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TECHNICAL NOTE 2236

SUPERSONIC FLOW AROUND CIRCULAR CONES AT ANGLES OF ATTACK

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SUMMARY

The flow around cones without axial symmetry and moving at supersonic velocity is analyzed. Singular points are shown to exist in the flow around the cone if no axial symmetry exists. The results of the analysis are applied to the determination of flow around circular cones at small angles of attack. The concept of a vortical layer around the cone at small angles of attack is introduced, and the correct values of the first-order terms of the velocity components are determined.

The method used is applied to cones at finite angles of attack, and it is shown that good agreement with experimental results can be obtained from the first-order theory if the complete equation for the pressure distribution is used.

INTRODUCTION

The flow around a cone having a circular cross section and moving at supersonic speed has been determined by means of the assumption of small disturbances or by means of more rigorous methods that consider the existence of the shocks. The latter methods can be applied for any Mach number larger than unity and have been developed by several authors, at first by assuming all the flow as potential flow (references 1 and 2) and later by also considering the variation of entropy due to the change in angle of attack (reference 3). By means of the development given in reference 3, values of flow properties around circular cones at an angle of attack have been tabulated in reference 4. The method has been extended in reference 5 to larger angles of attack.

In the method given in references 3, 4, and 5, the flow properties were considered continuous and were developed in Fourier series in terms of the angle of attack; however, the existence of a singular point at the surface of the cone was neglected. The derivatives of the flow properties were obtained by differentiating the Fourier series term by term, and the terms of the series that represent the derivatives were assumed to be of

the same order as the corresponding terms of the integral quantities. For this reason, an erroneous distribution of the entropy at the surface of the cone was obtained.

In this paper, the flow around the cone in the general case is discussed, the existence of singular points in the flow is proved, and a different procedure for determining the flow around cones at small, but finite, angles of attack is developed. This procedure shows the way in which the values tabulated in reference 4 can be used if a simple correction is introduced. The values obtained in this way are compared with experimental results at several values of angle of attack.

SYMBOLS

r,ψ,θ polar coordinates (see fig. 1)

v_r polar velocity component in radial direction (along r), referred to limiting velocity (see fig. 1)

v_n polar velocity component normal to v_r in meridian plane θ = Constant, referred to limiting velocity (fig. 1)

w polar velocity component normal to meridian plane θ = Constant, referred to limiting velocity (fig. 1)

t time

p pressure

ρ density

S entropy

γ ratio of specific heats $\left(c_p/c_v\right)$

$R = c_p - c_v$

c_p specific heat at constant pressure

c_v specific heat at constant volume

a	speed of sound
L	projection of streamline on sphere r = Constant with center at center of the polar coordinate system
V	local velocity
V_1	undisturbed velocity, referred to limiting velocity
V_l	limiting velocity (velocity for expansion in the vacuum)
σ	semiapex angle of conical shock
δ	inclination of axis of conical shock with respect to free-stream direction
η	inclination of axis of conical shock with respect to axis of body
α	angle of attack

Subscripts:

1	stream conditions
a	zero-order terms of Fourier series (part independent from angle of attack)
b	first-order terms of Fourier series (part proportional to angle of attack)
c	higher-order terms of Fourier series or quantities at surface of cone
e	quantities at external surface of vortical layer
s	quantities for polar coordinate system having axis coincident with axis of conical shock

A prime is used to designate the terms of zero order in the power series in $\Delta\theta$ for the quantities in the neighborhood of the meridian plane $\theta = \pi$; two primes are used to designate the factor of the term containing $\Delta\theta^2/2$ in the same power series.

THE FLOW FIELD FOR CONICAL FLOW WITHOUT AXIAL SYMMETRY

In order to analyze the flow field for conical flow without axial symmetry at supersonic speeds, assume a polar coordinate system (r,ψ,θ). Call v_r the velocity component in the radial direction, v_n the velocity component in the direction normal to r in the meridian plane θ = Constant, and w the component normal to the meridian plane (fig. 1); that is,

$$v_r = \frac{dr}{dt}$$

$$v_n = \frac{r\,d\psi}{dt}$$

$$w = \frac{r\,d\theta}{dt}\sin\psi$$

If the flow is conical,

$$\frac{\partial v_r}{\partial r} = 0 \qquad \frac{\partial v_n}{\partial r} = 0 \qquad \frac{\partial w}{\partial r} = 0$$

$$\frac{\partial p}{\partial r} = 0 \qquad \frac{\partial \rho}{\partial r} = 0 \qquad \frac{\partial S}{\partial r} = 0$$

For these conditions Euler's equations become

$$v_n\frac{\partial v_r}{\partial \psi} + \frac{w}{\sin\psi}\frac{\partial v_r}{\partial \theta} - v_n^2 - w^2 = 0 \tag{1a}$$

$$v_n\frac{\partial v_n}{\partial \psi} + \frac{w}{\sin\psi}\frac{\partial v_n}{\partial \theta} + \frac{1}{\rho}\frac{\partial p}{\partial \psi} + v_r v_n - w^2\cot\psi = 0 \tag{1b}$$

$$v_n\frac{\partial w}{\partial \psi} + \frac{w}{\sin\psi}\frac{\partial w}{\partial \theta} + \frac{1}{\rho\sin\psi}\frac{\partial p}{\partial \theta} + v_r w + v_n w\cot\psi = 0 \tag{1c}$$

and the continuity equation becomes

$$2\rho v_r \sin\psi + v_n \sin\psi \frac{\partial\rho}{\partial\psi} + \rho \sin\psi \frac{\partial v_n}{\partial\psi} + v_n\rho \cos\psi + w \frac{\partial\rho}{\partial\theta} + \rho \frac{\partial w}{\partial\theta} = 0 \quad (2)$$

Because the energy in the flow is constant, the following relations must apply:

$$\frac{\gamma}{\gamma - 1}\left(\frac{1}{\rho}\frac{\partial p}{\partial\theta} - \frac{p}{\rho^2}\frac{\partial\rho}{\partial\theta}\right) = -\left(v_r \frac{\partial v_r}{\partial\theta} + v_n \frac{\partial v_n}{\partial\theta} + w \frac{\partial w}{\partial\theta}\right) \quad (3a)$$

$$\frac{\gamma}{\gamma - 1}\left(\frac{1}{\rho}\frac{\partial p}{\partial\psi} - \frac{p}{\rho^2}\frac{\partial\rho}{\partial\psi}\right) = -\left(v_r \frac{\partial v_r}{\partial\psi} + v_n \frac{\partial v_n}{\partial\psi} + w \frac{\partial w}{\partial\psi}\right) \quad (3b)$$

Combining equations (1), (2), and (3) results in

$$v_r\left(2 - \frac{v_n^2 + w^2}{a^2}\right) + v_n \cot\psi + \frac{\partial v_n}{\partial\psi}\left(1 - \frac{v_n^2}{a^2}\right) + \frac{\partial w}{\sin\psi\,\partial\theta}\left(1 - \frac{w^2}{a^2}\right) -$$

$$\frac{w v_n}{a^2}\left(\frac{\partial v_n}{\sin\psi\,\partial\theta} + \frac{\partial w}{\partial\psi}\right) = 0 \quad (4)$$

The entropy at any point of the flow can be expressed as

$$S = \frac{R}{\gamma - 1} \log_e \frac{p}{p_1}\left(\frac{\rho_1}{\rho}\right)^{\gamma} + \text{Constant}$$

where p and ρ are the local quantities and p_1 and ρ_1 the stream quantities. Therefore,

$$\left.\begin{aligned} \frac{\gamma - 1}{R}\frac{\partial S}{\partial\theta} &= \frac{1}{p}\frac{\partial p}{\partial\theta} - \gamma\frac{1}{\rho}\frac{\partial\rho}{\partial\theta} \\ \frac{\gamma - 1}{R}\frac{\partial S}{\partial\psi} &= \frac{1}{p}\frac{\partial p}{\partial\psi} - \gamma\frac{1}{\rho}\frac{\partial\rho}{\partial\psi} \end{aligned}\right\} \quad (5)$$

