REPORT No. 733

CRITICAL COMPRESSIVE STRESS FOR FLAT RECTANGULAR PLATES SUPPORTED ALONG ALL EDGES AND ELASTICALLY RESTRAINED AGAINST ROTATION ALONG THE UNLOADED EDGES

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SUMMARY

A chart is presented for the values of the coefficient in the formula for the critical compressive stress at which buckling may be expected to occur in flat rectangular plates supported along all edges and, in addition, elastically restrained against rotation along the unloaded edges.

The mathematical derivations of the formulas required in the construction of the chart are given.

INTRODUCTION

In the design of stressed-skin structures for aircraft as well as in the design of compression members, it is desirable to know the compressive stress at which buckling occurs. In practice the structure is usually so imperfect or so eccentrically loaded that lateral deflection starts with the beginning of loading. When lateral deflection starts with the beginning of loading, however, there is usually a very pronounced increase in deflection at the critical compressive stress for which buckling would have occurred had the structure been perfectly shaped and centrally loaded. The evaluation of this critical compressive stress for a flat plate, with certain conditions of edge support, is discussed in this paper.

When a flat plate is loaded in compression, the two loaded edges are either simply supported or restrained in some manner. If the two unloaded edges are not supported, the plate is considered to be a column. When one or both of the unloaded edges are also supported or restrained in some manner, the critical compressive stress is greatly increased over that for the plate as a column. That the compressive stress is increased when one or both of the edges are supported or restrained has been recognized for years. Because of the importance of the edge conditions, formulas based on the assumption that each edge of the plate is free, simply supported, or fixed have been employed in design. (See the summary of these formulas given in reference 1.)

A study of the theory and the more reliable test data on the buckling of plate elements in stressed-skin structures and compression members revealed the necessity for a more careful consideration of the edge condition of plates than has been previously attempted. Accordingly studies were made of the critical compressive stress for I-, Z-, channel, and rectangular-tube sections in which proper consideration was given to the interaction between the individual parts of the cross section. (See references 2, 3, and 4.) In order to make the results of the work more generally applicable, studies were also made of the basic plate elements that constitute these sections. All the basic design charts resulting from this investigation were made available in 1938. The combination of the present paper with references 2, 3, 4, and 5 is a more complete presentation of all this material.

The basic element treated in this paper is a plate supported along the four edges, elastically restrained against rotation along the unloaded edges but with no restraint against rotation along the loaded edges. The loaded edges are therefore considered to be simply supported according to the usual terminology. This basic element is representative of the webs of I-, Z-, and channel-section columns, of the walls of a rectangular tube, and of the flat skin between the stiffeners of a stressed-skin structure. The basic element representative of an outstanding flange with elastic restraint against rotation along one unloaded edge is treated in reference 5.

The mathematical derivations required for the investigation of the present paper are given in appendixes A and B. The results of practical use are given in the body of the paper.

Bernard Rubenstein, formerly of the NACA staff, performed all the mathematical derivations required for appendix B, the presentation of which was adapted to the purposes of this paper.

EVALUATION OF CRITICAL STRESS

Within the elastic range.—Within the elastic range in which the effective modulus of elasticity is Young's modulus, the critical compressive stress f_{cr} for a thin flat rectangular plate is expressed as (reference 6, p. 331, equation (214)):

$$f_{cr}=\frac{k\pi^2Et^2}{12(1-\mu^2)b^2} \tag{1}$$

where

k nondimensional coefficient that depends upon conditions of edge restraint and shape of plate
E Young's modulus
t thickness of plate
μ Poisson's ratio
b width of plate

Beyond the elastic range.—When the plate is stressed in compression beyond the elastic range, the effective modulus of elasticity for the plate is less than Young's

modulus. If a single, over-all effective plate modulus ηE is substituted for Young's modulus E, the critical stress, when the material of the plate is loaded beyond its elastic range, can be obtained from equation (1). The nondimensional coefficient η has a value that lies between zero and unity and is determined by the stress. For stresses within the elastic range, $\eta=1$. For a more complete discussion and definition of η, see reference 2.

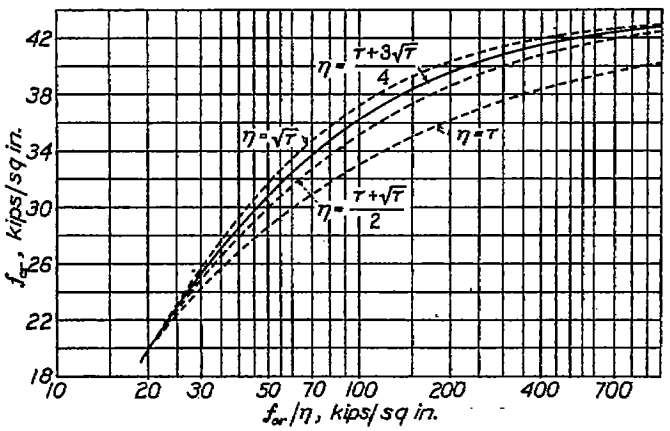

FIGURE 1.—Variation of f_{cr} with f_{cr}/η for 24S-T aluminum alloy of minimum required properties. (When $f_{cr}/\eta<19.6$ kips/sq in., $\eta=1$ and $f_{cr}=f_{cr}/\eta$.)

If ηE is substituted for E in equation (1), the resulting equation cannot be directly solved for f_{cr}. If the equation is divided by η, however, f_{cr}/η is given directly by the geometrical dimensions of the plate, Young's modulus E, and Poisson's ratio μ. Thus

$$\frac{f_{cr}}{\eta}=\frac{k\pi^2Et^2}{12(1-\mu^2)b^2} \tag{2}$$

For a given material, the relationship between f_{cr} and f_{cr}/η tends to be fixed by the compressive stress-strain curve. This relationship is discussed in reference 2, where it is shown how probable relationships between f_{cr} and f_{cr}/η are obtained from the column curve for the material because column curves are more readily available than compressive stress-strain curves. The question is, therefore, what column formula should be used? Equations (8) and (9) of reference 7 define column curves that apply when the material just satisfies the minimum requirements of Navy Department Specification 46A9a for 24S-T aluminum alloy. The relationships between f_{cr} and f_{cr}/η for this case are given in references 2, 3, and 4 and in figure 1 of this paper.

The 24S-T material delivered under specification 46A9a almost always has properties that are better than the minimum required properties. The relationships between f_{cr} and f_{cr}/η for the average 24S-T material delivered are given in figure 2. This figure has been prepared in the manner described in reference 2; the column curves for average 24S-T material as given in reference 8 were used.

Figures similar to 1 and 2 of this paper may be prepared for any material. The engineer using this paper must therefore decide whether the computation should be based on minimum required material properties or average material properties.

