REPORT No. 734

CRITICAL COMPRESSIVE STRESS FOR OUTSTANDING FLANGES

By Eugene E. Lundquist and Elbridge Z. Stowell

SUMMARY

A chart is presented for the values of the coefficient in the formula for the critical compressive stress at which buckling may be expected to occur in outstanding flanges. These flanges are flat rectangular plates supported along the loaded edges, supported and elastically restrained along one unloaded edge, and free along the other unloaded edge.

The mathematical derivations of the formulas required for the construction of the chart are given.

INTRODUCTION

In the design of stressed-skin structures for aircraft as well as in the design of compression members, it is desirable to know the compressive stress at which buckling occurs. In practice the structure is usually so imperfect or so eccentrically loaded that lateral deflection starts with the beginning of loading. When lateral deflection starts with the beginning of loading, however, there is usually a very pronounced increase in deflection at the critical compressive stress for which buckling would have occurred had the structure been perfectly straight and centrally loaded. The evaluation of this critical compressive stress for a flat plate, with certain conditions of edge support, is discussed in this paper.

When a flat plate is loaded in compression, the two loaded edges are either simply supported or restrained in some manner. If the two unloaded edges are not supported, the plate is considered to be a column. When one, or both, unloaded edges are also supported or restrained in some manner, the critical compressive stress is greatly increased over that for the plate as a column. That the compressive stress is increased when one, or both, edges are supported or restrained in some manner has been recognized for years. Because of the importance of the edge conditions, formulas based on the assumption that each edge of the plate is free, simply supported, or fixed have been employed in design. (See the summary of these formulas given in reference 1.)

A study of the theory and the more reliable test data on the buckling of plate elements in stressed-skin structures and compression members revealed the necessity for a more careful consideration of the edge conditions of plates than has been previously attempted. Accordingly studies were made of the critical compressive stress for I-, Z-, channel, and rectangular-tube sections in which proper consideration was given to the interaction between the individual parts of the cross section. (See references 2, 3, and 4.) In order to make the results of the work more generally applicable, studies were also made of the basic plate elements that comprise these sections. All the design charts resulting from this investigation were made available in 1938. The combination of the present paper with references 2, 3, 4, and 5 is a more complete presentation of all this material.

The basic element treated in this paper is a plate simply supported along the loaded edges, supported and elastically restrained against rotation along one unloaded edge and free along the remaining unloaded edge. This basic element is representative of the outstanding flange on the I-, Z-, and channel-section columns. In reference 5 is treated the basic element representative of the webs of these sections with elastic restraint along both unloaded edges.

The mathematical derivations required for the investigation of the present paper are given in appendixes A and B. The results of practical use are given in the body of the paper.

EVALUATION OF CRITICAL STRESS

Within the elastic range.—Within the elastic range in which the effective modulus of elasticity is Young's modulus, the critical compressive stress f_{cr} for a thin flat rectangular plate is expressed as (reference 6, p. 331, equation (214))

$$f_{cr}=\frac{k\pi^2Et^2}{12(1-\mu^2)b^2} \tag{1}$$

where

k nondimensional coefficient that depends upon conditions of edge restraint and shape of plate
E Young's modulus
t thickness of plate
μ Poisson's ratio
b width of plate

Beyond the elastic range.—When the plate is stressed in compression beyond the elastic range, the effective modulus of elasticity for the plate is less than Young's modulus. If a single, over-all effective plate modulus ηE is substituted for Young's modulus E, the critical stress, when the material of the plate is loaded beyond the elastic range, can be obtained from equation (1). The nondimensional coefficient η has a value that lies between zero and unity and is determined by the stress. For stresses within the elastic range, $\eta=1$. For a more complete discussion and definition of η, see reference 2.

If ηE is substituted for E in equation (1), the resulting equation cannot be directly solved for f_{cr}. If the equation is divided by η, however, f_{cr}/η is given directly by the geometrical dimensions of the plate, Young's modulus E, and Poisson's ratio μ. Thus

$$\frac{f_{cr}}{\eta}=\frac{k\pi^2Et^2}{12(1-\mu^2)b^2} \tag{2}$$

For a given material, the relationship between f_{cr} and f_{cr}/η tends to be fixed by the compressive stress-strain curve. This relationship is discussed in reference 2, where it is shown how probable relationships between f_{cr} and f_{cr}/η are obtained from the column curve of the material because column curves are more readily available than compressive stress-strain curves. The question is, therefore, what column formula should be used? Equations (8) and (9) of reference 7 define column curves that apply when the material just satisfies the minimum requirements of Navy Department Specification 46A9a for 24S–T aluminum alloy. The relationships between f_{cr} and f_{cr}/η for this case are given in references 2, 3, and 4 and in figure 1 of this paper.

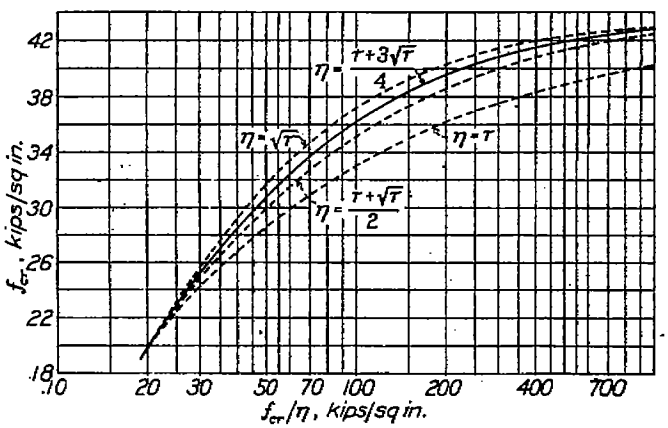

FIGURE 1.—Variation of f_{cr} with f_{cr}/η for 24S–T aluminum alloy of minimum required properties. (When $f_{cr}/\eta<19.6$ kips/sq in., $\eta=1$ and $f_{cr}=f_{cr}/\eta$.)

The 24S–T material delivered under specification 46A9a almost always has properties that are better than the minimum required properties. The relationships between f_{cr} and f_{cr}/η for the average 24S–T material delivered are given in figure 2. This figure has been prepared in the manner described in reference 2, the column curves for average 24S–T material as given in reference 8 being used.

Figures similar to 1 and 2 of this paper may be prepared for any material. The engineer using this paper must therefore decide whether the computation should be based on minimum required material properties or average material properties.

Regardless of whether figure 1 or 2 is used, if the restraint against the rotation of the flange at its base is near zero and λ/b is greater than approximately 2.5, it is recommended that the curve $\eta=\frac{\tau+\sqrt{\tau}}{2}$ be used. For all other values of the restraint, the curve $\eta=\frac{\tau+3\sqrt{\tau}}{4}$ should be satisfactory. In figures 1 and 2 the different equations involving τ merely identify different curves that result from the relationships indicated. The value of τ is $\overline{E}/E$, the ratio of the effective column modulus for bending failure at the stress f_{cr} to Young's modulus.

