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BENDING AND SHEAR STRESSES DEVELOPED BY THE INSTANTANEOUS ARREST OF THE ROOT OF A CANTILEVER BEAM ROTATING WITH CONSTANT ANGULAR VELOCITY ABOUT A TRANSVERSE AXIS THROUGH THE ROOT

By Elbridge Z. Stowell, Edward B. Schwartz
and John C. Houbolt

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NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

ADVANCE RESTRICTED REPORT

BENDING AND SHEAR STRESSES DEVELOPED BY THE INSTANTANEOUS ARREST OF THE ROOT OF A CANTILEVER BEAM ROTATING WITH CONSTANT ANGULAR VELOCITY ABOUT A TRANSVERSE AXIS THROUGH THE ROOT

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SUMMARY

A theoretical investigation was made of the behavior of a cantilever beam in rotational motion about a transverse axis through the root when the rotation of the root is suddenly stopped. Equations are given for determining the stresses, the deflections, and the accelerations that occur in the beam as a result of the arrest of motion.

The equations for bending and shear stress reveal that, at a given percentage of the distance from root to tip and at a given tip velocity, the bending stresses for a particular mode are independent of the length of the beam and the shear stresses vary inversely with the length. When examined with respect to a given angular velocity instead of a given tip velocity, the equations reveal that the bending stress is proportional to the length of the beam; whereas the shear stress is independent of the length.

Sufficient experimental verification of the theory has previously been given in connection with another problem of the same type.

INTRODUCTION

Stresses occur in the structure of an airplane as a result of the shocks experienced in landing. These

shocks involve a rather sudden change in the motion of the airplane or its component parts.

In reference 1 a theoretical and experimental investigation was made of the behavior of a cantilever beam in transverse translational motion when the root of the beam was suddenly brought to rest. Equations were given for determining the stresses, the deflections, and the accelerations that existed throughout the beam as a result of the impact. Experimental verification was presented for the theoretical equations.

Shocks involving changes in rotational motion, as well as shocks involving changes in translational motion, may be experienced by the airplane. The fundamental theory of reference 1 has therefore been applied herein to an arrested cantilever beam; the only change is that the initial motion arrested is rotation about a transverse axis through the root.

As in reference 1, the present paper is based on the usual engineering beam theory. In this theory, the deflections are considered to be the result of bending alone and shear deflections are neglected. The theory, as applied to ordinary beams, gives reasonably good results as long as the distance between inflection points is greater than a few times the depth of the beam. When this theory for beam action is used in vibration problems, such as that in the present paper, the results are satisfactory for those modes of vibration for which the nodes are not too close together. Because the theory of the present paper is basically the same as that of reference 1, the experimental verification given in reference 1 may reasonably be expected to apply to the present equations. Additional experimental check was therefore considered unnecessary.

SYMBOLS

E modulus of elasticity

γ weight density of material

λ coefficient of equivalent viscous damping of material

c	velocity of sound in material $\left(\sqrt{\frac{Eg}{\gamma}}\right)$
g	acceleration of gravity
L	length of beam
I	moment of inertia of cross section of beam about neutral axis
A	cross-sectional area of beam
ρ	radius of gyration of cross section of beam $\left(\sqrt{\frac{I}{A}}\right)$
x	coordinate along beam measured from root
y	distance from neutral axis of beam to any fiber
t	time, zero at impact
p	operator $\frac{\partial}{\partial t}$
n	integers 1, 2, 3, etc., designating a particular mode of vibration (used as subscript)
θ_n	nth positive root of $1 + \cos\ \theta \cosh \theta = 0$
ω_n	undamped natural angular frequency of nth mode, radians per second $\left(\rho c \frac{\theta_n^2}{L^2}\right)$
ω_n'	damped natural angular frequency of nth mode, radians per second $\left(\omega_n\sqrt{1 - \frac{\lambda^2\omega_n^2}{4E^2}}\right)$ $\left(\text{When } \frac{\lambda^2\omega_n^2}{4E^2} > 1, \text{ the "frequency" is defined by } \omega_n' = \omega_n\sqrt{\frac{\lambda^2\omega_n^2}{4E^2} - 1}.\right)$
Ω	angular velocity of beam prior to impact
$w(x,t)$	deflection of beam at station x and time t
$w_n(x,t)$	deflection of beam at station x and time t for nth mode of vibration

$a(x,t)$ acceleration of beam at station x and time t

$a_n(x,t)$ acceleration of beam at station x and time t for nth mode of vibration

$\sigma(x,y,t)$ bending stress in beam at station x, distance from neutral axis y, and time t

$\sigma_n(x,y,t)$ bending stress in beam at station x, distance from neutral axis y, and time t for nth mode of vibration

$\bar{\tau}(x,t)$ average shear stress over cross section of beam at station x and time t

$\bar{\tau}_n(x,t)$ average shear stress over cross section of beam at station x and time t for nth mode of vibration

A_n bending-stress coefficient

B_n shear-stress coefficient

C_n deflection coefficient

RESULTS AND CONCLUSIONS

Theoretical Results

When a cantilever beam rotating with uniform angular velocity Ω about the root is suddenly stopped at the root, theoretically an infinite number of modes of vibration are excited. With each successive mode, damping has an increasing influence upon the frequencies and amplitudes of vibration and, for sufficiently high modes, damping even changes the type of motion from oscillatory to nonoscillatory. In the lower modes, however, damping has little effect and only terms of the first order in damping need to be included in the equations. Only the equations applicable to the lower modes, which alone are of importance in any practical case, are presented in this section. A more complete treatment of damping is given in the appendix.

The angular frequencies (2π times the frequency in cps) are given by the equation

$$\omega_n = \rho c \frac{\theta_n^2}{L^2} \tag{1}$$

where θ_n has the following values for successive modes of vibration:

$$\theta_1 = 1.875104 \qquad \theta_5 = 14.137168$$
$$\theta_2 = 4.694098 \qquad \theta_6 = 17.278759$$
$$\theta_3 = 7.854757 \qquad \theta_n \approx \frac{1}{2}(2n-1)\pi,\ n > 6$$
$$\theta_4 = 10.995541$$

These values of θ_n, which are the same as those given in reference 1, are characteristic of a cantilever beam.