Regardless of whether figure 1 or 2 is used, it is recommended that the $\eta=\frac{\tau+3\sqrt{\tau}}{4}$ curve be used for all values of restraint against rotation until future experimental data indicate that a different curve should be used. In figures 1 and 2 the different equations involving τ merely identify different curves that result from the relationships indicated. The value of τ is $\bar{E}/E$, the ratio of the effective column modulus for bending failure at the stress f_{cr} to Young's modulus.

EVALUATION OF k

The value of f_{cr}/η at which buckling occurs is given by equation (2), in which all of the quantities are known except the value of the coefficient k.

Equal restraints on the side edges of the plate.—From figure 3 can be obtained the values of k for the case of equal restraint on each side edge of the plate, which is a special case of the general solution in appendix A for any restraint on either side edge of the plate. In the chart of figure 3, k is plotted against the ratio of half wave length to the plate width λ/b for different values of a parameter ϵ, termed the "restraint" coefficient. (Trayer and March in reference 9 refer

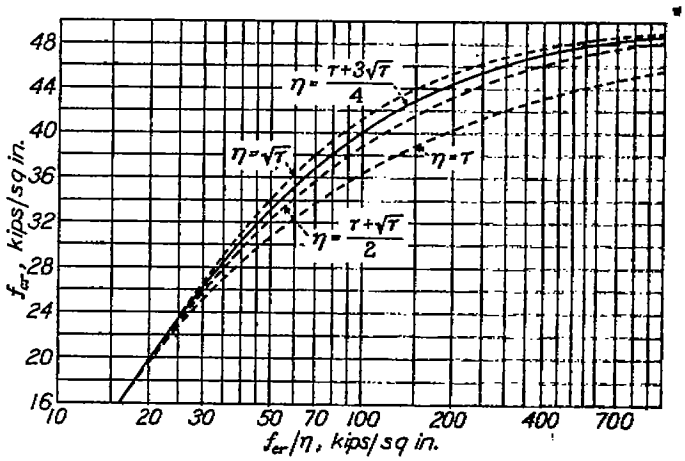

FIGURE 2.—Variation of f_{cr} with f_{cr}/η for 24S-T aluminum alloy of average properties. (When $f_{cr}/\eta<16.7$ kips/sq in., $\eta=1$ and $f_{cr}=f_{cr}/\eta$.)

to ϵ as the "fixity" coefficient. In the present paper restraint coefficient was chosen to avoid confusion with the fixity coefficient c for columns.)

The restraint coefficient ϵ depends upon the relative stiffness of the plate and the restraining element along the side edge of the plate. The simplest conception of ϵ is obtained when the restraining element, or stiffener, is assumed to be replaced by an elastic medium in which rotation at one point does not influence rotation at another point. For this type of restraining medium along the edge of the plate, within the elastic range

$$\epsilon=\frac{4S_0b}{D} \tag{3}$$

beyond the elastic range

$$\epsilon=\frac{4S_0b}{\eta D} \tag{4}$$

where

S_0 stiffness per unit length of elastic restraining medium or moment required to rotate a unit length of elastic medium through one-fourth radian

D flexural rigidity of plate, per unit length $\left[\frac{Et^3}{12(1-\mu^2)}\right]$

η coefficient to allow for a decrease in D due to the application of stresses beyond the elastic range

Inasmuch as η is a function of stress, its value for 24S–T material can be obtained from figure 4 or 5, depending upon whether minimum required properties or average properties are being used. The values of τ_1, τ_2, τ_3, also given in figures 4 and 5, occur in appendix A.

If S_0 is zero, ϵ is also zero and the condition of zero

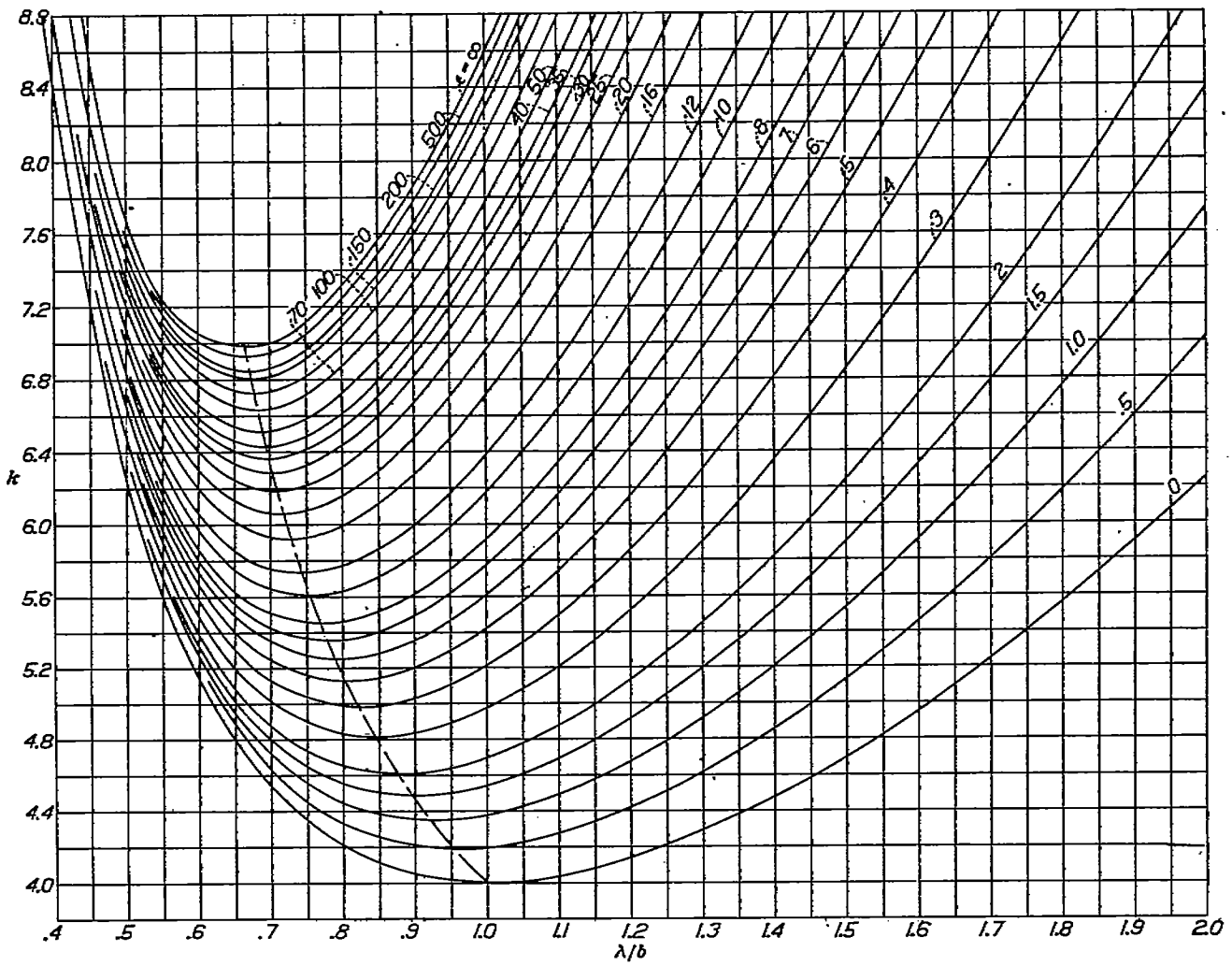

FIGURE 3.—General design chart giving values of k for equal restraint coefficients ϵ on each side edge of the plate. $\frac{f_{cr}}{\eta}=\frac{k\pi^2Et^3}{12(1-\mu^2)b^2}$