When the restraint against the rotation approaches zero, the $\eta=\frac{\tau+\sqrt{\tau}}{2}$ curve is recommended in recognition of the fact that the resistance of the plate elements to buckling arises largely from their torsional rigidity. The two curves recommended to show the relationship between f_{cr} and f_{cr}/η should be used until future experimental data indicate that different curves should be used.

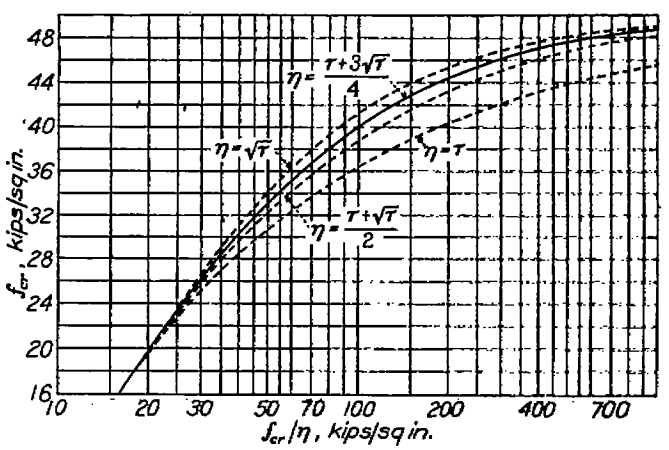

FIGURE 2.—Variation of f_{cr} with f_{cr}/η for 24S–T aluminum alloy of average properties. (When $f_{cr}/\eta<16.7$ kips/sq in., $\eta=1$ and $f_{cr}=f_{cr}/\eta$.)

EVALUATION OF k

The value of f_{cr}/η at which buckling occurs is given by equation (2), in which all of the quantities are known except the value of the coefficient k. The values of k can be obtained from figure 3; figure 3(b) is a portion of figure 3(a) plotted to a larger scale. In this chart, k is plotted against the ratio of the half wave length to the width λ/b for different values of a parameter ϵ, termed the "restraint" coefficient. (In reference 9 Trayer and March refer to ϵ as the "fixity" coefficient. In this paper ϵ is called the restraint coefficient to avoid confusion with the fixity coefficient c for columns.)

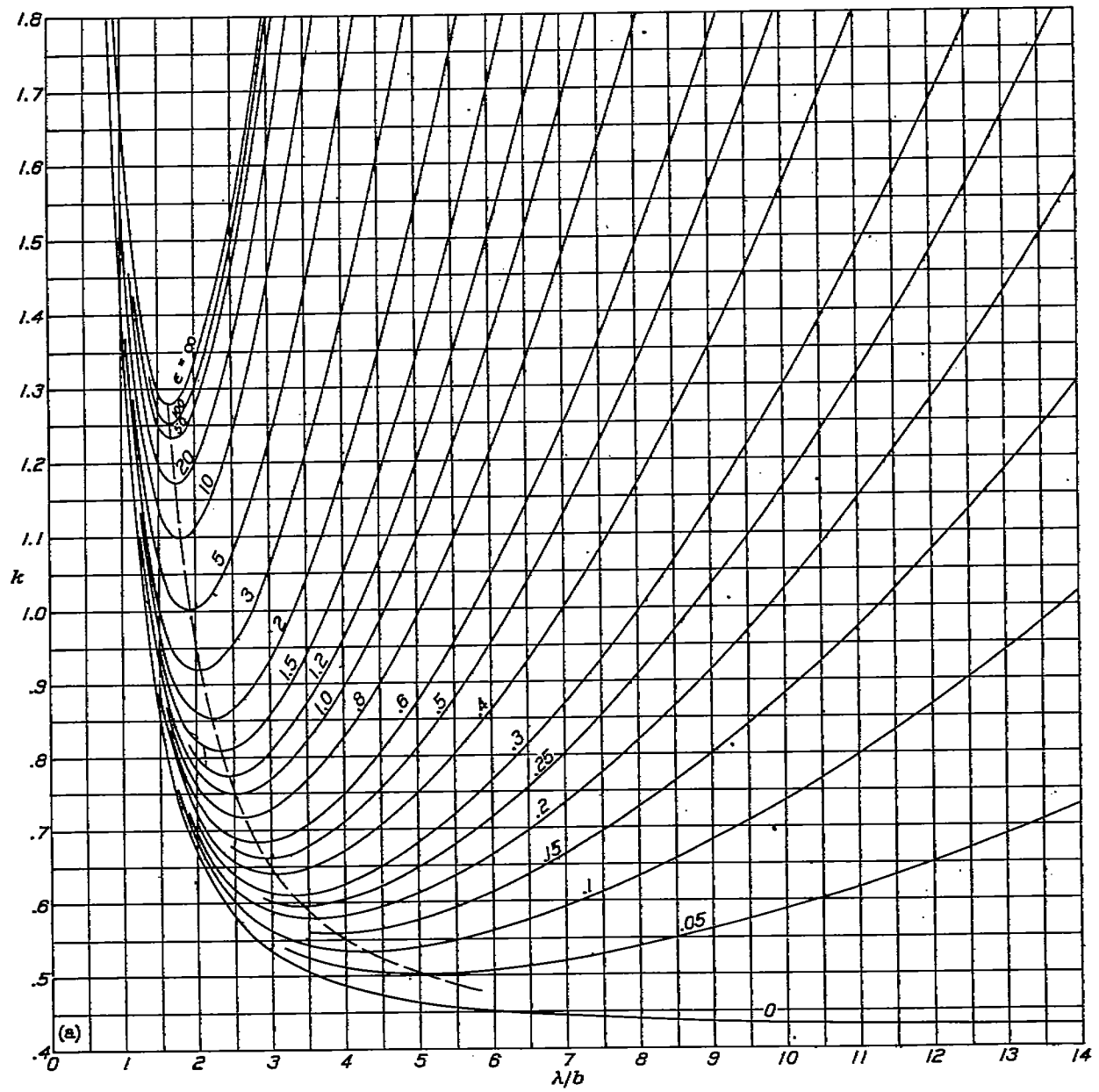

FIGURE 3.—General design chart giving values of k. $\frac{f_{cr}}{\eta}=\frac{k\pi^2E t^2}{12(1-\mu^2)b^2}$

The restraint coefficient ϵ depends upon the relative stiffness of the plate and the restraining element along the side edge of the plate. The simplest conception of ϵ is obtained when the restraining element, or stiffness, is assumed to be replaced by an elastic medium in which rotation at one point does not influence rotation at another point. For this type of restraining medium along the edge of the plate,

within the elastic range, $\epsilon=\frac{4S_0b}{D}$ (3)

beyond the elastic range, $\epsilon=\frac{4S_0b}{\eta D}$ (4)

where

S_0 stiffness per unit length of elastic restraining medium or moment required to rotate a unit length of elastic medium through one-fourth radian

D flexural rigidity of plate, per unit length $\left[\frac{Et^3}{12(1-\mu^2)}\right]$

η coefficient to allow for a decrease in D due to the application of stresses beyond the elastic range

Inasmuch as η is a function of stress, its value for 24S-T material can be obtained from figure 4 or 5, depending upon whether minimum required properties or average properties are being used. The values of τ_1, τ_2, and τ_3 also given in figures 4 and 5 occur in appendix A.