The energy that the beam possesses before the motion of the root is halted is consumed in exciting the various modes of vibration and is distributed among the modes as follows:

Mode, n	Percentage of energy
1	97.07
2	2.47
3	.31
4	.08
5 to ∞	.07

This distribution of energy among the various modes of vibration is presented graphically in figure 1.

Expressions for the bending stresses, shear stresses, deflections, and accelerations are of the same form as the expressions given in reference 1 for these quantities except that the velocity of translation v is replaced by the tip velocity ΩL and the coefficients A_n, B_n, and C_n have new values. For the nth mode of vibration these expressions are as follows:

For stresses and deflections,

$$\sigma_n(x,y,t) = A_n \frac{\Omega L}{c} \frac{Y_E}{\rho} e^{-\frac{\lambda\omega_n^2}{2E}t} \sin \omega_n t \tag{2}$$

$$\bar{\tau}_n(x,t) = B_n \frac{\Omega L}{c} \frac{\rho_E}{L} e^{-\frac{\lambda\omega_n^2}{2E}t} \sin \omega_n t \tag{3}$$

$$w_n(x,t) = C_n \frac{\Omega L}{c} \frac{L^2}{\rho} e^{-\frac{\lambda\omega_n^2}{2E}t} \sin \omega_n t \tag{4}$$

For accelerations, when damping is sufficiently small,

$$a_n(x,t) = -\omega_n^2 w_n(x,t) \tag{5}$$

The variations of the dimensionless coefficients A_n, B_n, and C_n with position along the beam x/L are given for the first three modes ($n = 1$, 2, and 3) in figures 2 to 4. The highest absolute values of A_n and B_n, and hence the highest stresses, occur at the root of the beam. These values, for the first three modes, are

Mode, n	A_n at root	B_n at root
1	1.1376	1.566
2	.1815	.8679
3	.0648	.5088

The foregoing values of A_n and B_n at the root are presented graphically in figure 5.

When the effects of damping are neglected, the maximum values with respect to time of $\sigma_n(x,y,t)$ and $\bar{\tau}_n(x,t)$ associated with the nth mode of vibration are

$$\sigma_n(x,y) = A_n \frac{\Omega L}{c} \frac{Y_E}{\rho} \tag{6}$$

and

$$\bar{\tau}_n(x) = B_n \frac{\Omega L}{c} \frac{\rho}{L} E \tag{7}$$

Equations (2) to (5) for stresses, deflections, and accelerations give the values associated with the nth mode of vibration. Since all modes of vibration occur simultaneously, the net results are the superposition of the effects of all modes. This superposition gives the following equations: For bending stress,

$$\sigma(x,y,t) = \frac{\Omega L}{c} \frac{y}{\rho} E \left(A_1 e^{-\frac{\lambda \omega_1^2}{2E} t} \sin \omega_1 t + A_2 e^{-\frac{\lambda \omega_2^2}{2E} t} \sin \omega_2 t + \ldots \right) \tag{8}$$

For average shear stress,

$$\bar{\tau}(x,t) = \frac{\Omega L}{c} \frac{\rho}{L} E \left(B_1 e^{-\frac{\lambda \omega_1^2}{2E} t} \sin \omega_1 t + B_2 e^{-\frac{\lambda \omega_2^2}{2E} t} \sin \omega_2 t + \ldots \right) \tag{9}$$

For deflection,

$$w(x,t) = \frac{\Omega L}{c} \frac{L^2}{\rho} \left(C_1 e^{-\frac{\lambda \omega_1^2}{2E} t} \sin \omega_1 t + C_2 e^{-\frac{\lambda \omega_2^2}{2E} t} \sin \omega_2 t + \ldots \right) \tag{10}$$

For acceleration, when damping is sufficiently small,

$$a(x,t) = \frac{\Omega L}{c} \frac{L^2}{\rho} \left(C_1 \omega_1^2 e^{-\frac{\lambda \omega_1^2}{2E} t} \sin \omega_1 t + C_2 \omega_2^2 e^{-\frac{\lambda \omega_2^2}{2E} t} \sin \omega_2 t + \ldots \right) \tag{11}$$

The equations for bending stress reveal that, at a given percentage of the distance from root to tip and at a given tip velocity, the bending stress for a particular mode is independent of the length of the beam. The equations for shear stress reveal that, for a given tip velocity, the shear stresses at any station vary inversely with the length of the beam. When examined with respect to a given angular velocity instead of a given tip velocity, the equations for bending and shear

stresses reveal that the bending stress is proportional to the length of the beam; whereas the shear stress is independent of the length of the beam.

Influence of Type of Shock

The frequencies and deflected shapes of the modes of vibration are characteristic of the cantilever and not of the initial shock. The type of shock, however, determines the relative amplitudes of the various modes. The amplitude of each mode, in turn, determines the energy and stresses developed by that mode. In figure 6, arrested translation (reference 1) and arrested rotation (present paper) are compared on the basis of the distribution of deflections, energies, and stresses among the first three modes. In the case of arrested translation, about 61 percent of the total energy is in the first mode; the first mode predominates as regards tip deflection and root bending stress, while the largest shear stress occurs in the second mode. In the case of arrested rotation, about 97 percent of the total energy is in the first mode, and the first mode predominates not only with respect to tip deflection and root bending stress but also with respect to maximum shear stress.

If the motion that is arrested is a combination of translation and rotation about the root, the deflections and stresses can be found by superposing the results for the translation and the rotation separately. If the cantilever is not perpendicular to the direction of root translation at the moment of impact, the translational velocity used should be the component of the root velocity perpendicular to the cantilever at the moment of impact. The component of root velocity parallel to the cantilever causes longitudinal vibrations but such vibrations are not considered herein.

The equations given in this report and in reference 1 apply also to the cases in which the root of a cantilever beam at rest is suddenly given a constant angular velocity or a constant linear velocity in a direction perpendicular to the length of the beam.

