REPORT No. 450

THE CALCULATION OF TAKE-OFF RUN

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SUMMARY

A comparatively simple method of calculating length of take-off run is developed from the assumption of a linear variation in net accelerating force with air speed and it is shown that the error involved is negligible. The run formula is reduced to

$$S=\frac{K_s V_s^2}{\left(\frac{T_1}{W}\right)}$$

Where V_s is the take-off speed, T_1 is the initial net accelerating force, W is the gross weight and K_s a coefficient depending only on the ratio of initial to final net accelerating force. Detailed instructions are given for application of the formula and for the calculation of all factors involved.

INTRODUCTION

Exact calculations may be made for the distance run during the take-off of a landplane if sufficient data are available on propeller, engine, and airplane characteristics, and if some assumption is made regarding the pilot's technique. The calculations are necessarily laborious since the relations are such that step-by-step integration is required. In most cases lack of exact data on power plant and airplane prevents an exact calculation even if such were desired, while for all practical purposes a close approximation is all that is needed.

A number of formulas and methods have been proposed and used extensively in the past for calculating take-off runs. Most of these methods are based on average thrusts obtainable with the power plants formerly in use, and under favorable conditions gave satisfactory results. However, the advent of high performance has brought about changes in propeller design, for example, that greatly affect the take-off run so that a complete revision is required to accommodate the new variables.

The present study is concerned with the development of a method suitable for routine take-off calculations in the Bureau of Aeronautics of the Navy Department. Such a method, to be satisfactory, was required to be reasonably simple without neglecting any important variable. The method derived in this or has been in use long enough to justify considerable confidence in its validity. All attempts to obtain a more simple method have resulted in unsatisfactory reliability. While the method here presented is intended as a practical approximation to a difficult problem, it is highly probable that greater accuracy would have no significance in view of lack of definite knowledge regarding basic data.

Unless otherwise specified the take-off run is to be the run required to attain a given air speed in a calm; lb.-ft.-sec. units will be used.

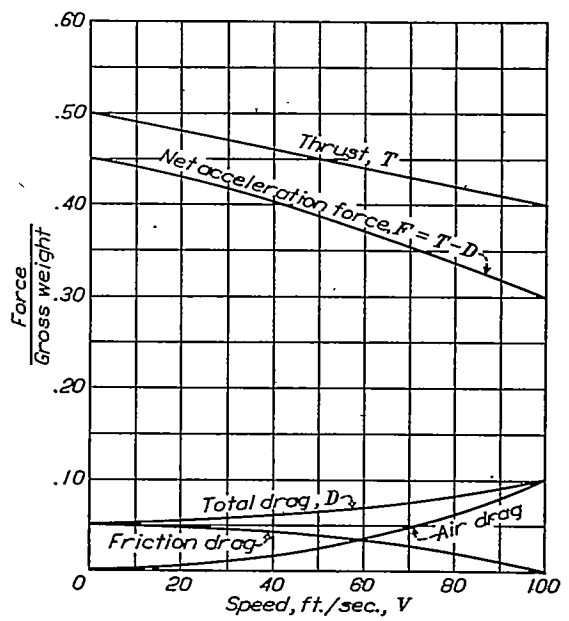

Figure 1.—Forces acting during take-off run

GENERAL CALCULATION OF TAKE-OFF RUN

The major forces acting on the airplane during take-off are thrust, air drag, and friction drag. In the simplest case, at constant angle of attack, the air drag will vary as the square of the air speed and the friction drag will vary with the load on the wheels. The net accelerating force will then be the difference between the thrust and the total drag as indicated on Figure 1. For convenience, the forces may be reduced to unit values by dividing each by the gross weight.

In general the take-off may be considered to consist of three stages: A short initial period during which the tail is raised to normal position, a relatively long

period of acceleration at a substantially constant angle of attack and a short period during which the angle is increased for take-off. It is possible for the third stage to be partially or wholly suppressed, depending on the airplane characteristics and pilot's efforts. If a definite sequence be assumed, the corresponding curves for air drag and friction drag may be determined. The variation of thrust with air speed may be calculated from engine and propeller test data. The normal tendency is for the thrust to decrease slightly with increase in speed and with a substantially linear relation, as indicated on Figure 1. In some cases, however, the thrust may remain constant, or even increase with air speed.

Assuming that the thrust and the drag are known as a function of air speed, the calculation of take-off run is made as illustrated by Figure 2 and Table I. The

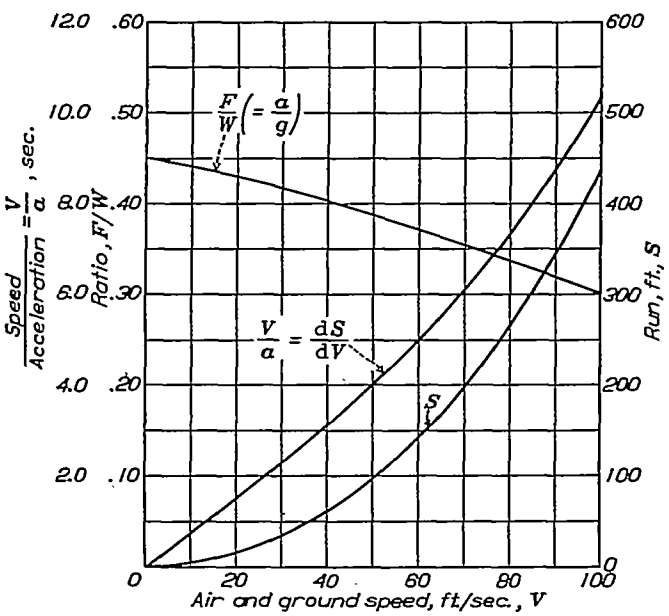

FIGURE 2

curve of net accelerating force F/W in Figure 2 is the same as that in Figure 1. The acceleration is given by

