of attack, i. e.,

$$\alpha_i = \frac{u'}{V} \qquad (59)$$

so that $\alpha_e = \alpha_g - \alpha_i$ . . .

Again, since the resultant force on the wing is perpendicular to the relative velocity, its direction is changed, thus giving rise to a component parallel to the direction of flight but in an opposite direction. This component therefore opposes the motion of the wing, and is called the "induced" drag, D_i, to distinguish it from the ordinary drag due to the viscosity of the

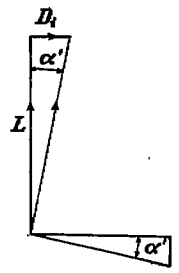

FIG. 14

air. The magnitude of the lift is not much affected by this change in the relative wind. It is evident from the geometry that if dx is an element along the straight line representing the span and dD_i and dL are the corresponding induced drag and lift,

$$dD_i = \frac{u'}{V}\, dL \qquad (60)$$

It is important to determine the connection between u^1, the induced down wash, and u, the final downward velocity in the vertical layers of air after the wing has passed through. In one second the wing moves forward a distance V, and therefore the kinetic energy imparted to the air in a layer of thickness V equals the product of the downward impulse (i. e., dL in this case) by the mean of the initial and final downward velocity, viz, $\int \frac{u}{2}\, dL$. Considered also in terms of the induced drag, this energy equals $\int V dD_i$ which, from (60), equals $\int u'\, dL$. Therefore

$$u' = \frac{u}{2} \qquad (61)$$

B. MINIMUM INDUCED DRAG

An important question in regard to the wing is: Assuming a definite total lift, what distribution of the lift along the span will give rise to a minimum induced drag? Or, calling downward momentum imparted to the air in one second G (i. e., the lift) and the kinetic energy T, what is the distribution of lift such that for a slight modification in the flow $\delta T = O$ while $G =$ constant? Let there be a slight change in the flow brought about by the addition of a flow defined by a velocity potential φ. The impulse per unit length along the span required to produce this flow is $-\rho\varphi$. Therefore the increase in momentum would be $-\rho \int \varphi\, dx$ along the span. This must equal zero, since G is constant. The impulse acts upon air already flowing downward with velocity $u' = \frac{1}{2} u$, hence the increase in kinetic energy is the sum of two terms, $-\frac{1}{2} \rho \int u \varphi\, dx$ and the energy of the added flow itself, which may be neglected since it is proportional to the square of the added velocity, which may be assumed small. Hence, since $\delta T = O$, $\frac{1}{2} \rho \int u \varphi\, dx = O$. Therefore, to satisfy both conditions, $u =$ const. along the span, and the induced angle of attack is the same at all points. It is easy to see that this is the condition for a minimum (and not a maximum). (In the case of a biplane without stagger, the same condition of $u =$ const. would be true over both wings.)

In one second the wing advances a distance V and imparts, therefore, a downward velocity u, constant along the span, and a momentum equal to the lift. Let $K\rho$ be the apparent transverse mass of the projection of the wing on a transverse vertical plane due to the flow in the plane (e. g., monoplane wing would give practically a straight line of length b, equal to the span); that is, it is the apparent transverse mass of a surface whose edge is the projection referred to and whose depth is unity. Since the surface described in one second as the wing advances a distance V has as its edge the projection mentioned and a depth V, the apparent

mass set in motion in one second is $K\rho V$, and the momentum imparted in one second is $K\rho V u$ This must equal the lift. Therefore

$$u=\frac{L}{\rho K V} \qquad (62)$$

for the case of minimum induced drag.

The kinetic energy imparted in one second is $\frac{1}{2} K\rho V\cdot u^2$ and this must equal $D_i V$; therefore,

$$D_{i_{min}}=\frac{1}{2}\rho K u^2$$

$$=\frac{L^2}{4\frac{\rho}{2}V^2\cdot K} \qquad (63)$$

Further

$$\alpha_i=\frac{u'}{V}=\frac{D_{i_{min}}}{L}=\frac{L}{4\frac{\rho}{2}V^2\cdot K} \qquad (64)$$

Since in the neighborhood of a minimum, properties change slowly, these values of $D_{i_{min}}$ and α_i may be used for other cases of lift distribution also.