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APPENDIX

THEORETICAL DERIVATION

General analysis.- Consider a beam of uniform cross section in equilibrium. If a portion of the beam is suddenly disturbed, as by a shock, the beam is set into damped bending oscillations. The equation of motion for these bending oscillations is given by the differential equation (reference 1)

$$E\rho^2 \frac{\partial^4 w}{\partial x^4} + \lambda\rho^2 \frac{\partial^5 w}{\partial x^4 \, \partial t} + \frac{\gamma}{g} \frac{\partial^2 w}{\partial t^2} = 0 \tag{A1}$$

By using $c^2 = \frac{gE}{\gamma}$, equation (A1) can be written

$$\frac{\partial^4 w}{\partial x^4} + \frac{\lambda}{E} \frac{\partial^5 w}{\partial x^4 \, \partial t} + \frac{1}{\rho^2 c^2} \frac{\partial^2 w}{\partial t^2} = 0 \tag{A2}$$

This partial differential equation is reduced to an ordinary differential equation of the fourth order by writing $p = \frac{\partial}{\partial t}$; thus

$$\left(1 + \frac{\lambda p}{E}\right) \frac{d^4 w}{dx^4} + \frac{p^2}{\rho^2 c^2} w = 0 \tag{A3}$$

The general solution of equation (A3) is

$$w = P \cosh \theta \frac{x}{L} + Q \sinh \theta \frac{x}{L} + R \cos \theta \frac{x}{L} + S \sin \theta \frac{x}{L} \tag{A4}$$

where

$$\theta = L \sqrt{\frac{ip}{\rho c \sqrt{1 + p \frac{\lambda}{E}}}}$$

The coefficients P, Q, R, and S are to be determined from the boundary conditions. The case under consideration

is that of a cantilever beam rotating about its root with constant angular velocity Ω and having the rotation suddenly stopped by clamping of the root in position as the beam coincides with the x-axis. The boundary conditions for this case are

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right)_{x=0} = p\left(\frac{\partial w}{\partial x}\right)_{x=0} = \Omega - \Omega 1$$

$$(w)_{x=0} = \left(\frac{\partial^2 w}{\partial x^2}\right)_{x=L} = \left(\frac{\partial^3 w}{\partial x^3}\right)_{x=L} = 0$$

The angular velocity of the root as given by the boundary condition $\Omega - \Omega 1$ (discontinuous function) is represented graphically in figure 7. By the procedure adopted in reference 1, the solution will be obtained for the boundary condition

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right)_{x=0} = -\Omega 1$$

and the constant angular velocity Ω will be added to the resulting angular velocity.

With the application of the boundary conditions to equation (A4), the operational solution for the angular velocity induced by $-\Omega 1$ is found to be

$$p\left(\frac{\partial w}{\partial x}\right) = \frac{-\Omega 1}{2(1 + \cosh\theta\cos\theta)}\left[(1 + \cos\theta\cosh\theta)\left(\cosh\theta\frac{x}{L} + \cos\theta\frac{x}{L}\right)\right.$$

$$+ \sin\theta\sinh\theta\left(\cos\theta\frac{x}{L} - \cosh\theta\frac{x}{L}\right)$$

$$\left. + (\cosh\theta\sin\theta - \sinh\theta\cos\theta)\left(\sinh\theta\frac{x}{L} + \sin\theta\frac{x}{L}\right)\right]$$

Interpretation of this operational expression by use of the Heaviside expansion theorem and addition of the constant angular velocity Ω gives for the total angular velocity

$$\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right) = \Omega - \Omega 1 + \Omega \sum_{n=1}^{\infty} F'\left(\theta_n \frac{x}{L}\right) e^{-\frac{\lambda \omega_n^2}{2E} t} \left(\cos \omega_n' t - \frac{\frac{\lambda \omega_n}{2E}}{\sqrt{1 - \frac{\lambda^2 \omega_n^2}{4E^2}}} \sin \omega_n' t \right) 1 \quad \text{(A5)}$$

where

θ_n — nth positive root of $1 + \cos \theta \cosh \theta = 0$ (all roots - namely, $\pm\theta_n$ and $\pm i\theta_n$ - have been considered in the interpretation)

$\omega_n = \rho c \frac{\theta_n^2}{L^2}$ — undamped natural angular frequency of nth mode, radians/sec

$\omega_n' = \omega_n \sqrt{1 - \frac{\lambda^2 \omega_n^2}{4E^2}}$ — damped natural angular frequency of nth mode, radians/sec

$$F'\left(\theta_n \frac{x}{L}\right) = 2 \frac{\sin \theta_n \sinh \theta_n \left(\cos \theta_n \frac{x}{L} - \cosh \theta_n \frac{x}{L}\right) + \left(\cosh \theta_n \sin \theta_n - \sinh \theta_n \cos \theta_n\right)\left(\sinh \theta_n \frac{x}{L} + \sin \theta_n \frac{x}{L}\right)}{\theta_n \left(\cosh \theta_n \sin \theta_n - \sinh \theta_n \cos \theta_n\right)}$$

Integration of equation (A5) with respect to x gives for velocity

$$\frac{\partial w}{\partial t} = (\Omega - \Omega \mathit{1})x + \Omega L \sum_{n=1}^{\infty} F\left(\theta_n \frac{x}{L}\right) e^{-\frac{\lambda \omega_n^2}{2E} t} \left(\cos \omega_n' t - \frac{\frac{\lambda \omega_n}{2E}}{\sqrt{1 - \frac{\lambda^2 \omega_n^2}{4E^2}}} \sin \omega_n' t\right) \mathit{1} \qquad \text{(A6)}$$

where

$$\left(\theta_n \frac{x}{L}\right) = 2 \frac{\sin \theta_n \sinh \theta_n \left(\sin \theta_n \frac{x}{L} - \sinh \theta_n \frac{x}{L}\right) + \left(\cosh \theta_n \sin \theta_n - \sinh \theta_n \cos \theta_n\right)\left(\cosh \theta_n \frac{x}{L} - \cos \theta_n \frac{x}{L}\right)}{\theta_n^2 \left(\cosh \theta_n \sin \theta_n - \sinh \theta_n \cos \theta_n\right)}$$