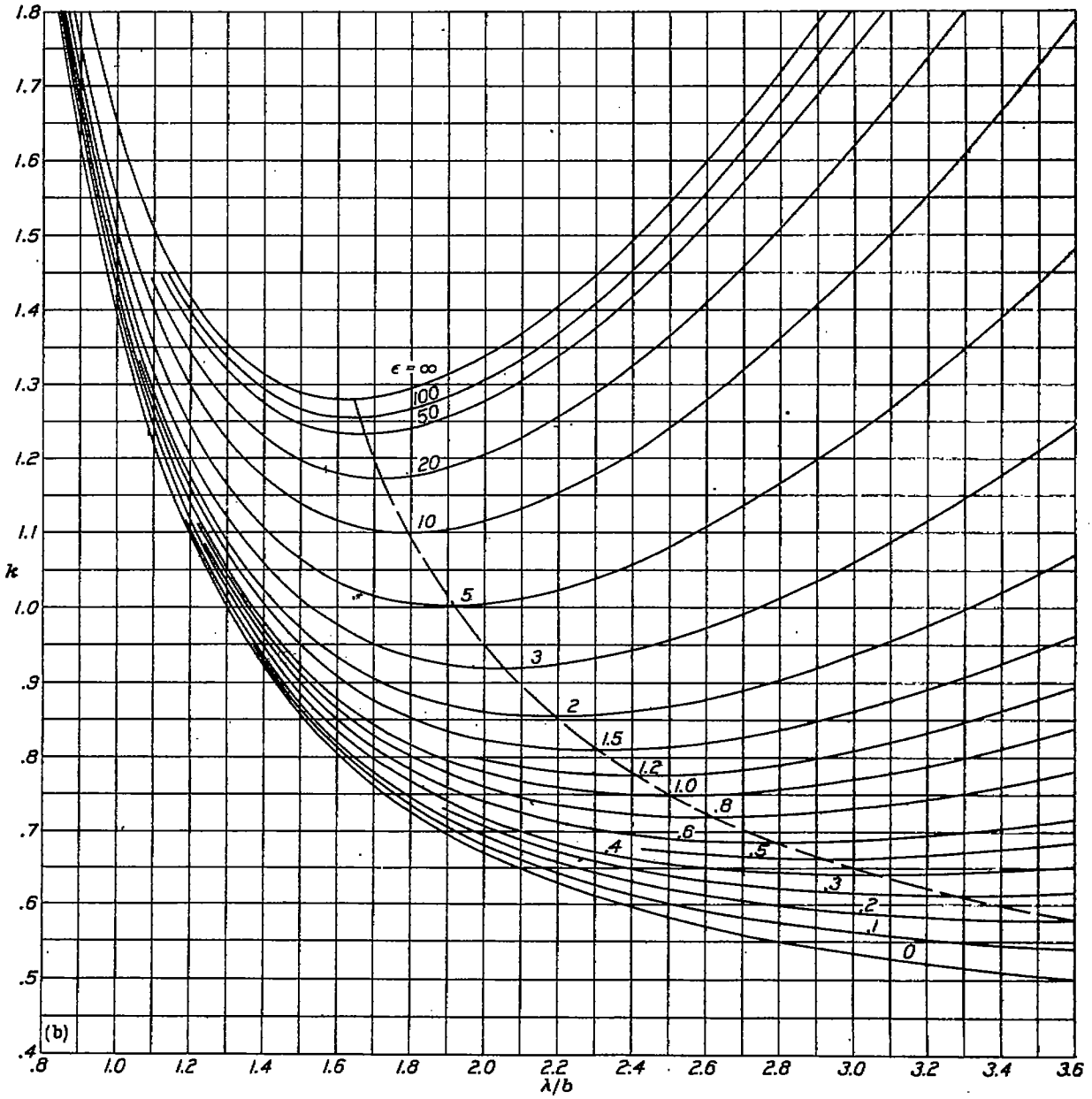

FIGURE 3.—Concluded.

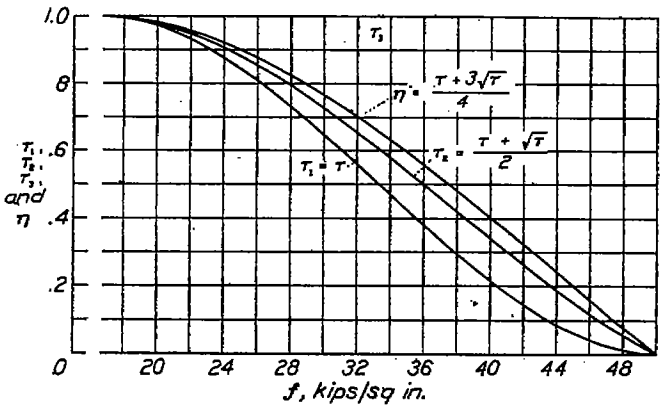

FIGURE 4.—Variation of τ_1, τ_2, τ_3, and η with the compressive stress f for 24S-T aluminum alloy of minimum required properties.

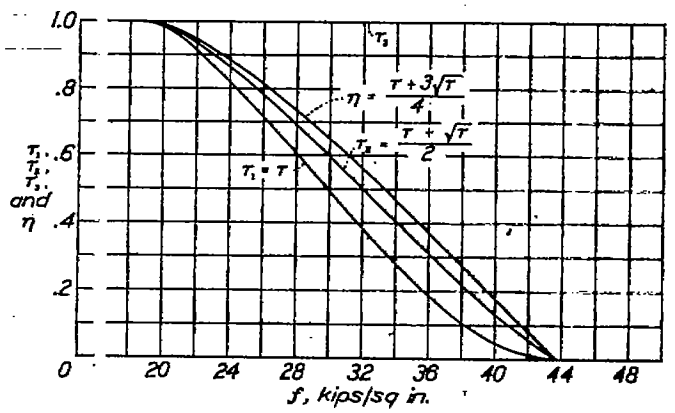

FIGURE 5.—Variation of τ_1, τ_2, τ_3, and η with the compressive stress f for 24S-T aluminum alloy of average properties.