The two dimensional flow in the transverse vertical plane about a line of length b equal to the span has already been discussed, viz, for the case of uniform velocity u downward at all points of the line, from (13′)

$$F=u\frac{b}{2}i\left(\frac{2}{b}z-i\sqrt{1-\left(\frac{2}{b}z\right)^2}\right)$$

Hence, for points on the line,

$$\varphi_0=\pm u\frac{b}{2}\sqrt{1-\left(\frac{2}{b}x\right)^2}$$

The difference in the potential on the two sides of the line at a point x is $2u\frac{b}{2}\sqrt{1-\left(\frac{2}{b}x\right)^2}$, which corresponds to an impulse per unit length (along the chord) of $\rho 2 u\frac{b}{2}\sqrt{1-\left(\frac{2}{b}x\right)^2}$. See (1a).

In one second this unit length advances a distance V and communicates a momentum L_1, where L_1 is the lift per unit length of the span; since this momentum is imparted over a length V, the momentum imparted per unit length is $\frac{L_1}{V}$. Therefore

$$\frac{dL}{dx}=L_1=2Vu\rho\frac{b}{2}\sqrt{1-\left(\frac{2}{b}x\right)^2}=2Vu\rho\frac{b}{2}\sin\delta_0$$

where

$$\cos\delta_0=\frac{2}{b}x.$$

Hence

$$L=\int_{-b/2}^{+b/2}\frac{dL}{dx}dx=2Vu\rho\left(\frac{b}{2}\right)\int_0^{\pi}\sin^2\delta_0\cdot d\delta_0=Vu\rho\pi\left(\frac{b}{2}\right)^2$$

and therefore, on substituting for u its value in terms of L,

$$\frac{dL}{dx}=\frac{4L\sin\delta_0}{\pi b} \qquad (65)$$

This particular distribution of lift along the span, corresponding to minimum induced drag, is called elliptical, because since

$$x=\frac{b}{2}\cos\delta_o \text{ and}$$

$$y=\frac{dL}{dx}=\frac{4L}{\pi b}\sin\delta_o$$

the points (x, y) lie on a semiellipse.

Returning to the values found for D_i and α_i, the value of K may be substitued, viz, $\pi\frac{b^2}{4}$. Therefore

$$D_{i_{min}}=\frac{L^2}{\pi b^2\frac{\rho}{2}V^2} \qquad (66)$$

$$\alpha_i=\frac{L}{\pi b^2\frac{\rho}{2}V^2} \qquad (67)$$

The effective angle of attack varies from point to point along the span. It has been proved that for an element of the wing, of area S, over which α is constant

$$L=2\pi\frac{\rho}{2}V^2S\alpha$$

If α_e is the effective angle of attack for an element dx of the span,

$$dL=2\pi\frac{\rho}{2}V^2\alpha_e\cdot c\,dx \text{ where } c \text{ is the chord}$$

Hence

$$\alpha_e=\frac{dL}{dx}\frac{1}{2\pi\frac{\rho}{2}V^2\cdot c}$$

Therefore substituting for $\frac{dL}{dx}$ from (65)

$$\alpha_e=\frac{2L\sin\delta_o}{b\pi^2c\frac{\rho}{2}V^2} \qquad (68)$$

Calling the geometrical angle of attack α_g, it is evident that

$$\alpha_e=\alpha_g-\alpha_i$$

Hence

$$\alpha_g=\frac{2L\sin\delta_o}{b\pi^2c\cdot\frac{\rho}{2}V^2}+\frac{L}{\pi b^2\frac{\rho}{2}V^2}=\alpha_e\left(1+\frac{\pi c}{2b\sin\delta_o}\right) \qquad (69)$$

C. GENERAL CASE OF CALCULATION OF INDUCED DRAG WHEN DISTRIBUTION OF LIFT ALONG THE SPAN IS KNOWN

In case, however, that the induced downwash is not constant along the span, the induced drag is not a minimum and the distribution of lift is not elliptical; so the formulas just deduced for α_e and α_g do not hold. For the case of a variable transverse velocity along the span use can be made of the general formula (27′)

$$\varphi_o=\frac{b}{2}(A_1\sin\delta_o+A_2\sin 2\delta_o+\cdots), \text{ where } \cos\delta_o=\frac{2}{b}x$$

Hence the difference of potential between opposite points is

$$b\,(A_1 \sin \delta_0 + A_2 \sin 2\delta_0 + \cdots)$$

and therefore, by the same argument as before,

$$\frac{dL}{dx} = bV\rho\,(A_1 \sin \delta_0 + A_2 \sin 2\delta_0 + \cdots) \qquad (70)$$

Consequently, if $\frac{dL}{dx}$ is specified at all points of the span, $\frac{1}{bV\rho}\frac{dL}{dx}$ may be expanded in a Fourier's series and, the values of the constants being thus determined, the flow is known, etc.