Combining equations (5) with equations (1), (2), and (3) results in the following expressions:

$$\left.\begin{aligned}\frac{a^2}{\gamma R}\frac{\partial S}{\partial \theta} &= v_n \sin\psi \frac{\partial w}{\partial \psi} - v_r \frac{\partial v_r}{\partial \theta} - v_n \frac{\partial v_n}{\partial \theta} + v_r w \sin\psi + v_n w \cos\psi \\ \frac{a^2}{\gamma R}\frac{\partial S}{\partial \psi} &= -v_r \frac{\partial v_r}{\partial \psi} - w \frac{\partial w}{\partial \psi} + \frac{w}{\sin\psi}\frac{\partial v_n}{\partial \theta} + v_r v_n - w^2 \cot\psi\end{aligned}\right\} \quad (6)$$

Equations (6) combined with equation (1a) give

$$v_n \sin\psi \frac{\partial S}{\partial \psi} = -w \frac{\partial S}{\partial \theta} \quad (7)$$

Equation (7) is general for any conical flow and defines the lines of constant entropy, which correspond to the streamlines. In fact, if L is the streamline projection on the sphere given by r = Constant,

$$\frac{dS}{dL} = \frac{\partial S}{\partial \psi}\frac{d\psi}{dL} + \frac{\partial S}{\partial \theta}\frac{d\theta}{dL} = 0$$

and, from equation (7),

$$\left(\frac{d\theta}{\partial \psi}\right)_L = \frac{w}{v_n \sin\psi} \quad (8)$$

At the surface of any conical body the component of velocity in the direction normal to the surface is zero and the stream moves tangentially to the body; therefore, the entropy at the surface of the body must be constant or must change in a discontinuous way (in which case equations (7) and (8) are not valid).

THE PROPERTIES OF CONICAL FLOW WITHOUT AXIAL SYMMETRY

In order to analyze the properties of conical flow without axial symmetry, consider first a polar coordinate system having its axis coincident with the axis of a circular cone at a small angle of attack (fig. 2) and assume that the direction of the undisturbed velocity V_1 is

in the plane $\theta = 0$, $\theta = \pi$. In this case, the plane $\theta = 0$, $\theta = \pi$ is a plane of symmetry of the flow and, in this plane,

$$w = 0$$

$$v_n \neq 0$$

and

$$\frac{\partial v_r}{\partial \theta} = 0 \qquad \frac{\partial v_n}{\partial \theta} = 0 \qquad \frac{\partial S}{\partial \theta} = 0$$

Therefore, equation (7) shows that in the plane $\theta = 0$, $\theta = \pi$ the entropy is constant. At the surface of the cone ($\psi = \psi_c$), the normal component v_n is zero and $w \neq 0$; therefore, equation (7) shows that the entropy remains constant also along the surface of the cone ($\psi = \psi_c$). Only at points A and B, (defined by $\theta = 0$, $\theta = \pi$, and $\psi = \psi_c$) $v_n \rightarrow 0$ and $w \rightarrow 0$; therefore, equation (7) is indeterminate. Because the body is at an angle of attack, the axis of conical shock does not remain coincident with the direction of the velocity V_1, and the entropy in the plane $\theta = 0$ must be different from the entropy in the plane $\theta = \pi$. The entropy at the cone surface therefore must be different from the entropy at the plane $\theta = 0$, from the entropy at the plane $\theta = \pi$, or from both. In this case, where $w = 0$ and $v_n = 0$, a discontinuity of entropy must exist either at A or B or both points.

In order to find a relation between the value of the entropy at the surface of the cone and the values of the entropy in the meridian plane $\theta = 0$, $\theta = \pi$, the following considerations can be used: In the meridian plane ($\theta = 0$, $\theta = \pi$), w, $\frac{\partial S}{\partial \theta}$, and $\frac{\partial v_n}{\partial \theta}$ are zero because the plane is a plane of symmetry of the flow field and, from equation (7), $\frac{\partial S}{\partial \psi} = 0$. Therefore, in the plane of symmetry in the zone outside of the singular points A and B,

$$\frac{\partial}{\partial \psi}\left(\frac{\partial^2 S}{\partial \theta^2}\right) = -\frac{2\,\frac{\partial w}{\partial \theta}}{v_n \sin \psi}\,\frac{\partial^2 S}{\partial \theta^2} \tag{9}$$

For the case considered, the velocity component v_n is negative at the shock or at the Mach cone and remains negative throughout the field, until it becomes zero at the surface of the cone; therefore, the value of $\frac{\partial^2 S}{\partial \theta^2}$ tends to increase in absolute value as ψ decreases from the

value corresponding to the value at the surface of the shock to the value at the surface of the cone when $\frac{\partial w}{\partial \theta}$ is negative and tends to decrease when $\frac{\partial w}{\partial \theta}$ is positive.

Because the entropy remains constant along each streamline, the decrease of the absolute value of $\frac{\partial^2 s}{\partial \theta^2}$ as ψ decreases corresponds to a departure of projection of the streamlines on the sphere $r = \text{Constant}$ from the plane of symmetry, but, if $\frac{\partial}{\partial \psi}\left(\frac{\partial^2 s}{\partial \theta^2}\right)$ is of sign opposite to $\frac{\partial^2 s}{\partial \theta^2}$, the projection of the streamlines on the sphere $r = \text{Constant}$ tends to converge toward the plane of symmetry as ψ decreases from the value at the shock to the value at the surface of the cone.

Now, with the convention used in figure 2, the component w is negative throughout the field and is zero at $\theta = 0$ and $\theta = \pi$. Therefore, $\frac{\partial w}{\partial \theta}$ is negative in the zone $\theta = 0$ but is positive in the zone $\theta = \pi$, and the streamline projection tends to converge toward the zone of point A and diverge from the zone of point B. Because of the departure of the streamlines from the plane $\theta = \pi$, the entropy in zone B remains constant, and, therefore, the entropy at the surface of the cone is equal to the entropy at the meridian plane $\theta = \pi$; at point A a discontinuity of entropy exists from the value corresponding to the plane $\theta = 0$ to the larger value existing in the plane $\theta = \pi$. All the projections of the streamlines converge at point A where the entropy is not single-valued. Because v_n approaches zero near the cone, equation (8) shows that all the streamline projections tend to become parallel to the line $\psi = \text{Constant}$ in the zone near the cone and converge at A. The value of v_n in the meridian plane $\theta = 0$ can change sign and can be positive in the neighborhood of the point A (case of large angles of attack). In this case the right-hand side of equation (9) changes sign and the singular point moves away from A in the meridian plane $\theta = 0$ and occurs at the other point of the meridian plane where v_n is also zero. (At the shock or at the Mach wave v_n is negative; therefore, another singular point where $v_n = 0$ must exist.)

It is interesting to observe that singular points must exist in any supersonic conical flow without axial symmetry. Considerations similar to those used for the case of circular cones at an angle of attack can be extended to other cases, and it can be shown that the streamlines that are tangent to the body start from points of the shock and meet the body

at points where the component of velocity perpendicular to the radius and tangent to the body vanishes and has a positive derivative in the streamline direction $\left(\text{equivalent to the condition of positive } \frac{\partial w}{\partial \theta}\right)$. Convengency of streamlines occurs and, therefore, the points are singular at the points where this component vanishes and has a negative derivative in the direction tangent to the body, while the component normal to the body also vanishes and has a negative derivative in the direction normal to the body $\left(\text{equivalent to } v_n = 0 \text{ and } \frac{\partial v_n}{\partial \psi} \text{ negative}\right)$.

For example, the conical body of figure 3 has two planes of symmetry, AA' and BB' when w and v_n are zero, but at the points BB', $\frac{\partial w}{\partial \theta}$ is positive, while at AA', it is negative. Therefore, because $\frac{\partial v_n}{\partial \psi}$ is negative at AA' and BB', the points AA' are singular points and the entropy at the surface of the body is determined by the shock strength at the points CC'.

DETERMINATION OF THE FLOW AROUND CIRCULAR CONES AT SMALL ANGLES OF ATTACK

In order to determine the flow around circular cones at small angles of attack, consider a polar coordinate system, the axis of which is coincident with the body axis. At the surface of the body the velocity component v_n is zero and in the neighborhood of the body is very small; therefore, the terms v_n^2/a^2 can be neglected with respect to unity.

