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TECHNICAL NOTE 2200

A STUDY OF SECOND-ORDER SUPERSONIC-FLOW THEORY

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SUMMARY

An attempt is made to develop a second approximation to the solution of problems of supersonic flow which can be solved by existing first-order theory. The method of attack adopted is an iteration process using the linearized solution as the first step.

For plane flow it is found that a particular solution of the iteration equation can be written down at once in terms of the first-order solution. The second-order problem is thereby reduced to an equivalent first-order problem and can be readily solved. At the surface of a single body, the solution reduces to the well-known result of Busemann. The plane case is considered in some detail insofar as it gives insight into the nature of the iteration process.

Again for axially symmetric flow the problem is reduced to a first-order problem by the discovery of a particular solution. For smooth bodies, the second-order solution can then be calculated by the method of Kármán and Moore. Bodies with corners are also treated by a slight modification of the method. The computing time required is several times that for a careful first-order solution. The second-order solution for pressures on cones represents a great improvement over the linearized result. Second-order theory also agrees well with several solutions calculated by the numerical method of characteristics.

For full three-dimensional flow, only a partial particular solution has been found. As an example of a more general problem, the solution is derived for an inclined cone. The possibility of treating other inclined bodies of revolution and three-dimensional wings is discussed briefly.

INTRODUCTION

As the linearized theory of supersonic flow approaches full development, the question arises as to whether more exact approximations are practical. If viscous effects are large, refinement of the perfect-fluid solution is useless. Otherwise, however, higher approximations are known to yield a closer approach to reality. In intermediate cases, an improved solution is desirable in order to assess the relative effects of viscosity and nonlinearity.

The prototype of a higher-order solution for supersonic flow is Busemann's series for the surface pressure in plane flow past a single body. This simple result is of considerable value in analyzing supersonic airfoil sections. Two terms of the series prove sufficient for almost all requirements; the extension to third and fourth order is chiefly of academic interest.

The aim of the present study is, therefore, to find a second approximation, analogous to Busemann's result, for supersonic flow past bodies which can be treated by existing first-order theory. The natural method of attack, and apparently the only practical one, is by means of an iteration process, taking the usual linearized result as the first step. Several writers have applied this procedure to subsonic flow. In supersonic flow, as usual, the solution is simpler, so that more general problems can be solved.

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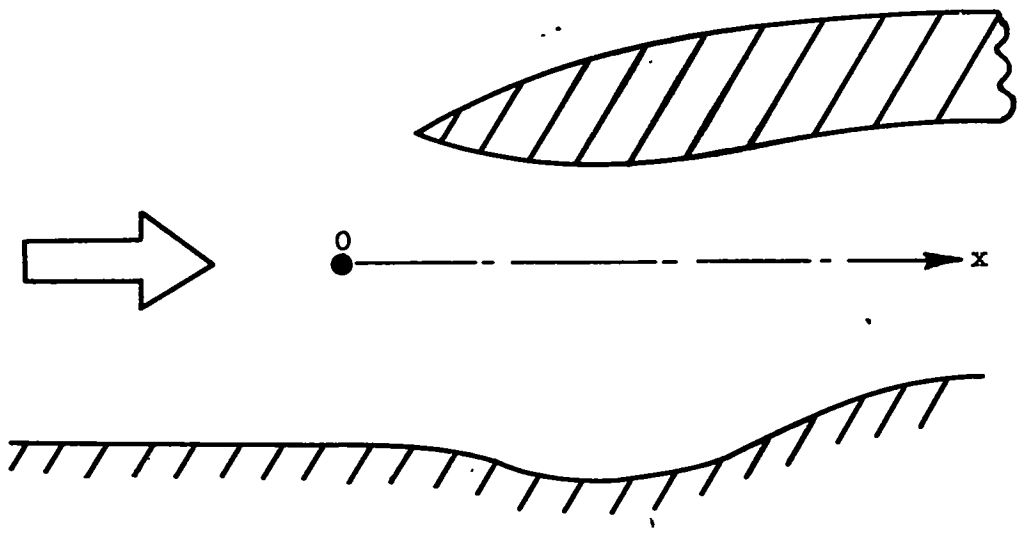
ITERATION PROCEDURE

Basic assumptions.- The problem to be considered is that of steady three-dimensional supersonic flow of a polytropic gas past one or more slender bodies. As indicated in the following diagram, the bodies are



The problem.

assumed either to be pointed or to extend upstream indefinitely as cylinders parallel to the free-stream direction. In either case, the origin

of coordinates can be chosen so that all variations in body shape are confined to the half-space $x > 0$. Wind axes are introduced, so that for $x \leqq 0$ the flow is uniform and parallel to the x-axis, with velocity U and Mach number M. (For definitions of all symbols, see appendix.)

The bodies are slender, which means that at any point the component of U normal to the surface is small compared with U itself. The symbol ϵ will be used throughout as a measure of this smallness. Thus the ordinates of a body will be written as ϵ times a function of order unity. Used in this way, ϵ serves to distinguish terms of various orders.

It will be assumed that the full linearized solution to the problem is available. Then the aim of this investigation is to provide a second approximation to the exact nonlinear solution. The linearized, or first-order, solution is defined as the result of keeping only linear perturbation terms in the equation of motion. Similarly, the second-order solution is the result of retaining products of perturbation quantities. In addition, however, certain of the triple products are in some cases found to be as important as one or more double products and are therefore also retained in the equation.

It may be noted that the second-order solution will not generally consist simply of terms of order ϵ and ϵ^2, though this is the case for plane flow. For example, the second-order solution for flow past a body of revolution contains terms as high as $\epsilon^4 \log^2 \epsilon$.

The flow is assumed to be irrotational and isentropic. This assumption is justified in the first- and second-order solutions, since the resulting error is found to be at most of the order of terms neglected elsewhere.

Exact perturbation equation.- Under the previous assumptions, there exists a velocity potential Ω. In Cartesian coordinates, the equation of motion is (reference 1, equation (39))

$$\left(c^2 - \Omega_x^2\right)\Omega_{xx} + \left(c^2 - \Omega_y^2\right)\Omega_{yy} + \left(c^2 - \Omega_z^2\right)\Omega_{zz} -$$

$$2\Omega_y\Omega_z\Omega_{yz} - 2\Omega_z\Omega_x\Omega_{zx} - 2\Omega_x\Omega_y\Omega_{xy} = 0 \tag{1}$$

Here the local speed of sound c is related to c_o, its value in the uniform stream, by

$$c^2 = c_o^2 - \frac{\gamma - 1}{2}\left(\Omega_x^2 + \Omega_y^2 + \Omega_z^2 - U^2\right) \tag{2}$$

where γ is the adiabatic exponent. The subscript notation is used to indicate differentiation.

A perturbation potential Φ is now introduced in the usual way. For convenience, however, Φ is normalized through division by the free-stream velocity, so that

$$\Omega = U(x + \Phi) \tag{3}$$

The perturbation velocity at any point is then the gradient of Φ multiplied by U.

Introducing the perturbation potential into the equation of motion gives, after some manipulation,

$$\Phi_{yy} + \Phi_{zz} - \beta^2\Phi_{xx} = M^2\begin{bmatrix} \frac{\gamma - 1}{2}\left(2\Phi_x + \Phi_x^2 + \Phi_y^2 + \Phi_z^2\right)\left(\Phi_{xx} + \Phi_{yy} + \Phi_{zz}\right) + \\ 2\Phi_{xx}\Phi_x + \Phi_{xx}\Phi_x^2 + \Phi_{yy}\Phi_y^2 + \Phi_{zz}\Phi_z^2 + \\ 2\Phi_{yz}\Phi_y\Phi_z + 2\Phi_{zx}\Phi_z(1 + \Phi_x) + 2\Phi_{xy}(1 + \Phi_x)\Phi_y \end{bmatrix} \tag{4}$$

where $\beta = \sqrt{M^2 - 1}$.