Integration of equation (A6) with respect to time with the condition $(w)_{t=0} = 0$ gives for the deflection (when $t \geqq 0$)

$$w(x,t) = \Omega L \sum_{n=1}^{\infty} \frac{F\left(\theta_n \frac{x}{L}\right)}{\omega_n'} e^{-\frac{\lambda \omega_n^2}{2E} t} \sin \omega_n' t \mathit{1}$$

$$= \frac{\Omega L}{c} \frac{L^2}{\rho} \sum_{n=1}^{\infty} C_n \frac{1}{\sqrt{1 - \frac{\lambda^2 \omega_n^2}{4E^2}}} e^{-\frac{\lambda \omega_n^2}{2E} t} \sin \omega_n' t \mathit{1} \qquad \text{(A7)}$$

where

$$C_n = \frac{F\left(\theta_n \frac{x}{L}\right)}{\theta_n^2}$$

The contribution of the nth mode to the deflection is

$$w_n(x,t) = \frac{\Omega L}{c}\frac{L^2}{\rho} C_n \frac{1}{\sqrt{1 - \frac{\lambda^2\omega_n^2}{4E^2}}} e^{-\frac{\lambda\omega_n^2}{2E}t} \sin \omega_n' t \qquad \text{(A8)}$$

When $\frac{\lambda\omega_n}{2E} > 1$, equation (A8) may be put in the form

$$w_n(x,t) = \frac{\Omega L}{c}\frac{L^2}{\rho} C_n \frac{1}{\sqrt{\frac{\lambda^2\omega_n^2}{4E^2} - 1}} e^{-\frac{\lambda\omega_n^2}{2E}t} \sinh \omega_n' t \qquad \text{(A9)}$$

where now

$$\omega_n' = \omega_n\sqrt{\frac{\lambda^2\omega_n^2}{4E^2} - 1}$$

The form indicated by equation (A8), in which $\frac{\lambda\omega_n}{2E} < 1$, is characteristic of the lower modes and represents damped oscillatory motion. The form indicated by equation (A9), in which $\frac{\lambda\omega_n}{2E} > 1$ (damping greater than critical), is characteristic of the higher modes and represents subsidence motion.

The complete behavior of the cantilever may be determined from equations (A6) for velocity and (A7) for deflection. The quantities of interest are the bending stress, the shear stress, and to some extent the accelerations. When damping is present, the equations representing the contribution of the nth mode to these quantities may be given in the two forms indicated by equations (A8) and (A9). In subsequent equations, however, only the form indicated by (A8) is given, because this form is characteristic of the modes that are of practical importance.

Bending stresses.- The bending stresses $\sigma(x,y,t)$ at any fiber located a distance y from the neutral axis are

$$\sigma(x,y,t) = Ey\frac{\partial^2 w}{\partial x^2} = E\frac{\Omega L}{c}\frac{y}{\rho}\sum_{n=1}^{\infty} A_n \frac{1}{\sqrt{1-\frac{\lambda^2\omega_n^2}{4E^2}}} e^{-\frac{\lambda\omega_n^2}{2E}t} \sin \omega_n' t \mathit{1}$$

where

$$A_n = 2\frac{\left(\cosh\theta_n \sin\theta_n - \sinh\theta_n \cos\theta_n\right)\left(\cosh\theta_n\frac{x}{L} + \cos\theta_n\frac{x}{L}\right) - \sin\theta_n \sinh\theta_n\left(\sinh\theta_n\frac{x}{L} + \sin\theta_n\frac{x}{L}\right)}{\theta_n^2\left(\cosh\theta_n \sin\theta_n - \sinh\theta_n \cos\theta_n\right)}$$

The contribution to bending stress of the nth mode is

$$\sigma_n(x,y,t) = E\frac{\Omega L}{c}\frac{y}{\rho} A_n \frac{1}{\sqrt{1-\frac{\lambda^2\omega_n^2}{4E^2}}} e^{-\frac{\lambda\omega_n^2}{2E}t} \sin \omega_n' t \mathit{1}$$

Average shear stresses.- The average shear stress on the cross section is

$$\bar{\tau}(x,t) = E\rho^2\frac{\partial^3 w}{\partial x^3} = E\frac{\Omega L}{c}\frac{\rho}{L}\sum_{n=1}^{\infty} B_n \frac{1}{\sqrt{1-\frac{\lambda^2\omega_n^2}{4E^2}}} e^{-\frac{\lambda\omega_n^2}{2E}t} \sin \omega_n' t \mathit{1}$$

where

$$B_n = 2\frac{\left(\cosh\theta_n \sin\theta_n - \sinh\theta_n \cos\theta_n\right)\left(\sinh\theta_n\frac{x}{L} - \sin\theta_n\frac{x}{L}\right) - \sin\theta_n \sinh\theta_n\left(\cosh\theta_n\frac{x}{L} + \cos\theta_n\frac{x}{L}\right)}{\theta_n\left(\cosh\theta_n \sin\theta_n - \sinh\theta_n \cos\theta_n\right)}$$

The contribution to shear stress of the nth mode is

$$\tau_n(x,t) = E\frac{\Omega L}{c}\frac{\rho}{L}B_n\frac{1}{\sqrt{1-\frac{\lambda^2\omega_n^2}{4E^2}}}e^{-\frac{\lambda\omega_n^2}{2E}t}\sin\omega_n' t\mathit{1}$$

<u>Accelerations</u>.- From equation (A6), with the aid of the relation

$$pF(t)\mathit{1} = F(0)p\mathit{1} + F'(t)\mathit{1}$$

the acceleration anywhere on the beam is found to be

$$a(x,t) = \frac{\partial^2 w}{\partial t^2} = \left[\sum_{n=1}^{\infty} F\left(\theta_n\frac{x}{L}\right) - \frac{x}{L}\right]\Omega L p\mathit{1}$$

$$- \frac{\Omega L}{c}\frac{L^2}{\rho}\sum_{n=1}^{\infty} C_n\frac{\omega_n^2\left(1-\frac{\lambda^2\omega_n^2}{2E^2}\right)}{\sqrt{1-\frac{\lambda^2\omega_n^2}{4E^2}}}e^{-\frac{\lambda\omega_n^2}{2E}t}\left(\sin\omega_n' t + \frac{\frac{\lambda\omega_n'}{E}}{1-\frac{\lambda^2\omega_n^2}{2E^2}}\cos\omega_n' t\right)\mathit{1}$$

With the aid of the orthogonal properties of the function $F\left(\theta_n \frac{x}{L}\right)$, it is possible to show that

$$\sum_{n=1}^{\infty} F\left(\theta_n \frac{x}{L}\right) - \frac{x}{L} = 0$$

The linear accelerations are therefore always finite.

