REPORT 986

THE REVERSIBILITY THEOREM FOR THIN AIRFOILS IN SUBSONIC AND SUPERSONIC FLOW

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SUMMARY

A method introduced by Munk is extended to prove that the lift-curve slope of thin wings in either subsonic or supersonic flow is the same when the direction of flight of the wing is reversed. It is also shown that the wing reversal does not change the thickness drag, damping-in-roll parameter C_{l_p}, or the damping-in-pitch parameter C_{m_q}.

INTRODUCTION

The present report makes use of and extends a paper by Munk (reference 1) in which simple dynamic concepts are used to prove that the lift-curve slope and thickness drag of supersonic airfoils with supersonic edges are the same when the airfoil is flown in a reversed direction. This extension of Munk's work provides a proof that the thickness drag, lift-curve slope, damping in roll, and the damping-in-pitch parameter C_{m_q} remain the same when any airfoil or system of airfoils is reversed in both subsonic and supersonic flow. The theorem applies to cases in which the trailing-edge velocities are finite; no restrictions are placed on plan form.

The reversibility theorem for drag was first obtained by different methods by Von Kármán (reference 2). Hayes has treated the lifting case for a restricted series of wing types at supersonic speeds. (See references 3 and 4.) Harmon (reference 5) has extensively treated the stability derivatives for a restricted group of plan forms at supersonic speeds.

PROOF

Under the assumptions of the linearized potential-flow theory, it becomes possible to obtain a great simplification of subsonic and supersonic lifting-surface problems. The use of the linear equations of motion allows the boundary conditions on a lifting surface to be satisfied on a plane near the wing surface and permits the use of the superposition principle. Consider a set of Cartesian coordinates x,y,z in which the x-axis is taken in the flight direction and the z-axis, in the vertical direction. The boundary conditions become a stipulation of the vertical-velocity distribution over the projection of the wing surface on the xy-plane. As a result of this simplification, the effects of camber, twist, angle of attack, and thickness may be treated separately.

For the complete comprehension of the analysis to follow, it is necessary to understand the manner in which drag ultimately appears in the flow field. Two distinct forms of drag may be found: one associated with a trailing vortex system, the other with the production of waves. In the case of a vortex wake, the drag shows up in the wake a great distance downstream in the form of a pressure defect which, when integrated over a plane normal to the flight path, yields the drag. This result is identical with that of incompressible flow. The drag produced by wave formation shows up in the field as a combined momentum and pressure defect; of course, the thin-airfoil theory predicts a wave drag only at supersonic speeds. In all cases, the total resistance may be obtained by integrating the momentum transport across the sides of a box enclosing the wing. It is often convenient to place the sides of the box at infinity and allow the top and bottom to approach the plane of the wing. This process yields for the drag

$$D=-2\rho\int\int\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial z}\,dy\,dx \tag{1}$$

where ρ is the stream density, ϕ is the disturbance potential, and the integration taken over both upper and lower sides extends to infinity. Note that the drag is independent of the main stream direction but depends only on the disturbance potential ϕ. In the usual problems, singularities occur on the wing leading edges and care must be taken with the integration if the quantities in the integrand of equation (1) are evaluated on the xy-plane. Neglect of the singular behavior leads to the omission of the leading-edge suction forces. For additional information on the fundamentals of the linear theory, see references 2 and 6.

Thickness drag.—Consider a symmetrical airfoil at zero angle of attack. The potential of the flow may be expressed as

$$\Phi_1=Vx+\phi_1 \tag{2}$$

where V is the stream velocity and ϕ_1 is the disturbance potential which satisfies the boundary condition

$$\frac{1}{V}\left(\frac{\partial\phi_1}{\partial z}\right)_{z=0}=\frac{dz}{dx} \tag{3}$$

where $\frac{dz}{dx}$ is the airfoil surface slope. In addition, the potential must satisfy the usual conditions for physical flows such as the vanishing of the perturbation velocities at infinity for subsonic flow and undisturbed flow ahead of the foremost Mach waves in supersonic flow. Assume the main flow direction to be reversed. The new potential would result:

$$\Phi_2=-Vx+\phi_2 \tag{4}$$

where ϕ_2 satisfies the condition

$$\frac{1}{V}\left(\frac{\partial\phi_2}{\partial z}\right)_{z=0}=-\frac{dz}{dx} \qquad (5)$$