If the angle of attack is small, the component w is also small and the terms w^2/a^2 can also be neglected. On the basis of this approximation, in the neighborhood of the body, equation (4) can be expressed as

$$2v_r + v_n \cot \psi + \frac{\partial v_n}{\partial \psi} + \frac{\partial w}{\sin \psi \, \partial \theta} = 0 \tag{10}$$

This equation permits a particular solution of this type chosen from physical considerations

$$\left.\begin{aligned} v_r &= v_{r_a} + v_{r_b} \cos\theta + \sum v_{r_c} \cos m\theta \\ v_n &= v_{n_a} + v_{n_b} \cos\theta + \sum v_{n_c} \cos m\theta \\ w &= w_b \sin\theta + \sum w_c \sin m\theta \end{aligned}\right\} \quad (11)$$

where $v_{r_{a,b,c}}$, $v_{n_{a,b,c}}$, and $w_{b,c}$ are functions only of ψ, are constant for constant values of ψ, and must be chosen in a form that satisfies the boundary conditions.

Consider now a conical shock having circular cross section and semi-apex angle σ (fig. 4). Consider the cone inclined at an angle δ with respect to the undisturbed velocity with a polar coordinate system, the axis of which is coincident with the axis of the conical shock. If v_{r_s}, v_{n_s}, and w_s are the velocity components referred to the limiting velocity V_l in the new coordinate system (r_s, ψ_s, θ_s), from the shock-wave relations the following equations result:

$$v_{r_s} = V_1 \cos\delta \cos\sigma + V_1 \sin\delta \sin\sigma \cos\theta_s$$

$$w_s = -V_1 \sin\delta \sin\theta_s$$

$$-v_{n_s} = \frac{\gamma - 1}{\gamma + 1} \frac{1 - v_{r_s}^2 - w_s^2}{V_1 \cos\delta \sin\sigma - V_1 \sin\delta \cos\sigma \cos\theta_s}$$

If δ is assumed to be small and terms of the order of δ^2 are neglected, these equations become

$$\left.\begin{aligned} v_{r_s} &= V_1 \cos\sigma + \delta V_1 \sin\sigma \cos\theta_s \\ w_s &= -V_1 \delta \sin\theta_s \\ v_{n_s} &= -\frac{\gamma - 1}{\gamma + 1}\left(\frac{1 - V_1^2 \cos^2\sigma}{V_1 \sin\sigma}\right) - \frac{\gamma - 1}{\gamma + 1} \delta \cos\sigma \left(-2V_1 + \frac{1 - V_1^2 \cos^2\sigma}{V_1 \sin^2\sigma}\right) \cos\theta_s \end{aligned}\right\} \quad (12)$$

The flow behind a circular conical shock inclined at a small angle δ with respect to the undisturbed stream can therefore be expressed in the form

$$\left.\begin{aligned} v_{r_s} &= v_{r_{a_s}} + \delta v_{r_{b_s}} \cos \theta_s \\ v_{n_{b_s}} &= v_{n_{a_s}} + \delta v_{n_{b_s}} \cos \theta_s \\ w &= \delta w_{b_s} \sin \theta_s \end{aligned}\right\} \tag{13}$$

where all the terms containing v_{r_b}, v_{n_b}, and w_b are small, that is, of the order of δ.

If the axis of the conical coordinates is rotated at an angle η of the same order as the angle δ and terms of the order of δ^2 are neglected, the velocity components v_r, v_n, and w referred to the new axis (fig. 4) become

$$\left.\begin{aligned} v_r &= v_{r_{a_s}} + \frac{\partial v_{r_{a_s}}}{\partial \psi_s} \eta + \delta v_{r_{b_s}} \cos \theta \\ v_n &= v_{n_{a_s}} + \frac{\partial v_{n_{a_s}}}{\partial \psi_s} \eta + \delta v_{n_{b_s}} \cos \theta \\ w &= \delta w_{b_s} \sin \theta \end{aligned}\right\} \tag{14}$$

where θ, the coordinate referred to the new axis, is given by

$$\cos \theta = \cos \theta_s - \eta \cot \psi_s \sin^2\theta_s$$

Equations (14) show that the flow behind a circular conical shock inclined at a small angle δ with respect to the direction of the undisturbed stream and at an angle η with respect to a conical body can also be expressed in the form given in equations (11), where, if the angles η and δ are small and the terms of the order of η^2 and δ^2 are neglected, only the terms having the subscripts a and b must be considered. Therefore, equations (11), when the terms with the subscript c are neglected, are valid for small angles of attack ($\alpha = \delta - \eta$), and a

conical circular shock is consistent with the solution chosen for the flow around the body. At the surface of the cone the assumption that w^2 is small corresponds to the assumption that only the first-order terms of angle of attack are considered. The conical shock is inclined at an angle η with respect to the circular cone.

This analysis is similar to the analysis of references 2 and 3. No assumption, however, has been introduced for the entropy distribution; only the velocity components have been considered to be in the form of equations (11), and no limiting assumption has been introduced for the derivatives. In reference 3, in addition to the equations

$$v_r = v_{r_a} + \alpha v_{r_b} \cos \theta \tag{15a}$$

$$v_n = v_{n_b} + \alpha v_{n_b} \cos \theta \tag{15b}$$

$$w = \alpha w_b \sin \theta \tag{15c}$$

the expressions

$$p = p_a + \alpha p_b \cos \theta \tag{16a}$$

$$\rho = \rho_a + \alpha \rho_b \cos \theta \tag{16b}$$

$$S = S_a + \alpha S_b \cos \theta \tag{16c}$$

have been used, and the derivatives of entropy, pressure, and density have been obtained from differentiation of equations (16). In this way a solution has been found which gives values of entropy variables along the cone surface and constant in each meridian plane, while the entropy actually remains constant along the cone surface and changes in the meridian plane. An incorrect entropy distribution has therefore been obtained at the surface of the body.

In order to analyze in more detail the significance of equations (16) and their inconsistency with the approximation considered in references 2 and 3, consider equation (7). In the plane of symmetry $\theta = 0$ or $\theta = \pi$, $\frac{\partial S}{\partial \psi} = 0$; therefore, $S_a + \alpha S_b$ or $S_a - \alpha S_b$ of equation (16c) must remain constant.

Consider now the plane $\theta = \pi$ and express the entropy S in the form

$$S = S' - S'' \frac{\Delta\theta^2}{2} \tag{17}$$

which satisfies the condition of symmetry. Because of equation (7),

$$\frac{dS'}{d\psi} = 0 \tag{18}$$

and, from equations (16) and (17),

$$S' = S_a - \alpha S_b \tag{19}$$

and

$$\alpha S_b = -S'' \tag{20}$$

From equations (7) and (17),

$$v_n \sin\psi\left(\frac{\partial S'}{\partial\psi} - \frac{\partial S''}{\partial\psi}\frac{\Delta\theta^2}{2}\right) = wS''\Delta\theta$$

However, $\frac{\partial S'}{\partial\psi} = 0$; therefore,

$$\frac{\Delta\theta}{2}\frac{\partial S''}{\partial\psi} = -\frac{wS''}{v_n \sin\psi}$$

or, since from equation (15c) $w = -\alpha w_b \Delta\theta$,

$$\frac{\partial S''}{\partial\psi} = \frac{2\alpha w_b S''}{v_n \sin\psi}$$

By use of equation (20), in the neighborhood of the plane $\theta = \pi$,

$$\alpha\frac{\partial S_b}{\partial\psi} = \frac{2w_b S_b \alpha^2}{v_n \sin\psi} = \frac{2wS''}{v_n \sin\psi}$$

In reference 3, the term $\frac{wS''}{v_n \sin \psi}$ has been considered everywhere to be of the order of α^2 and has been neglected; hence, $\frac{\partial S_b}{\partial \psi} = 0$ and $\frac{\partial S_a}{\partial \psi} = 0$. However, neglecting this term is correct only when the ratio w_b/v_n is of an order different from $1/\alpha$ and, therefore, when $|v_n| \gg 0$. Near the surface of the cone, v_n approaches zero, and, therefore, the term $\frac{\partial S''}{\partial \psi}$ becomes large and cannot be neglected. The extent of the field around the cone where the term $\frac{\partial S''}{\partial \psi}$ is of the order of α can be easily determined.