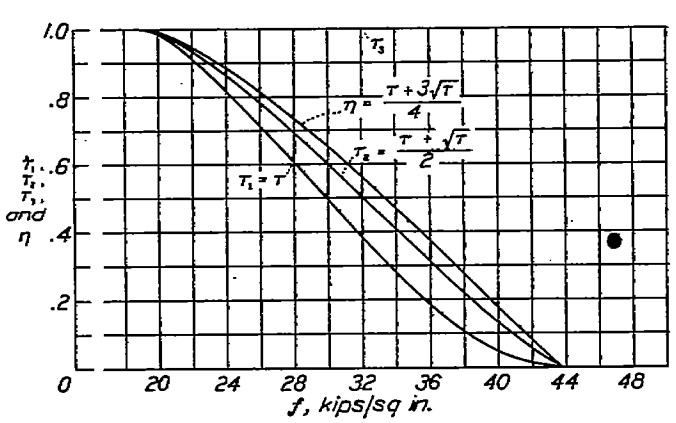

FIGURE 4.—Variation of τ_1, τ_2, τ_3, and η with the compressive stress f, for 24S–T aluminum alloy of minimum required properties.

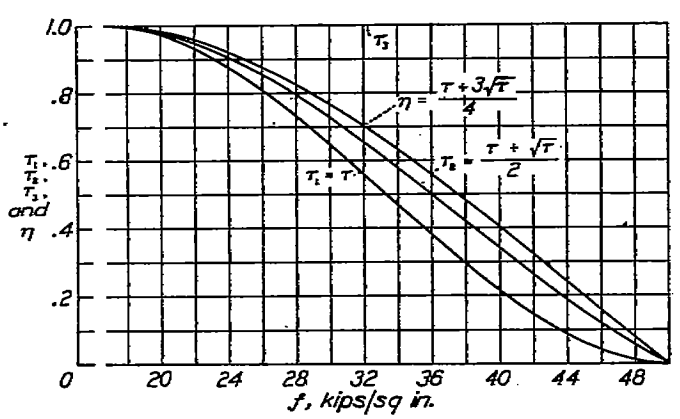

FIGURE 5.—Variation of τ_1, τ_2, τ_3, and η with the compressive stress f, for 24S–T aluminum alloy of average properties.

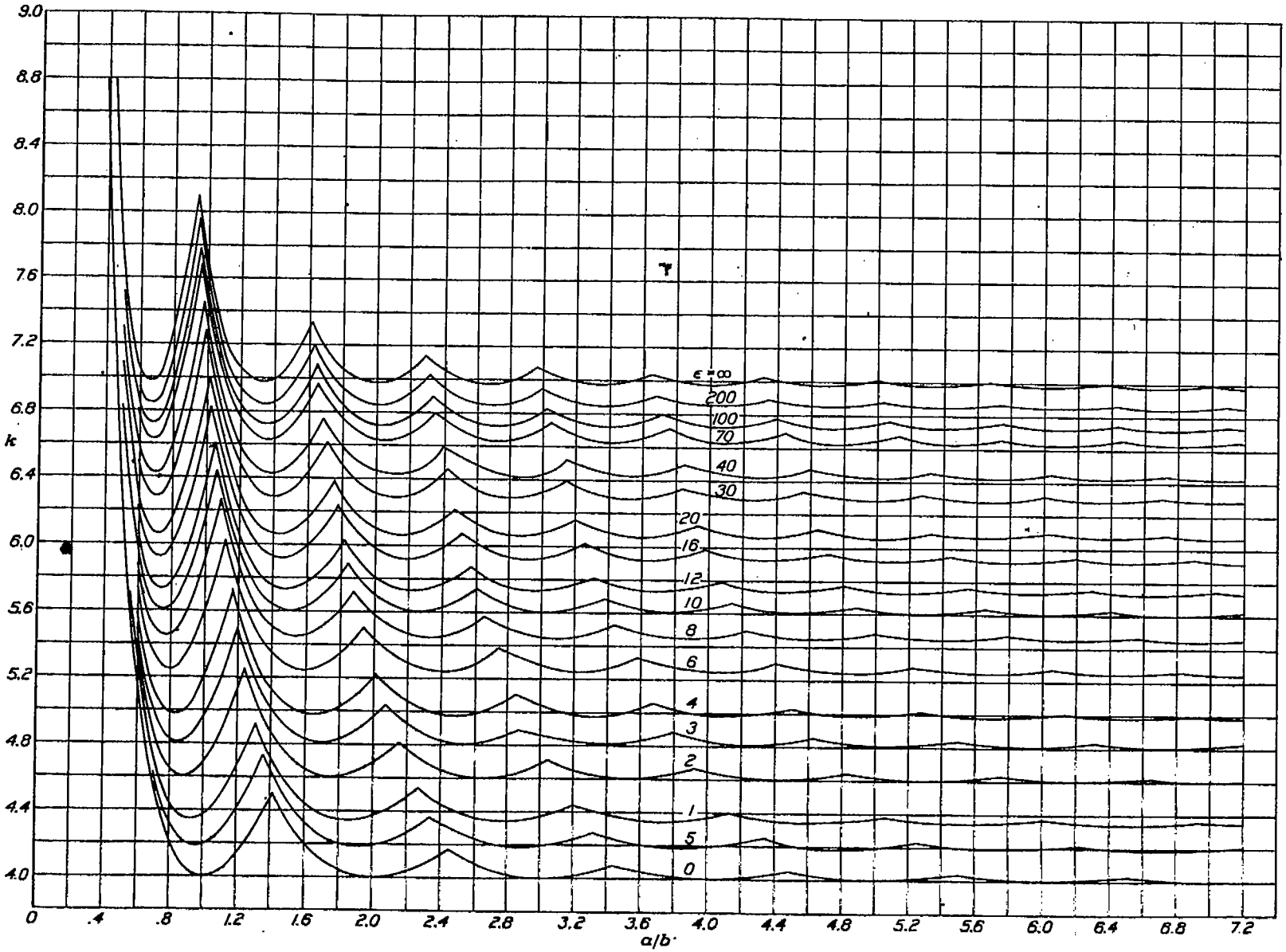

FIGURE 6.—Special design chart giving values of k for equal restraint coefficients ϵ on each side edge of the plate when the restraint is independent of the half wave length λ $\frac{f_{cr}}{\eta}=\frac{kw^2E\pi^2}{12(1-\mu^2)b^2}$

restraint, or simple support, is obtained. If S_0 is infinite, ϵ is also infinite and the condition of infinite restraint or of a fixed edge is obtained. Therefore a variation of ϵ from zero to infinity will cover all possible conditions of restraint at the side edge of the plate.

