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SUMMARY

The theory of elasticity shows that the maximum stress at a circular hole in an infinite plate in tension is three times the applied stress when the material remains elastic. The effect of plasticity of the material is to lower this ratio. The stress concentration factor is approximately $1 + 2\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty}$ where $(E_s)_{a,\pi/2}$ is the secant modulus at the point of maximum stress and $(E_s)_\infty$ is the secant modulus at points far removed from the hole, where the stress is applied. This relation must be solved by trial and error. Values of stress concentration obtained from the formula are in good agreement with limited tests on 24S-T3 aluminum-alloy tension panels. The strain concentration factor determined at the same time is also in agreement with these tests.

INTRODUCTION

Stress concentration factors have been universally computed on the basis of the theory of elasticity. If, however, the material is stressed into the plastic range, the theory of elasticity no longer applies and stress concentration factors computed on that basis are in error.

Experimental data on the stress and strain concentration at a circular hole in a large, wide sheet of 24S-T3 aluminum alloy in tension were published in reference 1. If the material remains elastic, the theory of elasticity predicts concentration factors of 3 for both stress and strain at the point of maximum stress. Values close to 3 were actually found experimentally when no part of the sheet was stretched beyond the elastic range. When the sheet was further stressed into the plastic range, the stress concentration factor (based on applied stress instead of net-section stress as was done in reference 1) decreased to 1.4 and the strain concentration factor increased to 8.6.

This paper considers the theoretical problem of the stress distribution in an infinitely large sheet with a circular hole for the general case where the material may have any stress-strain curve. The plate is assumed to be under uniform tension at a large distance from the hole. The material is taken to be isotropic and incompressible.

RESULTS AND CONCLUSIONS

The calculation, as presented in the appendix, gives the formula for the stress concentration at a circular hole in an infinite sheet as

$$1 + 2\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty}$$

where $(E_s)_{a,\pi/2}$ is the secant modulus at the point of maximum stress and $(E_s)_\infty$ is the secant modulus at points far removed from the hole, where the load is applied. A numerical trial-and-error procedure is required to solve for the stress concentration factor.

In reference 1, experimental data were given on the stress and strain concentration factors for a wide sheet of 24S-T3 aluminum alloy with a circular hole under tension. A curve of E_s/E for this material was determined and is shown in figure 1. From this curve, which was taken from the stress-strain curve ending at point E in figure 4 of reference 1, stress and strain concentration factors can be easily computed to compare with figure 5 of reference 1. (In the tests $(E_s)_\infty = E$.) Such a comparison is shown in figure 2 of the present paper. The stress concentration factor appears to be given by the formula with accuracy which is adequate. The factors for strain are somewhat lower than those reported in reference 1; the apparent discrepancy is probably due in part to the peculiarities of the stress-strain curve, which permit a slight error in stress to be enormously magnified in strain, and in part to the use of 1/2 for Poisson's ratio.

In order to compare the distribution of strain and stress given by the formulas of this paper with measured distributions, figures 3 and 4 have been prepared. Figures 6 and 8 of reference 1, which represent the measured distributions of strain and stress perpendicular to the direction of the tension, have been reproduced herein as figures 3 and 4, respectively, with the addition of circular points to represent computed values at three different stress levels (σ_∞ = 21, 30, and 37 ksi, corresponding to the net-section-stress levels of σ_{av} = 25, 35, and 45 ksi of reference 1). For the strain distribution (fig. 3), the computed points

agree with the measured curves at the hole and at a large distance away from it but fall appreciably below the curves between these two places. The disagreement may be considered as a measure of the amount by which stress equilibrium and strain compatibility are not satisfied by the theory. For the stress distribution (fig. 4), the agreement between the curves and the points is better and the theoretical values may be considered to represent the actual distribution fairly well.