Consider a polar coordinate system having its axis coincident with the axis of the cone. At the surface of the body v_n is zero; therefore, in the neighborhood of the body the velocity component v_n can be expressed in the form

$$v_n = \left(\frac{\partial v_n}{\partial \psi}\right)_{\psi_c} \Delta\psi$$

or, by use of equation (10), in the form

$$v_n = -\left(2v_r + \frac{\partial w}{\sin \psi \, \partial \theta}\right)(\Delta\psi)_V \tag{21}$$

Therefore, v_n is of the order of α when $(\Delta\psi)_V$ is of the order of α. In this conical layer of thickness $(\Delta\psi)_V$ of the order of α, $\frac{\partial S''}{\partial \psi}$ is also of the order of α or larger. In this layer, which can be called the vortical layer, the term $\frac{\partial S_a}{\partial \psi}$ also is of the order of α because from equations (18) and (19) $\frac{\partial S_a}{\partial \psi}$ can be shown to be of the same order as $\frac{\partial S''}{\partial \psi}$.

In order to investigate the effect of this vortical layer on the velocity and pressure distribution at the surface of the cone, consider

equations (2), (4), (6), and (7). If the velocity components in the neighborhood of the meridian plane ($\theta = \pi$) are expressed as

$$v_r = v_r' - \frac{\Delta\theta^2}{2} v_r'' \tag{22a}$$

$$v_n = v_n' - \frac{\Delta\theta^2}{2} v_n'' \tag{22b}$$

$$w = -w''\left(\Delta\theta - \frac{\Delta\theta^3}{3!}\right) \tag{22c}$$

the following expressions can be obtained:

$$\frac{\partial S''}{\partial\psi} = -\frac{2w''S''}{v_n' \sin\psi} \tag{23}$$

$$\frac{\partial v_r''}{\partial\psi} = v_n'' - \frac{2w''}{v_n'}\left(\frac{v_r''}{\sin\psi} + w''\right) \tag{24}$$

$$\frac{\partial w''}{\partial\psi} = \frac{1}{v_n' \sin\psi}\left(v_r'v_r'' + v_n'v_n'' + w''v_r' \sin\psi + v_n'w'' \cos\psi + \frac{S''a'^2}{\gamma R}\right) \tag{25}$$

$$\frac{\partial v_n''}{\partial\psi} = \frac{1}{1 - \frac{v_n'^2}{a'^2}}\left[2B\left(\frac{v_n'v_n''}{a'^2} - A\frac{v_n'^2}{a'^2}\right) - v_n'' \cot\psi - v_r''\left(2 - \frac{v_n'^2}{a'^2}\right) - \right.$$

$$\left. \frac{w''}{\sin\psi}\left(1 + \frac{2w''^2}{a'^2} - \frac{2v_n'v_n''}{a'^2} + \frac{2v_n' \sin\psi}{a'^2}\frac{\partial w''}{\partial\psi}\right) - \frac{2v_r'w''^2}{a'^2}\right] \tag{26}$$

where

$$A = \frac{-v_r'v_r'' - v_n'v_n'' + w''^2}{a'^2}\frac{\gamma - 1}{2} \tag{27}$$

$$B = -\frac{v_r' + v_n' \cot \psi + \frac{w''}{\sin \psi}}{1 - \frac{v_n'^2}{a'^2}} \tag{28}$$

and

$$\frac{\partial v_{r_a}}{\partial \psi} = v_{n_a} - \frac{a'^2}{v_{r_a}} \frac{1}{\gamma R} \frac{\partial S_a}{\partial \psi} \tag{29}$$

Therefore, all the derivatives of the velocity components of zero and first order are affected by the entropy variation in the meridian plane. If the terms $\frac{\partial S_a}{\partial \psi}$ and $\frac{\partial S''}{\partial \psi}$ are of the order of α, the correction required for the entropy variation is of the order of α^2 because the thickness of the vortical layer is of the order of α and, therefore, must be considered only if higher-order terms are considered, as in reference 5. A correction is also required in this approximation when $\frac{\partial S_a}{\partial \psi}$ and $\frac{\partial S''}{\partial \psi}$ become of zero order.

The only place where the terms $\frac{\partial S_a}{\partial \psi}$ and $\frac{\partial S''}{\partial \psi}$ are of zero order is is near the surface of the cone in a layer of thickness α^2 where v_n tends to the order of α^2 (equation (9)). Because this layer is of the order of α^2, it can be considered in this approximation, which neglects terms of the order of α^2 or higher, that across this layer the pressure distribution remains constant, but an abrupt variation of entropy occurs; therefore, in this approximation the phenomenon can be represented as in references 3 and 4, where the entropy remains constant in every meridian plane until a vortical layer of infinitesimal thickness is reached at the surface of the cone across which a variation of entropy occurs from the value $S_a + \alpha S_b \cos \theta$ to the value $S_a - \alpha S_b$ that exists at the surface of the cone. Across the layer a variation of density and velocity components occurs and can be easily determined.

Call v_{r_e}, w_e, S_e, p_e, ρ_e, and a_e the quantities at the external surface of the layer (these are the quantities tabulated in reference 4), and v_{r_c}, w_c, S_c, p_c, and ρ_c the quantities at the surface of the cone. Because it has been shown that

$$p_e = p_c \tag{30}$$

then

$$v_{r_e}^2 + w_e^2 - v_{r_c}^2 - w_c^2 = -\frac{2}{\gamma - 1} a_e^2 \frac{S_e - S_c}{c_p} \tag{31}$$

where $S_e - S_c$ is the entropy jump across the layer.

Now,

$$v_r = v_{r_a} + \alpha v_{r_b} \cos \theta$$

$$w = \alpha w_b \sin \theta$$

and in the plane $\theta = \pi$

$$v_{r_e} = v_{r_c}$$

$$w_e = w_c = 0$$

$$S_e = S_c$$

or

$$\left.\begin{aligned} \left(v_{r_a} - \alpha v_{r_b}\right)_e &= \left(v_{r_a} - \alpha v_{r_b}\right)_c = v_r' \\ \left(S_a - \alpha S_b\right)_e &= \left(S_a - \alpha S_b\right)_c = S' \end{aligned}\right\} \tag{32}$$

Therefore, if terms of the order of α^2 are neglected, equation (31) becomes

$$v_r'\left(v_{r_b}\right)_e - v_r'\left(v_{r_b}\right)_c = -\frac{1}{\gamma - 1} a^2 \frac{S_b}{c_p} \tag{33}$$

or

$$\left(v_{r_b}\right)_c - \left(v_{r_b}\right)_e = \frac{a^2}{(\gamma - 1)v_r'} \frac{S_b}{c_p} \tag{34}$$

However, from equation (1a), at the surface of the cone,

$$w = \frac{\partial v_r}{\sin \psi \, \partial \theta}$$

or

$$v_{r_b} = -\frac{w_b}{\sin \psi_c} \tag{35}$$

therefore, the values of v_{r_b} and w_b can be determined from equation (34) and from the tables of reference 4 where

$$S_b = c_v\left(\frac{p_b}{p_a} - \gamma \frac{\rho_b}{\rho_a}\right) \tag{36}$$

and

$$a_e{}^2 = \frac{\gamma - 1}{2}\left(1 - v_r{}'^2\right) \tag{37}$$

The values v_{r_a}, v_{r_b}, and w_b having been determined, the values of v_r and w can be obtained at any part of the cone from equations (15).

From the velocity components, the correct pressure distribution can be determined since the entropy remains constant and is equal to $S_a - \alpha S_b$ at the surface of the cone.