In this general case, by (29),

$$u = A_1 \frac{\sin \delta_0}{\sin \delta_0} + 2A_2 \frac{\sin 2\delta_0}{\sin \delta_0} + \text{etc.}$$

and

$$\alpha_i = \frac{u'}{V} = \frac{\frac{1}{2}u}{V} = \frac{1}{2V \sin \delta_0}\,(A_1 \sin \delta_0 + 2A_2 \sin 2\delta_0 + \text{etc.}) \qquad (71)$$

Further, since $\alpha_i = \frac{dD_i}{dL}$,

$$D_i = \int_{-b/2}^{+b/2} \alpha_i \frac{dL}{dx}\,dx = \rho\left(\frac{b}{2}\right)^2 \int_0^\pi (A_1 \sin \delta_0 + 2A_2 \sin 2\delta_0 + \text{etc.})\,(A_1 \sin \delta_0 + A_2 \sin 2\delta_0 + \text{etc.})\,d\delta_0$$

$$= \frac{\pi}{2}\rho\left(\frac{b}{2}\right)^2 (A_1^2 + 2A_2^2 + \cdots) \qquad (72)$$

(In the case of minimum induced drag, we found, (66) and preceding, the values

$$D_{i_{min}} = \frac{L^2}{\pi b^2 \frac{\rho}{2} V^2}$$

$$L = Vu\rho\pi\left(\frac{b}{2}\right)^2$$

where $u = A_1$ in the general formula, i. e.

$$L = A_1 \rho V \pi \left(\frac{b}{2}\right)$$

and therefore

$$D_{i_{min}} = \frac{\pi}{2}\rho\left(\frac{b}{2}\right)^2 A_1^2$$

which shows, since A_2, A_3, etc., are small compared with A_1, that $D_{i_{min}}$—and therefore the corresponding α_i—may be used even in the general case of variable downwash).

D. EFFECT OF INDUCTION UPON LIFT AND ROLLING MOMENTS

The problem of deducing D_i and α_i has been solved, then, for the case when $\frac{dL}{dx}$ is given for all points of the span. Consider now the problem of the angle of attack being known at each point, is there any simple plan form which will permit a solution?

It has been proved that

$$\alpha_e = \frac{\frac{dL}{dx}}{2\pi c \cdot \frac{\rho}{2} V^2}$$

and that $\frac{dL}{dx} = b\ V\rho\ (A_1 \sin \delta_o + A_2 \sin 2\delta_o + \ . \ . \ .)$

Hence the general term is

$$\alpha_e = \frac{bA_n \sin n\delta_o}{\pi c V} \qquad (73)$$

Further, the general term in α_i, as given in (71), is

$$\alpha_i = \frac{nA_n \sin n\delta_o}{2V \sin \delta_o} \qquad (74)$$

Therefore one is a constant times the other if c is propertional to $\sin \delta_o$ at each point of the span.

For an ellipse of semiaxes $\frac{b}{2}$ and $\frac{C}{2}$ formed as the plan of the wing

$$\frac{\left(\frac{c}{2}\right)^2}{\left(\frac{C}{2}\right)^2} + \frac{x^2}{\left(\frac{b}{2}\right)^2} = 1;$$

hence, since

$$\frac{2}{b} x = \cos \delta_o$$

$$c = C \sin \delta_o$$

Therefore such an elliptical wing makes α_e proportional to α_i.

Further, since the area

$$S = \pi \frac{b}{2} \cdot \frac{C}{2};\ \frac{\pi}{4} b\ c = S \sin \delta_o.$$

The same formula holds for a semiellipse.