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Langley Air Force Base, Va., February 1, 1950

APPENDIX

DERIVATION OF CONCENTRATION FACTORS

Figure 5 shows the coordinate system used in the derivation of the concentration factors. A tensile stress σ_∞ is applied to the sheet at a large distance from the hole. The radial stress is denoted by σ_r, the circumferential stress, by σ_θ, and the shear stress, by τ. At the hole, where $r = a$, the stresses σ_r and τ must vanish. At infinity, the stresses must be

$$\sigma_r = \sigma_\infty \frac{1 + \cos 2\theta}{2}$$

$$\sigma_\theta = \sigma_\infty \frac{1 - \cos 2\theta}{2}$$

$$-\tau = \sigma_\infty \frac{\sin 2\theta}{2}$$

Assumption of stress system.- Assume a set of stresses at any point (r,θ) as follows:

$$\sigma_r = \frac{\sigma_\infty}{2}\left[1 - \frac{a^2}{r^2} + G\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\sigma_\theta = \frac{\sigma_\infty}{2}\left[1 + \frac{a^2}{r^2} - G\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\tau = -\frac{\sigma_\infty}{2} G\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta$$

where G is a function of $E_s/(E_s)_\infty$, E_s being the secant modulus at the point (r,θ) and $(E_s)_\infty$ being the secant modulus at $r = \infty$. This stress

system satisfies the boundary conditions both at the hole and at infinity. When $\frac{E_s}{(E_s)_\infty} = 1$, the stresses are elastic everywhere, and from the known elastic solution $G(1) = 1$ is required as a limiting condition on the function G. See, for instance, reference 2.

The point of highest stress is at $\left(r = a,\ \theta = \frac{\pi}{2}\right)$. The stress concentration must reduce to unity there when the material becomes very plastic because of flow. This requirement gives a second limiting condition on the function G; namely, $G(0) = 0$ when $\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty} \rightarrow 0$.

Equilibrium of stresses.- The equations of equilibrium are

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = 0$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau}{\partial r} + \frac{2\tau}{r} = 0$$

When the assumed stresses are substituted into the equations of equilibrium, the results are

$$\frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau}{\partial\theta} + \frac{\sigma_r - \sigma_\theta}{r} = \frac{\sigma_\infty}{2}\left(1 - \frac{a^2}{r^2}\right)\left[\left(1 - 3\frac{a^2}{r^2}\right)\frac{\partial\frac{E_s}{(E_s)_\infty}}{\partial r}\cos 2\theta - \frac{1 + 3\frac{a^2}{r^2}}{r}\frac{\partial\frac{E_s}{(E_s)_\infty}}{\partial\theta}\sin 2\theta\right]\frac{dG}{d\frac{E_s}{(E_s)_\infty}}$$

$$\frac{1}{r}\frac{\partial\sigma_\theta}{\partial\theta} + \frac{\partial\tau}{\partial r} + \frac{2\tau}{r} = \frac{\sigma_\infty}{2}\left[-\frac{1 + 3\frac{a^4}{r^4}}{r}\frac{\partial\frac{E_s}{(E_s)_\infty}}{\partial\theta}\cos 2\theta + \left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\frac{\partial\frac{E_s}{(E_s)_\infty}}{\partial r}\sin 2\theta\right]\frac{dG}{d\frac{E_s}{(E_s)_\infty}}$$

Calculation of G-function.- The error in the satisfaction of these equations is proportional to $\dfrac{dG}{d\dfrac{E_s}{(E_s)_\infty}}$. It is desired that the mean square of the error be made a minimum; that is, $\displaystyle\int_0^1 \left[\frac{dG}{d\dfrac{E_s}{(E_s)_\infty}}\right]^2 d\frac{E_s}{(E_s)_\infty}$ should be a minimum. This expression represents a kind of mean square error in which the averaging is weighted heavily in the vicinity of the hole, since here the variation of the modulus is most rapid. The calculus of variations yields the Euler equation for this case as

$$-\frac{d}{d\dfrac{E_s}{(E_s)_\infty}}\left[2\frac{dG}{d\dfrac{E_s}{(E_s)_\infty}}\right] = 0$$