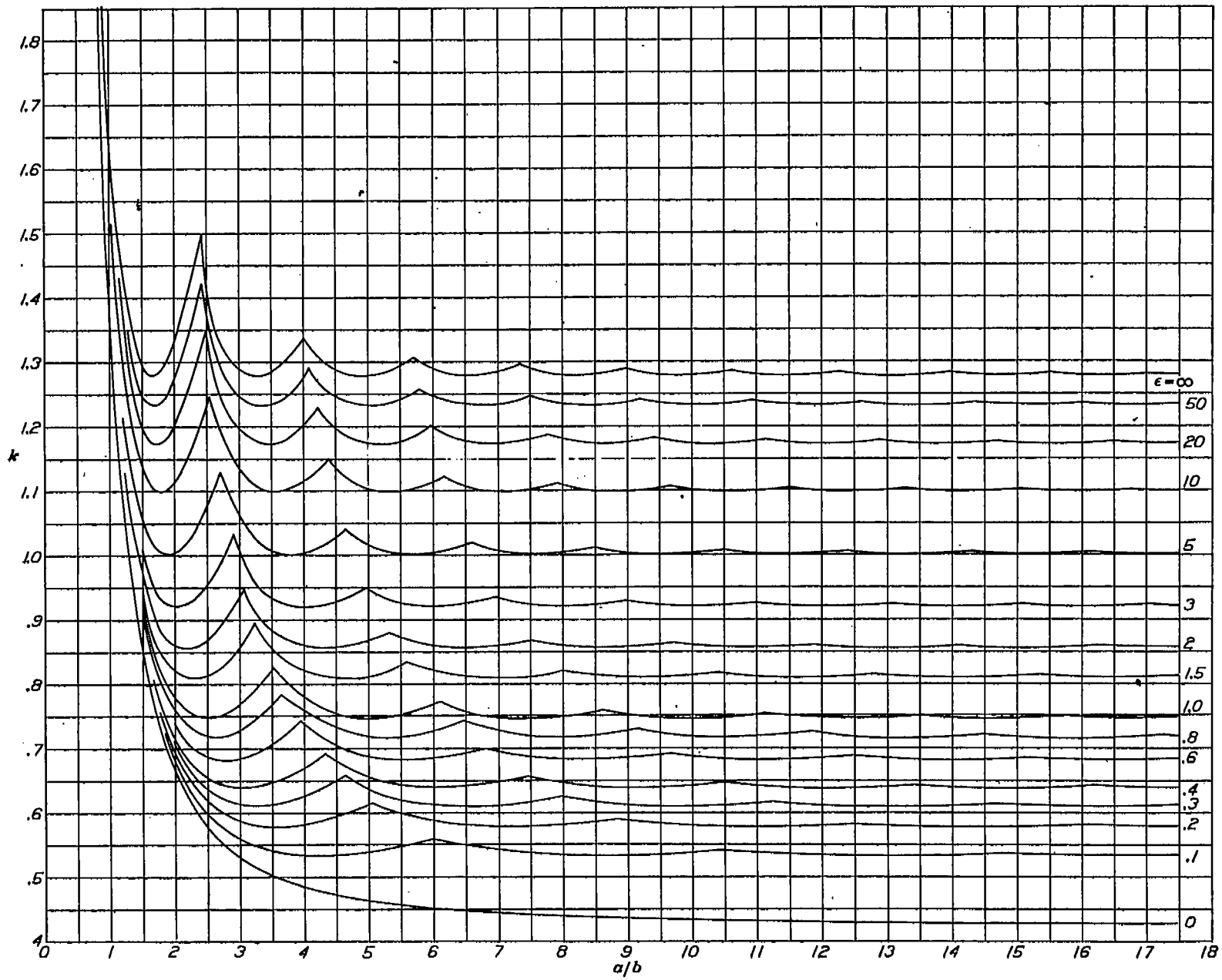

FIGURE 6.—Special design chart giving values of k when the restraint coefficient ϵ is independent of the half wave length λ. $\frac{f_{cr}}{\eta}=\frac{k\pi^2 E t^2}{12(1-\mu^2)b^2}$

If S_0 is zero, ϵ is also zero and the condition of simple support, or zero restraint, is obtained. If S_0 is infinite, ϵ is also infinite and the condition of a fixed edge or of infinite restraint is obtained. Therefore a variation of ϵ from zero to infinity will cover all possible conditions of restraint at the side edge of the plate.

Figure 3 shows that for each value of ϵ there is a value of λ/b for which k is a minimum. Strictly, a whole number m of half wave lengths λ must exist in the length of the plate a. Hence

$$\frac{\lambda}{b}=\frac{a}{mb} \qquad (5)$$

Thus, to read a value of k from figure 3, it is necessary to substitute $m=1$, 2, 3, etc. in equation (5) until a value for λ/b is obtained that gives the smallest value of k in figure 3. This smallest value of k is the one to be used in equations (1) or (2). This general procedure will always give the correct value of k for use in equations (1) or (2) regardless of whether or not S_0, and hence ϵ, is a function of the half wave length λ.

For the special case in which S_0, and hence ϵ, is independent of the half wave length λ, the general procedure described for obtaining a value for k can be used to construct a new chart, with the abscissa λ/b replaced by a/b. This new chart is given in figure 6.

When S_0, and hence ϵ, varies with λ or λ/b, figure 6 should not be used, but the general procedure as applied to figure 3 should be used to obtain the correct value of k for equations (1) and (2).

EVALUATION OF S_0 AND ϵ

Before it is possible to determine k from figure 3 or 6, it is necessary first to evaluate the restraint coefficient ϵ. The value of S_0 to be substituted in equation (3) or (4) will depend upon the characteristics of the structural member or members that provide the restraint. In this paper it is assumed that the restraint is provided by a specially defined elastic restraining medium. As a result of this assumption, it has been possible to derive the general chart of figure 3, which is independent of the structure that provides the restraint.

The basic property of the elastic-restraining medium is that rotation at one point of the medium does not affect rotation at another point of the medium. In many practical problems the elastic restraint is provided by a stiffener, a plate, or some other structure for which rotation at one point affects rotation at another point. Consequently, the evaluation of S_0 in any given problem must take into account the effect of this interaction within the elastic restraining structure.

The formula for S_0 to be used in any given problem will depend upon the type of structural member that provides the restraint. Because this entire subject of restraint supplied to the side edge of a plate has been rather superficially treated in the literature, it is being made the subject of a series of papers by the NACA, the first of which is reference 10.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *March 14, 1941.*

APPENDIX A

SOLUTION BY DIFFERENTIAL EQUATION

The procedure for obtaining the critical stress of a plate uniformly compressed along two opposite, simply supported edges is given in reference 6 (p. 337). In this method, which was also used by Dunn in reference 11, the critical stress is found by solving the differential equation expressing the equilibrium of the buckled plate. The same method is applied in this paper to the case in which an elastic restraint against rotation is present along one unloaded edge of the plate while the other unloaded edge remains free to deflect and to rotate. For generality, the elastic restraint is assumed to arise from an elastic medium distributed along the unloaded edge; this medium has the basic property that rotation at one point within it does not influence the rotation at any other point.