The contribution to acceleration of the nth mode is

$$a_n(x,t) = -\frac{\Omega L}{c} \frac{L^2}{\rho} \omega_n^2 C_n \frac{1 - \frac{\lambda^2 \omega_n^2}{2E^2}}{\sqrt{1 - \frac{\lambda^2 \omega_n^2}{4E^2}}} e^{-\frac{\lambda \omega_n^2}{2E} t} \left(\sin \omega_n' t + \frac{\frac{\lambda \omega_n'}{E}}{1 - \frac{\lambda^2 \omega_n^2}{2E^2}} \cos \omega_n' t \right) \tag{1}$$

Comparison with the expression for $w_n(x,t)$ from equation (A8) shows that the acceleration for each mode is out of phase with the deflection. When damping is sufficiently small, however, the relation between the acceleration and the deflection reduces to the well-known result for undamped vibration

$$a_n(x,t) = -\omega_n^2 \, w_n(x,t)$$

REFERENCE

1. Stowell, Elbridge Z., Schwartz, Edward B., and Houbolt, John C.: Bending and Shear Stresses Developed by the Instantaneous Arrest of the Root of a Moving Cantilever Beam. NACA ARR No. L4I27, 1944.

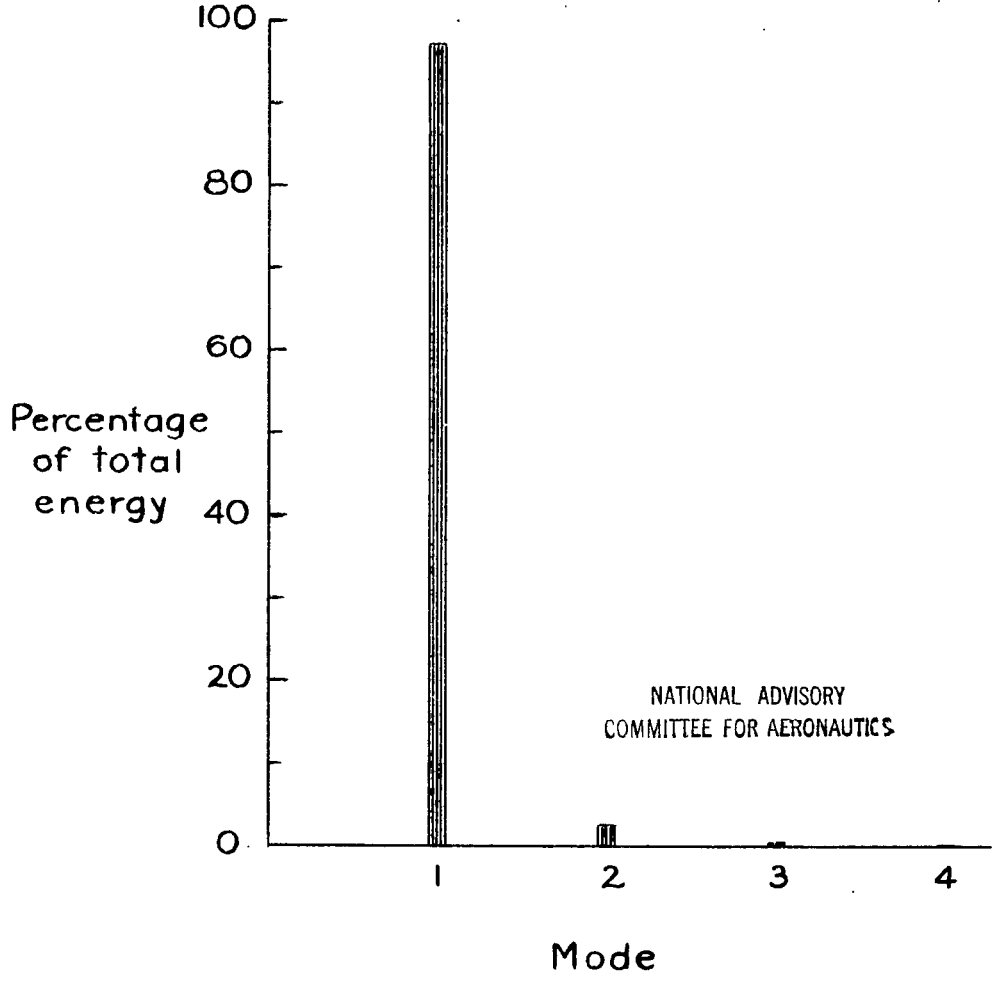

Figure 1.- Distribution of energy among the modes of vibration.