By superposing the solutions Φ_1 and ϕ_2 a new potential is formed:

$$\Phi_3=\Phi_1+\phi_2 \qquad (6)$$

Such a step is quite permissible inasmuch as the differential equation governing the flows is independent of the sign of the stream velocity. The vertical velocity $\left(\frac{\partial\Phi_3}{\partial z}\right)_{z=0}$ becomes zero and thus a boundary condition for a plate of zero thickness is satisfied. Inasmuch as there are no infinite induced velocities at the edges of the resultant wing and therefore no edge forces, the flat plate can produce no changes in stream momentum; hence, the momentum or pressure defects a great distance downstream in the flow must be equal to those upstream. Any momentum or pressure defects at infinity upstream arise from the reversed airfoil potential, and the momentum or pressure defects a great distance downstream arise only from the original airfoil potential. Since the drag of each airfoil is equal to the momentum or pressure defects in its wake, the drag of the two airfoils must necessarily be equal. It is well known that the drag of symmetrical bodies in subsonic potential flow is zero; hence, the reversibility of drag is most pertinent to supersonic flows. The preceding proof and discussion follows essentially that of Munk (reference 1).

Lift-curve slope.—Inasmuch as the lift-curve slope of a wing is independent of camber and twist, it is sufficient to treat a flat-plate airfoil at an angle of attack α. Unlike the symmetrical-drag case, however, a certain indeterminacy exists in the potential whenever subsonic trailing edges are present. Subsonic edges occur when the component of stream velocity normal to the edge is subsonic. In order to remove this indeterminacy it is necessary to specify the circulation. The use of the Kutta condition is an appropriate means for this process because, in effect, an additional boundary condition is imposed. This requirement, that the velocities at the trailing edge be finite, is indeed a physical condition arising from the fact that the boundary layer, always present at trailing edges, would separate from the edge rather than accommodate the high adverse accelerations around the edge. It is exactly the Kutta condition which leads to unique solutions and which is necessary to prove the reversibility theorem.

The potential of the flat-plate airfoil may now be written:

$$\Phi_1=Vx+\phi_1 \qquad (7)$$

where ϕ_1 is the disturbance potential satisfying the conditions that the trailing-edge velocities are finite and the boundary condition

$$\frac{1}{V}\frac{\partial\phi_1}{\partial z}=-\alpha \qquad (8)$$

The drag of the wing can be written:

$$D_1=L_1\alpha-F_1 \qquad (9)$$

where F is the component parallel to the surface of the resultant force, usually known in linear theory as the leading-edge suction force, and L is the lift force.

As in the drag case, the reversed-stream velocity produces the potential

$$\Phi_2=-Vx+\phi_2 \qquad (10)$$

where ϕ_2 satisfies the Kutta condition and the boundary condition

$$\frac{1}{V}\frac{\partial\phi_2}{\partial z}=\alpha \qquad (11)$$

The drag is now

$$D_2=L_2\alpha-F_2 \qquad (12)$$

The superposition of the potential ϕ_2 on Φ_1 results in the flow over a flat plate of zero angle of attack.

The drag of the combined airfoils is now

$$D_3=F_2-F_1 \qquad (13)$$

provided the superposition has not changed the leading-edge suction forces F_1 and F_2. Those suction forces have been shown (references 7 to 9) to be dependent on the asymptotic distribution of vorticity as the edge is approached; suction forces are obtained only when the vortex strength approaches infinity at the edge, this condition corresponding to infinite upwash velocity around the edge. Inasmuch as the superposition of a solution having finite-edge velocities does not alter the asymptotic strength of the singularities at the edge, it follows that the edge forces will be unchanged by the superposition.