Figure 7 shows the coordinate system and the plate dimensions. The differential equation for the equilibrium of a plate element is

$$ft\frac{\partial^2 w}{\partial x^2}=-D\left(\tau_1\frac{\partial^4 w}{\partial x^4}+2\tau_2\frac{\partial^4 w}{\partial x^2\partial y^2}+\tau_3\frac{\partial^4 w}{\partial y^4}\right) \quad \text{(A–1)}$$

where

f uniformly distributed compressive stress
t thickness of plate
w deflection normal to plate
x longitudinal coordinate in direction of applied stress
D flexural rigidity of plate, per unit length
y transverse coordinate across width of plate
τ_1, τ_2, and τ_3 coefficients equal to or less than unity

In equation (A–1) the term $ft(\partial^2 w/\partial x^2)$ is concerned with the external forces on the plate that cause buckling; whereas the term

$$-D\left(\tau_1\frac{\partial^4 w}{\partial x^4}+2\tau_2\frac{\partial^4 w}{\partial x^2\partial y^2}+\tau_3\frac{\partial^4 w}{\partial y^4}\right)$$

is concerned with the internal resistance of the plate to buckling. The terms involving τ_1 and τ_3 in equation (A–1) are concerned with the longitudinal and the transverse bending, respectively; whereas the term involving τ_2 is concerned principally with the torsional stiffness. The coefficients τ_1, τ_2, and τ_3 allow for the change in the magnitude of the various terms as the plate is stressed beyond the elastic range. In the elastic range, $\tau_1=\tau_2=\tau_3=1$.

The loaded edges are simply supported and are not displaced in the direction w. Of the several forms

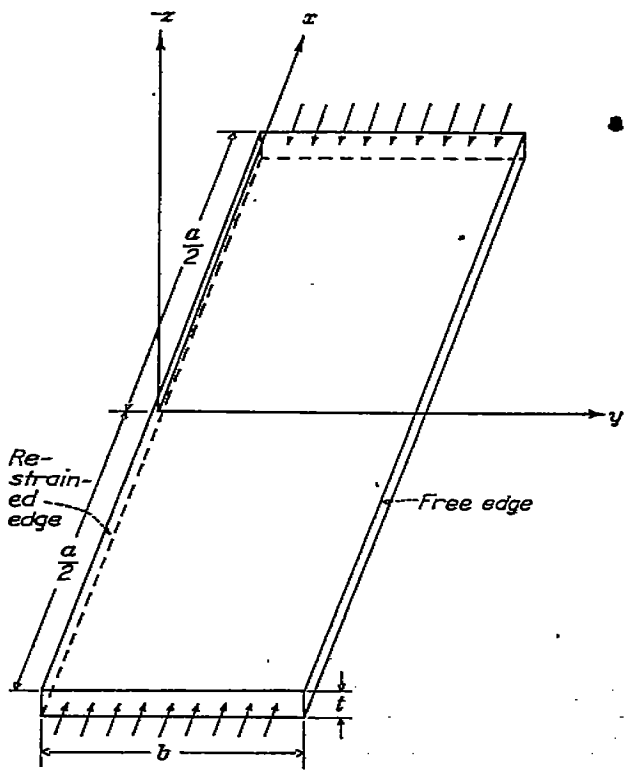

FIGURE 7.—Outstanding flange under edge compression.

of the general solution of equation (A–1) the following form was selected as appropriate for this problem:

$$w=\left(C_1\cosh\frac{\alpha y}{b}+C_2\sinh\frac{\alpha y}{b}+C_3\cos\frac{\beta y}{b}+C_4\sin\frac{\beta y}{b}\right)\cos\frac{\pi x}{\lambda} \quad \text{(A–2)}$$

where

$$\alpha=\pi\sqrt{\frac{b}{\lambda}}\sqrt{\frac{\tau_2}{\tau_3}\frac{b}{\lambda}+\sqrt{\frac{k}{\tau_3}+\left(\frac{b}{\lambda}\right)^2\left(\frac{\tau_2^2}{\tau_3^2}-\frac{\tau_1}{\tau_3}\right)}} \quad \text{(A–3)}$$

$$\beta=\pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{\tau_2}{\tau_3}\frac{b}{\lambda}+\sqrt{\frac{k}{\tau_3}+\left(\frac{b}{\lambda}\right)^2\left(\frac{\tau_2^2}{\tau_3^2}-\frac{\tau_1}{\tau_3}\right)}} \quad \text{(A–4)}$$

and

$$k=\frac{12(1-\mu^2)b^2f}{\pi^2Et^2} \quad \text{(A–5)}$$

Equation (A–2) satisfies the boundary conditions at the loaded edges and gives real values for both α and β near the buckling stress $f=f_{cr}$.

The values of the coefficients C_1, C_2, C_3, and C_4 are to be found from the boundary conditions along the side edges of the plate. The value of λ, the half wave length of the buckle pattern, is found from the condition that there must be an integral number of half wave lengths in the length a of the plate; thus

$$\lambda=\frac{a}{m} \tag{A-6}$$

where $m=1, 2, 3$, etc.

In the elastic range, where $\tau_1=\tau_2=\tau_3=1$, the values of α and β are

$$\alpha=\pi\sqrt{\frac{b}{\lambda}}\sqrt{\frac{b}{\lambda}+\sqrt{k}} \tag{A-7}$$

$$\beta=\pi\sqrt{\frac{b}{\lambda}}\sqrt{-\frac{b}{\lambda}+\sqrt{k}} \tag{A-8}$$

The solution given by equation (A-2) was selected to satisfy the boundary conditions of no deflection and simple support (no moment) along the loaded edges. The boundary conditions along the unloaded side edges have also to be satisfied. The boundary conditions along the unloaded side edges are:

$$(w)_{y=0}=0 \tag{A-9}$$

$$D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=0}=4S_0\left(\frac{\partial w}{\partial y}\right)_{y=0} \tag{A-10}$$

$$D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=b}=0 \tag{A-11}$$

$$D\left[\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}\right]_{y=b}=0 \tag{A-12}$$

where S_0 is the stiffness per unit length of the elastic restraining medium or the moment required to rotate a unit length of the medium through one-fourth radian.