Solution by iteration.- The exact perturbation equation (equation (4)) is completely equivalent to the original nonlinear potential equation (equation (1)). Simplifying assumptions must therefore be introduced in order to solve it. If it is assumed that squares and products of the derivatives of Φ can be neglected, the right-hand side of equation (4) disappears, leaving the wave equation

$$\Phi_{yy}^{(1)} + \Phi_{zz}^{(1)} - \beta^2\Phi_{xx}^{(1)} = 0 \tag{5}$$

This equation is the basis of the linearized or first-order perturbation theory, so that its solution is designated by $\Phi^{(1)}$.

More exact solution of equation (4) by means of iteration was first suggested by Prandtl (reference 2). The procedure has been applied to plane subsonic flow by Görtler (reference 3), Hantsche and Wendt (references 4 and 5), Imai and Oyama (references 6 and 7), and Kaplan (references 8 to 10). Schmieden and Kawalki (reference 11) applied it to subsonic flow past an ellipsoid of revolution. Most of these writers have considered the stream function rather than the potential, which restricts the method to plane or axially symmetric flows. The procedure is clearly described by Sauer (reference 1, p. 140) for the case of plane flow.

The linearized solution $\Phi^{(1)}$, subject to proper boundary conditions, is taken as the first approximation. Substituting this known solution into the right-hand side of equation (4) gives

$$\Phi_{yy}^{(2)} \quad \Phi_{zz}^{(2)} - \beta^2\Phi_{xx}^{(2)} = F_1(x,y,z) \tag{6}$$

where F_1 is a known function of the independent variables. This is again a linear equation, the nonhomogeneous wave equation. A second-order solution $\Phi^{(2)}$, subject to proper boundary conditions, can be sought by standard methods. The procedure can be repeated by substituting $\Phi^{(2)}$ into the right-hand side of equation (4) and solving again. Continuing this process yields a sequence of solutions $\Phi^{(n)}$ which, under proper conditions, presumably converges to the exact solution.

This procedure bears a superficial resemblance to the Picard process for hyperbolic equations in two independent variables (reference 12, p. 317). There is, however, an essential difference. In the Picard process, the characteristic lines of the differential equation are known at the outset, since the functions F_n do not depend on the highest-order derivatives. Here, on the other hand, the characteristic surfaces (the Mach cones) are initially unknown. Because of the fundamental role played by the characteristics in the theory of hyperbolic equations (see, for example, reference 13, ch. II), it might be anticipated that the characteristics should be revised at each step of the iteration process. Each step but the first would then involve equations with nonconstant coefficients. The subsonic counterpart of such a procedure is known to converge under proper conditions (reference 12, pp. 288-289).

However, the procedure outlined previously makes no provision for such revision. At each stage of the iteration process, the equation has the original characteristics of the undisturbed flow. As a result, the equation has constant coefficients, which greatly facilitates solution. Fortunately, it will be found that this procedure nevertheless gives an improved solution nearly everywhere in the flow field.

Second-order iteration equation.- Henceforth, only the first two steps of the iteration process will be considered in detail. It is therefore convenient to regard the second-order solution as consisting of the first-order solution plus a smaller additional term. Also, for simplicity, $\Phi^{(1)}$ will henceforth be replaced by ϕ. Then

$$\Phi^{(2)} = \phi + \varphi \tag{7a}$$

where

$$\left.\begin{aligned} \phi &= \Phi^{(1)} \\ \varphi &= \Phi^{(2)} - \Phi^{(1)} \end{aligned}\right\} \tag{7b}$$

Now $\phi = \Phi^{(1)}$ is a solution of the homogeneous wave equation (equation (5)), so that substituting into the exact perturbation equation (equation (4)) shows that φ, as well as $\Phi^{(2)}$, is a solution of the following second-order iteration equation:

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} \frac{\gamma - 1}{2}\left(2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2\right)\left(\phi_{xx} + \phi_{yy} + \phi_{zz}\right) + \\ 2\phi_{xx}\phi_x + \phi_{xx}\phi_x^2 + \phi_{yy}\phi_y^2 + \phi_{zz}\phi_z^2 + \\ 2\phi_{yz}\phi_y\phi_z + 2\phi_{zx}\phi_z\left(1 + \phi_x\right) + 2\phi_{xy}\left(1 + \phi_x\right)\phi_y \end{bmatrix} \tag{8}$$

Since ϕ satisfies equation (5), the term $\left(\phi_{xx} + \phi_{yy} + \phi_{zz}\right)$ in the right-hand side of equation (8) can be replaced by $M^2\phi_{xx}$, and the equation for φ becomes

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} \frac{\gamma - 1}{2} M^2\phi_{xx}\left(2\phi_x + \phi_x^2 + \phi_y^2 + \phi_z^2\right) + 2\phi_{xx}\phi_x + \\ \phi_{xx}\phi_x^2 + \phi_{yy}\phi_y^2 + \phi_{zz}\phi_z^2 + \\ 2\phi_{yz}\phi_y\phi_z + 2\phi_{zx}\phi_z\left(1 + \phi_x\right) + 2\phi_{xy}\left(1 + \phi_x\right)\phi_y \end{bmatrix} \tag{9}$$

Here the right-hand side contains not only products of perturbation quantities but also triple products. The latter can be omitted for plane flow, since they contribute terms of smaller order (equal to those found in the next iteration). Otherwise, certain of the triple products should be retained, since their contribution is as great as that of one or more of the double products and greater than any contribution from a third approximation. It will be seen later that triple products should be retained if they involve only derivatives normal to the free stream. Those which involve x-derivatives can apparently be neglected, so that the equation becomes

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2\begin{Bmatrix} \left[2 + (\gamma - 1)M^2\right]\phi_{xx}\phi_x + 2\phi_{xy}\phi_y + 2\phi_{zx}\phi_z + \\ \phi_{yy}\phi_y^2 + 2\phi_{yz}\phi_y\phi_z + \phi_{zz}\phi_z^2 \end{Bmatrix} \tag{10}$$

Here the triple products which may be important are grouped in the second line of the right-hand side.

The adiabatic exponent γ will be found to occur always in the combination

$$N = \frac{(\gamma + 1)M^2}{2\beta^2} \tag{11}$$

Making this substitution, the second-order iteration equation becomes finally

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} 2(N - 1)\beta^2\phi_{xx}\phi_x + 2\phi_{xy}\phi_y + 2\phi_{zx}\phi_z + \\ \phi_{yy}\phi_y{}^2 + 2\phi_{yz}\phi_y\phi_z + \phi_{zz}\phi_z{}^2 \end{bmatrix} \tag{12}$$

<u>Iteration equation in other coordinates</u>.- In cylindrical coordinates, equation (12) becomes

$$\varphi_{rr} + \frac{\varphi_r}{r} + \frac{\varphi_{\theta\theta}}{r^2} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} 2(N - 1)\beta^2\phi_{xx}\phi_x + 2\phi_{xr}\phi_r + 2\phi_{x\theta}\frac{\phi_\theta}{r^2} + \\ \phi_{rr}\phi_r{}^2 + 2\phi_{r\theta}\frac{\phi_r\phi_\theta}{r^2} - \frac{\phi_r\phi_\theta{}^2}{r^3} + \phi_{\theta\theta}\frac{\phi_\theta{}^2}{r^4} + \\ O\left(\phi_{xx}\phi_x{}^2,\ \phi_{xx}\phi_r{}^2,\ \phi_{xx}\frac{\phi_\theta{}^2}{r^2},\ \phi_{x\theta}\frac{\phi_x\phi_\theta}{r^2},\ \phi_{xr}\phi_x\phi_r\right) \end{bmatrix} \tag{13}$$

The terms whose form is indicated in the last line are the triple products which will be found to be negligible.