THE NUMERICAL DETERMINATION OF THE FLOW FIELD AROUND CONES AT SMALL ANGLES OF ATTACK

The method presented permits the determination of the pressure distribution around the cones with the assumption that the terms of the order of α^2 can be neglected. The pressure at any point can be obtained from the equation

$$\frac{p}{p_1} = \left(\frac{1 - V^2}{1 - V_1{}^2}\right)^{\frac{\gamma}{\gamma-1}} e^{\frac{1}{\gamma-1}\frac{\Delta S}{c_v}} \tag{38}$$

where V is the local velocity corresponding to the pressure p and ΔS is the increase of entropy with respect to the stream conditions where p_1 and V_1 exist. From equations (15), V^2 is found to be

$$V^2 = {v_{r_a}}^2 + \alpha^2 {v_{r_b}}^2 \cos^2\theta + 2\alpha v_{r_a} v_{r_b} \cos\theta + \alpha^2 {w_b}^2 \sin\theta \tag{39}$$

However, in equation (39) the terms of the order of α^2 having the form $2v_{r_a}v_{r_c} \cos m\theta$ have been neglected (see equation (11)). If all the terms of the order of α^2 are neglected, V^2 becomes

$$V^2 = {v_{r_a}}^2 + 2\alpha v_{r_a} v_{r_b} \cos\theta \tag{40}$$

Equations (39) and (40) are different in terms of the order of α^2; however, for finite angles of attack good agreement is obtained only if equation (39) in which some of the α^2 terms are retained is used in equation (38).

The reason for the better approximation given by equation (39) can be understood if the magnitude of the terms containing α^2 in the expression of V^2 is considered. Along the surface of the cone, αw_b is given by

$$\alpha w_b = -\frac{\alpha v_{r_b}}{\sin\psi_c} \tag{41}$$

For a finite value of α and a small value of ψ_c, $\alpha^2 {w_b}^2$ is of the same order as αv_{r_b} because $\sin^2\psi_c$ is also small; therefore, $\alpha^2 {w_b}^2$ can have an effect on the velocity and pressure distribution of the same order as the term $\alpha v_{r_a} v_{r_b}$ which is the only term retained in equation (40). Because the term $\alpha^2 w^2$ is correct and significant, it can still also be retained if other terms in α^2 are neglected, and, therefore, equation (39) is the expression that must be used for finite angle of attack. $\left(\text{For example, for a } 10^\circ \text{ cone } (\psi_c = 10^\circ)\ \sin^2\psi_c = \frac{1}{33}\right.$ and $\frac{\alpha}{\sin^2\psi} = 1$ for $\alpha = 1.75^\circ$, which can be considered a small angle.$\left.\right)$

In reference 3, equation (16a) has been used in the derivation of the method; however, the use of this equation is not necessary, as is briefly shown in the following paragraph.

Consider a conical circular shock having the semiapex angle σ and inclined at an angle δ with respect to the undisturbed stream. In the neighborhood of the plane $\theta = \pi$ the velocity components can be expressed as in equations (22) and at the shock are given by (from equations (12))

$$v_r' = V_1 \cos \sigma - V_1\delta \sin \sigma \tag{42}$$

$$v_r'' = -\delta V_1 \sin \sigma \tag{43}$$

$$w'' = -V_1\delta \tag{44}$$

$$v_n' = -\frac{\gamma - 1}{\gamma + 1}\frac{1 - V_1^2\cos^2\sigma}{V_1 \sin \sigma} + \frac{\gamma - 1}{\gamma + 1}\delta \cos \sigma\left(-2V_1 + \frac{1 - V_1^2\cos^2\sigma}{V_1 \sin^2\sigma}\right) \tag{45}$$

$$v_n'' = \frac{\gamma - 1}{\gamma + 1}\delta \cos \sigma\left(-2V_1 + \frac{1 - V_1^2\cos^2\sigma}{V_1 \sin^2\sigma}\right) \tag{46}$$

and S' and S'' are given by

$$S' = c_v \log_e \left\{1 + \left(\frac{\gamma - 1}{2} M_1^2 + 1\right)\left[V_1^2\sin^2(\sigma + \delta) - v_n'^2\right]\right\} + (c_p - c_v)\log_e \frac{-v_n'}{V_1 \sin(\sigma + \delta)} \tag{47}$$

$$S'' = 2\delta\frac{V_1^2\cos \sigma \sin \sigma - v_n' \frac{v_n''}{\delta}}{\frac{1}{1 + \frac{\gamma - 1}{2} M_1^2}\left(V_1^2\sin^2\sigma - v_n'^2\right)} + (c_p - c_v)\delta\left(\frac{v_n''/\delta}{v_n'} - \frac{\cos \sigma}{\sin \sigma}\right) \tag{48}$$

All the velocity components are referred to the limiting velocity, and V_1 is the undisturbed velocity also referred to the limiting velocity.

In the meridian plane $\theta = \pi$ the entropy S' is constant; therefore when $w = 0$, from equation (1a),

$$\frac{\partial v_r'}{\partial \psi} = v_n' \tag{49}$$

If equation (4) is applied to the meridian plane $\theta = \pi$ and equation (21) is used, the radius of the hodograph diagram at any value of ψ smaller than σ is given by (reference 6)

$$(R)_\psi = -\left(\frac{v_r' + v_n' \cot \psi + \frac{w''}{\sin \psi}}{1 - \frac{v_n'^2}{a'^2}}\right) \tag{50}$$

where

$$\frac{2}{\gamma - 1} a'^2 = 1 - v_r'^2 - v_n'^2 \tag{51}$$

Therefore, at the point $\psi - \Delta\psi$ in the meridian plane $\theta = \pi$,

$$(v_r')_{\psi-\Delta\psi} = -(v_n')_\psi \sin \Delta\psi + (v_r' - R)_\psi \cos \Delta\psi + (R)_\psi \tag{52}$$

and

$$(v_n')_{\psi-\Delta\psi} = (v_n')_\psi \cos \Delta\psi + (v_r' - R)_\psi \sin \Delta\psi \tag{53}$$

From equations (50) to (53), the velocity components v_r' and v_n' can be determined if the component w'' is known in the meridian plane $\theta = \pi$. Since $\frac{\partial w''}{\partial \psi}$ is given by equation (25), the value of w'' can be determined for any value of ψ. But v_{r_a} and v_{n_a} are the quantities obtained for zero angle of attack at the same coordinate ψ of a coordinate system in axis with the conical shock, and

$$v_r'' = v_r' - v_{r_a}$$

$$v_n'' = v_n' - v_{n_a}$$

therefore, the entire flow field can be determined until v_n' becomes 0. The value of ψ for which $v_n' = 0$ corresponds to $\psi_c + \alpha - \delta$ and, therefore, gives the value of α.

Considerations similar to those used for cones can be used for the characteristics method presented in reference 6. In this case the pressure at the surface of the body can be obtained from the complete equations, and the vortical layer must be considered in order to obtain

the correct distribution of entropy. The application is the same because the entropy does not change along each streamline.

COMPARISON WITH EXPERIMENTAL RESULTS

In order to have an indication of the accuracy that can be expected from the first-order theory, theoretical results have been compared with some experimental results available. The theoretical results have been obtained by using the values of reference 4 for the conditions outside the vortical layer and the pressure at the surface of the cone has been determined in the following way:

From tables of reference 4 the value of δ/α has been determined (δ/ϵ of tables). The position ψ in the plane $\theta = \pi$ of the conical body in the shock coordinate system is

$$\psi = \psi_c + \alpha - \delta \tag{54}$$

The value of v_{r_a} at $\psi_c + \alpha - \delta$ has been determined from the tables of reference 4 $\left(v_r \text{ at } \psi \text{ is given by } u_S - 2u_S \frac{(\alpha - \delta)^2}{2}\right)$. The value of v_r'' has been obtained from the tables in shock coordinates

$$\left[\left(v_{r_b}\right)_e\right]_\psi = -(x)_\psi + \left(\frac{\partial v_{r_a}}{\partial \psi}\right)\frac{\alpha - \delta}{\alpha} = -\left(x - v_n \frac{\alpha - \delta}{\alpha}\right) \tag{55}$$

Because $\frac{\partial v_r}{\partial \psi} = 0$ at $\psi = \psi_c$, the value of $\left[\left(v_{r_b}\right)_e\right]_\psi$ can be taken equal to the value of v_{r_b} at ψ and, therefore, to $-x$ at ψ_S of the tables. Then $\left(v_{r_a}\right)_c$, $\left(v_{r_b}\right)_c$, and $\left(w_b\right)_c$ in the body coordinate system have been obtained by means of equations (32), (34), and (35).