Figure 3 shows that for each value of ϵ there is a value of λ/b for which k is a minimum. Strictly, a whole number m of half wave lengths λ must exist in the length of the plate a. Hence

$$\frac{\lambda}{b}=\frac{a}{mb} \tag{5}$$

Thus, to read a value of k from figure 3, it is necessary to substitute $m=1$, 2, 3, etc. in equation (5) until a value for λ/b is obtained that gives the smallest value of k in figure 3. This smallest value of k is the one to be used in equation (1) or (2). This general procedure will always give the correct value of k for use in equation (1) or (2) regardless of whether or not S_0, and hence ϵ, is a function of the half wave length λ.

For the special case in which S_0, and hence ϵ, is independent of the half wave length λ, the general procedure described for obtaining a value for k can be used to construct a new chart, with the abscissa λ/b replaced by a/b. This new chart is given in figure 6.

When S_0, and hence ϵ, varies with λ or λ/b, figure 6 should not be used but the general procedure as applied to figure 3 should be used to obtain the correct value of k for equations (1) and (2).

Unequal restraints on the side edges of the plate.—The charts of figures 3 and 6 were drawn on the assumption that equal restraints exist along each side edge of the plate. If unequal restraints exist along each side edge, the method for equal restraints is applied, and one side restraint is used first and then the other. The average of the two values of k thus obtained is a reasonably good approximation of the true value of k. This average may be either the arithmetic mean, $(k_1+k_2)/2$, or the geometric mean, $\sqrt{k_1k_2}$. The value of k as given by each of these averages is compared with the true value of k in table I for a number of special cases. For all of the cases except the last three in table I, the values of k_1 and k_2 were read at the value of λ/b that gave the minimum k. In the last three cases the values of k_1 and k_2 were read at the same value of λ/b.

Inspection of table I shows that, when the values of k_1 and k_2 are read near the minimum points of the curves for ϵ_1 and ϵ_2, respectively, the arithmetic mean generally gives smaller errors than the geometric mean although either one could be used without serious error in practical problems. On the other hand, if the values of k_1 and k_2 are read at the same value of λ/b, the arithmetic mean gives definitely larger errors. It is therefore recommended that the geometric mean be used when the value of λ/b is fixed. Either method may be used when the values of k_1 and k_2 are read near the minimum points.

When the critical compressive stress for unequal restraints is found by the method of the geometric mean, the error in k, and hence the error in the critical stress for problems in the elastic range, is less than 3 percent. Beyond the elastic range it is more conservative to average the two critical compressive stresses than to average the corresponding values of k and then to compute the critical compressive stress.

EVALUATION OF S_0 AND ϵ

Before it is possible to determine k from figure 3 or 6, it is necessary first to evaluate the restraint coefficient ϵ. The value of S_0 to be substituted in equation (3) or (4) will depend upon the characteristics of the structural member, or members, that provide the restraint. In this paper it is assumed that the restraint is provided by a specially defined elastic restraining medium. As a result of this assumption, it has been possible to derive the general chart of figure 3, which is independent of the structure that provides the restraint.

The basic property of the elastic restraining medium is that rotation at one point of the medium does not affect rotation at another point of the medium. In many practical problems the elastic restraint is provided by a stiffener, a plate, or some other structure for which rotation at one point affects rotation at another point. Consequently, the evaluation of S_0 in any given problem must take into account the effect of this interaction within the elastic restraining structure.

The formula for S_0 to be used in any given problem will depend upon the type of structural member that provides the restraint. Because this entire subject of the restraint supplied to the side edge of a plate has been rather superficially treated in the literature, it is being made the subject of several papers by the NACA, the first of which is reference 10.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *March 8, 1941.*

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APPENDIX A

SOLUTION BY DIFFERENTIAL EQUATION

The procedure for obtaining the critical stress of a plate uniformly compressed along two opposite, simply supported edges is given in reference 6 (p. 337). In this method, which was also used by Dunn in reference 11, the critical stress is found by solving the differential equation expressing the equilibrium of the buckled plate. The same method is applied in the present paper to the case in which unequal elastic restraints against rotation are present along the unloaded side edges of the plate. For generality, the elastic restraint is assumed to arise from an elastic medium distributed along the unloaded edges; this medium has the basic property that rotation at one point within it does not influence the rotation at any other point.

Figure 7 shows the coordinate system and the plate dimensions. The differential equation for equilibrium of a plate element is

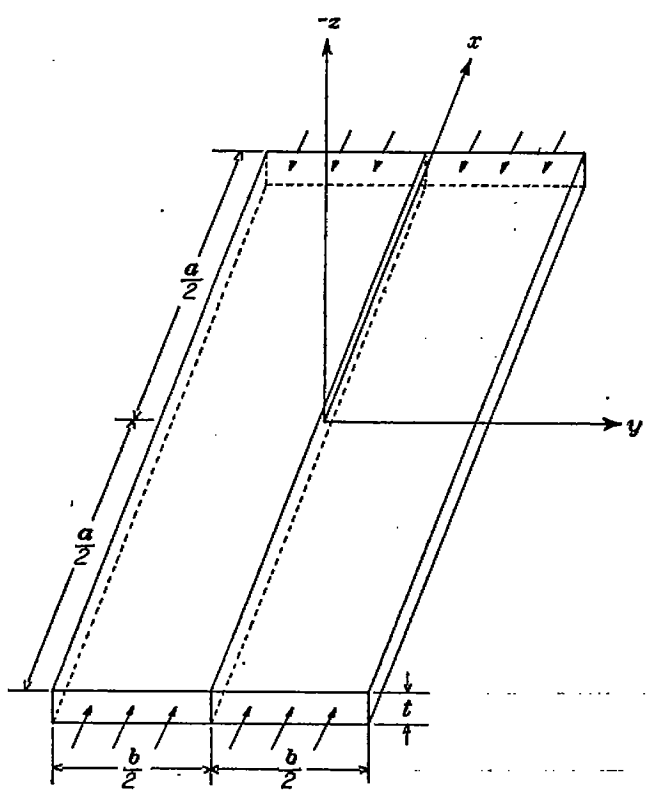

FIGURE 7.—Rectangular plate under edge compression.

$$ft\frac{\partial^2 w}{\partial x^2}=-D\left(\tau_1\frac{\partial^4 w}{\partial x^4}+2\tau_2\frac{\partial^4 w}{\partial x^2\partial y^2}+\tau_3\frac{\partial^4 w}{\partial y^4}\right) \quad \text{(A–1)}$$

where

f uniformly distributed compressive stress
t thickness of plate
w deflection normal to plate
x longitudinal coordinate in direction of applied stress
D flexural rigidity of plate, per unit length
y transverse coordinate across width of plate
τ_1, τ_2, and τ_3 coefficients equal to or less than unity