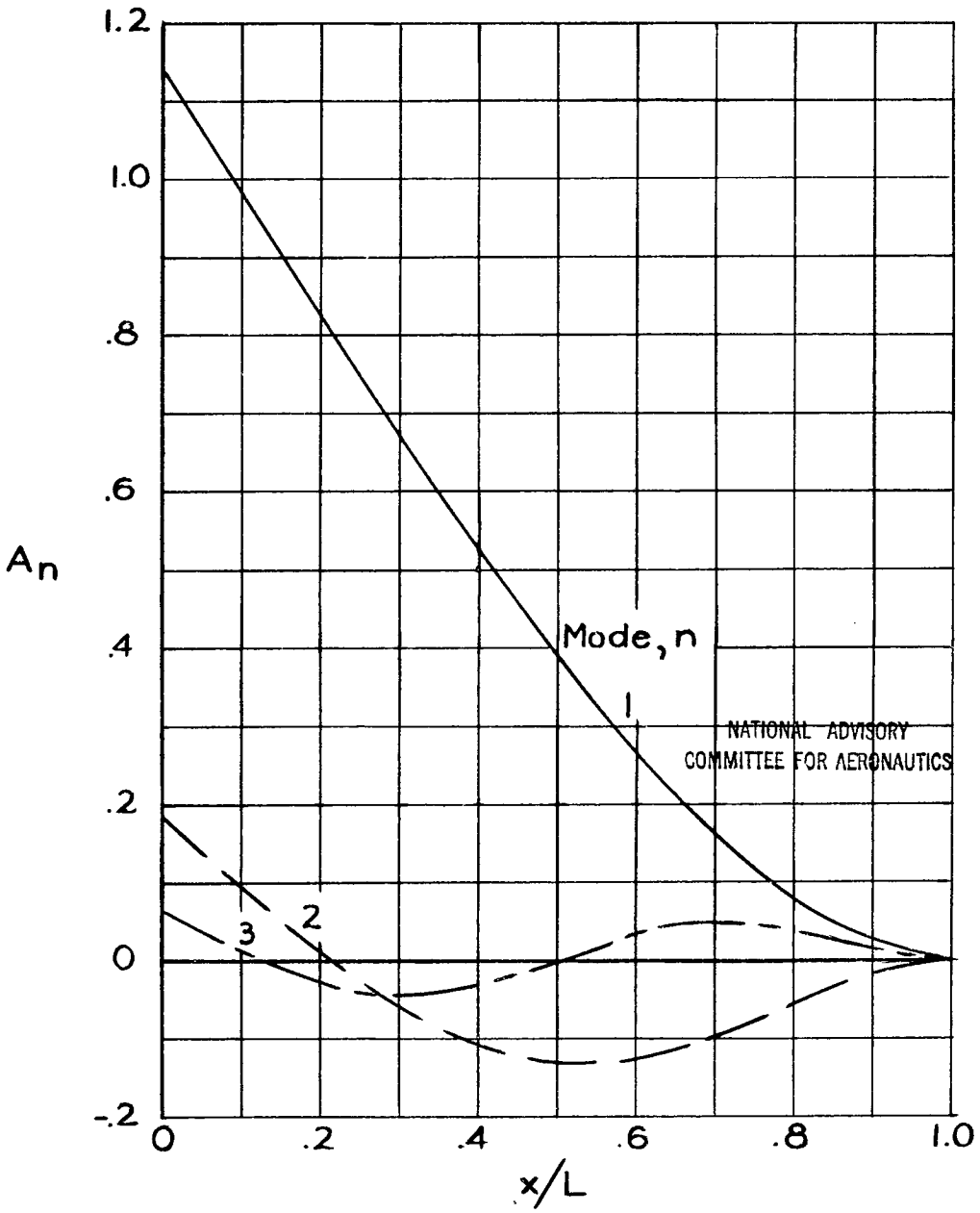

Figure 2.- Variation of bending-stress coefficient A_n with $\frac{x}{L}$.

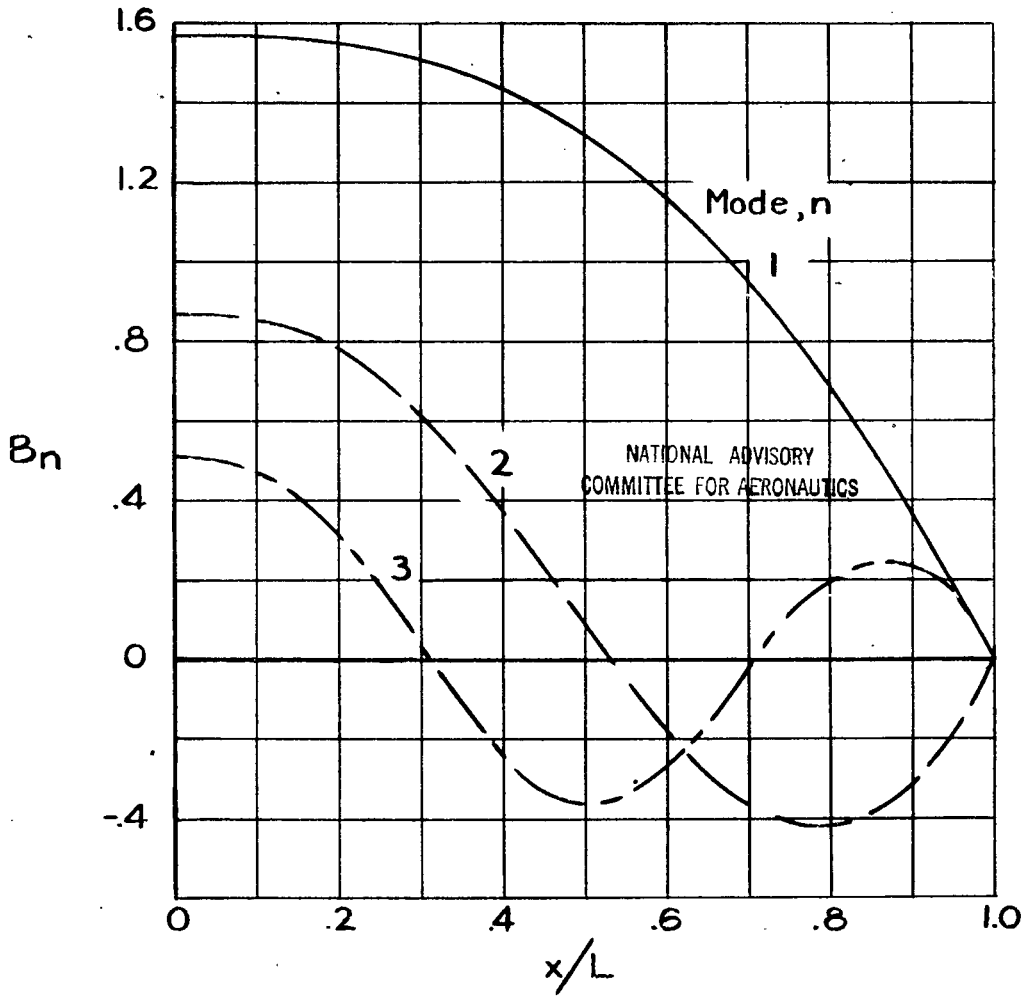

Figure 3.- Variation of shear-stress coefficient B_n with $\frac{x}{L}$.