For such a plan form, then, the general terms are

$$\alpha_e = \frac{A_n \sin n\ \delta_o}{2\ V \cdot \sin \delta_o \frac{2S}{b^2}}$$

and

$$\frac{\alpha_i}{\alpha_e} = \frac{n\pi c}{2b} = 2n \frac{S}{b^2}$$

For the first term, i. e., $n=1$

$$\alpha_g = \alpha_e + \alpha_i = \alpha_e \left(1 + \frac{2S}{b^2}\right) = \left(1 + \frac{2S}{b^2}\right) \frac{A_1 \sin \delta_o}{2V \frac{2S}{b^2} \sin \delta_o}$$

and for the general case

$$\alpha_g = \frac{1}{2V \cdot \frac{2S}{b^2} \sin \delta_o} \left\{ \left(1 + \frac{2S}{b^2}\right) A_1 \sin \delta_o + \left(1 + \frac{4S}{b^2}\right) A_2 \sin 2\delta_o + \ . \ . \ . \right\} \qquad (75)$$

or

$$2V \cdot \frac{2S}{b^2} \sin \delta_o \cdot \alpha_g = \left(1 + \frac{2S}{b^2}\right) A_1 \sin \delta_o + \left(1 + \frac{4S}{b^2}\right) A_2 \sin 2\delta_o + \quad . \ . \qquad (75a)$$

Therefore, if α_g is specified over the span of an elliptic wing, the Fourier expansion gives values for the A's, and these may be substituted in the formulas previously found, viz:

$$\frac{dL}{dx}=b\,V\rho\,(A_1 \sin\delta_0 + A_2 \sin 2\delta_0 + \text{etc.})$$

$$\alpha_i=\frac{1}{2V\sin\delta_0}\,(A_1 \sin\delta_0 + 2A_2 \sin 2\delta_0 + \text{etc.})$$

$$D_1=\frac{\pi}{2}\,\rho\left(\frac{b}{2}\right)^2 (A_1^2 + 2A_2^2 + \text{etc.})$$

The entire lift

$$L=\int_{-b/2}^{+b/2}\frac{dL}{dx}\,dx=\int_0^{\pi}\frac{dL}{dx}\sin\delta_0\,d\delta_0.$$

So only the first term has any effect, and

$$L=b\,V\rho A_1\int_{-b/2}^{+b/2}\sin\delta_0\,dx$$

or replacing $\sin\delta_0$ by its value $\frac{\pi}{4}\frac{bc}{S}$ and A_1 by

$$2V\cdot\frac{2S}{b^2}\,\frac{1}{1+\frac{2S}{b^2}}\alpha_g$$

(since only the first term counts),

$$L=\frac{\rho}{2}V^2\cdot\frac{1}{1+\frac{2S}{b^2}}\cdot 2\pi\int_{-b/2}^{+b/2}\alpha_g\cdot c\,dx \qquad (76)$$

Expressed in terms of the effective angle of attack,

$$L=\frac{\rho}{2}V^2\cdot 2\pi\int_{-b/2}^{+b/2}\alpha_e\cdot c\cdot dx$$

hence

$$a_e=\frac{\alpha_g}{1+\frac{2S}{b^2}} \qquad (77)$$

If there were no induction, α_g would equal α_e. So the effect of induction is to reduce α_e in the ratio $1 : 1+\frac{2S}{b^2}$.

The rolling moment $M=\int_{-b/2}^{+b/2}\frac{dL}{dx}\cdot x\cdot dx$ along the span.

Hence

$$M=-\int_{-b/2}^{+b/2}\frac{dL}{dx}\cos\delta_0\cdot\sin\delta_0\cdot d\delta_0=-\frac{1}{2}\int_{-b/2}^{+b/2}\frac{dL}{dx}\sin 2\delta_0\cdot d\delta_0$$

Therefore only the second term in $\frac{dL}{dx}$, as given in (70), has any effect and, substituting in the original formula for M,

$$M=b\,V\rho A_2\int_{-b/2}^{+b/2}\sin 2\delta_0\cdot x\cdot dx$$

Further, using only the second term in (75a) for α_g,

$$2V\cdot\frac{2S}{b^2}\sin\delta_0\cdot\alpha_g=\left(1+\frac{4S}{b^2}\right)A_2\sin 2\delta_0$$

hence

$$A_2\sin 2\delta_0=2V\cdot\frac{2S}{b^2}\sin\delta_0\cdot\frac{1}{1+\frac{4S}{b^2}}$$

or, since

$$S\sin\delta_0=\frac{\pi}{4}bc,\ A_2\sin 2\delta_0=\pi V\frac{c}{b}\frac{1}{1+\frac{4S}{b^2}}\cdot\alpha_g$$