From equations (A-9) and (A-10) are obtained

$$C_1=\frac{\epsilon}{\alpha^2+\beta^2}(\alpha C_2+\beta C_4) \tag{A-13}$$

$$C_3=-\frac{\epsilon}{\alpha^2+\beta^2}(\alpha C_2+\beta C_4) \tag{A-14}$$

where

$$\epsilon=\frac{4S_0 b}{D} \tag{A-15}$$

From equations (A-11) and (A-12) are obtained

$$C_2\left[p\sinh\alpha+\frac{\alpha\epsilon}{\alpha^2+\beta^2}(p\cosh\alpha+q\cos\beta)\right]$$
$$-C_4\left[q\sin\beta-\frac{\beta\epsilon}{\alpha^2+\beta^2}(p\cosh\alpha+q\cos\beta)\right]=0 \tag{A-16}$$

$$C_2\left[q\alpha\cosh\alpha+\frac{\alpha\epsilon}{\alpha^2+\beta^2}(q\alpha\sinh\alpha-p\beta\sin\beta)\right]$$
$$-C_4\left[p\beta\cos\beta-\frac{\beta\epsilon}{\alpha^2+\beta^2}(q\alpha\sinh\alpha-p\beta\sin\beta)\right]=0 \tag{A-17}$$

where

$$p=\alpha^2-\mu\left(\frac{\pi b}{\lambda}\right)^2 \tag{A-18}$$

$$q=\beta^2+\mu\left(\frac{\pi b}{\lambda}\right)^2 \tag{A-19}$$

The buckled form of equilibrium of the plate is obtained when the determinant formed by the coefficients of C_2 and C_4 in equations (A-16) and (A-17) equals zero.

Thus,

$$(\alpha^2+\beta^2)(p^2\beta\sinh\alpha\cos\beta-q^2\alpha\cosh\alpha\sin\beta)$$
$$+\epsilon[(p^2+q^2)\alpha\beta\cosh\alpha\cos\beta+2pq\alpha\beta$$
$$+(p^2\beta^2-q^2\alpha^2)\sinh\alpha\sin\beta]=0 \tag{A-20}$$

This equation establishes the critical compressive stress for an outstanding flange elastically restrained against rotation at one unloaded side edge. Thus equation (A-20) was used to establish the exact values of k given in table I.

The condition of simple support (no restraint) along the supported edge is described by $\epsilon=0$. For this special case, the problem is to find the smallest value of $k\neq 0$ that will satisfy equation (A-20) when $\epsilon=0$. A convenient method for determining this value of k is first to solve for ϵ:

$$\epsilon=-\frac{(\alpha^2+\beta^2)(p^2\beta\sinh\alpha\cos\beta-q^2\alpha\cosh\alpha\sin\beta)}{(p^2+q^2)\alpha\beta\cosh\alpha\cos\beta+2pq\alpha\beta+(p^2\beta^2-q^2\alpha^2)\sinh\alpha\sin\beta} \tag{A-21}$$

When $\epsilon=0$, either

$$\alpha^2+\beta^2=0 \tag{A-22}$$

or

$$p^2\beta\sinh\alpha\cos\beta-q^2\alpha\cosh\alpha\sin\beta=0 \tag{A-23}$$

or

$$(p^2+q^2)\alpha\beta\cosh\alpha\cos\beta+2pq\alpha\beta$$
$$+(p^2\beta^2-q^2\alpha^2)\sinh\alpha\sin\beta=\infty \tag{A-24}$$

Equation (A-22) is true only if $k=0$, which can be true only if the compressive stress f is zero. Equation (A-24) applies only if $k=\infty$, which can be true only if the compressive stress f is infinite. Consequently if a finite value of $k\neq 0$ for which $\epsilon=0$ exists, equation (A-23) must be satisfied.

The special case of a fixed side edge (infinite restraint along the supported edge) is described by $\epsilon=\infty$. Equation (A-21) shows that, if $\epsilon=\infty$, either

$$\alpha^2+\beta^2=\infty \tag{A-25}$$

or

$$p^2\beta\sinh\alpha\cos\beta-q^2\alpha\cosh\alpha\sin\beta=\infty \tag{A-26}$$

or

$$(p^2+q^2)\alpha\beta\cosh\alpha\cos\beta+2pq\alpha\beta$$
$$+(p^2\beta^2-q^2\alpha^2)\sinh\alpha\sin\beta=0 \tag{A-27}$$

Equation (A-25) is true only for $k=\infty$, which can be true only if the compressive stress f is infinite. Equation (A-26) cannot be true for a finite value of k. Hence if a finite value of k for which $\epsilon=\infty$ exists, equation (A-27) must be satisfied.

APPENDIX B

SOLUTION BY ENERGY METHOD

Because the exact solution of the differential equation given in appendix A does not lend itself to a direct calculation of k as in the case of the energy method of solution, an energy solution was made to aid in the construction of the chart of figure 3. The energy method gives approximate values for k, the accuracy of which depends upon how closely the assumed deflection surface describes the true deflection surface.

The energy method as applied to the calculation of critical compressive stress is given in reference 6 (p. 327). The plate is stable when $(V_1+V_2)>T$ and unstable when $(V_1+V_2)<T$, where T is the work done by compressive forces on the plate, V_1 is the strain energy in the plate, and V_2 is the strain energy in the elastic restraining medium along one side edge of the plate. The critical stress is obtained from the condition of neutral stability:

$$T=V_1+V_2 \tag{B-1}$$

If w is the deflection normal to the plate at any point x, y in the plane of the plate shown in figure 7 and S_0 is the stiffness per unit length of the elastic restraining medium or moment required to rotate a unit length of elastic medium through one-fourth radian, then T, V_1, and V_2 are given by the following equations (see reference 6, equations (199) and (201) and reference 9, equation (73)):

$$T=\frac{1}{2}\int_0^b\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}} ft\left(\frac{\partial w}{\partial x}\right)^2 dxdy \tag{B-2}$$

$$V_1=\frac{D}{2}\int_0^b\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2+2(1-\mu)\left[\left(\frac{\partial^2 w}{\partial x\partial y}\right)^2-\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right]\right\}dxdy \tag{B-3}$$

$$V_2=\frac{4S_0}{2}\int_{-\frac{\lambda}{2}}^{\frac{\lambda}{2}}\left[\left(\frac{\partial w}{\partial y}\right)_{y=0}\right]^2 dx \tag{B-4}$$

In order to evaluate T, V_1, and V_2, it is necessary to assume a deflected surface w consistent with the boundary conditions. These boundary conditions at the side edges of the plate are, in the coordinate system of figure 7,

$$(w)_{y=0}=0 \tag{B-5}$$

$$D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=0}=4S_0\left(\frac{\partial w}{\partial y}\right)_{y=0} \tag{B-6}$$

$$D\left(\frac{\partial^2 w}{\partial y^2}+\mu\frac{\partial^2 w}{\partial x^2}\right)_{y=b}=0 \tag{B-7}$$

$$D\left[\frac{\partial^3 w}{\partial y^3}+(2-\mu)\frac{\partial^3 w}{\partial x^2\partial y}\right]_{y=b}=0 \tag{B-8}$$