$$a=g\left(\frac{F}{W}\right) \tag{1}$$

Since $V=\frac{dS}{dt}$ and $a=\frac{dV}{dt}$

$$\frac{V}{a}=\frac{dS}{dV}; \tag{2}$$

if $\frac{V}{a}$ is plotted against V as in Figure 2 the area under the curve is proportional to the run S. This area may be found by any convenient method, such as by a planimeter, the trapezoidal rule or Simpson's rule. The trapezoidal rule used in Table I is ordinarily the simplest method. By this method the area under a curve, between any number of equally spaced ordinates, is equal to the product of the sum of the inside ordinates plus one-half of the end ordinates, by the spacing between the ordinates. It is usually given in the form

$$A=(\tfrac{1}{2}y_1+y_2+y_3 \ldots . \tfrac{1}{2}y_n)\ \Delta X$$

where $y_1, y_2, y_3 \ldots . y_n$ are the ordinates of the curve equally spaced at the distance ΔX apart.

EQUATION FOR TAKE-OFF RUN

An extended analysis of calculated take-off runs in comparison with observed runs leads to the following conclusions:

1. The effects of thrust variation, gross weight, take-off air speed, and wind speed predominate to such an extent that in practice all other variables may be neglected.
2. The effect of normal variations in power is so great that considerable latitude is allowable in simplifying assumptions.

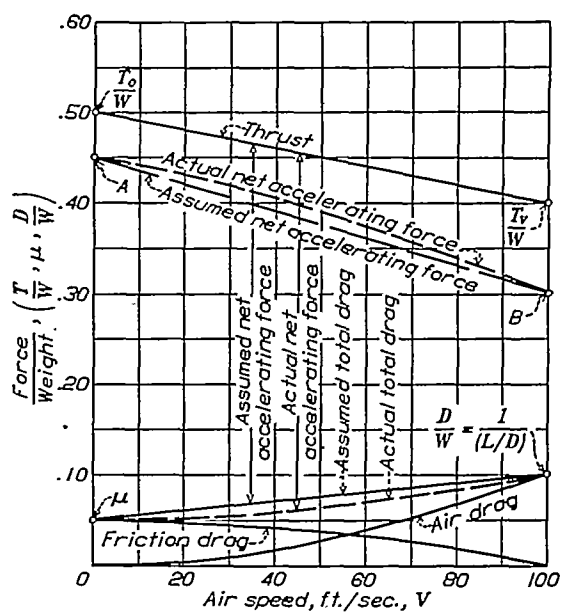

FIGURE 3.—Major forces acting during take-off run

3. The effect of thrust variation from the static condition to take-off is of such primary importance that proper allowance must be made for the propeller characteristics.
4. The effects of wind speed and actual take-off speed are best handled in the form of corrections to the run required to attain stalling speed in a calm.

The most promising method of simplifying the calculations appears to be in the assumption that the net accelerating force varies linearly with air speed as indicated on Figure 3. The calculations of Table II, based on this assumption, shows a run of 448.8 feet, as compared with 439.3 feet obtained in Table I. This is an error of less than 2.2 per cent, for what may be considered an average case. Such an error is entirely negligible in comparison with the uncertainty in power or propeller characteristics, and the variation in piloting.

With the assumption of a linear variation in net accelerating force the relations are as shown on Figure 3. The thrust per pound varies from T_o/W at the start, T_o being the static thrust, to T_v/W at take-off, T_v being the thrust at take-off speed V. The total drag per unit weight varies from $\mu = f/W$ at the start to $\frac{D}{W} = 1/\left(\frac{L}{D}\right) = \frac{D}{L}$ at the take off. The net accelerating force per unit weight at the start is

$$\frac{T_1}{W} = \frac{T_o}{W} - \mu \tag{3}$$

The net accelerating force at take-off is

$$\frac{T_F}{W} = \frac{T_v}{W} - \frac{D}{L} \tag{4}$$

Since the net accelerating force varies linearly with air speed it follows that

$$\frac{F}{W} = \frac{T_1}{W}\left(1 - K\frac{V}{V_s}\right) \tag{5}$$

where $(1-K)$ is the ratio of the final to the initial accelerating force. That is,

$$(1-K) = \frac{T_F}{T_1} \text{ or } K = \frac{T_1 - T_F}{T_1}.$$

The resulting equation of motion is

$$\frac{dV}{dt} = g\frac{F}{W} = g\frac{T_1}{W}\left(1 - K\frac{V}{V_s}\right) \tag{6}$$

or

$$\frac{V\,dV}{\left(1 - K\frac{V}{V_s}\right)} = g\frac{T_1}{W}dS \tag{7}$$

Integrating this gives

$$S = \frac{1}{g\left(\frac{T_1}{W}\right)}\left[-\frac{V\,V_s}{K} - \left(\frac{V_s}{K}\right)^2 \log\left(1 - K\frac{V}{V_s}\right)\right] \tag{8}$$