Consequently

$$M=\frac{\frac{\rho}{2}V^2\cdot 2\pi}{1+\frac{4S}{b^2}}\int_{-b/2}^{+b/2}c\cdot x\cdot\alpha_g\,dx \qquad (78)$$

Expressed in terms of effective angle of attack

$$\frac{dL}{dx}=2\pi\cdot\frac{\rho}{2}V\cdot c\cdot\alpha_e$$

Therefore the rolling moment, by its original definition,

$$M=\frac{\rho}{2}\cdot V^2 2\pi\int_{-b/2}^{+b/2}c\cdot x\cdot\alpha_e\,dx$$

hence

$$\alpha_e=\frac{\alpha_g}{1+\frac{4S}{b^2}}$$

and it is seen that so far as such moments are concerned, the effect of induction is to reduce α_e in the ratio $1:1+\frac{4S}{b^2}$.

E. NOTE CONCERNING BIPLANES

Since $D_i=\dfrac{L^2}{4\frac{\rho}{2}V^2\cdot K}$ where $K\rho$ is the apparent mass for a two-dimensional flow in the transverse plane; and, since, for a biplane, K is greater than for a single wing, D_i is less, other things being equal. Thus, if K applies to a biplane of a certain total area and span b, and K_1 to a monoplane of the same total area and of span b_1, the lift is the same for the two, and if the induced drag is to be the same

$$K=K_1=\pi\left(\frac{b_1}{2}\right)^2$$

or if

$$b_1=kb,\ k^2=\frac{K}{\pi\left(\frac{b}{2}\right)^2}$$

The value of K is known for different combinations of wings, and k may thus be deduced.

CONCLUSION

For a wing of span b and chord c, the area being S, if α_g is the geometrical angle of attack at a point x of the span,

$$L=2\pi\frac{\rho}{2}V^2\cdot\frac{1}{1+\frac{2S}{b^2}}\int_{-b/2}^{+b/2}\alpha_g\cdot c\cdot dx$$

i. e., the effect of induction on the lift is to reduce the effect of the angle of attack in the ratio of $1:1+\frac{2S}{b^2}$.

The rolling moment

$$M=2\pi\frac{\rho}{2}V^2\frac{1}{1+\frac{4S}{b^2}}\int_{-b/2}^{+b/2}\alpha_g\cdot c\cdot x\cdot dx$$

i. e., the effect of induction on rolling moment is to reduce the effect of the angle of attack in the ratio $1:1+\frac{4S}{b^2}$.

PROPELLER THEORY

INTRODUCTION

The purpose of a theory of the action of a propeller is to combine with Froude's slip-stream theory a theory of the action of the elements of the blades as airfoils. These elements actually move along spiral paths; but it is possible to simplify the treatment by considering the blades as a single element of area S. Often one can treat the blades as having a definite section, and the blade area as concentrated at one point, say 70 per cent of the radius from the axis. In Munk's treatment of the subject he assumes that, as the flight velocity V and the tip velocity U of the blades are varied, the "shape" of the slip stream does not vary, although its velocity v does. Under these circumstances v is obviously a linear function of V and U so long as the aerodynamic properties of the blade elements remain unchanged.

Under these circumstances, not simply can the efficiency of the propeller be calculated in terms of known quantities, but also a formula for $\frac{dv}{dU}$ which enables one to compute the thrust for any value of $\frac{U}{V}$.

References.—Munk—Analysis of W. F. Durand's and E. P. Lesley's Propeller Tests. N. A. C. A. Technical Report No. 175. Notes on Propeller Design. N. A. C. A. Technical Notes 91, 92, 93, 94.

FROUDE'S SLIPSTREAM THEORY

If the aircraft is moving with a velocity V through air otherwise at rest, the propeller sets in motion backward a slip-stream whose final mean velocity may be called v. The air actually passing through the propeller has already had imparted to it a portion of this velocity, and, by general principles of mechanics, this additional velocity may be proved to be approximately one-half of v. For, imagine the aircraft at rest—as in a wind-tunnel experiment—and placed in a stream of air having the velocity V. Let the propeller be revolving as usual, and let the velocity of the air through the propeller be called $V+w$. Let the final velocity of the slip-stream be called v as above. If m is the mass of air passing per unit time, the thrust of the propeller is mv. This force acts on air moving with velocity $V+w$, hence the work done per unit time is $mv(V+w)$. This is equal to the increase of the kinetic energy of the air, viz.:

$$\tfrac{1}{2}m(V+v)^2-\tfrac{1}{2}mV^2=mv\left(V+\frac{v}{2}\right)$$

Therefore

$$w=\frac{v}{2} \qquad (79)$$

The mass passing the propeller disk per unit time is

$$D^2\frac{\pi}{4}\left(V+\frac{v}{2}\right)\rho$$

and therefore the thrust

$$T=D^2\frac{\pi}{4}\left(V+\frac{v}{2}\right)\rho v$$

or

$$\frac{T}{D^2\frac{\pi}{4}V^2\frac{\rho}{2}}=\left(1+\frac{v}{V}\right)^2-1\equiv C_T \qquad (80)$$

an absolute coefficient.

If the ratio $\frac{v}{V}$ is small, $C_T=\frac{2v}{V}$, or $\frac{v}{V}=\frac{1}{2}C_T$; but this approximation can be used only for small values of C_T.

THE SLIP CURVE

Since, as explained above, the assumptions made justify one in writing v as a linear function of V, the velocity of flight, and of U $(=\pi nD)$, the tip velocity, we may write

$$\frac{v}{V}=m\left\{\frac{U}{V}-\left(\frac{U}{V}\right)_0\right\} \qquad (81)$$

where $\left(\frac{U}{V}\right)$ is the magnitude of the "relative" tip velocity for which the slip-stream velocity, and therefore the thrust, is zero. Therefore, if $\frac{v}{V}$ is plotted against $\frac{U}{V}$, the result is a straight line. (If in any actual propeller test, this plot is not such a straight line, it proves that the assumptions made above do not hold for this test.) This prediction is well supported by actual tests. The curve is called the "slip curve" and $m\left(=\frac{dv}{dU}\text{ for constant }V\right)$ is called the slip modulus. In plotting the curves the experimental values are formed by writing

$$\frac{U}{V}=\pi n\frac{D}{V}$$

$$\frac{v}{V}=\sqrt{1+\frac{T}{D^2\frac{\pi}{4}\frac{\rho}{2}V^2}}-1$$

Munk discusses these actual curves very fully in his papers. One consequence to be noted is that, as a result of tests already made, m is known for propellers of various types and of different blade width, and that its value does not differ greatly from one-eighth for ordinary propellers. Munk also shows how the effective pitch may be calculated.

THE SLIP MODULUS

With certain assumptions, the slip modulus may be calculated. Consider a propeller with narrow blades whose sections are "ideal" and whose pitch ratio is small. With such a propeller the influence of the slip stream on the effective angle of attack may be neglected. Consider the total effective blade area S concentrated at the distance $0.7r$ from the axis.

For small angles of attack, the thrust (i. e., the lift) $T=2\pi S\frac{\rho}{2}V_1^2\alpha$, where V_1 is the relative velocity of the air and α is the small angle of attack. This may be calculated as follows:

$$V_1^2=V^2+U'^2=V^2\left\{1+\left(\frac{U'}{V}\right)^2\right\}$$

if U' is the tangential velocity of the propeller at the point where S is considered, i. e., $U'=0.7\,U$.

Let U'_0 be such a tangential velocity as causes zero thrust at velocity V. If $\tan\varphi$ equals $\frac{V}{U'}$ and if U is increased slightly, the resulting angle of attack on the blade area S is

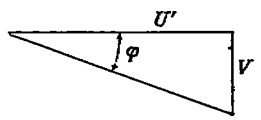

FIG. 15

$$\alpha=-d\varphi=\frac{dU'}{V}\frac{1}{1+\left(\frac{U'}{V}\right)^2} \qquad (82)$$

Therefore

$$T=2\pi S\frac{\rho}{2}V^2\frac{dU'}{V}$$

and

$$C_T=8\frac{S}{D^2}\frac{dU'}{V}$$

Since C_T is small, it equals $2\frac{v}{V}$ so that $\frac{v}{V}=4\frac{S}{D^2}\frac{dU'}{V}$. But $dU'=0.7dU$; and, since before the change dU, the thrust, and therefore v were zero, the slip stream which results from dU is such that $v=mdU=m\frac{dU'}{0.7}$,

$$m=\frac{\frac{v}{V}0.7}{\frac{dU'}{V}}=0.7\frac{4S}{D^2} \qquad (83)$$

The fact that m is greater for propellers of greater mean blade width is confirmed by experiment.