From equation (36) S_b has been obtained (it has a negative value), and from tables of reference 7 S_a can be determined - for example, from the value of the angle of the shock obtained from reference 7 - and

$$\Delta S = S_a - \alpha S_b \tag{56}$$

The pressure has been obtained from equation (38).

In figure 5 a comparison is presented for a cone of $\psi_c = 7.5^\circ$ at $M = 1.6$ and four angles of attack. The experimental data are obtained from tests performed in the Langley 4- by 4-foot supersonic tunnel. For comparison the values given by references 4 and 5 and by linear theory are also shown. In figure 6 a comparison is presented for a cone of $\psi_c = 10^\circ$ at $M = 6.86$ and two angles of attack. The experimental data have been obtained from tests performed at the Langley 11-inch hypersonic tunnel. The agreement in both cases is good, even at angles of attack where it would be expected that higher-order terms would be important.

CONCLUSIONS

The flow around cones without axial symmetry at supersonic velocity has been analyzed. Singular points which complicate the analysis of the flow field were shown to exist in the flow. The results of the analysis were applied to the determination of the flow around circular cones at an angle of attack. The concept of a vortical layer around the cone at small angles of attack has been introduced, and the correct values of the first-order terms of the velocity components were determined.

The method determined was applied to cones at finite angle of attack, and it is shown that good agreement with experimental results can be obtained from the first-order theory if the complete equation for the pressure distribution is used. The analysis can be extended to the application of the characteristics method around bodies of revolution at small angles of attack.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., September 26, 1950

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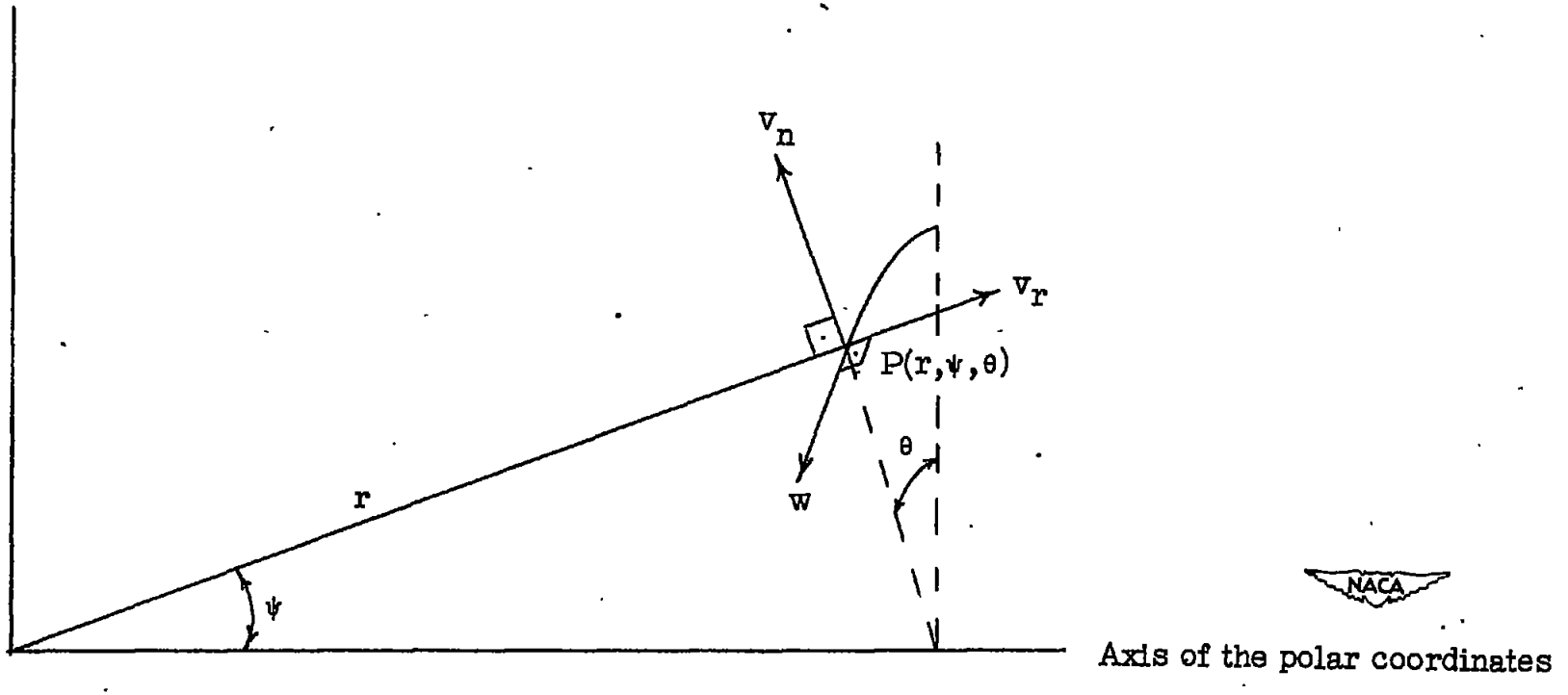

Figure 1.- The polar coordinate system.

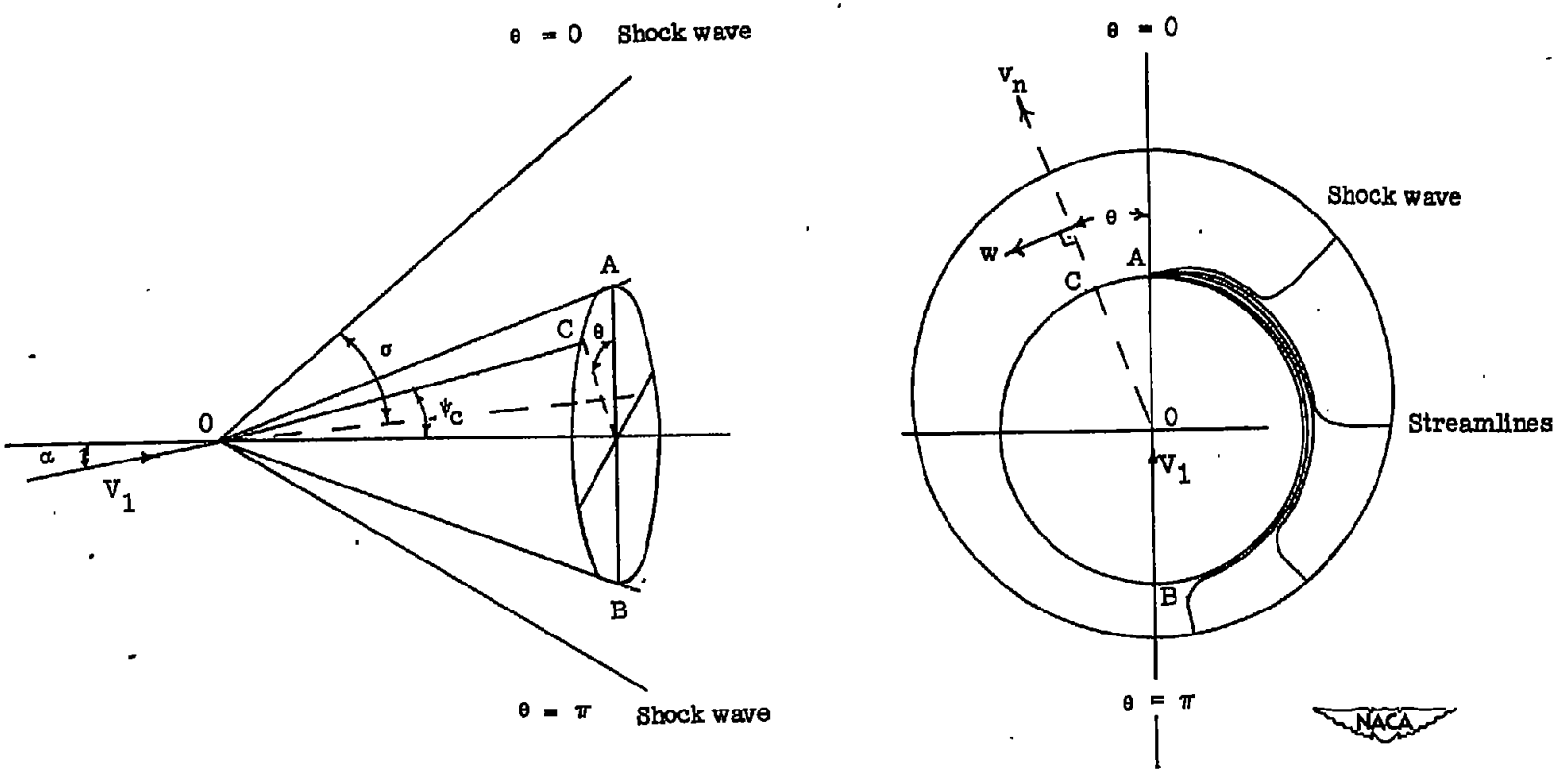

Figure 2.- The singular points at the surface of the cone.