For take-off at speed Vs equation (8) reduces to

$$S = \frac{V_s^2}{g\frac{T_1}{W}}\left[\frac{1}{K}\left(-1 - \frac{1}{K}\log(1-K)\right)\right] \tag{9}$$

The term within the brackets on the right hand side of equation (9) depends only on the ratio of the final net thrust to the initial net thrust T_F/T_1. Hence we may write

$$S_o = \frac{K_s\,V_s^2}{\left(\frac{T_1}{W}\right)} \tag{10}$$

where S_o is the total run to attain the speed V_s in a calm and $K_s = \frac{1}{g}\left[\frac{1}{K}\left(-1 - \frac{1}{K}\log(1-K)\right)\right]$.

Calculated values of K_s are given in Table III and plotted on Figure 4.

In order to use equation (10), the static thrust and the thrust at take-off speed must be known. Methods for calculating the thrust will be given.

TIME REQUIRED FOR TAKE-OFF

In some cases the time required for take-off is of importance. This may be obtained from equation (6) written in the form

$$\frac{dV}{1 - K\frac{V}{V_s}} = g\frac{T_1}{W}dt \tag{6a}$$

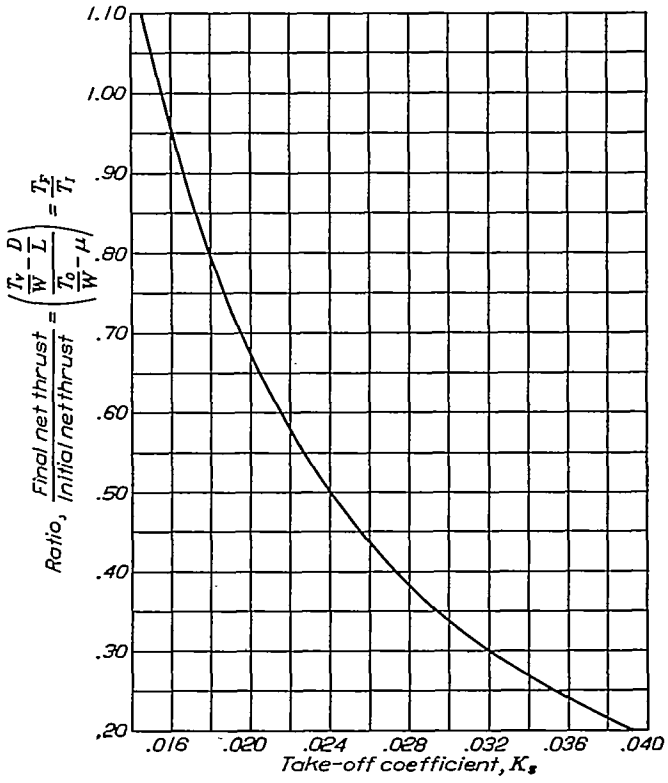

FIGURE 4.—Take-off run in a calm

$S_o = \frac{K_s\ V_s^2}{\left(\frac{T_o}{W} - \mu\right)}$ ft.

V_s = take-off air speed-ft./sec.
T_o = static thrust—pounds.
W = gross weight—pounds.
μ = coefficient of traction.

from which, by integration

$$t = \frac{-V_s}{gK\left(\frac{T_1}{W}\right)}\log\left(1 - K\frac{V}{V_s}\right) \tag{11}$$

For time to take-off speed $V = V_s$, this becomes

$$t = \frac{-V_s}{g\left(\frac{T_1}{W}\right)}\left[\frac{1}{K}\log(1-K)\right] \tag{12}$$

As in equation (9), the term in the brackets depends only on the ratio of the net thrusts. Hence

$$t_o = \frac{K_t\,V_s}{\left(\frac{T_1}{W}\right)} \tag{13}$$

where t is the time in seconds to attain the take-off speed V_s (in ft./sec.) in a calm and $K_t = \frac{-1}{gK} \log (1-K)$.