In the calculation given above it is assumed that the only change when the tip velocity is increased is dU'; but, as a matter of fact, there is an additional velocity $\frac{dv}{2}$ at right angles to U', which affects the angle of attack. Writing $\cot\varphi=\frac{U'}{V}$,

$$-\left\{1+\left(\frac{U'}{V}\right)^2\right\}d\varphi=\frac{dU'}{V}-\frac{U'}{V^2}\frac{dv}{2}.$$

Further,

$$dv=mdU=m\frac{U}{U'}dU'$$

Hence the angle of attack

$$\alpha=-d\varphi=\frac{1}{1+\left(\frac{U'}{V}\right)^2}\frac{dU'}{V}\left(1-\frac{m}{2}\frac{U}{V}\right) \qquad (82a)$$

and

$$m=0.7\frac{4S}{D^2}\left(1-\frac{m}{2}\frac{U}{V}\right)$$

The ratio $\frac{U}{V}$ appearing here is the value at zero thrust, and this should be indicated by writing $\left(\frac{U}{V}\right)_o$. Solving the equation for m,

$$m = \frac{0.7\frac{4S}{D^2}}{1 + .35\,\frac{4S}{D^2}\left(\frac{U}{V}\right)_o} \qquad (83a)$$

The "nominal blade width ratio," $\frac{2S}{D^2}$, is known for a propeller under test, and $\left(\frac{U}{V}\right)_o$ may be determined; so $m = \frac{dv}{dU}$ may be calculated. (In one test, calculation gave *0.13*, and observations of the slip curve gave *0.133*.)

Since this constant m may thus be considered known, $\frac{v}{V}$ may be calculated for any value of $\frac{U}{V}$ and therefore C_T is known and hence T, the thrust.

TORQUE

The propeller efficiency is the ratio of TV to the power delivered, that is, to the product of the torque Q by ω, the angular velocity.

$$\eta = \frac{TV}{Q\omega} \qquad (84)$$

But $T = C_T \cdot D^2 \frac{\pi}{4} \cdot V^2 \frac{\rho}{2}$, and a new coefficient C_Q may be defined such that

$$Q = C_Q \cdot \frac{D}{2} \cdot D^2 \frac{\pi}{4} \cdot V^2 \frac{\rho}{2} \qquad (85)$$

Then

$$\eta = \frac{C_T V}{C_Q\,\omega\,\frac{D}{2}} = \frac{C_T}{C_Q}\frac{V}{U} \qquad (86)$$

The power delivered may be thought of as being spent in three ways: (1) As absorbed in thrust, i. e., $Q_1\omega = TV$, and therefore, since $\eta_1 = 1$, the corresponding $C_{Q_1} = C_T \cdot \frac{V}{U}$; (2) as absorbed in building up the slip stream, i. e., $Q_2\omega = T\frac{v}{2}$, and, since $\eta_2 = \frac{2V}{v}$, the corresponding $C_{Q_2} = C_T \frac{V}{U} \cdot \frac{v}{2V}$; (3) as absorbed by friction, etc. Hence its corresponding $C_{Q_3} = C_Q - (C_{Q_1} + C_{Q_2})$, or $C_{Q_3} = C_Q - C_T\left(1 + \frac{v}{2V}\right)\frac{V}{U}$.

This is equivalent to a drag coefficient of the blades which may be calculated as follows: If S is the effective blade area, placed at a distance $0.7r$ from the axis, its tangential velocity is $.7\,U$; therefore, calling the drag coefficient C_D, the drag is $C_D\,S\,\frac{\rho}{2}\,(.7\,U)^2$, and the power spent in overcoming this is

$$C_D S\,\frac{\rho}{2}\,(.7\,U)^2\,(.7\,U)$$

This must equal

$$Q_3\omega = C_{Q_3}\left(\frac{D}{2}\,\omega\right) \cdot D^2\,\frac{\pi}{4}\,V^2\,\frac{\rho}{2}$$

$$= C_{Q_3}\,U\,D^2\,\frac{\pi}{4}\,V^2\,\frac{\rho}{2}$$

Therefore

$$C_D = C_{Q_2} \frac{D^2 \frac{\pi}{4}}{S \cdot \left(\frac{.7\,U}{V}\right)^2} \frac{1}{.7} \qquad (87)$$

For actual propellers Munk states that $C_D = 0.025$ approximately.