For conical flows it is convenient to introduce nonorthogonal conical coordinates (x,t,θ) where

$$t = \frac{\beta r}{x} \tag{14}$$

If the body itself is conical, the perturbation potentials are reduced to functions of two variables by introducing conical perturbation potentials (reference 14) so that

$$\phi(x,t,\theta) = x\bar{\phi}(t,\theta) \tag{15}$$

with corresponding definitions for $\bar{\Phi}^{(2)}$ and $\bar{\varphi}$. The derivatives are given by

$$\left.\begin{array}{lll}
\phi_x = \bar{\phi} - t\bar{\phi}_t & \phi_{xx} = \frac{t^2}{x}\bar{\phi}_{tt} & \phi_{r\theta} = \beta\bar{\phi}_{t\theta} \\
\phi_r = \beta\bar{\phi}_t & \phi_{rr} = \frac{\beta^2}{x}\bar{\phi}_{tt} & \phi_{\theta x} = \bar{\phi}_\theta - t\bar{\phi}_{t\theta} \\
\phi_\theta = x\bar{\phi}_\theta & \phi_{\theta\theta} = x\bar{\phi}_{\theta\theta} & \phi_{xr} = -\frac{\beta t}{x}\bar{\phi}_{tt}
\end{array}\right\} \tag{16}$$

with the same relations connecting $\Phi^{(2)}$ and $\bar{\Phi}^{(2)}$, and φ and $\bar{\varphi}$. The iteration equation becomes

$$(1-t^2)\bar{\varphi}_{tt} + \frac{\bar{\varphi}_t}{t} + \frac{\bar{\varphi}_{\theta\theta}}{t^2} = M^2\left\{\begin{array}{l}
2(N-1)t^2\bar{\phi}_{tt}\left(\bar{\phi} - t\bar{\phi}_t\right) - 2t\bar{\phi}_{tt}\bar{\phi}_t + \frac{2}{t^2}\left(\bar{\phi}_\theta - t\bar{\phi}_{t\theta}\right)\bar{\phi}_\theta + \\
\beta^2\bar{\phi}_{tt}\bar{\phi}_t^{\,2} + 2\,\frac{\beta^2}{t^2}\,\bar{\phi}_{t\theta}\bar{\phi}_t\bar{\phi}_\theta - \frac{\beta^2}{t^3}\,\bar{\phi}_t\bar{\phi}_\theta^{\,2} + \frac{\beta^4}{t^4}\,\bar{\phi}_{\theta\theta}\bar{\phi}_\theta^{\,2} + \\
O\left[t^2\bar{\phi}_{tt}\left(\bar{\phi} - t\bar{\phi}_t\right)^2,\ t^2\bar{\phi}_{tt}\bar{\phi}_t^{\,2},\ \bar{\phi}_{tt}\bar{\phi}_\theta^{\,2},\right. \\
\left.\ \frac{1}{t^2}\left(\bar{\phi}_\theta - t\bar{\phi}_{t\theta}\right)\left(\bar{\phi} - t\bar{\phi}_t\right)\bar{\phi}_\theta,\ t\bar{\phi}_{tt}\bar{\phi}_t\left(\bar{\phi} - t\bar{\phi}_t\right)\right]
\end{array}\right\} \tag{17}$$

Here the grouping of terms corresponds to that in equation (13).

Boundary conditions.- Physical considerations suggest that the flow should satisfy the following conditions:

(1) The resultant velocity is tangent to the surface of the body

(2) All flow perturbations vanish identically everywhere upstream of the plane $x = 0$

The theory of hyperbolic differential equations shows that these two requirements are sufficient to determine the solution. The first imposes one condition along the timelike surface of the body, and the second imposes two conditions on a spacelike surface. This corresponds mathematically to the case of mixed boundary conditions (reference 12, p. 172) and leads to a determinate solution (see reference 13, p. 85).

The tangency condition may be written formally as

$$\nabla\Omega\cdot\nabla S = 0 \tag{18}$$

where $S(x,y,z) = 0$ is the equation of the surface of the body. In a more useful form it becomes, for the first- and second-order problems,

$$\frac{\phi_c}{1 + \phi_x} = \text{Slope} \qquad \text{(on the surface)} \tag{19a}$$

$$\frac{\phi_c + \varphi_c}{1 + \phi_x + \varphi_x} = \text{Slope} \qquad \text{(on the surface)} \tag{19b}$$

Here ϕ_c means the cross-wind component of the normal derivative of ϕ at the surface of the body. In plane flow or for planar systems ϕ_c is ϕ_y, and in axially symmetric flow ϕ_c is ϕ_r. The slope of the body is measured with respect to the free-stream direction. If the first-order tangency condition (equation (19a)) is satisfied exactly, the second-order condition can be simplified to

$$\frac{\varphi_c}{\varphi_x} = \text{Slope} \qquad \text{(on the surface)} \tag{19b'}$$

In linearized theory, the tangency condition (equation (19a)) is frequently approximated by neglecting Φ_x in comparison with unity. If the corresponding approximation is made in the second-order problem, the two tangency conditions become

$$\phi_c = \text{Slope} \qquad \text{(on the surface)} \tag{20a}$$

$$\frac{\phi_c + \varphi_c}{1 + \phi_x} = \text{Slope} \qquad \text{(on the surface)} \tag{20b}$$

This approximation will not be made except for plane flow, since otherwise it apparently causes unnecessary loss in accuracy.

A planar system is defined to be a system for which the first-order tangency condition can be imposed at a plane parallel to the free stream rather than on the surface of the body (reference 15, p. 52). Thin flat wings are planar systems, while slender pointed bodies of revolution are not. For planar systems the second-order tangency condition can also be imposed at the plane, provided that the value of ϕ_y is calculated at the surface of the body (ϕ_x and φ_x may be calculated at the plane). That is, for planar systems the tangency conditions are

$$\left(\phi_y\right)_{\text{plane}} = (\text{Slope})\left(1 + \phi_x\right)_{\text{plane}} \tag{21a}$$

$$\left(\varphi_y\right)_{plane} = (\text{Slope})\left(1 + \phi_x + \varphi_x\right)_{plane} - \left(\phi_y\right)_{surface} \tag{21b}$$

Corresponding results hold for quasi-cylindrical bodies, which are bodies of revolution whose radius varies so slightly that the tangency conditions can be imposed at a circular cylinder parallel to the free stream.

The other two boundary conditions are that

$$\left.\begin{aligned} \phi(0,y,z) + \varphi(0,y,z) &= 0 \\ \phi_x(0,y,z) + \varphi_x(0,y,z) &= 0 \end{aligned}\right\} \tag{22}$$

These conditions must be satisfied by the first-order solution alone and must therefore be satisfied also by the additional second-order potential alone. Consequently, the conditions are

$$\left.\begin{aligned} \phi(0,y,z) &= 0 \\ \phi_x(0,y,z) &= 0 \end{aligned}\right\} \tag{23a}$$

$$\left.\begin{aligned} \varphi(0,y,z) &= 0 \\ \varphi_x(0,y,z) &= 0 \end{aligned}\right\} \tag{23b}$$

Determination of pressure.- When the potential field has been determined, the net velocity q at any point is given by

$$q^2 = (U + u)^2 + v^2 + w^2 \tag{24}$$

where

$$\left.\begin{aligned} \frac{u}{U} &= \Phi_x^{(2)} \\ \frac{v}{U} &= \begin{cases} \Phi_y^{(2)} \\ \Phi_r^{(2)} \end{cases} \\ \frac{w}{U} &= \begin{cases} \Phi_z^{(2)} \\ \frac{1}{r}\,\Phi_\theta^{(2)} \end{cases} \end{aligned}\right\} \tag{25}$$

in both Cartesian and cylindrical coordinates. Because the flow was assumed to be isentropic, the pressure coefficient is given by

$$C_p = \frac{p - p_o}{\frac{1}{2}\rho_o U^2} = \frac{2}{\gamma M^2}\left\{\left[1 + \frac{\gamma - 1}{2} M^2\left(1 - \frac{q^2}{U^2}\right)\right]^{\frac{\gamma}{\gamma-1}} - 1\right\} \tag{26}$$

where p_o and ρ_o are the free-stream pressure and density.