When buckling occurs, a restraining moment will be applied to the plate along the edge $y=0$; the magnitude of the moment will depend upon the stiffness of the elastic restraining medium. If the elastic medium offers no restraint against rotation, this moment will be zero and the plate will swing about the edge $y=0$, as about a hinge. In this case the plate will remain essentially flat across its width. On the other hand, if the elastic medium offers infinite restraint against rotation, the plate will not rotate along the edge $y=0$ and the plate will deflect across its width into a shape similar to that for a cantilever beam. For any restraint of the elastic medium between zero and infinity the deflection curve across the width of the plate is taken as the sum of the straight line and the cantilever-deflection curve. In the direction of the length the usual sine curve indicated by the solution of the differential equation is used. Thus the deflection surface assumed for the plate is, in the coordinate system of figure 7,

$$w=\left\{A\frac{y}{b}+B\left[\left(\frac{y}{b}\right)^5+a_1\left(\frac{y}{b}\right)^4+a_2\left(\frac{y}{b}\right)^3+a_3\left(\frac{y}{b}\right)^2\right]\right\}\cos\frac{\pi x}{\lambda} \tag{B-9}$$

where A and B are arbitrary deflection amplitudes and $a_1=-4.963$, $a_2=9.852$, and $a_3=-9.778$. These values of a_1, a_2, and a_3 were selected by taking the proportion of two deflection curves that gave the lowest critical compressive stress for a fixed-edge flange for which $\mu=0.3$. These two deflection curves were for a cantilever beam with lateral uniform load and for a lateral load proportional to y.

The condition $B=0$ represents the case of a simply

supported or hinged edge at $y=0$. The case of $A=0$ represents the condition of a clamped edge at $y=0$. The ratio A/B is therefore a measure of edge restraint and is related to the restraint coefficient ϵ through the boundary condition given in equation (B–6). Substitution of w as given by equation (B–9) in equation (B–6) gives

$$B=A\frac{\epsilon}{2a_3} \tag{B–10}$$

where, by definition,

$$\epsilon=\frac{4S_0b}{D} \tag{B–11}$$

Substitution of the value of B as given in expression (B–10) in the deflection equation (B–9) gives

$$w=A\left\{\frac{y}{b}+\frac{\epsilon}{2a_3}\left[\left(\frac{y}{b}\right)^5+a_1\left(\frac{y}{b}\right)^4+a_2\left(\frac{y}{b}\right)^3+a_3\left(\frac{y}{b}\right)^2\right]\right\}\cos\frac{\pi x}{\lambda} \tag{B–12}$$

Equation (B–12) shows how the shape of the deflection surface is affected by the restraint coefficient ϵ. This equation is used in the evaluation of V_1, V_2, and T. Thus,

$$T=A^2\frac{\pi^2bt}{4\lambda}\left(\frac{1}{3}+\frac{c_8\epsilon}{2a_3}+\frac{c_9\epsilon^2}{4a_3^2}\right)f \tag{B–13}$$

$$V_1=A^2\frac{\pi^2D}{2b\lambda}\left\{1-\mu+\frac{1}{6}\left(\frac{\pi b}{\lambda}\right)^2+\frac{\epsilon}{2}\left[\frac{c_1}{2}\left(\frac{\pi b}{\lambda}\right)^2+c_2-\mu c_3\right]+\frac{\epsilon^2}{4}\left[\frac{c_4}{2}\left(\frac{\pi b}{\lambda}\right)^2+\frac{\frac{c_5}{2}}{\left(\frac{\pi b}{\lambda}\right)^2}+c_6-\mu c_7\right]\right\} \tag{B–14}$$

$$V_2=A^2\frac{D\lambda\epsilon}{8b^3} \tag{B–15}$$

where

$$c_1=\frac{a_3}{2}\left(\frac{2}{7}+\frac{a_1}{3}+\frac{2a_2}{5}+\frac{1a_3}{2}\right)=0.23694$$

$$c_2=\frac{2}{a_3}(1+a_1+a_2+a_3)=0.79546$$

$$c_3=\frac{1}{a_3}(6+5a_1+4a_2+3a_3)=0.89395$$

$$c_4=\frac{1}{a_3^2}\left(\frac{1}{11}+\frac{a_1^2}{9}+\frac{2a_2}{9}+\frac{a_2^2}{7}+\frac{2a_1a_2}{7}+\frac{a_2^2}{5}+\frac{a_1}{5}+\frac{a_3}{4}+\frac{a_1a_2}{4}+\frac{a_2a_3}{3}\right)=0.04286$$

$$c_5=\frac{1}{a_3^2}\left(\frac{400}{7}+\frac{144}{5}a_1^2+48a_2+12a_2^2+16a_1a_3+4a_3^2+80a_1+20a_3+36a_1a_2+12a_2a_3\right)=0.56712$$

$$c_6=\frac{1}{a_3^2}\left(\frac{25}{9}+\frac{16}{7}a_1^2+\frac{30}{7}a_2+\frac{9}{5}a_2^2+\frac{16}{5}a_1a_2+\frac{4}{3}a_3^2+5a_1+\frac{10}{3}a_3+4a_1a_2+3a_2a_3\right)=0.17564$$

$$c_7=\frac{1}{a_3^2}(5+9a_1+8a_2+7a_3+4a_1^2+7a_1a_2+6a_1a_3+3a_2^2+5a_2a_3+2a_3^2)=0.19736$$

$$c_8=a_3c_1=-2.3168$$

$$c_9=a_3^2c_4=4.0982$$

It is permissible to substitute the values of T, V_1, and V_2 as given by equations (B–14) to (B–16) in equation (B–1) only when the applied stress f has its critical value f_{cr}. After this substitution it is found that

$$f_{cr}=\frac{k\pi^2Et^2}{12(1-\mu^2)b^2} \tag{B–16}$$

where

$$k=\frac{2}{\pi^2}\left\{\frac{1-\mu+\frac{1}{6}\left(\frac{\pi b}{\lambda}\right)^2+\frac{\epsilon}{2}\left[\frac{c_1}{2}\left(\frac{\pi b}{\lambda}\right)^2+c_2-\mu c_3\right]+\frac{\epsilon^2}{4}\left[\frac{c_4}{2}\left(\frac{\pi b}{\lambda}\right)^2+\frac{c_5}{2\left(\frac{\pi b}{\lambda}\right)^2}+c_6-\mu c_7\right]+\frac{\epsilon}{2\left(\frac{\pi b}{\lambda}\right)^2}}{\frac{1}{3}+\frac{c_8\epsilon}{2a_3}+\frac{c_9\epsilon^2}{4a_3^2}}\right\} \tag{B–17}$$

Equation (B–17) was used to calculate the values of k listed in the columns designated (a) of table I. With these values of k as a guide, a number of correct values of k were obtained by satisfying equation (A–20) of appendix A. In this manner the errors in k as given by equation (B–17) were established at isolated points. From this knowledge of the errors, corrections were made to all the values of k given in columns (a) of table I. These corrected values of k, which are recommended, are listed in the columns designated (b) of table I. The recommended values of k were used in the construction of figures 3 and 6.