from which

$$G = c_1 \frac{E_s}{(E_s)_\infty} + c_2$$

where c_1 and c_2 are constants. From the conditions $G(1) = 1$ and $G(0) = 0$, it is found that

$$c_1 = 1$$

$$c_2 = 0$$

so that

$$G = \frac{E_s}{(E_s)_\infty}$$

Final stress system.- The final stress system, obtained by inserting the expression for G into the assumed expressions for stress, is

$$\sigma_r = \frac{\sigma_\infty}{2}\left[1 - \frac{a^2}{r^2} + \frac{E_s}{(E_s)_\infty}\left(1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\sigma_\theta = \frac{\sigma_\infty}{2}\left[1 + \frac{a^2}{r^2} - \frac{E_s}{(E_s)_\infty}\left(1 + 3\frac{a^4}{r^4}\right)\cos 2\theta\right]$$

$$\tau = -\frac{\sigma_\infty}{2}\frac{E_s}{(E_s)_\infty}\left(1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right)\sin 2\theta$$

Stress concentration factor.- At the hole $\sigma_r = \tau = 0$; the stress σ_θ is a maximum for $\theta = \frac{\pi}{2}$ and has the value

$$\left(\sigma_\theta\right)_{a,\pi/2} = \frac{\sigma_\infty}{2}\left[2 + 4\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty}\right]$$

and the stress concentration factor is

$$\frac{(\sigma_\theta)_{a,\pi/2}}{\sigma_\infty} = 1 + 2\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty}$$

Strain concentration factor.- The strains ϵ_r, ϵ_θ, and γ are found from the stresses through the stress-strain relations

$$\epsilon_r = \frac{\sigma_r - \frac{\sigma_\theta}{2}}{E_s}$$

$$\epsilon_\theta = \frac{\sigma_\theta - \frac{\sigma_r}{2}}{E_s}$$

$$\gamma = \frac{3\tau}{E_s}$$

The strains are:

$$\epsilon_r = \frac{\sigma_\infty}{2E_s}\left(\frac{1 - 3\frac{a^2}{r^2}}{2} + \frac{E_s}{(E_s)_\infty}\,\frac{3 - 8\frac{a^2}{r^2} + 9\frac{a^4}{r^4}}{2}\cos 2\theta\right)$$

$$\epsilon_\theta = \frac{\sigma_\infty}{2E_s}\left(\frac{1 + 3\frac{a^2}{r^2}}{2} - \frac{E_s}{(E_s)_\infty}\,\frac{3 - 4\frac{a^2}{r^2} + 9\frac{a^4}{r^4}}{2}\cos 2\theta\right)$$

$$\gamma = -\frac{3\sigma_\infty}{2(E_s)_\infty}\left(1 - \frac{a^2}{r^2}\right)\left(1 + 3\frac{a^2}{r^2}\right)\sin 2\theta$$

At the hole where $\theta = \frac{\pi}{2}$, the strain concentration factor is

$$\frac{(\epsilon_\theta)_{a,\pi/2}}{\sigma_\infty/(E_s)_\infty} = \frac{1 + 2\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty}}{\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty}}$$

The strain concentration factor is thus the stress concentration factor divided by $\frac{(E_s)_{a,\pi/2}}{(E_s)_\infty}$.

REFERENCES

1. Griffith, George E.: Experimental Investigation of the Effects of Plastic Flow in a Tension Panel with a Circular Hole. NACA TN 1705, 1948.