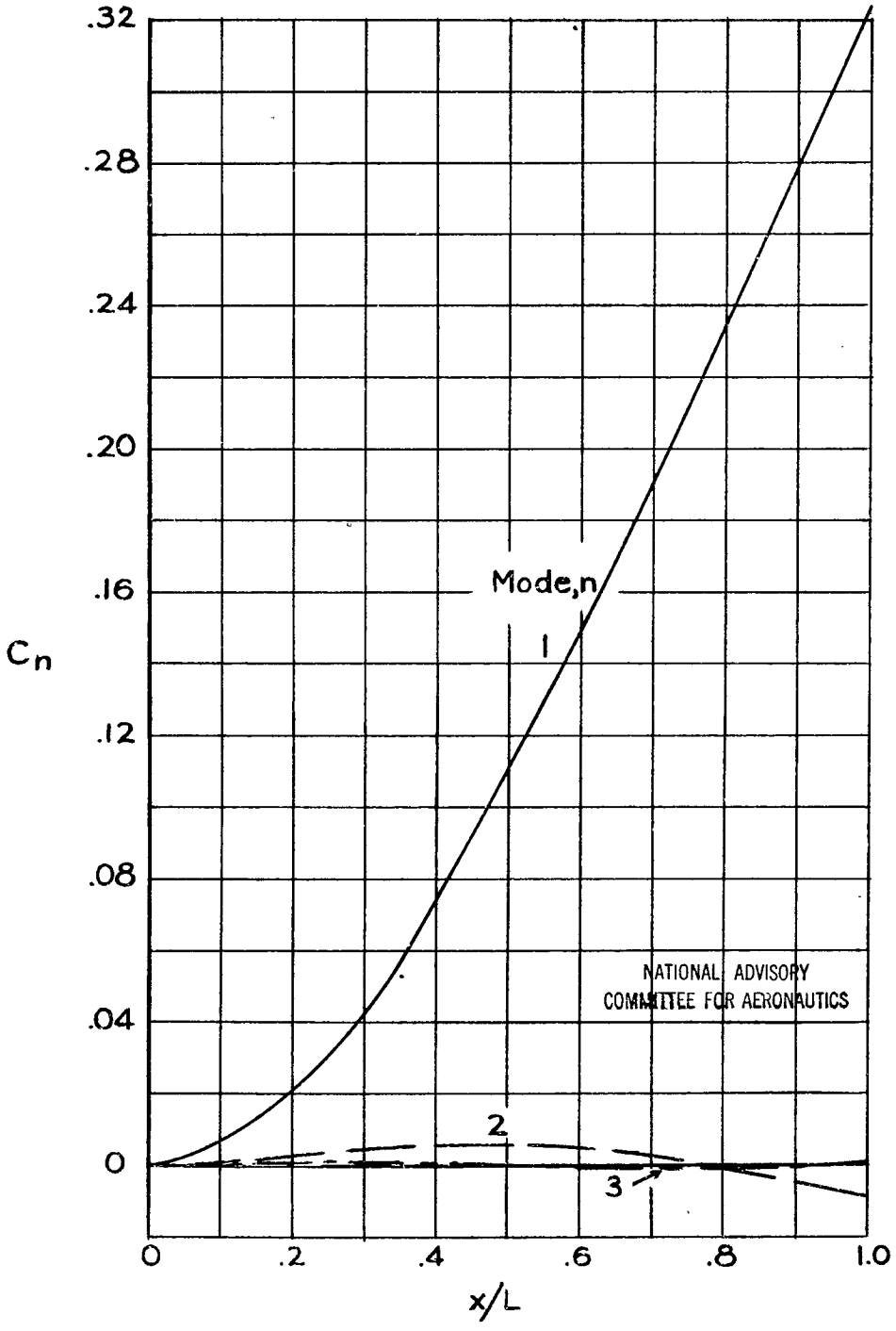

Figure 4.- Variation of deflection coefficient C_n with $\frac{x}{L}$.

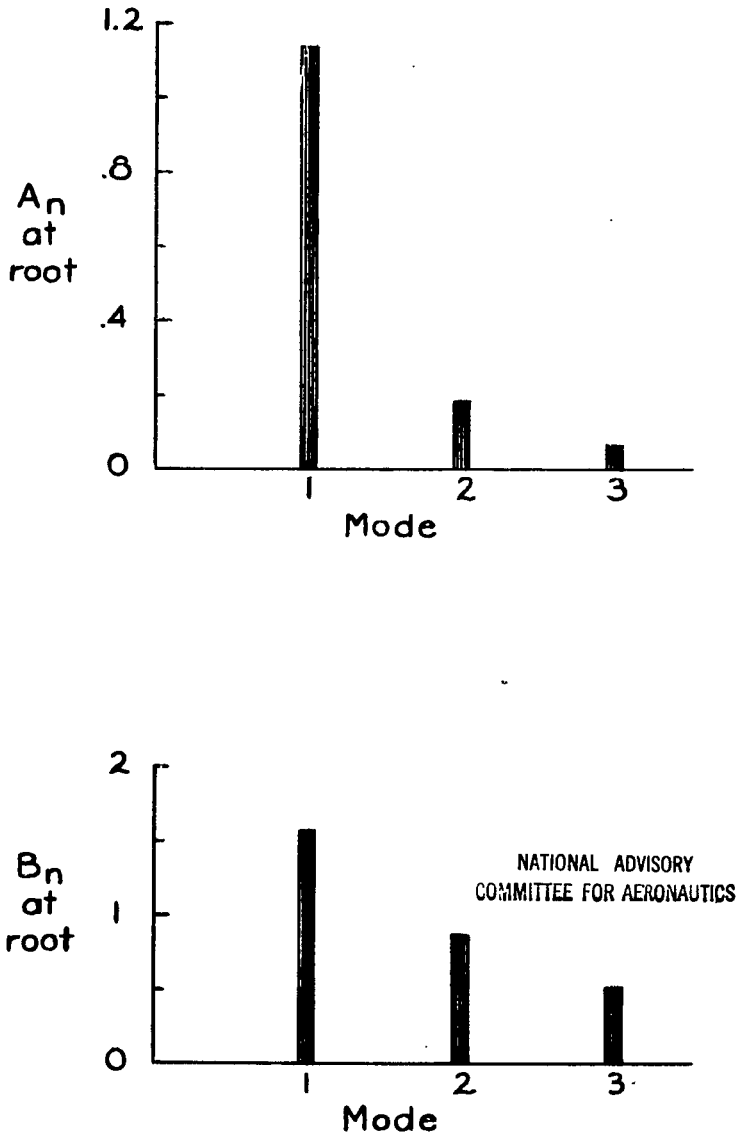

Figure 5.- Values of bending-stress coefficient A_n and shear-stress coefficient B_n at root, $\frac{x}{L} = 0$.