If there were no frictional loses, the efficiency would equal

$$\frac{TV}{(Q_1 + Q_2)\,\omega} = \frac{TV}{T\left(V + \frac{v}{2}\right)} = \frac{1}{1 + \frac{1}{2}\frac{v}{V}}$$

Therefore

$$\eta_{max} = \frac{1}{1 + \frac{1}{2}\frac{v}{V}} \qquad (88)$$

and, since C_T is known in terms of $\frac{v}{V}$, and T is known in terms of C_T, this maximum efficiency may be expressed in terms of T.

THE TORQUE SLIP CURVE

The slip curve described previously is derived from knowledge of the thrust, and is therefore more useful in the discussion of data obtained from model tests than in the case of tests in actual flight, for in the latter the thrust is an indefinite quantity—so far as theory is concerned. The theoretical value of the slip modulus, m, is derived only by making obvious assumptions, and, rather than trying to improve the theory, it is better to compare the theoretical value with observed values, obtained from the study of actual slip curves for propellers in flight. Again, in studying the properties of different propellers in flight, it is better to start with the knowledge of the torque or power and to deduce a different type of slip curve, because the power is much more definite than the thrust. Further, propellers are designed to absorb a given horsepower at a certain number of revolutions. Consequently, Munk describes a new slip modulus referring to the torque as modified by the interference of the fuselage, etc.

Define a power coefficient

$$C_P = \frac{P}{V \frac{\rho}{2} V^2 D^2 \frac{\pi}{4}} \qquad (89)$$

Then, since

$$C_T = \frac{T}{\frac{\rho}{2} V^2 D^2 \frac{\pi}{4}}$$

and, since in the absence of viscosity the efficiency

$$\frac{TV}{P} = \frac{1}{1 + \frac{1}{2}\frac{v}{V}}$$

the "ideal" coefficient would have the value

$$C_T\left(1 + \frac{1}{2}\frac{v}{V}\right)$$

The actual coefficient is of course larger. Then in terms of the actual coefficients C_P and C_T define a slip velocity v' by the equation

$$C_P = C_T\left(1 + \frac{1}{2}\frac{v'}{V}\right) \qquad (90)$$

In other words, v' is derived from the knowledge of the torque, as v is from that of the thrust. v' is slightly greater than v.

The curve of $\frac{v'}{V}$ plotted against $\frac{U}{V}$ is called the "torque slip curve," and by studying the two curves, that of $\frac{v}{V}$ and that of $\frac{v'}{V}$, for actual propellers in flight, knowledge may be obtained which will enable the better application of data from model tests.

In order to obtain the values of $\frac{v'}{V}$ from measured quantities, it is necessary to derive a relation between it and C_P. From the two equations

$$C_T=\left(1+\frac{v'}{V}\right)^2-1 \text{ and } C_P=C_T\left(1+\frac{1}{2}\frac{v'}{V}\right),\ C_P=\frac{1}{2}\left(\frac{v'}{V}\right)^3+2\left(\frac{v'}{V}\right)^2+2\left(\frac{v'}{V}\right) \quad \text{(91)}$$

In N. A. C. A. Report No. 183, Munk gives values of the solution of this equation, so that values of $\frac{v'}{V}$ may be obtained, and then the corresponding torque slip curves,

$$\frac{v'}{V}=m'\left\{\frac{U}{V}-\left(\frac{U}{V}\right)\right\}$$

may be plotted. These may then be compared with the thrust slip curves,

$$\frac{v}{V}=m\left\{\frac{U}{V}-\left(\frac{U}{V}\right)_o\right\}$$

in which $\frac{v}{V}$ is obtained from a knowledge of T.

Munk made a detailed study of the performance of certain propellers as published in the British R. & M. Nos. 586 and 704, and deduced for comparative purposes the following data:

1. Curves for m and m'.
2. Calculation of m from the theoretical formula, and the value of the correction factor $\frac{m_{obs.}}{m_{calc.}}$. (This varied from 0.97 to 1.13.)
3. Mean effective angle of attack, at 0.7 radius, $\alpha=\cot^{-1}\left(\frac{0.7\,U}{V}\right)_o$. Also the observed value. The difference is to be attributed to the effect of the camber of the blade section and to the elastic torsion of the blades.

These formulas and test data are, of course, most important in the study of sets of propellers and of the same propeller when attached to different engines; but they also are of direct use to the engineer who wishes to design a new propeller.