In equation (A–1) the term $ft(\partial^2 w/\partial x^2)$ is concerned with the external forces on the plate that cause buckling; whereas the term $-D\left(\tau_1\frac{\partial^4 w}{\partial x^4}+2\tau_2\frac{\partial^4 w}{\partial x^2\partial y^2}+\tau_3\frac{\partial^4 w}{\partial y^4}\right)$ is concerned with the internal resistance of the plate to buckling. The terms involving τ_1 and τ_3 in equation (A–1) are concerned with the longitudinal and the transverse bending, respectively; whereas the term involving τ_2 is concerned principally with the torsional stiffness. The coefficients τ_1, τ_2, and τ_3 allow for the change in the magnitude of the various terms as the plate is stressed beyond the elastic range. In the elastic range $\tau_1=\tau_2=\tau_3=1$.

The loaded edges are simply supported and are not displaced in the direction w. Of the several forms of the general solution of equation (A–1) the following form was selected as appropriate for this problem:

$$w=\left(C_1\cosh\frac{\alpha y}{b}+C_2\sinh\frac{\alpha y}{b}+C_3\cos\frac{\beta y}{b}+C_4\sin\frac{\beta y}{b}\right)\cos\frac{\pi x}{\lambda} \quad \text{(A–2)}$$

where

$$\alpha=\pi\sqrt{\frac{b}{\lambda}}\sqrt{\frac{\tau_2}{\tau_3}\frac{b}{\lambda}+\sqrt{\frac{k}{\tau_3}+\left(\frac{b}{\lambda}\right)^2\left(\frac{\tau_2^2}{\tau_3^2}-\frac{\tau_1}{\tau_3}\right)}} \quad \text{(A–3)}$$

$$\beta=\pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{\tau_2}{\tau_3}\frac{b}{\lambda}+\sqrt{\frac{k}{\tau_3}+\left(\frac{b}{\lambda}\right)^2\left(\frac{\tau_2^2}{\tau_3^2}-\frac{\tau_1}{\tau_3}\right)}} \quad \text{(A–4)}$$

$$k=\frac{12(1-\mu^2)b^2f}{\pi^2Et^2} \quad \text{(A–5)}$$

Equation (A–2) satisfies the boundary conditions at the loaded edges and gives real values for both α and β near the buckling stress $f=f_{cr}$.

The values of the coefficients C_1, C_2, C_3, and C_4 are to be found from the boundary conditions along the side edges of the plate. The value of λ, the half wave length of the buckle pattern, is found from the condition that there must be an integral number of half wave lengths in the length a of the plate; thus

$$\lambda=\frac{a}{m} \quad \text{(A–6)}$$

where $m=1, 2, 3$, etc.

In the elastic range, where $\tau_1=\tau_2=\tau_3=1$, the values of α and β are

$$\alpha=\pi\sqrt{\frac{b}{\lambda}}\sqrt{\frac{b}{\lambda}+\sqrt{k}} \tag{A-7}$$

$$\beta=\pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{b}{\lambda}+\sqrt{k}} \tag{A-8}$$

The solution given by equation (A-2) was selected to satisfy the boundary conditions of no deflection and simple support (no moment) along the loaded edges. The boundary conditions along the unloaded side edges have also to be satisfied. The boundary conditions along the unloaded side edges are:

$$(w)_{y=-\frac{b}{2}}=0 \tag{A-9}$$

$$(w)_{y=\frac{b}{2}}=0 \tag{A-10}$$

$$D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=-\frac{b}{2}}=4S_1\left(\frac{\partial w}{\partial y}\right)_{y=-\frac{b}{2}} \tag{A-11}$$

$$-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=\frac{b}{2}}=4S_2\left(\frac{\partial w}{\partial y}\right)_{y=\frac{b}{2}} \tag{A-12}$$

where S_1 and S_2 are the respective stiffnesses per unit length of the elastic restraining mediums or the moments required to rotate a unit length of the medium through one-fourth radian.

From equations (A-9) and (A-10) are obtained

$$C_3=-C_1\frac{\cosh\frac{\alpha}{2}}{\cos\frac{\beta}{2}} \tag{A-13}$$

$$C_4=-C_2\frac{\sinh\frac{\alpha}{2}}{\sin\frac{\beta}{2}} \tag{A-14}$$

From equations (A-11) and (A-12) are obtained

$$C_1\left[(\alpha^2+\beta^2)\cosh\frac{\alpha}{2}+\frac{4S_1b}{D}\left(\alpha\sinh\frac{\alpha}{2}+\beta\cosh\frac{\alpha}{2}\tan\frac{\beta}{2}\right)\right]$$
$$-C_2\left[(\alpha^2+\beta^2)\sinh\frac{\alpha}{2}+\frac{4S_1b}{D}\left(\alpha\cosh\frac{\alpha}{2}-\beta\sinh\frac{\alpha}{2}\cot\frac{\beta}{2}\right)\right]=0 \tag{A-15}$$

$$C_1\left[(\alpha^2+\beta^2)\cosh\frac{\alpha}{2}+\frac{4S_2b}{D}\left(\alpha\sinh\frac{\alpha}{2}+\beta\cosh\frac{\alpha}{2}\tan\frac{\beta}{2}\right)\right]$$
$$+C_2\left[(\alpha^2+\beta^2)\sinh\frac{\alpha}{2}+\frac{4S_2b}{D}\left(\alpha\cosh\frac{\alpha}{2}-\beta\sinh\frac{\alpha}{2}\cot\frac{\beta}{2}\right)\right]=0 \tag{A-16}$$

The buckled form of equlibrium of the plate is obtained when the determinant formed by the coefficient of C_1 and C_2 in equations (A-15) and (A-16) equals zero. Thus

$$\left[(\alpha^2+\beta^2)+\epsilon_1\left(\alpha\tanh\frac{\alpha}{2}+\beta\tan\frac{\beta}{2}\right)\right]$$
$$\cdot\left[(\alpha^2+\beta^2)+\epsilon_2\left(\alpha\coth\frac{\alpha}{2}-\beta\cot\frac{\beta}{2}\right)\right]$$
$$+\left[(\alpha^2+\beta^2)+\epsilon_2\left(\alpha\tanh\frac{\alpha}{2}+\beta\tan\frac{\beta}{2}\right)\right]$$
$$\left[(\alpha^2+\beta^2)+\epsilon_1\left(\alpha\coth\frac{\alpha}{2}-\beta\cot\frac{\beta}{2}\right)\right]=0 \tag{A-17}$$

where

$$\epsilon_1=\frac{4S_1b}{D} \tag{A-18}$$

$$\epsilon_2=\frac{4S_2b}{D} \tag{A-19}$$

Equal restraints on the side edges of the plate.—When $\epsilon_1=\epsilon_2=\epsilon$, equation (A-17) reduces to

$$\left[\alpha^2+\beta^2+\epsilon\left(\alpha\tanh\frac{\alpha}{2}+\beta\tan\frac{\beta}{2}\right)\right]$$
$$\left[\alpha^2+\beta^2+\epsilon\left(\alpha\coth\frac{\alpha}{2}-\beta\cot\frac{\beta}{2}\right)\right]=0 \tag{A-20}$$