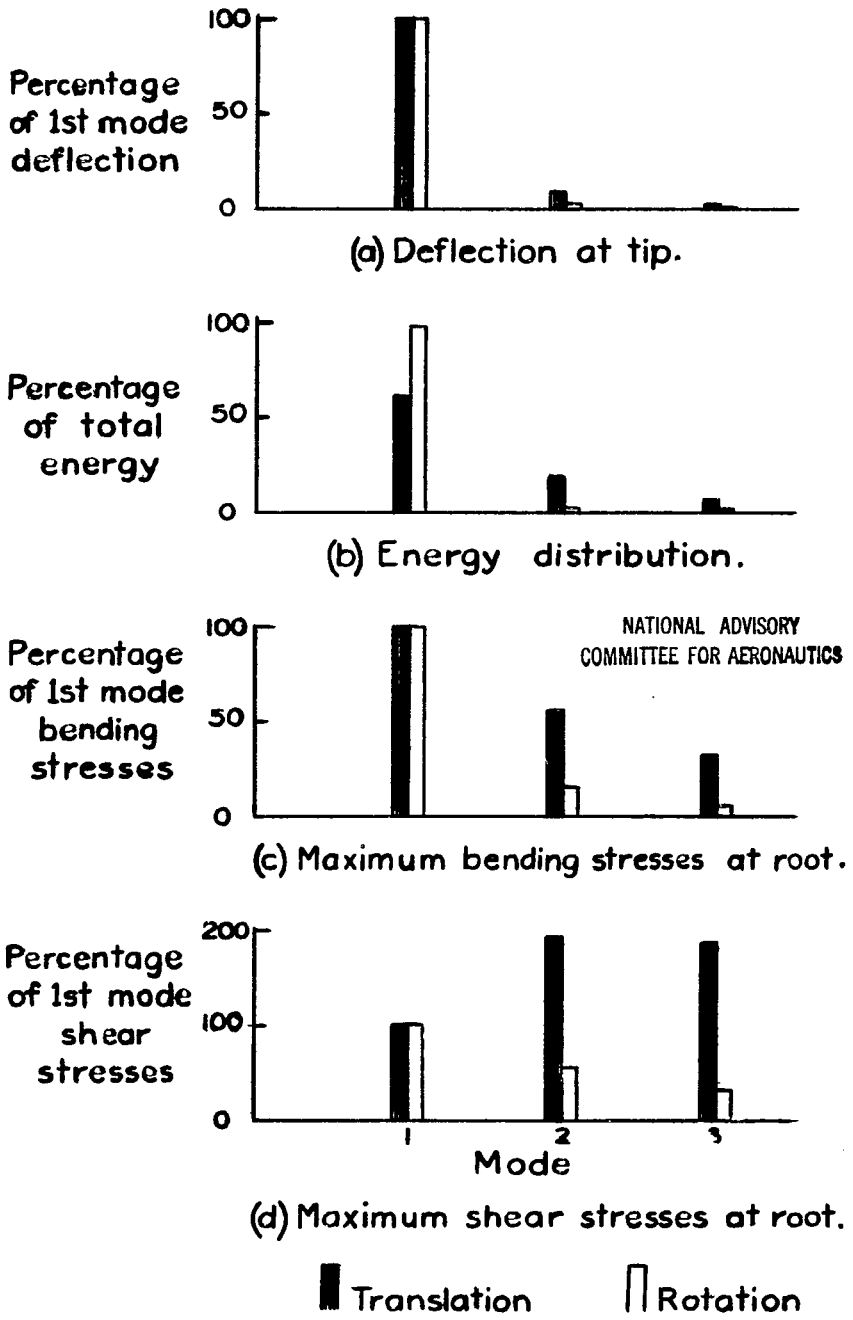

Figure 6.- Comparison of principal results of reference 1 (cantilever in translation) and the present paper (cantilever in rotation).

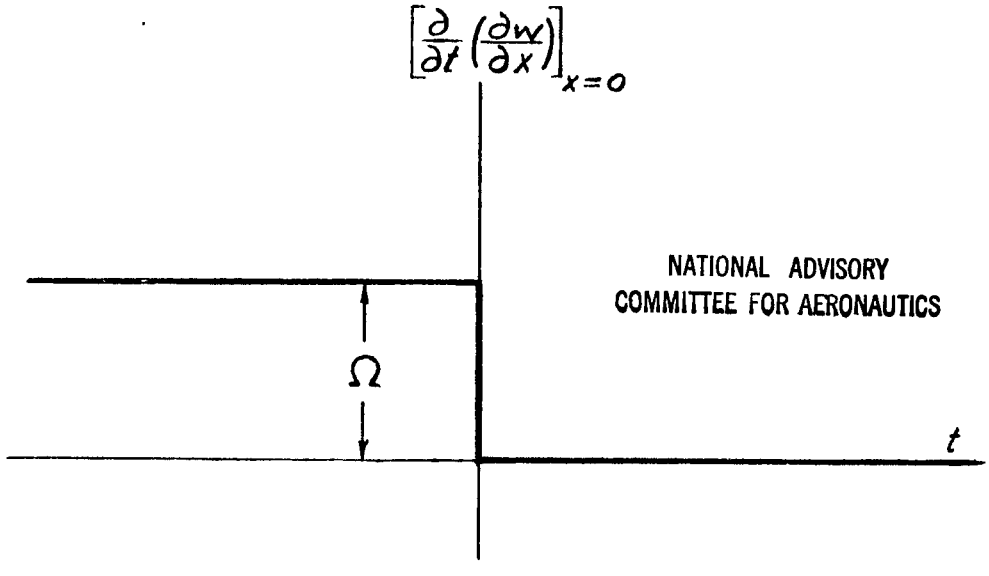

Figure 7. - Graphical representation of the discontinuous function $\Omega - \Omega 1$.

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