When a momentum balance in the stream is formed, the upstream momentum and pressure defects in the combined-airfoil case differ from the downstream momentum and pressure defects by the difference in the suction forces D_3. The upstream momentum and pressure defects are, however, equal to D_2, whereas those downstream are equal to D_1. Therefore,

$$D_1-D_2=F_2-F_1 \qquad (14)$$

or from equations (9) and (12)

$$L_1\alpha=L_2\alpha \qquad (15)$$

The lifts L_1 and L_2 are equal and, therefore, the lift-curve slopes are equal. Obviously, the lift-curve slopes of cambered and twisted wings are also unchanged when the airfoil is reversed. It is important to note that the drags are not equal unless the suction forces are zero or cancel.

In reference 4, the conclusion is reached that the lift theorem cannot be a general one; however, it appears that this conclusion was deduced from an equation of insufficient generality. Indeed, the analysis of the present report shows the lift theorem to apply to all plan forms so long as the Kutta condition is applied to subsonic trailing edges.

Damping in roll.—The proof for the reversibility of damping in roll proceeds in the same manner as that for the lift.

The rolling moment of the thin wing may be expressed as follows:

$$L_1'=\int_S \Delta p_1 y\, dS \tag{16}$$

where Δp is the pressure difference between the upper and lower surface and S is the wing area.

The drag of the linearly twisted wing used to represent the rolling flat plate is

$$D=\int_S \Delta p_1 \alpha\, dS-F_1 \tag{17}$$

The drag may be expressed as a function of the rolling moment inasmuch as $\alpha=\frac{py}{V}$ where p is the angular velocity in roll. For the twisted wing,

$$D_1=\frac{p}{V} L_1'-F_1 \tag{18}$$

The drag of the reversed airfoil is then

$$D_2=\frac{p}{V} L_2'-F_2 \tag{19}$$

Superposing the disturbance potential of the reversed airfoil again cancels the wing slopes, and the resulting momentum change at the combined airfoils becomes

$$D_3=F_2-F_1 \tag{20}$$

Establishing the conservation of momentum in the flow, as was done for the lifting case, gives the result:

$$\frac{p}{V} L_1'=\frac{p}{V} L_2' \tag{21}$$

Therefore, the rolling moment for the reversed airfoil is the same as that of the unreversed airfoil. It follows then that the rolling-moment derivative C_{l_p} for any wing is unchanged by reversal.

Steady pitching moment.—The pitching moment of a wing undergoing a steady pitching velocity q about the point x_0 may be written

$$M=\int_S (x-x_0)\Delta p\, dS \tag{22}$$

where x_0 is the reference point about which moments are taken. The drag of the cambered-wing surface representing the steady pitching motion is

$$D=\int_S \Delta p \alpha\, dS-F \tag{23}$$

and the local angle of attack for such a wing is

$$\alpha=q\,\frac{x-x_0}{V} \tag{24}$$

Hence, the drag may be expressed from equations (22) to (24) as follows:

$$D=\frac{q}{V} M-F \tag{25}$$

Performing the superposition of reversed potential and original potentials yields an airfoil of zero angle of attack; the momentum balance, as for the steady rolling case, cancels the suction forces to leave:

$$\frac{q}{V} M_1=\frac{q}{V} M_2 \tag{26}$$

The pitching moments of the two airfoils are the same and, therefore, the damping-in-pitch parameter C_{m_q} is unchanged by a reversal of the wing.

DISCUSSION

Inasmuch as the analysis presented is unrestricted as to plan form, it follows that any system of airfoils will obey the reversibility theorem; this does not allow for the reversal of the individual airfoils but only for the reversal of the complete system. Indeed, the same result holds for groups of airfoils in different horizontal planes, provided the boundary conditions for each wing are satisfied in the plane of the wing. It should be noticed that the pitching-moment coefficients, lift coefficients due to pitching, and the constants arising from camber such as $\alpha_{L=0}$ are not generally the same when the wing is reversed.

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