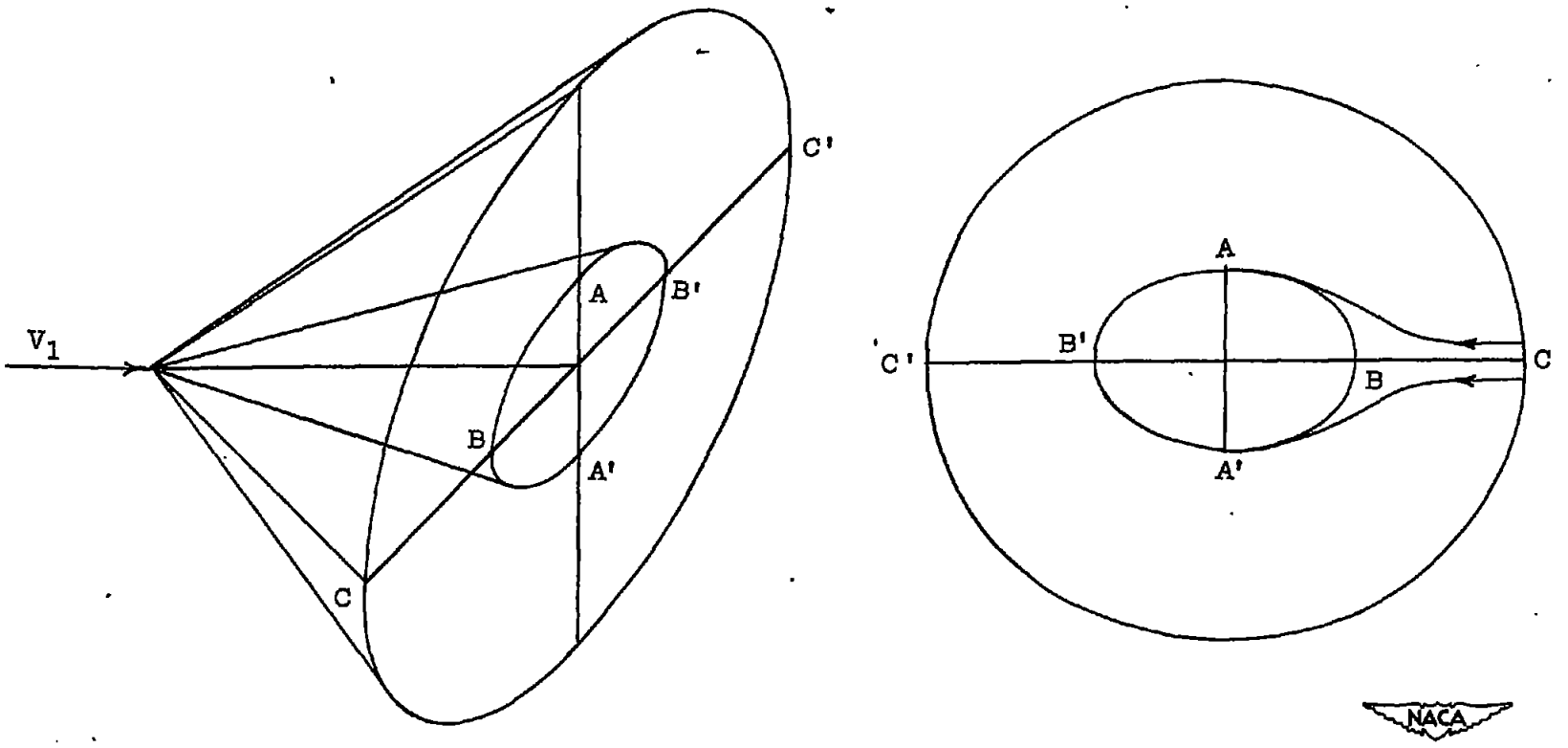

Figure 3.- The singular points at the surface of a cone in supersonic flow without axial symmetry.

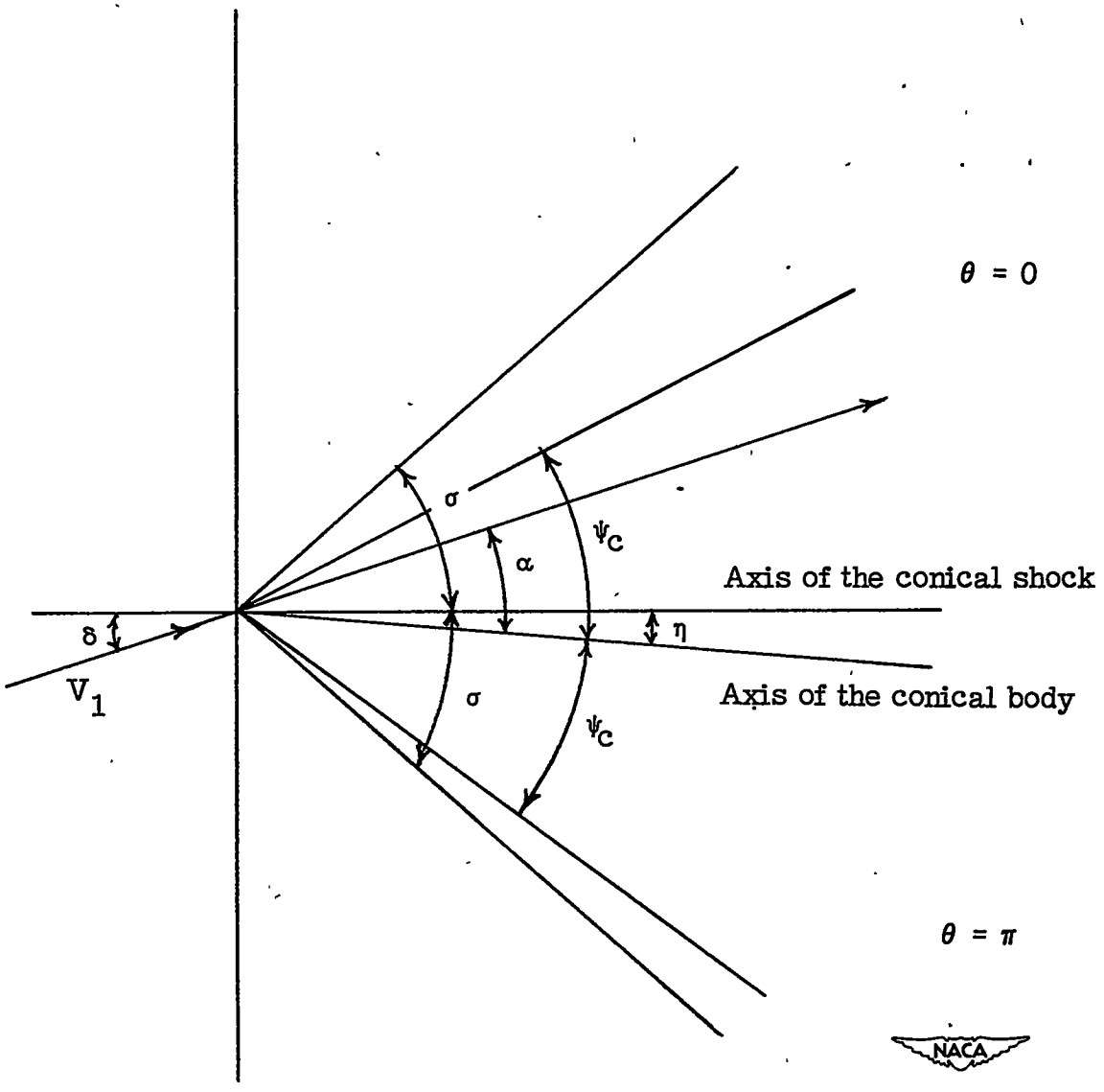

Figure 4.- The position of the conical shock with respect to the conical body.