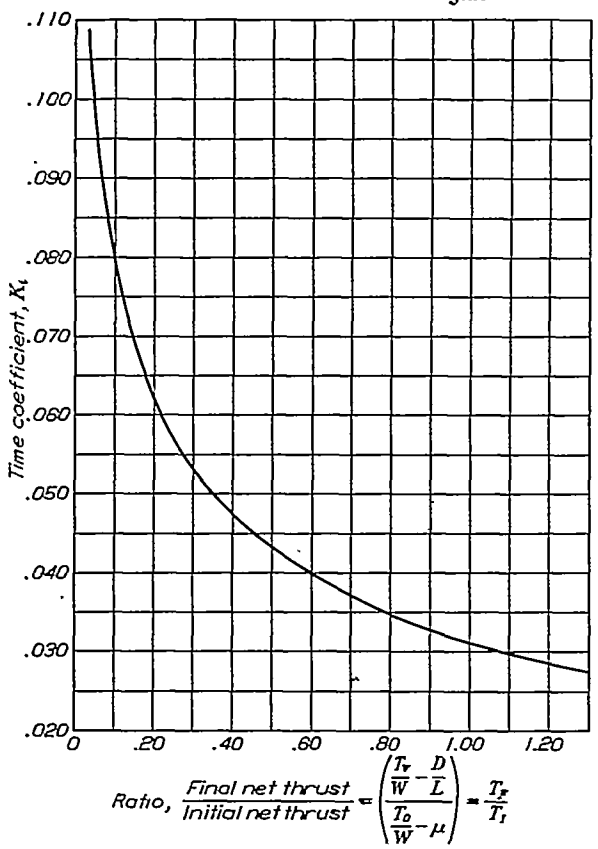

FIGURE 5.—Coefficient for time to take-off

$$t_{sec} = \frac{K_t \; V_s}{\left(\frac{T_1}{W}\right)}$$

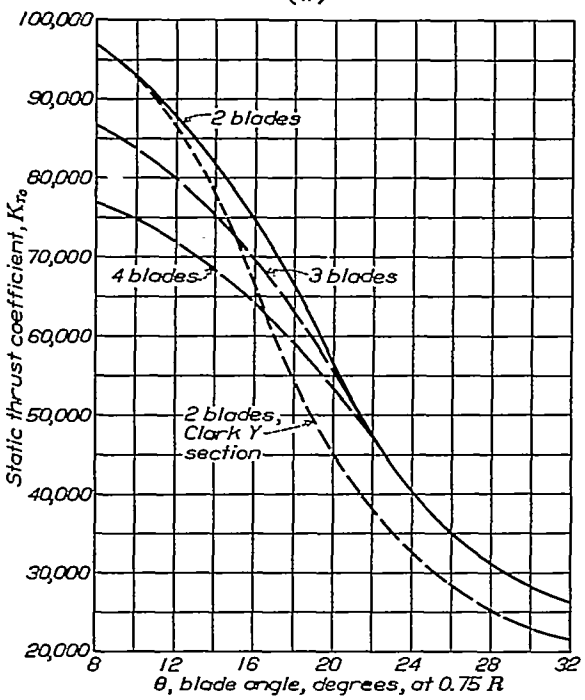

FIGURE 6.—Static thrust coefficient

Calculations for K_t are given in Table III and K_t is plotted as a function of the ratio T_F/T_1 on Figure 5.

INITIAL NET ACCELERATING FORCE

The initial net accelerating force per unit weight is defined by equation (3)

$$\frac{T_1}{W} = \frac{T_o}{W} - \mu \tag{3}$$

In reference 4 it is shown that the static thrust may be calculated by the equation

$$T_o = \frac{K_{T_o} \times \text{b. hp}}{\text{r. p. m.} \times D}$$

The static thrust coefficient K_{T_o} may be plotted as a function of blade angle at 0.75 R as in Figure 6.

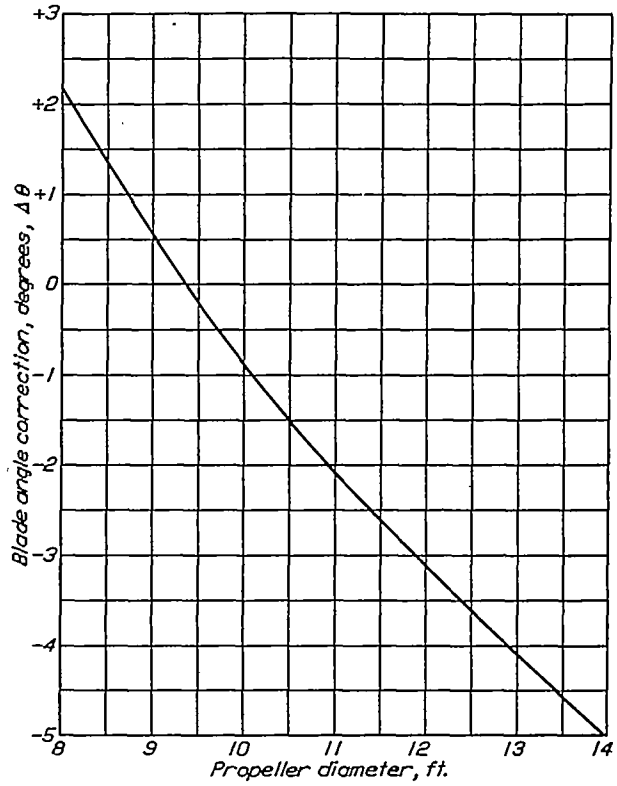

FIGURE 7.—Correction for obtaining blade angle at 0.75 radius from blade setting at 42″ station. Positive (+) values of $\Delta\theta$ to be added to setting at 42″ station, negative (−) values to be subtracted, to get blade angle at 0.75 radius.

Example: For 10 ft. dia. prop.

$\Delta\theta = -0.9°$

$\theta_{0.75R} = \theta_{42''} - .9°$

Since it is common practice to adjust or read the blade angle at a given station regardless of the propeller diameter, a correction must be applied to obtain the setting at 0.75 R. Figure 7 gives the correction $\Delta\theta$ to convert the blade angle at the 42-inch station to the corresponding value at 0.75 R. The 42-inch station is 75 per cent of 56 inches, so that the correction is zero for a diameter of 9 feet 4 inches. For diameters less than 9.33 feet the 0.75 R station will therefore lie inboard of the 42-inch station and consequently will have greater blade angles, i. e., $\Delta\theta$ is positive. For diameters greater than 9.33 feet the 0.75 R station blade setting will be less than at the 42-inch station, i. e., $\Delta\theta$ is negative.