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TABLE I.—VALUES OF k IN THE BUCKLING FORMULA FOR OUTSTANDING FLANGES ELASTICALLY RESTRAINED AT THE BASE

λ/b	0.8		0.9		1.0		1.25		1.5		1.75		2.0		2.25		2.5		3.0	
ϵ	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
0	1.983	[c] 1.954	1.660	1.631	1.426	1.402	1.065	1.047	0.870	[c] 0.858	0.752	0.742	0.676	0.669			0.586	0.582	0.537	[c] 0.533
.05																				
.1	1.991	1.957	1.663	1.634	1.430	[c] 1.405	1.071	1.053	.878	.866	.762	.753	.689	.682			.606	[c] .601	.565	.562
.15																				
.2	1.994	1.959	1.667	1.639	1.433	1.408	1.077	[c] 1.059	.885	.873	.772	.763	.701	.694			.624	.620	.592	.589
.25																			.606	.602
.3													.713	.706			.643	.638	.618	.615
.4	1.999	1.965	1.673	1.643	1.441	[c] 1.416	1.087	1.069	.899	[c] .887	.791	.782	.725	.718			.660	.655	.643	[c] .640
.5																	.677	.672	.666	.663
.6	2.004	1.968	1.679	1.649	1.443	1.424	1.097	1.079	.912	.900	.808	.799	.747	.740			.694	.689	.690	.687
.8	2.009	1.972	1.685	1.654	1.454	1.430	1.106	1.088	.925	.913	.824	.815	.767	.760			.724	.719	.733	.729
1.0	2.014	1.974	1.690	1.658	1.460	1.436	1.115	1.097	.936	.924	.839	.830	.786	.779			.753	.748	.774	.769
1.2													.804	.797			.780	.775	.812	.807
1.5	2.024	1.983	1.702	1.669	1.475	1.449	1.134	1.116	.963	.951	.873	.864	.829	.822			.818	.813	.865	.860
2.0	2.034	1.992	1.714	[c] 1.680	1.487	1.461	1.151	1.133	.985	[c] .974	.902	.893	.866	.859	0.859	[c] 0.855	.873	.867	.942	.936
3.0	2.051	2.008	1.732	1.696	1.508	1.481	1.180	1.162	1.023	1.012	.950	.941	.926	.919	.933	.928	.962	.956	1.067	1.060
5.0	2.076	2.033	1.760	1.724	1.540	1.513	1.221	1.203	1.076	1.065	1.018	1.009	1.010	1.003	1.036	1.030	1.086	1.079	1.239	1.230
10	2.113	2.069	1.801	1.763	1.585	1.557	1.279	1.261	1.149	1.138	1.109	1.100	1.122	1.115	1.172	1.165	1.248	1.239	1.463	1.451
20							1.327	1.309	1.209	1.198	1.192	1.173	1.211	1.204	1.279	1.271	1.375	1.364	1.636	1.619
50							1.368	1.350	1.258	1.247	1.243	1.235	1.284	1.277	1.365	1.356	1.477	1.463	1.773	1.747
100					1.671	1.643	1.384	1.366	1.278	1.267	1.267	1.260	1.312	1.305	1.399	1.387	1.516	1.500		
∞	2.200	2.150	1.895	[c] 1.852	1.686	1.658	1.403	[c] 1.385	1.300	1.289	1.293	[c] 1.287	1.344	1.337	1.436	[c] 1.424	1.559	1.541	1.884	[c] 1.846

λ/b	3.5		4.0		5.0		6.0		7.0		8.0		9.0		10.0		11.0		12.5		14.0	
ϵ	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)	(a)	(b)
0	0.507	0.505	0.488	0.486	0.466	0.464	0.453	0.451	0.446	[c] 0.445			0.438	0.438			0.434	0.434			0.431	[c] 0.430
.05			.513	.511	.504	.502	.509	.507	.521	.520	0.539	0.539	.562	.562	0.588	0.587	.619	.618	0.670	0.669	.730	.728
.1	.545	.542	.537	.535	.542	.540	.563	.561	.595	.594	.635	[c] .635	.683	.682	.738	.737	.800	.798	.904	[c] .903	1.023	1.020
.15	.563	.560	.561	.559	.579	.577	.616	.614	.667	.665	.729	.729	.802	.801	.885	.883	.977	.974	1.133	1.130	1.309	1.304
.2	.581	.578	.584	.582	.615	.613	.668	.666	.737	.735	.821	.820	.918	.917	1.028	1.026	1.150	1.145	1.357	1.352	1.590	1.582
.25	.599	.596	.607	.605	.651	.648	.719	.717	.806	.803	.911	.910	1.032	1.030	1.169	1.166	1.320	1.314	1.576	1.569	1.865	1.853
.3	.616	.613	.630	.627	.685	.682	.769	.767	.874	.871	1.000	.998	1.144	1.141	1.306	1.302	1.487	1.480	1.791	1.781	2.135	2.117
.4	.650	.647	.674	.671	.753	.750	.866	.864	1.006	[c] 1.003	1.171	1.168	1.361	1.357	1.574	1.569	1.810	[c] 1.802	2.209	2.195		
.5			.715	.711	.818	.814	.959	.956	1.132	1.128	1.336	1.332	1.569	1.563	1.831	1.824	2.121	2.112	2.610	2.591		
.6	.713	.710	.756	.752	.880	.875	1.048	1.043	1.254	1.248	1.494	1.487	1.770	1.761	2.078	2.067						
.8	.772	.768	.832	.828	.999	.992	1.218	1.210	1.484	1.474	1.794	1.782	2.148	2.134								
1.0	.827	.823	.903	.897	1.109	1.099	1.375	1.363	1.697	1.681	2.072	2.051										
1.2			.969	.962	1.211	1.197	1.522	1.505	1.896	1.872												
1.5	.949	.943	1.060	1.052	1.352	1.335	1.724	1.700	2.170	2.136												
2.0	1.052	1.044	1.194	[c] 1.184	1.559	1.539	2.019	[c] 1.986														
3.0	1.219	1.209	1.409	1.395	1.890	1.863																
5.0	1.448	1.434	1.705	1.687																		
10	1.745	1.726	2.085	2.060																		
20																						
50																						
100																						
∞	2.296	2.237																				

[c] Values obtained from the energy method.
[b] Recommended values.
[c] Values obtained from the exact solution of the differential equation.