It is the practice in linearized theory also to linearize the pressure relation. Substituting equation (24) into equation (26) and expanding gives

$$C_p = -2\frac{u}{U} - \frac{v^2 + w^2}{U^2} + \beta^2\frac{u^2}{U^2} + M^2\frac{u}{U}\frac{v^2 + w^2}{U^2} +$$

$$\frac{M^2}{4}\left(\frac{v^2 + w^2}{U^2}\right)^2 + O\left[\frac{u^3}{U^3}, \frac{u^2}{U^2}\frac{v^2 + w^2}{U^2}, \frac{u}{U}\left(\frac{v^2 + w^2}{U^2}\right)^2, \left(\frac{v^2 + w^2}{U^2}\right)^3\right] \tag{27}$$

In linearized theory only the first term is ordinarily retained. This is satisfactory for plane flow or flow past planar systems, since the contribution of the remaining terms is truly of higher order. In fact, for plane flow past a single body it happens that the next two terms cancel identically.

For slender bodies such as a cone, however, orders of magnitude are not so clearly distinguished. Busemann suggests (reference 14) that the second term is then sufficiently large compared with the first that it should be retained also. This view is supported by Lighthill (reference 16), who shows that the resulting solution is correct up to the order of the quantities contributed by the second term. Again, the third term, which also involves squares of perturbation quantities, is comparable with the second at high Mach numbers and might logically be retained. Having gone this far, it may be simpler to use the exact relation.

Each of these four possibilities for the first-order flow past a 5° cone is compared with the exact solution (reference 17) in figure 1. The series (equation (27)) is seen to alternate in this case. It converges so slowly, however, that linearizing the pressure relation introduces much greater errors than linearizing only the equation of motion. Adding each of the quadratic terms in turn causes fluctuations nearly as great as the error due directly to nonlinearity in the equation.

The point of view to be adopted here is that calculating the velocities and calculating the pressure are two essentially distinct

operations. A certain degree of approximation may be necessary in order to solve for the velocities, but the pressure relation need not then be approximated to the same extent simply for the sake of consistency. For it may happen that the resulting errors (though of the same mathematical order) are greater than those due to the original approximation. Indeed, this is evidently the case in the first-order solution for a cone and will be found true to a greater extent in the second-order solution.

Moreover, in the second-order solution so many terms of equation (27) must be retained that it is usually simpler to use the exact relation. For these reasons, the exact pressure equation (equation (26)) will be used throughout except in the case of plane flow.

PLANE FLOW

The second-order solution for conditions at a single surface in plane supersonic flow was given by Busemann (references 18 and 19). By using the iteration procedure, the solution will now be found throughout the flow field, including the case when several bodies interact.

The solution for plane flow is of interest chiefly insofar as it serves as a guide in more complicated problems. In particular, it provides insight into such details of the iteration process as the question of its success and the effect of sharp corners.

Role of a particular solution.- The second-order iteration equation can be attacked by standard methods, and in the case of plane flow a solution can be found directly. For plane and axially symmetric flows, however, a particular solution of the iteration equation can be written down at once in terms of the first-order solution. This solves the problem, because the complete solution consists of a particular solution plus a solution of the homogeneous equation, and the latter can be obtained by existing methods. That is, the additional second-order potential may be written as

$$\varphi = \psi + \chi \tag{28}$$

where

ψ any particular solution of nonhomogeneous iteration equation

χ a correction potential which is a solution of corresponding homogeneous wave equation and serves to correct the tangency condition

The problem for χ is the usual first-order problem whose solution is assumed to be known.

The role of the particular solution is to transfer the nonhomogeneity in the problem from the equation, where it is troublesome, to the boundary conditions, where it can be handled by existing theory. For linear partial differential equations it is always possible in principle to transfer nonhomogeneities in this way from the equation to the boundary conditions, and vice versa, by adding a suitable function to the dependent variable (see reference 20, p. 236).

Since the particular solution ψ will be found in terms of the first-order solution, it will vanish upstream of the plane $x = 0$. Then the correction potential must also vanish there, so that two boundary conditions are given by

$$\chi(0,y,z) = \chi_x(0,y,z) = 0 \tag{29}$$

From equation (19b'), the tangency condition for χ is found to be

$$\frac{\psi_c + \chi_c}{\psi_x + \chi_x} = \text{Slope} \qquad \text{(on the surface)} \tag{30}$$

or, in the case of planar systems, from equation (21b)

$$\left(\psi_y + \chi_y\right)_{\text{plane}} = (\text{Slope})\left(1 + \phi_x + \psi_x + \chi_x\right)_{\text{plane}} - \left(\phi_y\right)_{\text{surface}} \tag{31}$$

It should be noted that, although φ is small compared with ϕ, this is not necessarily true of either ψ or χ alone.

<u>Particular solution for plane flow</u>.- The first-order equation for plane flow is

$$\phi_{yy} - \beta^2\phi_{xx} = 0 \tag{32}$$

The general solution is

$$\phi(x,y) = H(x - \beta y) + J(x + \beta y) \tag{33}$$

where H and J are functions chosen so as to satisfy the boundary conditions.

In the iteration equation, all triple products can be neglected, and equation (12) becomes

$$\varphi_{yy} - \beta^2\varphi_{xx} = 2M^2\left[(N - 1)\beta^2\phi_{xx}\phi_x + \phi_{xy}\phi_y\right] \tag{34}$$

It can be verified directly that a particular solution of this equation is given by

$$\psi = M^2\phi_x\left[\left(1 - \frac{N}{2}\right)\phi + \frac{N}{2}\,y\phi_y\right] \tag{35}$$

To this must be added a solution χ of the homogeneous equation (equation (32)), which has the form

$$\chi = h(x - \beta y) + j(x + \beta y) \tag{36}$$

where h and j are functions determined by the second-order boundary conditions.

For flow past a single boundary (such as one surface of an airfoil) the first-order potential (equation (33)) contains only one or the other of the functions H and J. In this case $\phi_{xy}\phi_y = \beta^2\phi_{xx}\phi_x$ so that the iteration equation reduces to

$$\varphi_{yy} - \beta^2\varphi_{xx} = 2M^2\beta^2N\phi_{xx}\phi_x \tag{37}$$

The particular solution may then be simplified to

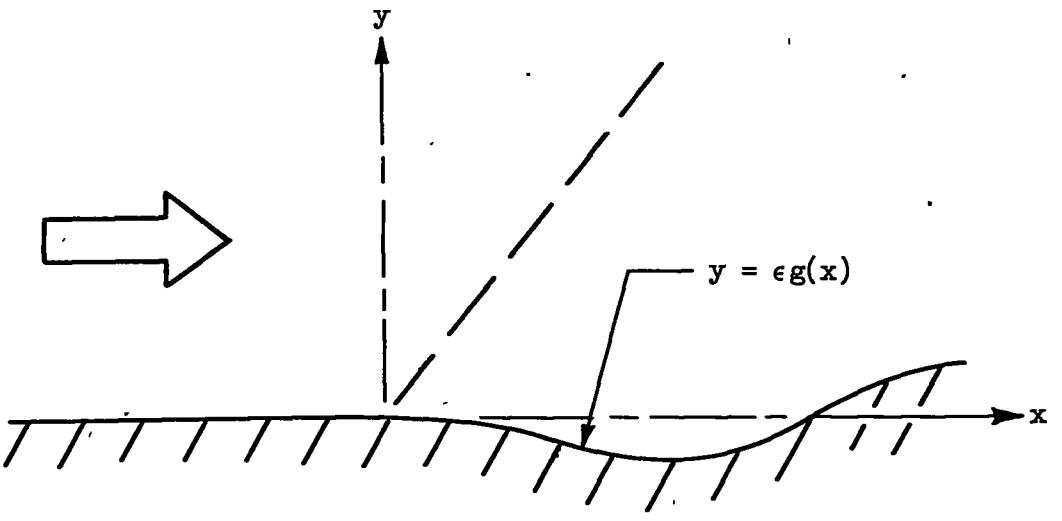
$$\psi = M^2\,\frac{N}{2}\,y\phi_x\phi_y \tag{38}$$

and the correction potential contains only h or j, according as the first-order solution contains only H or J.