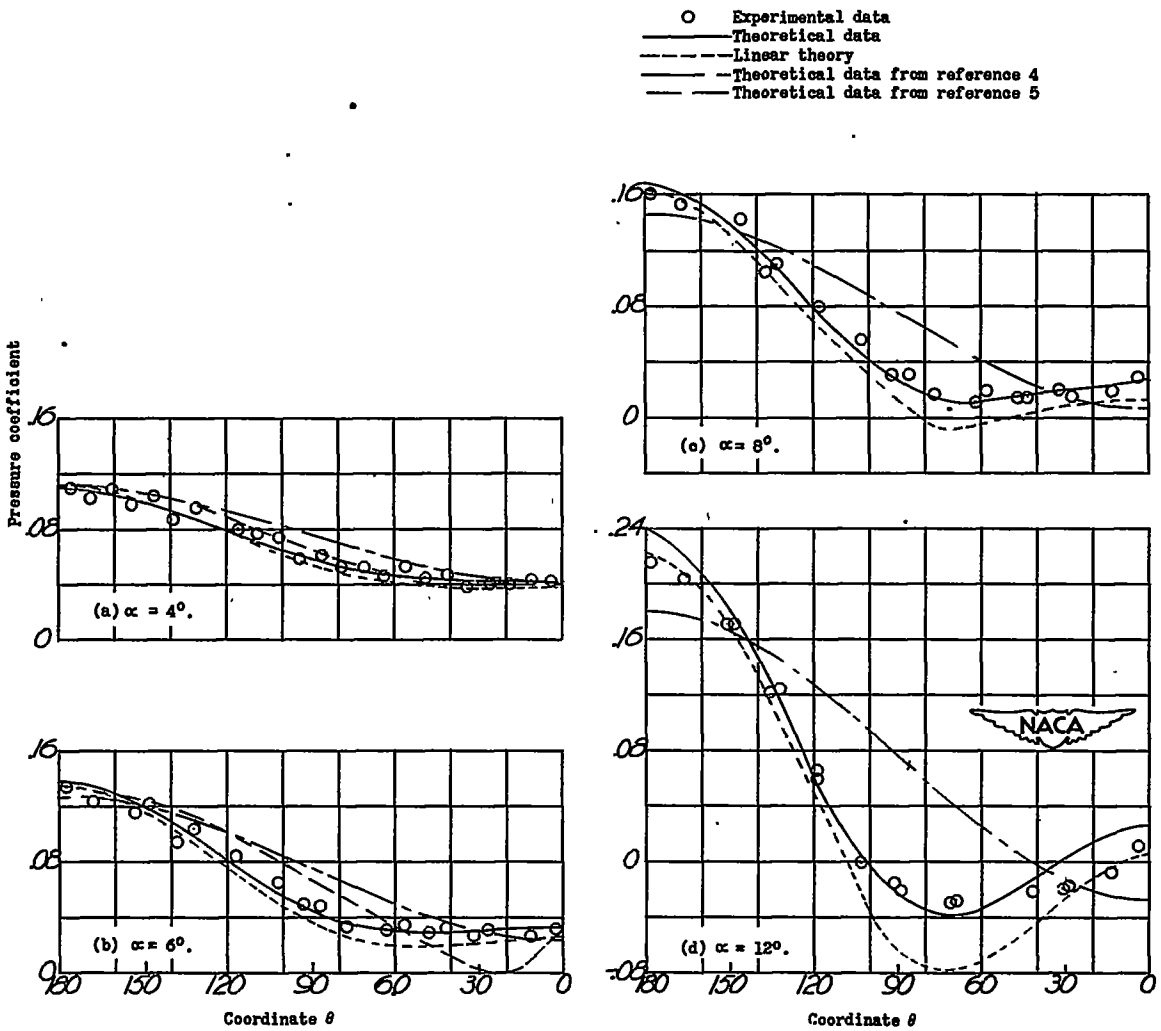

Figure 5.- Comparison of experimental pressure distribution over the surface of a cone with the theoretical pressure distribution. Semi-apex angle of cone, 7.5°; M = 1.6. (Experimental data obtained from the Langley 4- by 4-foot supersonic tunnel.)

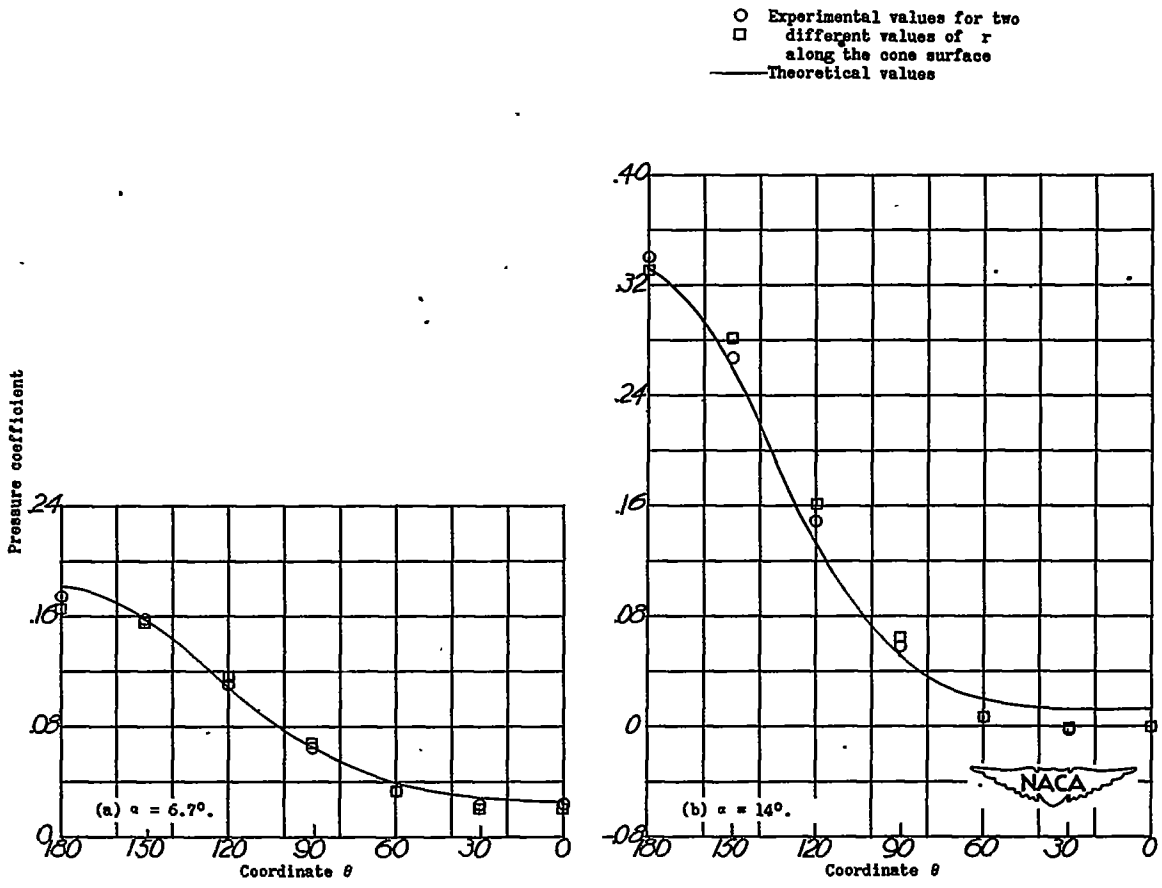

Figure 6.- Comparison of experimental pressure distribution over the surface of a cone with the theoretical pressure distribution. Semi-apex angle of cone, 10°; M = 6.86. (Experimental values for two different values of r along the cone surface; data obtained from the Langley 11-inch hypersonic tunnel.)

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