K_{T_0} may also be plotted against the V/nD for maximum efficiency as in Figure 8. In design studies it is problably more convenient to use V/nD than blade angle. Under normal conditions the two methods give identical results, but in general the blade angle is slightly preferable owing to the practice of setting the pitch instead of changing diameter to obtain the desired revolutions.

The coefficient of traction μ must be selected according to ground conditions. It varies from about 0.02 for a smooth surface such as a concrete runway or a flight deck to as much as 0.30 for a sandy surface. In the absence of exact data, the following values may be used:

Smooth deck or hard surface	$\mu=0.02$
Good field, hard turf	$\mu=0.04$
Average field, short grass	$\mu=0.05$
Average field, long grass	$\mu=0.10$
Soft ground, gravel or sand	$\mu=0.10$ to 0.30

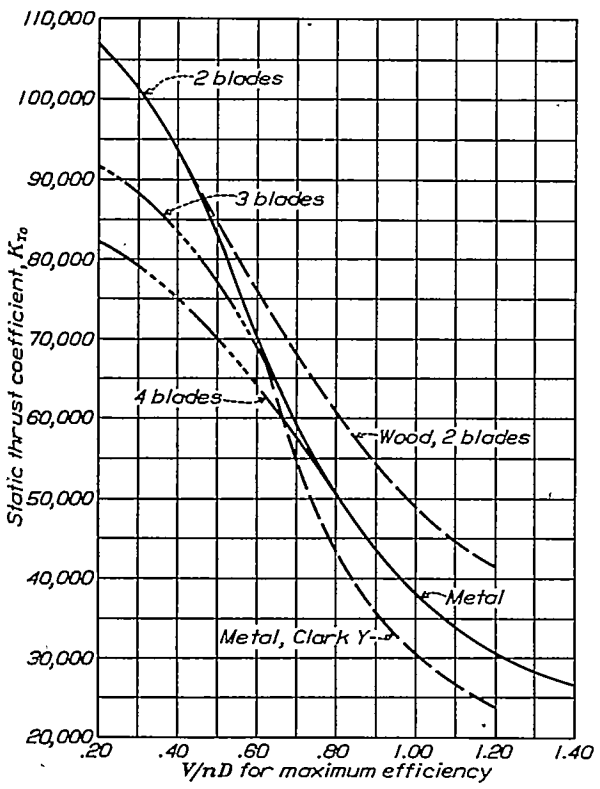

FIGURE 8.—Static thrust coefficient

FINAL NET ACCELERATING FORCE

The final net accelerating force per unit weight is defined by equation (4)

$$\frac{T_F}{W}=\frac{T_s}{W}-\frac{D}{L} \tag{4}$$

where T_s is the thrust at take-off speed and D/L is the ratio of drag to lift.

T_s may be obtained from the thrust horsepower at take-off speed V_s by the use of Figure 9, which is a plot of t.hp/t.hp$_m$ against V/V_m. The thrust horsepower at take-off speed is the product of the maximum t.hp by the ratio t.hp/t.hp$_m$ from the curve, or

$$\text{t.hp}=\text{t.hp}_\text{m}\times\frac{\text{t.hp}}{\text{t.hp}_\text{m}} \tag{15}$$

The corresponding thrust is given by

$$T_s=\frac{550\ \text{t.hp}}{V_{f.p.s.}} \tag{16}$$

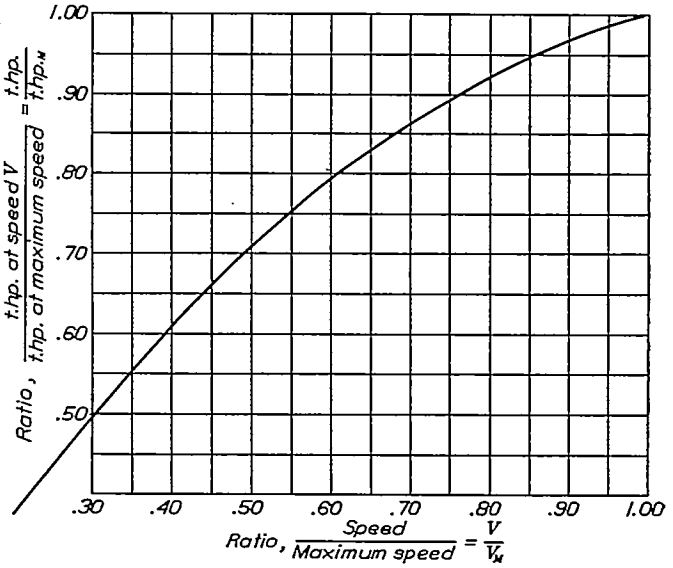

FIGURE 9.—General full-throttle t.hp. curve

The value of D/L may be taken as the reciprocal of the maximum value of L/D for the airplane, which is usually known. Figure 10 is a plot of $(L/D)_{max}$ as a function of aspect ratio and total parasite coefficient