Flow past a curved wall.- As an example of the application of the particular solution, consider flow past a wall which at some point begins to deviate slightly from a plane (see the following figure). The wall can be represented by

$$y = \epsilon g(x) \tag{39}$$

where ϵ is a parameter small compared with unity and $g(x)$ is a continuous function of order unity which vanishes for $x \leqq 0$.



Flow past a curved wall.

This is a planar system, so that the tangency conditions are given by equation (21). The approximation of equation (20) can also be made. Consequently, the first-order problem is

$$\left.\begin{aligned} \phi_{yy} - \beta^2\phi_{xx} &= 0 \\ \phi_y(x,0) &= \epsilon g'(x) \\ \phi(0,y) &= \phi_x(0,y) = 0 \end{aligned}\right\} \tag{40}$$

The solution is

$$\phi = -\frac{\epsilon}{\beta} g(x - \beta y) \tag{41}$$

Substituting into the right-hand side of equation (34) gives the iteration equation

$$\varphi_{yy} - \beta^2\varphi_{xx} = 2M^2N\epsilon^2 g'(x - \beta y)g''(x - \beta y) \tag{42}$$

According to equations (38) and (36), the solution is

$$\varphi = \psi + \chi = -\frac{M^2N}{2\beta}\epsilon^2 y\left[g'(x - \beta y)\right]^2 + h(x - \beta y) \tag{43}$$

Imposing the approximate second-order planar tangency condition (equation (21b)) gives

$$h'(x) = \epsilon^2\left\{\frac{2 - M^2N}{2\beta^2}\left[g'(x)\right]^2 - g(x)g''(x)\right\} \tag{44}$$

so that

$$h(x) = -\epsilon^2\left\{g(x)g'(x) + \frac{M^2(N - 2)}{2\beta^2}\int_0^x \left[g'(\xi)\right]^2 d\xi\right\} \tag{45}$$

The complete second-order perturbation potential is therefore

$$\Phi^{(2)} = -\frac{\epsilon}{\beta}g(x - \beta y) - \epsilon^2\left\{g(x - \beta y)g'(x - \beta y) + \frac{M^2N}{2\beta}y\left[g'(x - \beta y)\right]^2 + \right.$$

$$\left.\frac{M^2(N - 2)}{2\beta^2}\int_0^{x-\beta y}\left[g'(\xi)\right]^2 d\xi\right\} \tag{46}$$

The same result can be found by solving equation (42) directly, using the impulse method (reference 12, p. 164).

On the surface of the wall, the streamwise velocity perturbation is given by

$$\frac{u}{U} = -\frac{\epsilon}{\beta}g'(x) - \frac{M^2N - 2}{2\beta^2}\epsilon^2\left[g'(x)\right]^2 \tag{47}$$

The pressure coefficient at the wall can now be calculated from equation (27) which, upon replacing N by its value from equation (11), gives

$$C_p = \frac{2}{\beta}\epsilon g'(x) + \frac{(\gamma + 1)M^4 - 4\beta^2}{2\beta^4}\left[\epsilon g'(x)\right]^2 \tag{48}$$

This is the well-known result of Busemann (references 18 and 19). To second order, the surface pressure coefficient depends only upon the local slope.

Role of characteristics.- It was pointed out previously that, because of the underlying significance of the characteristic surfaces for solutions of hyperbolic equations, it might be expected that the characteristics would have to be revised successively at each stage of the iteration. However, an iteration process was chosen which permits no such revision. It is therefore pertinent to inquire in this simple example what role has been played by the original and the revised characteristics.

Only one of the two families of characteristics will be considered. The original characteristics of this family are the lines of slope

$$\frac{dy}{dx} = \frac{1}{\beta} \tag{49}$$

These are the Mach lines of the undisturbed flow which run downstream from the wall (see the preceding diagram). They are also characteristics of equation (32) in the mathematical sense (reference 12, ch. 5; reference 13, ch. II).

It can readily be shown that, if the first-order streamwise perturbation velocity at any point in a flow is $u^{(1)}$, then the revised local values of Mach number and β are given to first order by

$$M^{(1)} = M\left[1 + \beta^2(N - 1)\frac{u^{(1)}}{U}\right] \tag{50a}$$

$$\beta^{(1)} = \sqrt{M^{(1)2} - 1} = \beta\left[1 + M^2(N - 1)\frac{u^{(1)}}{U}\right] \tag{50b}$$

By using this result together with the first-order solution (equation (41)), the revised downstream Mach lines are found to have the slope

$$\frac{dy}{dx} = \frac{1}{\beta}\left[1 + \frac{M^2N}{\beta}\epsilon g'(x - \beta y)\right] \tag{51}$$

These are not the mathematical characteristics of the iteration equation (equation (42)) for the reason that fractions of the highest-order derivatives have there been transferred to the right-hand side and regarded as known. Mathematically, the characteristics continue to be given by equation (49).

Physically, the characteristics are lines along which discontinuities in velocity derivatives are propagated, and this definition is

completely equivalent to the mathematical one (reference 12, p. 297). Therefore, in the second-order solution derived above, discontinuities in acceleration must occur along the original characteristics.

Suppose, however, that no such discontinuities occur. For flow past a single body the downstream characteristics are also lines along which the velocity is constant, provided that shock waves do not appear. Setting

$$\left.\begin{aligned} d\Phi_x^{(2)} &= \Phi_{xx}^{(2)}\,dx + \Phi_{xy}^{(2)}\,dy = 0 \\ d\Phi_y^{(2)} &= \Phi_{xy}^{(2)}\,dx + \Phi_{yy}^{(2)}\,dy = 0 \end{aligned}\right\} \tag{52}$$

it is seen that the velocity is constant if

$$\frac{dy}{dx} = -\frac{\Phi_{xx}^{(2)}}{\Phi_{xy}^{(2)}} = -\frac{\Phi_{xy}^{(2)}}{\Phi_{yy}^{(2)}} \tag{53}$$

For the second approximation (equation (46)) the velocity is constant along lines of slope

$$\frac{dy}{dx} = \frac{1}{\beta}\left[1 + \frac{M^2N}{\beta}\,\epsilon g'(x - \beta y)\right] \tag{54}$$

which, according to equation (50b), are the revised characteristics. Consequently, although the characteristics have not been revised in the mathematical sense, the solution behaves physically as if they had, so long as discontinuities do not occur. The question of discontinuities will be considered in the next section.

The connection between the original and revised characteristics can be interpreted physically. The right-hand side of the iteration equation may be regarded as representing the effect of a known distribution of supersonic sources throughout the flow field. The influence of this source distribution spreads downstream along both families of original characteristics. The resulting velocity changes are just such that the second-order velocities become constant along the revised rather than the original characteristics.

Finally, it is interesting to note that the second-order potential is constant on lines which bisect the original and revised characteristics. For, setting

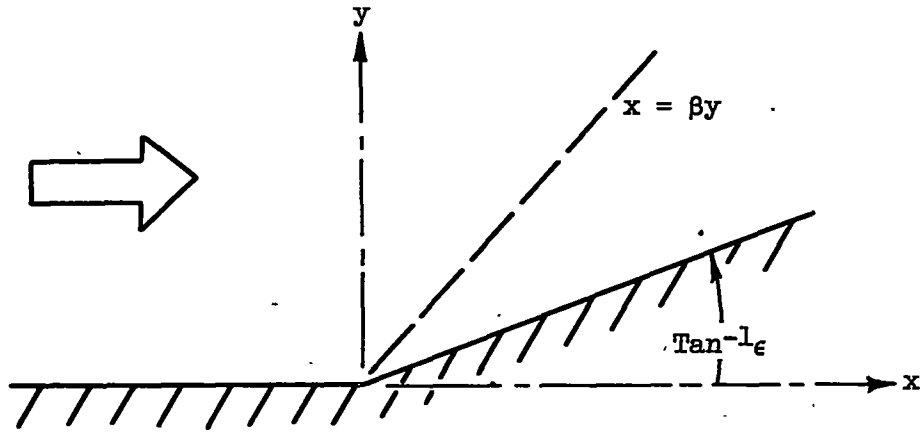
$$d\Phi^{(2)} = \Phi_x^{(2)}\,dx + \Phi_y^{(2)}\,dy = 0 \tag{55}$$

$\phi^{(2)}$ is found to be constant along lines of slope

$$\frac{dy}{dx} = \frac{1}{\beta}\left[1 + \frac{M^2N}{2\beta}\,\epsilon g'(x - \beta y)\right] \tag{56}$$

Flow past a corner and a parabolic bend.- A simple case in which discontinuities may occur is that of flow past a sharp corner. The exact solution is known to involve an oblique shock wave with attendant velocity discontinuities for compression and a continuous Prandtl-Meyer fan for expansion.