2. Timoshenko, S.: Theory of Elasticity. First ed., McGraw-Hill Book Co., Inc., 1934, p. 77.

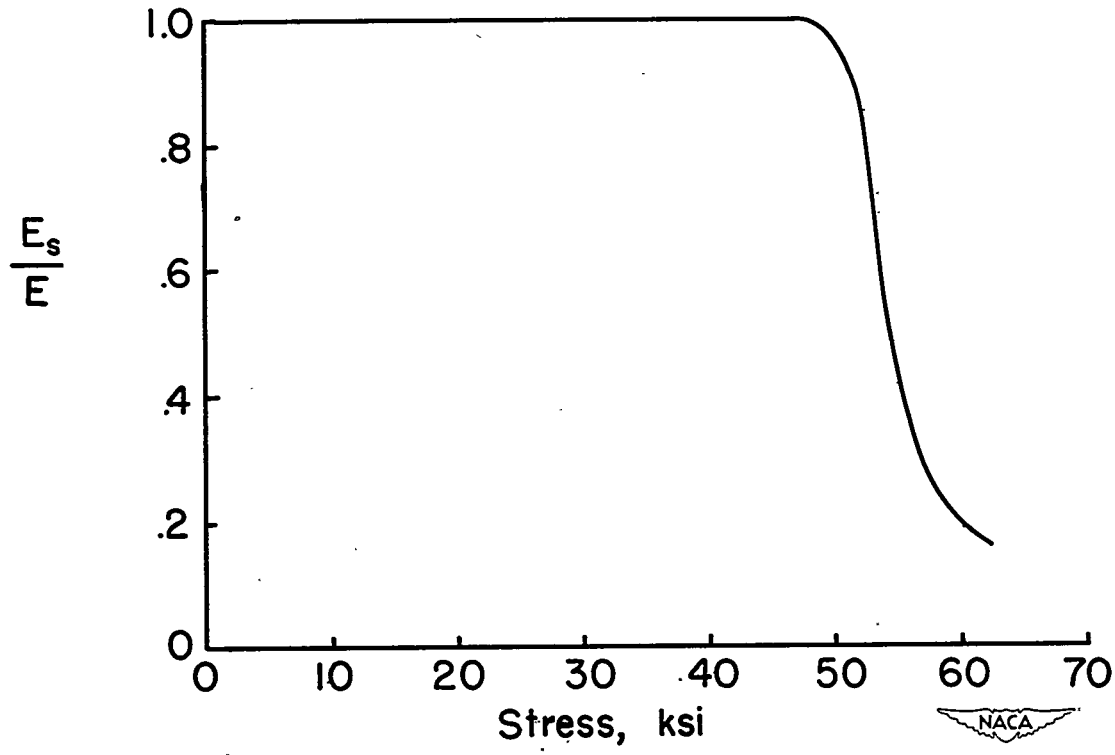

Figure 1.- Values of $\frac{E_s}{E}$ for 24S-T3 aluminum alloy used in reference 1.