The symmetrical buckled form of equilibrium is obtained by setting the first of these factors equal to zero:

$$\alpha^2+\beta^2+\epsilon\left(\alpha\tanh\frac{\alpha}{2}+\beta\tan\frac{\beta}{2}\right)=0 \tag{A-21}$$

This equation is the same as equation (14) of reference 11. The antisymmetrical buckled form of equilibrium is obtained by setting the second factor in equation (A-20) equal to zero:

$$\alpha^2+\beta^2+\epsilon\left(\alpha\coth\frac{\alpha}{2}-\beta\cot\frac{\beta}{2}\right)=0 \tag{A-22}$$

Of these two buckled forms the symmetrical form given by equation (A-21) will occur at the lower critical stress. Therefore equation (A-21) was used to establish the exact values of k given in table II.

The condition of both side edges fixed is described by $\epsilon=\infty$, for which case equation (A-21) becomes

$$\alpha\tanh\frac{\alpha}{2}+\beta\tan\frac{\beta}{2}=0 \tag{A-23}$$

It is of interest to compare this equation with the equation given by Timoshenko in reference 6 (p. 345). In Timoshenko's equation, the symmetrical and the antisymmetrical factors have not been separated as they have been in this paper.

The condition of both side edges simply supported (no restraint) is described by $\epsilon=0$. For this special case, the problem is to find the smallest value of $k\neq 0$ that will satisfy equation (A-21) when $\epsilon=0$. A convenient method for determining this value of k is first to solve for ϵ:

$$\epsilon=-\frac{\alpha^2+\beta^2}{\alpha\tanh\frac{\alpha}{2}+\beta\tan\frac{\beta}{2}} \tag{A-24}$$

When $k=0$, it is observed that $\alpha^2+\beta^2=0$ and hence

$\epsilon=0$. All values of k greater than zero give a finite, positive value for $\alpha^2+\beta^2$ as well as for $\alpha \tanh \alpha/2$. Consequently, the only values of k greater than zero that can make $\epsilon=0$ are those values that render $\beta \tan \beta/2$ infinite. The smallest value of β that causes $\beta \tan \beta/2$ to be infinite is $\beta=\pi$. Therefore the smallest value of k that gives $\beta=\pi$ is obtained by substituting $\beta=\pi$ in equation (A-4) or

$$\pi=\pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{\tau_2}{\tau_3}\frac{b}{\lambda}+\sqrt{\frac{k}{\tau_3}+\left(\frac{b}{\lambda}\right)^2\left(\frac{\tau_2^2}{\tau_3^2}-\frac{\tau_1}{\tau_3}\right)}} \qquad \text{(A-25)}$$

from which the value of k for $\epsilon=0$ is

$$k=\tau_3\left\{\left[\frac{1}{b/\lambda}+\frac{\tau_2}{\tau_3}\frac{b}{\lambda}\right]^2-\left(\frac{b}{\lambda}\right)^2\left[\left(\frac{\tau_2}{\tau_3}\right)^2-\frac{\tau_1}{\tau_3}\right]\right\} \qquad \text{(A-26)}$$

This equation shows that k is a function of the half wave length, λ. If the plate is long, λ/b will adjust itself so as to cause k to have its minimum value. This value of λ/b is $\sqrt[4]{\frac{\tau_1}{\tau_3}}$, which gives

$$k_{min}=2(\tau_2+\sqrt{\tau_1\tau_3}) \qquad \text{(A-27)}$$

In the elastic range in which $\tau_1=\tau_2=\tau_3=1$, these equations give $\lambda/b=1$ and $k_{min}=4$.

Unequal restraints on the side edges of the plate.—When the restraint coefficients ϵ_1 and ϵ_2 at the two side edges of the plate are unequal, equation (A-17) must be used to establish the value of k and hence the critical value of the compressive stress f. This method of establishing k is long and cumbersome. A much shorter and more easily applied method is therefore desirable for practical application. The recommended short method given in the main part of this paper gives good approximate values.

APPENDIX B

SOLUTION BY ENERGY METHOD FOR EQUAL SIDE RESTRAINTS

Because the exact solution of the differential equation given in appendix A does not lend itself to a direct calculation of k, as in the case of the energy method of solution, an energy solution was made to aid in the construction of the chart of figure 3. The energy method gives approximate values for k, the accuracy of which depends upon how closely the assumed deflection surface describes the true deflection surface.

The energy method as applied to the calculation of critical compressive stresses is given in reference 6 (p. 327). The plate is stable when $(V_1+V_2)>T$ and unstable when $(V_1+V_2)<T$ where T is the work done by the compressive forces on the plate, V_1 is the strain energy in the plate, and V_2 is the strain energy in two elastic restraining mediums at the edges of the plate. The critical stress is obtained from the condition of neutral stability:

$$T=V_1+V_2 \tag{B-1}$$

If w is the deflection normal to the plate at any point x, y in the plane of the plate shown in figure 7 and $S_0=S_1=S_2$ (see appendix A), then T, V_1, and V_2 are given by the following equations. (See reference 6, equations (199) and (201) and reference 9, equation (73).)

$$T=\frac{1}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} ft\left(\frac{\partial w}{\partial x}\right)^2 dxdy \tag{B-2}$$

$$V_1=\frac{D}{2}\int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 +2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}dxdy \tag{B-3}$$

$$V_2=\frac{1}{2}\left\{4S_0\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\left[\left(\frac{\partial w}{\partial y}\right)_{y=-\frac{b}{2}}\right]^2 dx+4S_0\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\left[\left(\frac{\partial w}{\partial y}\right)_{y=\frac{b}{2}}\right]^2 dx\right\} \tag{B-4}$$

In order to evaluate T, V_1, and V_2, it is necessary to assume a deflected surface w consistent with the boundary conditions. These conditions specify that along the two side edges of the plate there be (1) no deflection and (2) equal restraint against rotation. The side edges will therefore be subjected to equal and opposite edge moments. A plate with no restraining moments at its edges buckles in the form of a sine curve across the plate. A beam with equal and opposite end moments deflects into a circular arc. Both the sine curve and the circular arc satisfy the condition of zero deflection at the side edges of the plate. Consequently, for the deflection curve across the width of the plate, a curve given by the sum of a circular arc and a sine curve was selected. In the direction of the length, the usual sine curve indicated by the solution of the differential equation is used. Thus the deflection surface assumed for the plate is, in the coordinate system of figure 7,

$$w=\left[\frac{4A}{b^2}\left(y^2-\frac{b^2}{4}\right)+\left(\frac{4A}{\pi}+B\right)\cos\frac{\pi y}{b}\right]\cos\frac{\pi x}{\lambda} \tag{B-5}$$

where A and B are arbitrary deflection amplitudes.