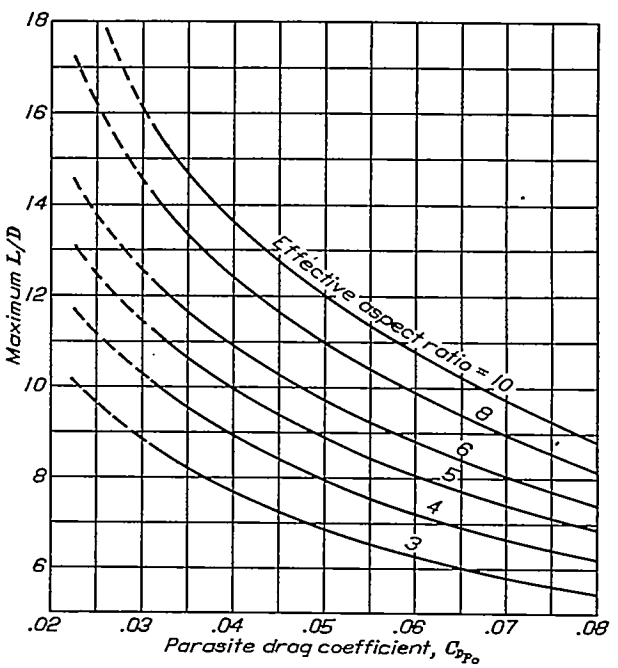

FIGURE 10

$C_{D_{P_0}}$ for use in estimating $(L/D)_{max}$. The total parasite coefficient $C_{D_{P_0}}$ includes the wing profile drag.

EFFECT OF WIND ON TAKE-OFF RUN

While the effect of wind may be obtained directly from the integrated equations of motion, it is always

more convenient to apply a correction to the run in a calm. Analysis of a series of take-off runs under various assumed conditions shows that a single curve (fig. 11) is sufficient to give the correction factor. In Figure 11 the ratio of the run S_w in a wind V_w to the run S_o in a calm is plotted against the ratio of

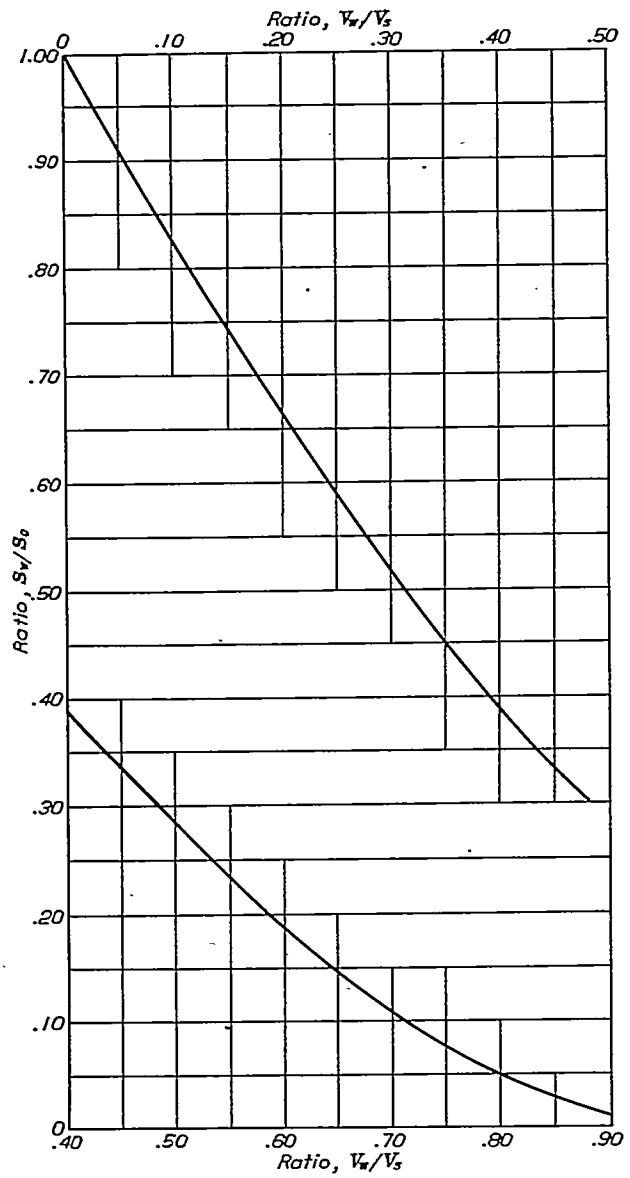

FIGURE 11.—Effect of wind on take-off run

wind speed V_w to the take-off speed V_s. When the run in a calm is known, the run in any wind is

$$S_w = S_o \times \frac{S_w}{S_o} \tag{17}$$

the value of S_w/S_o being read from Figure 11 for the value of V_w/V_s corresponding to the desired V_w.

EFFECT OF CHANGES IN WEIGHT AND TAKE-OFF SPEED

The effect of a change in weight is to change both the take-off coefficient K_s and the take-off speed V_s. Since W/V_s^2 is assumed constant

$$\frac{S_1}{S_2} = F\left(\frac{W_1}{W_2}\right)^2 \tag{18}$$

where F is a correction factor to allow for the variation in K_s. Figure 12 gives F as a function of W_1/W_2 and T_F/T_I. It will be noted that for small changes in weight (± 25 per cent) and for normal values of T_F/T_I (i. e., 0.60 to 1.00) the value of F may be taken as unity for most purposes, the average error probably being of the order of 3 per cent.