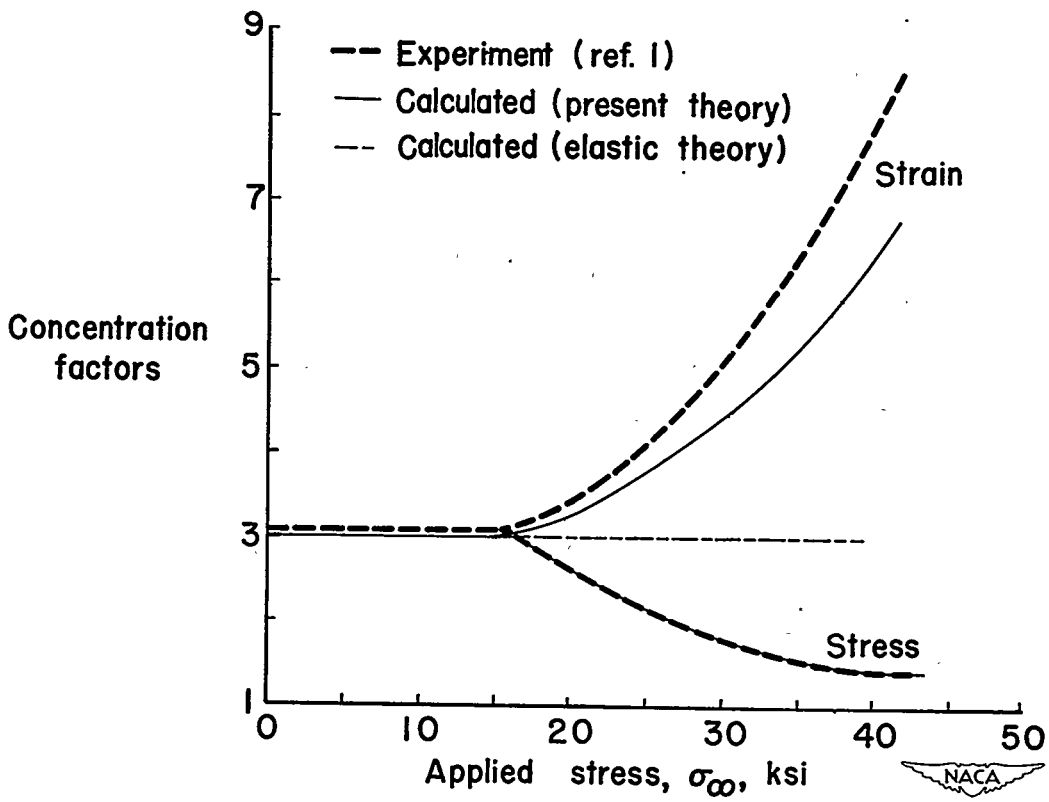

Figure 2.- Comparison between calculated stress and strain concentration factors and tests of reference 1.