Denoting the tangent of the deflection angle by ϵ, positive for compression (see the following figure), the function $g(x)$ appearing



Flow past a corner.

in equation (39) is

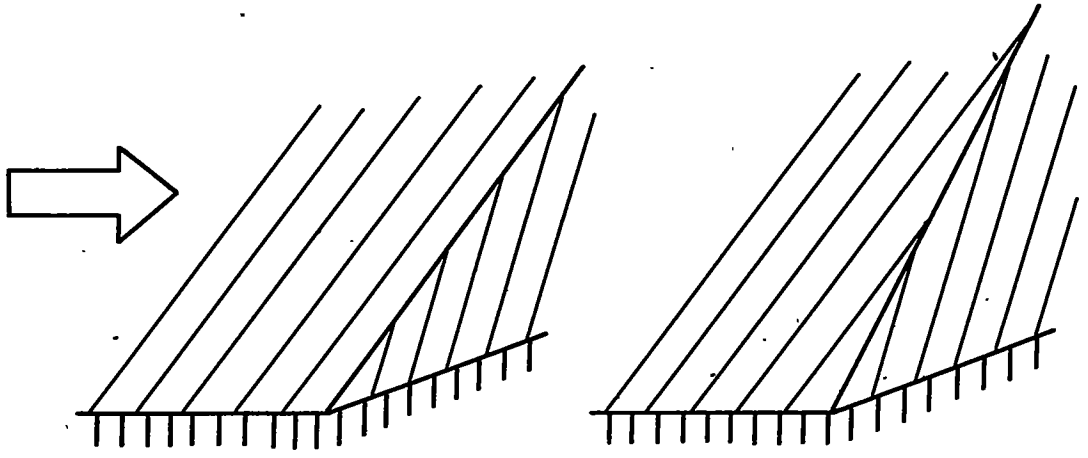
$$g(x) = \begin{cases} 0 & x \leqq 0 \\ x & x \geqq 0 \end{cases} \tag{57}$$

From equation (46) the second-order perturbation potential is found to be

$$\phi^{(2)}(x,y) = -\frac{\epsilon}{\beta}(x - \beta y) + \frac{\epsilon^2}{\beta^2}(x - \beta y) - \frac{M^2N}{2\beta^2}\,\epsilon^2 x \tag{58}$$

to the right of the line $x = \beta y$ and zero to the left. Consequently, in either compression $(\epsilon > 0)$ or expansion $(\epsilon < 0)$ the second-order potential suffers a discontinuous drop along the Mach line from the corner, of strength proportional to the distance from the corner. Such a discontinuity cannot be admitted, which indicates that the iteration process fails in this region.

In the case of compression, the solution can be corrected by analytically continuing the perturbation potential upstream until it can be joined continuously to the free-stream potential. (This is permissible since the line of discontinuity is not actually a characteristic.) From the result of equation (56) the juncture is seen to occur along the line from the corner which bisects the upstream and downstream Mach directions, as indicated in the following diagram:

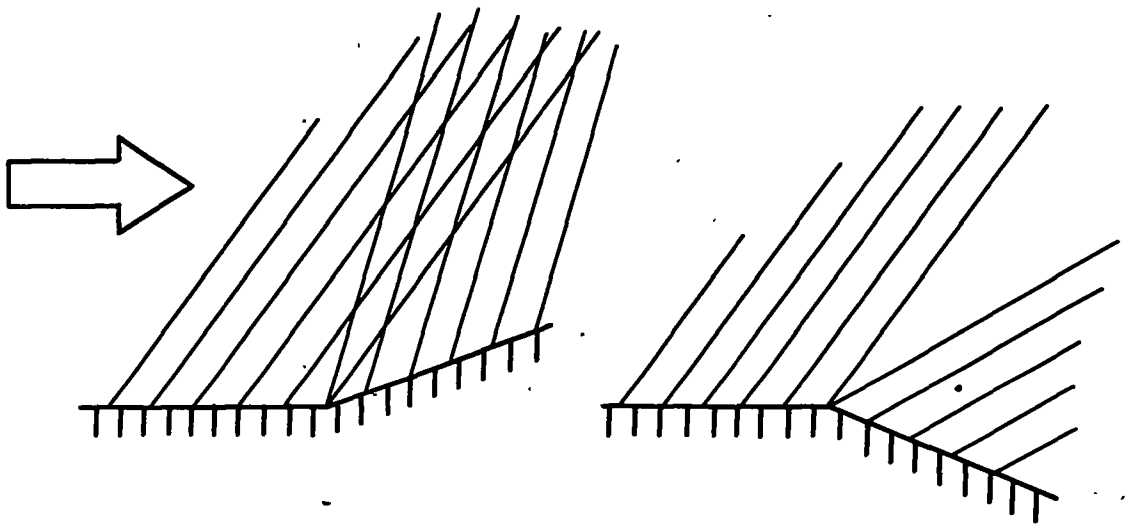


Mach lines before and after adjustment
of potential discontinuity.

The adjusted discontinuity corresponds to a shock wave, for it is known that an oblique shock bisects the Mach lines to first order (reference 13, p. 354). In the case of expansion, this type of correction cannot be justified, since it would involve continuation of the free-stream potential across a true characteristic. Instead, a Prandtl-Meyer fan must be inserted.

Evidently the iteration process is successful except within an angular region of order ϵ lying near the Mach line from the corner. In particular, the pressure is given correctly everywhere on the surface of the wall.

It is enlightening to observe that the alternative method of iteration, in which the characteristics are successively revised, fails in the same region. The potential is double-valued over a fan-shaped region in the case of compression and is left undefined over a similar region in the case of expansion (see the following diagram). The same artificial



Second-order flow past a corner using revised characteristics.

corrections are necessary to complete the solution.

Consider next flow past a parabolic bend which is represented by

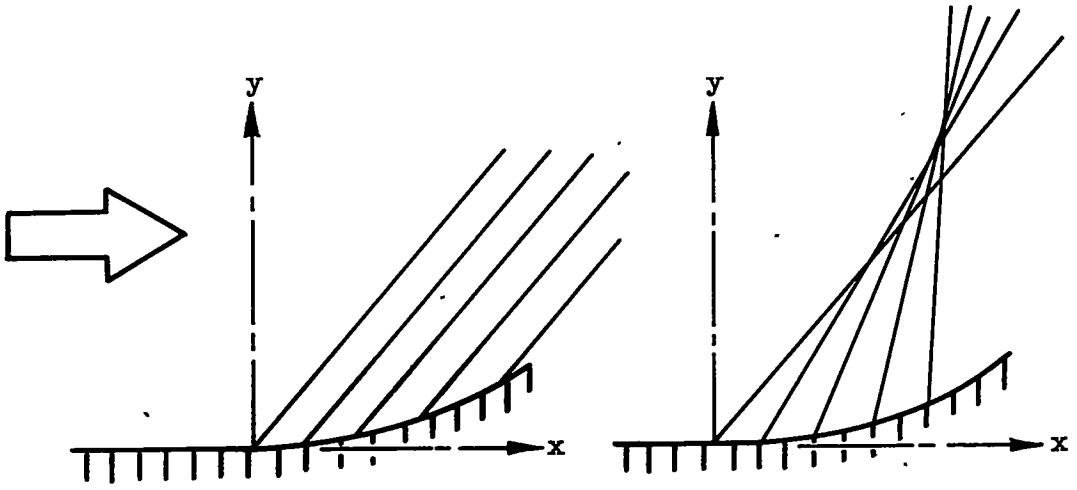
$$y = \frac{1}{2}\epsilon x^2 \qquad x \geqq 0 \tag{59}$$

From equation (46) the second-order perturbation potential is found to be

$$\Phi^{(2)}(x,y) = -\frac{\epsilon}{2\beta}(x-\beta y)^2 - \frac{M^2(N+1)-3}{6\beta^2}\epsilon^2(x-\beta y)^2 - \frac{M^2N}{2\beta}\epsilon^2 y(x-\beta y)^2 \tag{60}$$

The potential and also the velocities are continuous, so that the previous difficulties do not occur. The acceleration is discontinuous across the original characteristic $x = \beta y$, which in this case happens to be also a revised characteristic. However, a new complication arises. It is well-known that, in the exact solution for the compressive case,

the characteristics form an envelope, as shown in the accompanying figure.