When the stalling speed is varied, with weight held constant, there will be a slight change in the ratio T_F/T_I and a corresponding change in K_s. In general these changes are negligible so that for all practical purposes

$$\frac{S_1}{S_2} = \left(\frac{V_{s1}}{V_{s2}}\right)^2. \tag{19}$$

INSTRUCTIONS FOR CALCULATION OF TAKE-OFF RUN

In general the use of equation (10) requires the following steps:

1. Obtain the static thrust coefficient K_{T_O} from Figure 6 or Figure 8 and calculate the static thrust T_o using equation (14);
2. Divide the static thrust by the gross weight, and subtract the proper value of the coefficient of traction μ to obtain the initial net accelerating force per unit weight $\frac{T_I}{W}$ (equation (3));
3. Calculate the thrust available at take-off speed T_v using the method described above, equations (15) and (16) with Figure 9, and divide T_v by the gross weight;
4. Calculate or estimate the maximum L/D of the airplane (using Figure 10, if necessary) and subtract the reciprocal from T_v/W to obtain the final net accelerating force per unit weight T_F/W (equation (4));
5. Take the ratio of T_F/T_I and read the corresponding value of the take-off coefficient K_s from Figure 4.
6. Calculate the take-off run in a calm, using this value of K_s with equation (10);
7. Apply corrections as required; for wind using equation (17) with Figure 11, or for a different gross weight using equation (18) with Figure 12.

An example of a calculation for take-off run, showing method of tabulating work, is given in Table IV.

BUREAU OF AERONAUTICS,
NAVY DEPARTMENT,
WASHINGTON, D. C., *Sept., 1932.*

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TABLE I

CALCULATED TAKE-OFF RUN USING NET ACCELERATING FORCE SHOWN ON FIGURE 2

Air speed V ft./sec.	Net accelerating force $\frac{F}{W}$	Acceleration a ft./sec.²	$\frac{V}{a}\frac{dS}{dV}$	$\Sigma\left(\frac{V}{a}\right)$	Distance run S ft.
0	0.4500	14.48	0	0	0
10	.4395	14.13	0.71	0.35	3.5
20	.4280	13.77	1.45	1.43	14.3
30	.4155	13.36	2.25	3.28	32.8
40	.4020	12.92	3.09	5.95	59.5
50	.3875	12.46	4.01	9.50	95.0
60	.3720	11.97	5.02	14.02	140.2
70	.3555	11.43	6.12	19.59	195.9
80	.3380	10.87	7.35	26.32	263.2
90	.3195	10.27	8.76	34.38	343.8
100	.3000	9.65	10.35	43.93	439.3

TABLE II

CALCULATED TAKE-OFF RUN USING ASSUMED LINEAR NET ACCELERATING FORCE SHOWN ON FIGURE 3

Air speed V ft./sec.	Net accelerating force $\frac{F}{W}$	Acceleration a ft./sec.²	$\frac{V}{a}\frac{dS}{dV}$	$\Sigma\left(\frac{V}{a}\right)$	Distance run S ft.
0	0.450	14.48	0	0	0
10	.435	13.98	0.71	0.35	3.5
20	.420	13.50	1.48	1.45	14.5
30	.405	13.02	2.30	3.34	33.4
40	.390	12.55	3.19	6.08	60.8
50	.375	12.06	4.14	9.75	97.5
60	.360	11.59	5.18	14.41	144.1
70	.345	11.10	6.30	20.15	201.5
80	.330	10.62	7.53	27.06	270.6
90	.315	10.13	8.88	35.27	352.7
100	.300	9.65	10.35	44.88	448.8