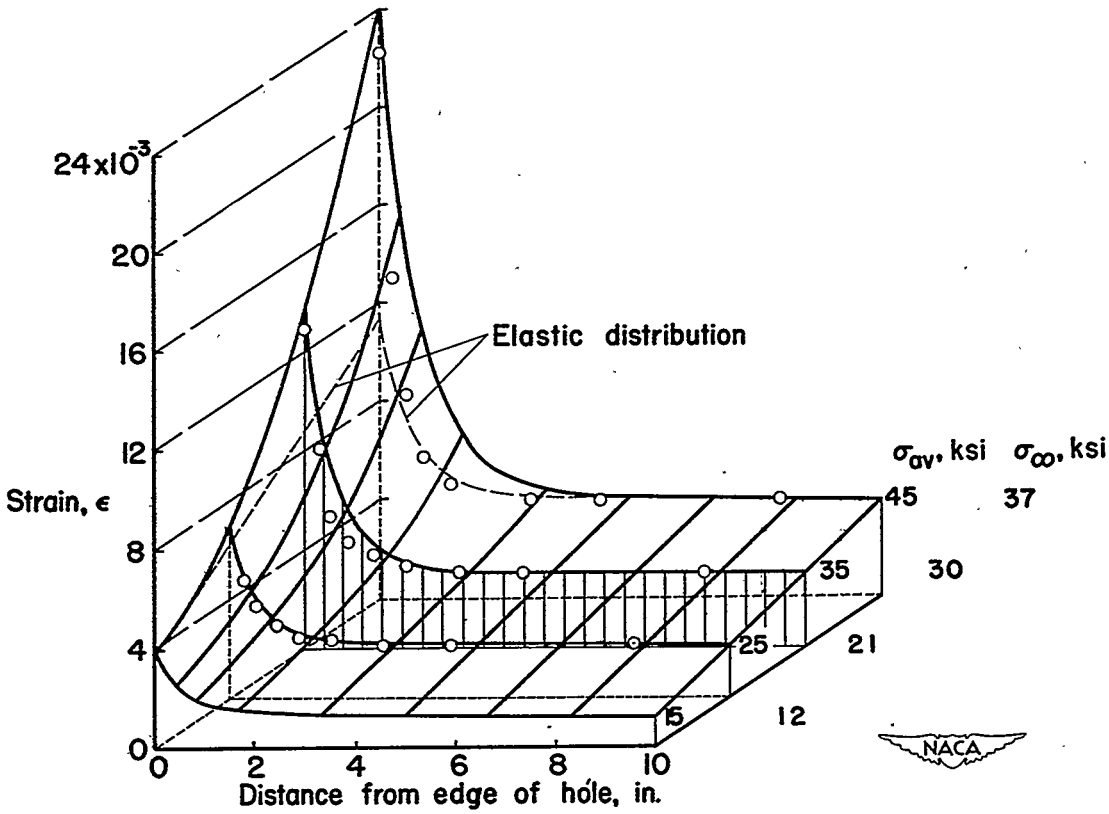

Figure 3.- Comparison between calculated points, for three values of applied stress, and experimental strain distribution from reference 1.

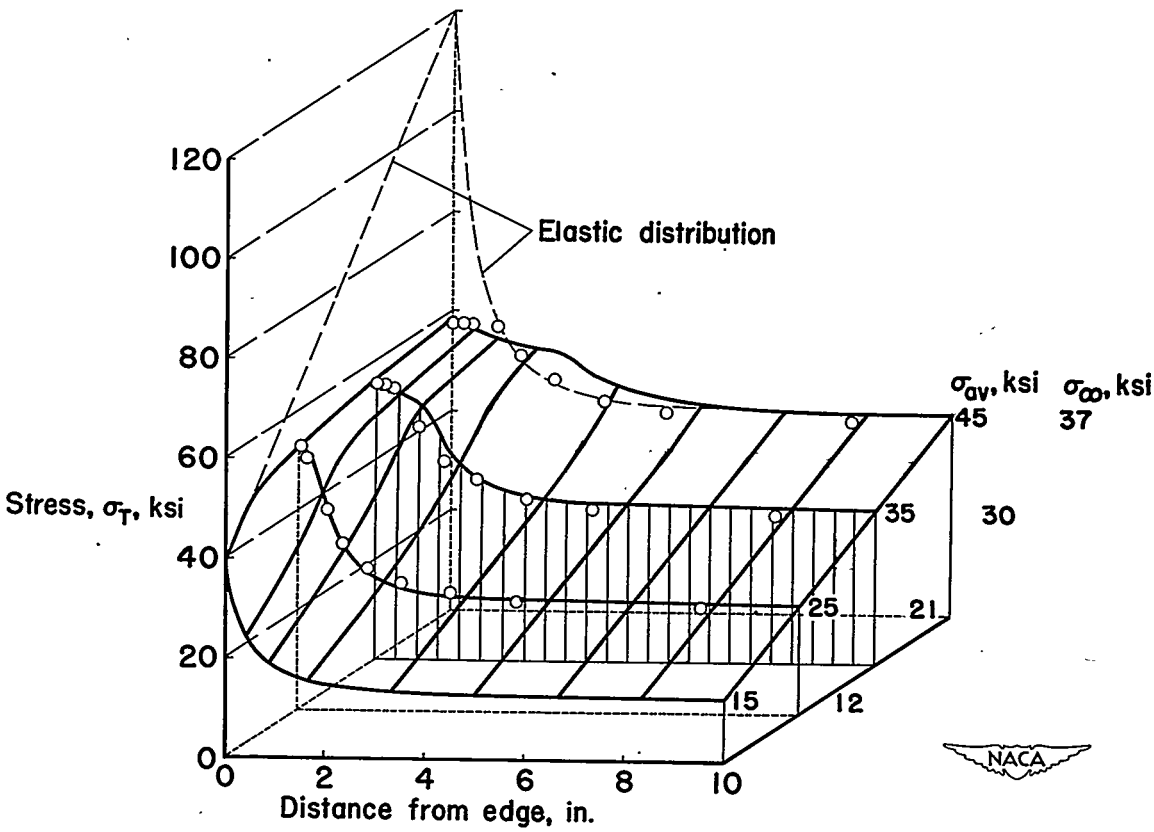

Figure 4.-Comparison between calculated points, for three values of applied stress, and experimental stress distribution from reference 1.