First- and second-order flow past a parabolic bend.

Inside the cusp the potential is triple-valued (reference 13, p. 111), so that a shock wave must be inserted. This envelope must also arise in the second approximation, since the characteristics are no longer parallel. However, the second-order potential given by equation (60) is single-valued, so that it cannot predict the formation of an envelope. Again the iteration process fails in a part of the flow field.

It can be seen that the alternative iteration process, using revised characteristics, will produce an envelope.

Convergence for plane flow.- The examples just considered indicate that the success of the iteration procedure should be carefully investigated. A step of an iteration process may be considered successful if, in some sense, it significantly improves the solution. In particular, one is interested in the success of the second-order solution.

It should be noted that a divergent process may be successful for many steps and that, on the other hand, convergence does not necessarily imply success. In practice, however, one would expect a convergent process to be successful. As used here, success is a subjective notion, not amenable to analysis. Consequently, only the convergence of the iteration procedure can be considered in any detail.

Unfortunately, proofs of sufficient conditions for convergence have not been obtained, even in the case of plane flow. However, the above

examples suggest certain conjectures regarding convergence. These will be stated and some arguments for their plausibility will be advanced.

For flow past a slightly curved plane wall represented by $y = \epsilon g(x)$ the solution obtained by iteration using the revised characteristics is conjected to converge in any bounded region adjacent to the wall provided that

(a) ϵ is sufficiently small

(b) $g(x)$ is continuously differentiable

If $g(x)$ has only a piecewise continuous derivative, the convergence holds except possibly in fan-shaped regions springing from each corner, which lie near the original Mach line and subtend an angle of order ϵ.

For the iteration process actually adopted, in which the characteristics are not revised, the first n steps are conjectured to form part of a convergent process provided that

(a) ϵ is sufficiently small

(b') $g(x)$ has continuous derivatives up to $(n - 1)$st order if the potential is required, nth order if the velocities are required

If condition (b') is satisfied only piecewise, the result holds except possibly in fan-shaped regions springing from each corner.

In the first case, condition (a) is necessary in order to insure that the solution be unique, as is clear from the example of the parabolic wall. The above examples also show that condition (b) is necessary.

If the sufficiency of these two conditions is assumed, their connection with condition (b') in the second case can be illustrated by analogy with a mathematical model (suggested by Dr. C. R. DePrima) which retains the essential difference between the two iteration processes - namely, that the correct characteristics are not used in the method actually adopted. Consider the first-order problem given by equation (40):

$$\left.\begin{aligned} \phi_{yy} - \phi_{xx} &= 0 \\ \phi_y(x,0) &= \epsilon g'(x) \\ \phi(0,y) &= \phi_x(0,y) = 0 \end{aligned}\right\} \qquad (61)$$

where $\beta = 1$ has been taken for convenience. The solution (equation (41)) was

$$\phi = -\epsilon g(x - y) \tag{62}$$

Now it is attempted to solve this problem using characteristics which differ from the true characteristics by $O(\epsilon)$. Thus consider the equivalent problem

$$\left.\begin{aligned} \phi_{yy} - (1 - \epsilon)\phi_{xx} &= \epsilon\phi_{xx} \\ \phi_y(x,0) &= \epsilon g'(x) \\ \phi(0,y) = \phi_x(0,y) &= 0 \end{aligned}\right\} \tag{63}$$

and solve by iteration. In the first approximation the right-hand side can be neglected, so that

$$\phi_{yy} - (1 - \epsilon)\phi_{xx} = 0 \tag{64}$$

which has the solution, subject to the boundary conditions,

$$\phi^{(1)} = -\epsilon g(x - \sqrt{1 - \epsilon}y) \tag{65}$$

Substituting this into the right-hand side of equation (63) gives the iteration equation for the second approximation:

$$\phi_{yy} - (1 - \epsilon)\phi_{xx} = -\epsilon^2 g''(x - \sqrt{1 - \epsilon}y) \tag{66}$$

Using the impulse method (reference 12, p. 164) gives the solution, subject to the boundary conditions,

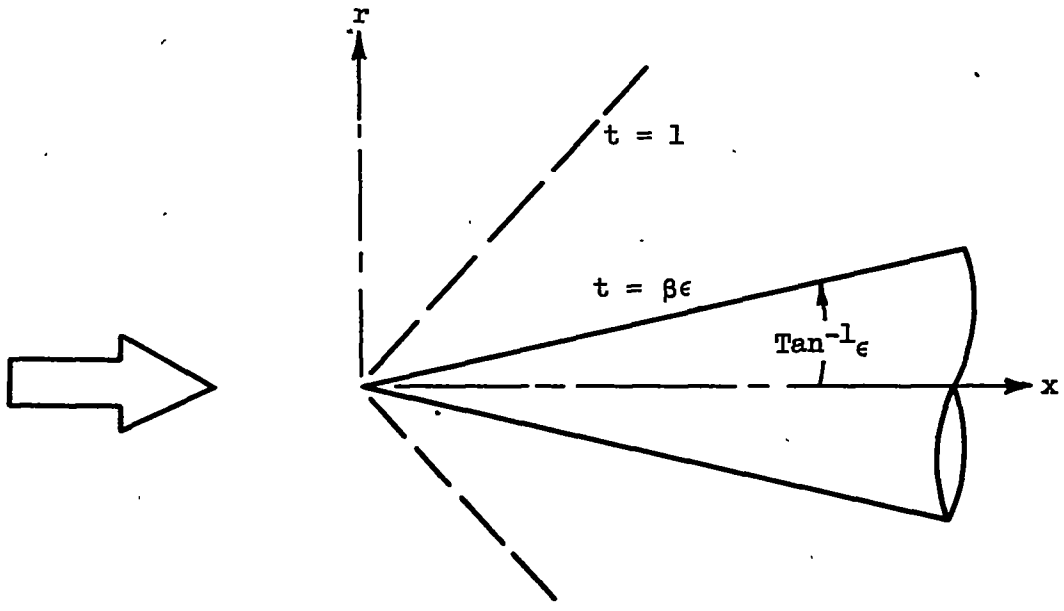
$$\phi^{(2)} = -\epsilon g(x - \sqrt{1 - \epsilon}y) + \frac{1}{2}\epsilon^2 y g'(x - \sqrt{1 - \epsilon}y) \tag{67}$$

But this is just the Taylor series expansion, correct to $O(\epsilon^2)$, of the true solution (equation (62)). Subsequent iterations add additional terms to the expansion. Hence, despite the use of slightly incorrect characteristics, the iteration process converges to the correct solution. The connection between conditions (b) and (b') is thus seen to be that the existence of sufficiently many continuous derivatives compensates for the fact that the wrong characteristics are used.

AXIALLY SYMMETRIC FLOW

Before discussing the general solution for bodies of revolution, it is convenient to consider the simple problem of a cone. In this case the second-order solution can be found directly. The results will be useful in indicating which triple products should be retained in the general case.