The combination of A and B in equation (B-5) was selected so that $A=0$ would represent the condition of simply supported side edges and $B=0$, the condition of fixed side edges. The ratio A/B is therefore a measure of edge restraint and is related to the restraint coefficient ϵ through the boundary condition:

$$-D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=\frac{b}{2}}=4S_0\left(\frac{\partial w}{\partial y}\right)_{y=\frac{b}{2}} \tag{B-6}$$

Substitution of w as given by equation (B-5) in equation (B-6) gives

$$A=\frac{\pi\epsilon}{8}B \tag{B-7}$$

where, by definition,

$$\epsilon=\frac{4S_0 b}{D} \tag{B-8}$$

Substitution of the value of A as given by expression (B-7) in equation (B-5) gives

$$w=B\left[\frac{\pi\epsilon}{2b^2}\left(y^2-\frac{b^2}{4}\right)+\left(1+\frac{\epsilon}{2}\right)\cos\frac{\pi y}{b}\right]\cos\frac{\pi x}{\lambda} \tag{B-9}$$

This new equation for w shows how the shape of the deflection surface is affected by the restraint coefficient ϵ. This equation is used in the evaluation of T, V_1, and V_2.

Thus

$$T=B^2\frac{\pi^2bt}{4\lambda}\left[\frac{\pi^2\epsilon^2}{120}-\frac{4\epsilon}{\pi^2}\left(1+\frac{\epsilon}{2}\right)+\frac{1}{2}\left(1+\frac{\epsilon}{2}\right)^2\right]f \quad \text{(B-10)}$$

$$V_1=B^2\frac{\pi^4D}{4b\lambda}\left\{\left[\frac{\pi^2}{120}\left(\frac{b}{\lambda}\right)^2+\frac{1}{\pi^2}\left(\frac{\lambda}{b}\right)^2+\frac{1}{6}\right]\epsilon^2+\left(1+\frac{\epsilon}{2}\right)\left(\frac{b}{\lambda}+\frac{\lambda}{b}\right)^2\left[\frac{1}{2}\left(1+\frac{\epsilon}{2}\right)-\frac{4\epsilon}{\pi^2}\right]\right\} \quad \text{(B-11)}$$

$$V_2=B^2\frac{\pi^2D\lambda\epsilon}{2b^3} \quad \text{(B-12)}$$

It is permissible to substitute these values in equation (B-1) only when the compressive stress f has its critical value f_{cr}. From this substitution,

$$f_{cr}=\frac{k\pi^2Et^2}{12(1-\mu^2)b^2} \quad \text{(B-13)}$$

where

$$k=\frac{\left[\frac{\left(\frac{\pi b}{\lambda}\right)^2}{120}+\frac{1}{\left(\frac{\pi b}{\lambda}\right)^2}+\frac{1}{6}\right]\epsilon^2+\left(1+\frac{\epsilon}{2}\right)\left(\frac{b}{\lambda}+\frac{\lambda}{b}\right)^2\left[\frac{1}{2}\left(1+\frac{\epsilon}{2}\right)-\frac{4\epsilon}{\pi^2}\right]+\frac{2\epsilon}{\left(\frac{\pi b}{\lambda}\right)^2}}{\frac{\pi^2\epsilon^2}{120}-\frac{4\epsilon}{\pi^2}\left(1+\frac{\epsilon}{2}\right)+\frac{1}{2}\left(1+\frac{\epsilon}{2}\right)^2} \quad \text{(B-14)}$$

Equation (B-14) was used to calculate the values of k listed in the columns (a) of table II. With these values of k as a guide, a number of correct values of k were obtained by satisfying equation (A-21) of appendix A. In this manner the errors in k as given by equation (B-14) were established at isolated points. From this knowledge of the errors, corrections were made to all the values of k given in columns (a) of table II. These corrected values of k, which are recommended, are listed in the columns (b) of table II. The recommended values of k were used in the construction of figures 3 and 6.

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TABLE I.—COMPARISON OF APPROXIMATE AND EXACT VALUES OF k FOR A PLATE WITH UNEQUAL ELASTIC RESTRAINTS AT THE UNLOADED EDGES

ϵ_1	ϵ_2	$\frac{\lambda}{b}$ and k for ϵ_1		$\frac{\lambda}{b}$ and k for ϵ_2		True value of k from equation (A-17)	Arithmetic mean		Geometric mean	
		λ/b	k_1	λ/b	k_2		$k=\frac{k_1+k_2}{2}$	Error (percent)	$k=\sqrt{k_1k_2}$	Error (percent)
0	0	1.000	4.00	1.000	4.00	4.00	4.00	0	4.00	0
0	.5	1.000	4.00	.805	5.12	4.57	4.56	−.22	4.53	−.88
1	5	.922	4.34	.805	5.12	4.74	4.73	−.21	4.71	−.63
2	5	.876	4.60	.805	5.12	4.86	4.86	0	4.85	−.21
3	5	.845	4.81	.805	5.12	4.96	4.97	.20	4.96	0
4	5	.823	4.98	.805	5.12	5.05	5.05	0	5.05	0
5	5	.805	5.12	.805	5.12	5.12	5.12	0	5.12	0
0	20	1.000	4.00	0.719	6.06	5.03	5.03	0	4.92	−2.19
2	20	.876	4.60	.719	6.06	5.31	5.33	.38	5.28	−.56
5	20	.805	5.12	.719	6.06	5.58	5.59	.18	5.57	−.18
10	20	.755	5.60	.719	6.06	5.83	5.83	0	5.83	0
20	20	.719	6.06	.719	6.06	6.06	6.06	0	6.06	0
0	∞	1.00	4.00	0.666	6.98	5.42	5.49	1.29	5.28	−2.58
3	∞	.845	4.81	.666	6.98	5.82	5.90	1.38	5.79	−.52
10	∞	.755	5.60	.666	6.98	6.25	6.29	.64	6.25	0
40	∞	.696	6.43	.666	6.98	6.69	6.71	.30	6.70	.15
∞	∞	.666	6.98	.666	6.98	6.98	6.98	0	6.98	0
0	∞	0.65	4.79	0.65	6.98	5.67	5.89	3.88	5.78	1.94
0	∞	1.00	4.00	1.00	8.89	5.74	6.29	9.58	5.86	2.09
0	8	1.50	4.69	1.50	8.90	6.51	6.79	4.30	6.46	−.77