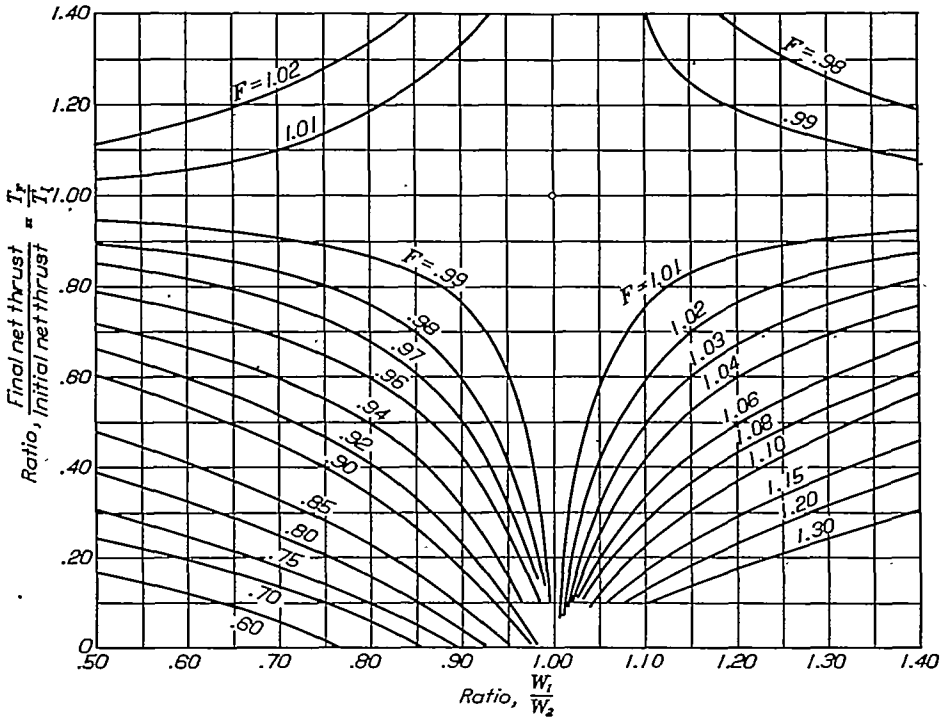

FIGURE 12.—Effect of weight on take-off run

$$\frac{S_1}{S_2}=F\left(\frac{W_1}{W_2}\right)^2$$

TABLE III

CALCULATION FOR K_s AND K_t

$\frac{T_F}{T_I}=(1-K)$	K	$\log(1-K)$	$\frac{\log(1-K)}{K^2}$ A	$\frac{1}{K}$ B	$\frac{F}{g}=-B-A$	$\frac{K_s}{F}$	$\frac{-\log(1-K)}{K}$	$\frac{K_t}{-\log(1-K)}=\frac{}{gK}$
1.30	−0.30	0.262364	2.915159	−3.3333	0.41817	0.01300	0.8745	0.02718
1.20	−.20	.182321	4.558036	−5.0000	.44196	.01374	.9116	.02833
1.10	−.10	.095310	9.531021	−10.0000	.46898	.01458	.9531	.02962
1.05	−.05	+.048790	19.51606	−20.0000	.48394	.01504	.9758	.03033
1.00	0	0	----------	----------	(.50000)	(.01554)	(1.0000)	.03108
.95	+0.05	−.051293	−20.51732	+20.0000	.51732	.01609	1.0259	.03189
.90	.10	−.105360	−10.53603	10.0000	.53603	.01666	1.0536	.03275
.85	.15	−.162519	−7.223053	6.66667	.55639	.01729	1.0835	.03368
.80	.20	−.223143	−5.578587	5.0000	.57859	.01798	1.1157	.03467
.75	.25	−.287682	−4.602912	4.0000	.60291	.01874	1.1507	.03577
.70	.30	−.356675	−3.963053	3.3333	.62972	.01957	1.1889	.03695
.60	.40	−.510826	−3.192659	2.5000	.69266	.02153	1.2771	.03969
.50	.50	−.693147	−2.772589	2.0000	.77259	.02401	1.3863	.04309
.40	.60	−.916291	−2.545252	1.6666	.87859	.02731	1.5272	.04747
.30	.70	−1.203973	−2.457087	1.428571	1.02852	.03197	1.7200	.05346
.20	.80	−1.609438	−2.514746	1.25000	1.26475	.03931	2.0118	.06253
.10	.90	−2.302585	−2.842695	1.1111	1.73158	.05382	2.5584	.07952

TABLE IV

EXAMPLE OF CALCULATIONS FOR TAKE-OFF RUN

		Remarks
Airplane type	Biplane	
Gross weight W lb	3,000	
b. hp/prop. r.p.m	400/1900	
Wing area S sq. ft	250	
Wing loading W/S lb./sq. ft	12.00	
Wing section	Clark Y	
Maximum lift coefficient $C_L=$	1.50	
Stalling speed V_s ft./sec	82	
Maximum speed V_m ft./sec	220	
Propeller diameter D ft	9.0	
V/nD	.77	
$\eta=$	.80	
t.hp=	320	$=\eta=\times$b.hp.
Propeller blade setting 42″ R	20.0°	Setting given.
Propeller blade setting 0.75 R	20.5°	Figure 7.
Propeller blade section, or design	normal	
Static thrust coefficient K_{T_0}	53,000	Figure 6 or Figure 6-8.
Static thrust $T_0=\frac{K_{T_0}\times\text{b.hp.}}{\text{r.p.m.}\times D}$	1,240	Equation (14).
$\frac{T_0}{W}$	.413	

TABLE IV

EXAMPLE OF CALCULATIONS FOR TAKE-OFF RUN—Continued

		Remarks
$\frac{T_1}{W}=\left(\frac{T_0}{W}-\mu\right)$ $\mu=.02$	.393	Equation (3).
Ratio V_s/V_m	.373	
t.hp/t.hp at V_s	.578	From Figure 9.
t.hp at V_s	185	Equation (15).
Thrust at V_s $T_s=$	1,240	Equation (16).
T_s/W	.413	
Maximum L/D	9.0	See Figure 10.
$\frac{T_F}{W}=\left(\frac{T_s}{W}-\frac{D}{L}\right)$	.302	Equation (4).
Ratio T_F/T_I	.770	
Take-off run coefficient K_s	.0183	From Figure 4.
Run $S_s=\frac{K_s V_s^2}{(T_I/W)}$ ft	313	Equation (10).
Wind velocity V_w ft./sec	40	
Ratio V_w/V_s	.49	
Ratio S_w/S_s	.295	From Figure 11.
S_w ft	92	Equation (17).