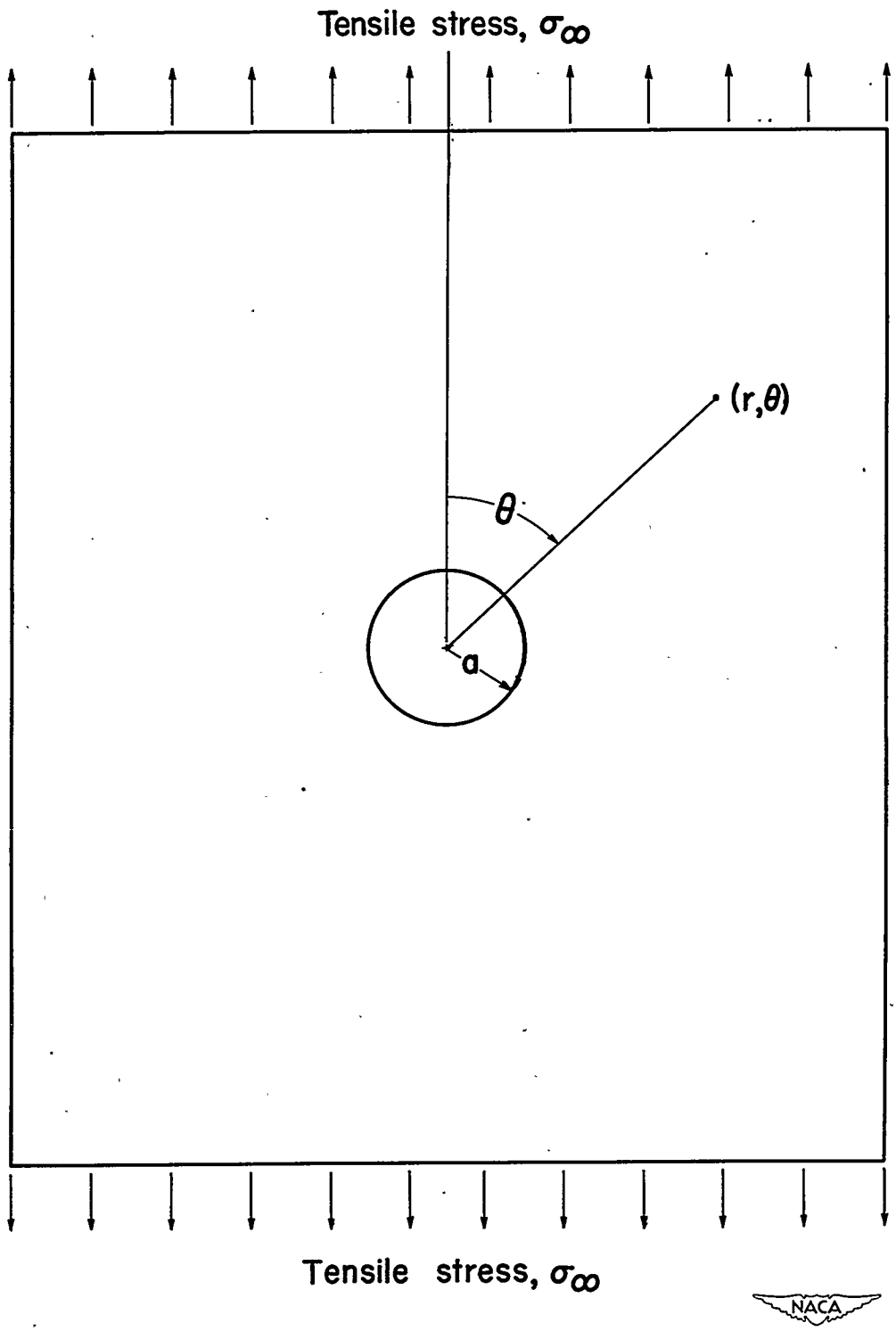

Figure 5.- Coordinate system for sheet with hole.

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