TABLE II.—VALUES OF k IN THE BUCKLING FORMULA FOR A LONGITUDINALLY COMPRESSED RECTANGULAR PLATE WITH EQUAL ELASTIC RESTRAINTS ON TWO OPPOSITE SIDE EDGES

λ/b	0.4		0.5		0.6		0.7		0.8		0.9		1.0		1.1	
c	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
0	8.410	[c]8.410	6.250	[c]6.250	5.138	[c]5.138	4.531	[c]4.531	4.203	[c]4.203	4.045	[c]4.045	4.000	[c]4.000	4.036	[c]4.036
0.5	8.442	8.438	6.299	6.291	5.208	5.206	4.627	4.627	4.327	4.326	4.202	4.202	4.194	4.194	4.271	4.271
1	8.473	8.468	6.346	6.333	5.275	5.271	4.716	4.715	4.443	4.441	4.348	4.347	4.374	4.373	4.488	4.487
1.5	8.503	8.497	6.390	6.375	5.337	5.331	4.799	4.797	4.550	4.547	4.483	4.481	4.539	4.538	4.687	4.685
2	8.531	[c]8.525	6.432	6.414	5.395	5.387	4.876	4.873	4.650	4.646	4.608	4.605	4.693	4.691	4.872	4.870
3	8.585	8.572	6.510	6.487	5.501	5.491	5.017	5.012	4.831	4.825	4.833	4.829	4.968	4.965	5.203	5.200
4	8.634	8.611	6.579	6.551	5.596	5.584	5.141	5.134	4.969	4.962	5.030	5.025	5.208	5.205	5.490	5.487
5	8.679	8.649	6.642	6.611	5.680	5.666	5.251	5.243	5.128	5.120	5.203	5.198	5.419	5.415	5.743	5.739
6	8.720	[c]8.694	6.699	6.665	5.756	[c]5.741	5.369	5.359	5.253	5.244	5.357	5.351	5.605	5.601	5.965	5.960
7	-------	-------	6.750	6.714	5.825	5.809	5.438	5.427	5.364	5.355	5.494	5.487	5.771	5.766	6.164	6.159
8	8.793	8.746	6.797	6.758	5.886	5.868	5.517	5.505	5.464	5.454	5.617	5.609	5.920	5.914	6.341	6.335
10	8.854	8.801	6.879	6.837	5.994	5.974	5.655	5.640	5.636	5.625	5.829	5.820	6.176	6.169	6.646	6.639
12	-------	-------	6.949	6.904	6.085	6.063	5.770	5.752	5.780	5.768	6.005	5.995	6.388	6.380	6.897	6.889
16	8.991	8.927	7.060	7.012	6.228	6.203	5.951	5.929	6.005	5.992	6.279	6.268	6.718	6.708	7.289	7.280
20	9.057	[c]8.989	7.145	[c]7.095	6.336	6.309	6.087	6.063	6.173	6.159	6.484	6.472	6.963	[c]6.952	7.579	7.569
25	-------	-------	7.226	7.174	6.439	6.410	6.216	6.190	6.331	6.316	6.676	6.663	7.193	7.181	7.850	7.839
30	9.168	9.092	7.288	7.234	6.517	6.487	6.314	6.287	6.451	6.435	6.821	6.807	7.367	7.354	8.056	8.044
35	-------	-------	7.337	7.281	6.579	6.548	6.391	6.363	6.546	6.529	6.935	6.920	7.503	7.489	8.216	8.203
40	9.239	9.158	7.377	7.320	6.629	6.597	6.453	6.424	6.622	6.605	7.027	7.011	7.612	7.597	8.345	8.332
50	9.287	9.204	7.438	7.380	6.705	6.671	6.547	6.517	6.737	6.719	7.166	7.149	7.777	7.761	8.538	8.524
70	-------	-------	7.515	7.456	6.802	6.766	6.656	6.635	6.882	6.863	7.340	7.322	7.984	7.967	8.782	8.767
100	9.401	9.312	7.580	7.519	6.882	6.845	6.765	6.733	7.001	6.981	7.484	7.465	8.155	8.137	8.983	8.967
150	-------	-------	7.634	7.572	6.950	6.912	6.848	6.815	7.102	7.082	7.604	7.584	8.298	8.279	9.150	9.134
200	-------	-------	7.663	7.600	6.985	6.946	6.891	6.858	7.155	7.134	7.668	7.648	8.373	8.353	9.238	9.221
500	-------	-------	7.719	7.655	7.053	7.013	6.974	6.940	7.255	7.234	7.788	7.767	8.515	8.494	9.405	9.387
∞	9.545	[c]9.448	7.757	[c]7.692	7.101	7.059	7.032	6.997	7.326	[c]7.304	7.872	7.850	8.615	[c]8.593	-------	-------

λ/b	1.2		1.3		1.4		1.5		1.6		1.8		2.0	
c	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
0	4.134	[c]4.134	4.282	[c]4.282	4.470	[c]4.470	4.694	[c]4.694	4.951	[c]4.951	5.549	[c]5.549	6.250	[c]6.250
0.5	4.414	4.414	4.610	4.610	4.850	4.850	5.131	5.131	5.447	5.447	6.176	6.176	7.025	7.025
1	4.671	4.670	4.911	4.911	5.199	5.199	5.530	5.530	5.901	5.901	6.751	6.751	7.733	7.732
1.5	4.908	4.906	5.188	5.187	5.520	5.520	5.898	5.897	6.320	6.320	7.279	7.278	8.384	8.383
2	5.126	5.124	5.444	5.443	5.816	5.816	6.237	6.236	6.705	6.704	7.785	7.784	8.984	8.983
3	5.518	5.516	5.901	5.899	6.345	6.344	6.843	6.842	7.392	7.391	8.632	8.631	10.051	10.050
4	5.858	5.855	6.298	6.296	6.803	6.802	7.366	7.365	7.986	7.984	9.380	9.378		
5	6.155	6.151	6.645	6.643	7.203	7.202	7.824	7.822	8.504	8.502				
6	6.418	6.414	6.951	[c]6.948	7.555	7.553	8.226	8.224	8.961	8.959	10.607	[c]10.605		
7	6.651	6.646	7.222	7.219	7.865	7.863	8.584	8.582	9.366	9.364				
8	6.860	6.855	7.465	7.461	8.147	8.144	8.903	8.900						
10	7.218	7.212	7.881	7.877	8.626	8.623	9.448	9.444						
12	7.513	7.507	8.224	8.220	9.020	9.017	9.897	9.893						
16	7.972	7.965	8.755	8.750	9.630	9.626								
20	8.311	8.303	9.148	[c]9.143										
25	8.628	8.619	9.515	9.509										
30	8.868	8.859	9.791	9.785										
35	9.055	9.046												
40	9.205	9.195												
50	9.430	9.420												
70	9.709	9.698												
100														
150														
200														
500														
∞														

[a] Values obtained from the energy method. [b] Recommended values. [c] Values obtained from the exact solution of the differential equation.