Flow past a cone.- Consider flow past a slender cone of semivertex angle $\tan^{-1}\epsilon$, as shown in the following diagram:



Flow past a cone.

The flow is conical and axially symmetric, so that the iteration equation is given by equation (17) with θ-derivatives omitted. Including the boundary conditions from equations (19a) and (23a), the first-order problem is

$$\left.\begin{aligned}
&(1 - t^2)\bar{\phi}_{tt} + \frac{\bar{\phi}_t}{t} = 0 \\
&\beta\bar{\phi}_t(\beta\epsilon) = \epsilon\left[1 + \bar{\phi}(\beta\epsilon) - \beta\epsilon\bar{\phi}_t(\beta\epsilon)\right] \\
&\bar{\phi}(\infty) = \bar{\phi}_t(\infty) = 0
\end{aligned}\right\} \tag{68}$$

Using the integrating factor $t/\sqrt{1 - t^2}$, the equation can be integrated to give the well-known result that

$$\bar{\phi} = -A\left(\operatorname{sech}^{-1}t - \sqrt{1 - t^2}\right) \tag{69}$$

which is understood to vanish except within the downstream Mach cone, where $t \leqq 1$. The tangency condition is satisfied by putting

$$A = \frac{\epsilon^2}{\sqrt{1 - \beta^2\epsilon^2} + \epsilon^2\operatorname{sech}^{-1}(\beta\epsilon)} \tag{70}$$

At the Mach cone $(t = 1)$, all velocity perturbations vanish, so that the first-order solution predicts no deflection of the shock wave from the Mach cone (see reference 13, p. 403).

Substituting the first-order solution into the iteration equation (equation (17)) gives

$$\left(1 - t^2\right)\bar{\varphi}_{tt} + \frac{\bar{\varphi}_t}{t} = A^2M^2\left[2\,\frac{1}{t^2} + 2(N - 1)\,\frac{\operatorname{sech}^{-1}t}{\sqrt{1 - t^2}} - \beta^2A\,\frac{\sqrt{1 - t^2}}{t^4}\right] \tag{71a}$$

and from equations (19b') and (23b) the corresponding boundary conditions are

$$\beta\bar{\varphi}_t(\beta\epsilon) = \epsilon\left[\bar{\varphi}(\beta\epsilon) - \beta\epsilon\bar{\varphi}_t(\beta\epsilon)\right] \tag{71b}$$

$$\bar{\varphi}(\infty) = \bar{\varphi}_t(\infty) = 0 \tag{71c}$$

Equation (71a) can again be solved using the integrating factor $t/\sqrt{1 - t^2}$. The various integrals encountered can invariably be treated by integrating by parts one or more times. Using the second boundary condition, the complete conical second-order perturbation potential is found to be

$$\begin{aligned}\bar{\Phi}^{(2)}(t) = {} & -A\left(\operatorname{sech}^{-1}t - \sqrt{1 - t^2}\right) + A^2M^2\left[B\left(\operatorname{sech}^{-1}t - \sqrt{1 - t^2}\right) + \right.\\ & \left.\left(\operatorname{sech}^{-1}t\right)^2 - (N + 1)\sqrt{1 - t^2}\,\operatorname{sech}^{-1}t - \frac{\beta^2A}{4}\,\frac{(1 - t^2)^{3/2}}{t^2}\right] + \\ & O\left[\epsilon^6\left(\operatorname{sech}^{-1}t\right)^3\right]\end{aligned} \tag{72}$$

From equation (16), the streamwise and radial velocity perturbations are

$$\frac{u}{U} = -A \operatorname{sech}^{-1}t + A^2M^2\left[B \operatorname{sech}^{-1}t + \left(\operatorname{sech}^{-1}t\right)^2 - (N - 1)\frac{\operatorname{sech}^{-1}t}{\sqrt{1 - t^2}} - (N + 1) - \frac{3}{4}\beta^2 A \frac{\sqrt{1 - t^2}}{t^2}\right] \tag{73a}$$

$$\frac{1}{\beta}\frac{v}{U} = A\frac{\sqrt{1 - t^2}}{t} + A^2M^2\left[-B\frac{\sqrt{1 - t^2}}{t} - 2\frac{\sqrt{1 - t^2}\operatorname{sech}^{-1}t}{t} + (N + 1)\frac{1}{t} + (N - 1)\frac{t \operatorname{sech}^{-1}t}{\sqrt{1 - t^2}} + \frac{1}{4}\beta^2 A\frac{(2 + t^2)\sqrt{1 - t^2}}{t^3}\right] \tag{73b}$$

The constant B must be adjusted so as to satisfy the tangency condition, equation (71b). In actual computation it is easier to adjust B numerically in exactly this fashion rather than to calculate it from the cumbersome expression which could be written down. The pressure coefficient at any point can then be calculated from equation (26).

The last term in the bracket in equation (71a) is the triple product $\beta^2\bar{\phi}_{tt}\bar{\phi}_t^2$ which is retained in iteration equation (17). Its retention is now justified by noting that its contribution - the last term in equation (72) - is of the same order as other terms near the surface of the cone $(t = \beta\epsilon)$. Actually, it contributed a second term, which has been neglected since it is at most of order $\epsilon^6\operatorname{sech}^{-1}\beta\epsilon$. It can also be verified that the other triple products, whose form is indicated at the end of equation (17), are in fact negligible, since they contribute at most terms of order $\epsilon^6(\operatorname{sech}^{-1}\beta\epsilon)^2$. Consideration of a further iteration indicates that a third approximation would add terms no greater than $\epsilon^6(\operatorname{sech}^{-1}\beta\epsilon)^3$, which is greater than the terms just neglected.

The second-order result for surface pressure coefficient is compared in figure 2 with the exact solution (reference 17) for cones of 5°, 10°, 15°, and 20° semivertex angles. Also shown for comparison are the first-order results based upon the exact expression (equation (26)) for the pressure coefficient. The second-order solution is seen to provide a much better approximation throughout the range of Mach numbers up to the point at which the Mach angle equals the cone angle, beyond which the perturbation solutions have no physical meaning.

Shock-wave angle.- The solution for plane flow past a corner suggests that the second-order solution for the cone may fail near the Mach cone. However, if it is valid there, a first approximation to the shock-wave deflection and, consequently, the entropy change can be calculated from the fact that to first order an oblique shock bisects the Mach lines. It was noted before that first-order theory predicts no difference between the shock position and the Mach cone.

Assume provisionally that the solution is valid at the Mach cone, while indicating by ? the possibility that it is not. From equation (73) the velocity perturbations just behind the Mach cone are

$$\left.\begin{aligned}\left(\frac{u}{U}\right)_{t=1} &\overset{?}{=} -2M^2N\epsilon^4 \\ \left(\frac{v}{U}\right)_{t=1} &\overset{?}{=} 2\beta M^2N\epsilon^4\end{aligned}\right\} \tag{74}$$

so that the perturbation is normal to the Mach cone. Here A (equation (70)) has been approximated by ϵ^2. From equation (50b) the cotangent of the revised Mach angle just behind the cone is found to be

$$\beta^{(1)} \overset{?}{=} \beta\left[1 - 2M^4N(N - 1)\epsilon^4\right] \tag{75}$$

The upward stream inclination there is approximately $\left(\frac{v}{U}\right)_{t=1}$, so that the Mach lines have the slope

$$\frac{dr}{dx} \overset{?}{=} \frac{1}{\beta}\left(1 + 2M^4N^2\epsilon^4\right) \tag{76}$$

Therefore, the slope of the shock wave differs from that of the original Mach cone by

$$\tan\lambda - \frac{1}{\beta} \overset{?}{=} \frac{M^4N^2}{\beta}\epsilon^4 = \frac{(\gamma + 1)^2M^8}{4\beta^5}\epsilon^4 \tag{77}$$

This problem has been treated rigorously by Lighthill (reference 21) and by Broderick (reference 22), who find that actually

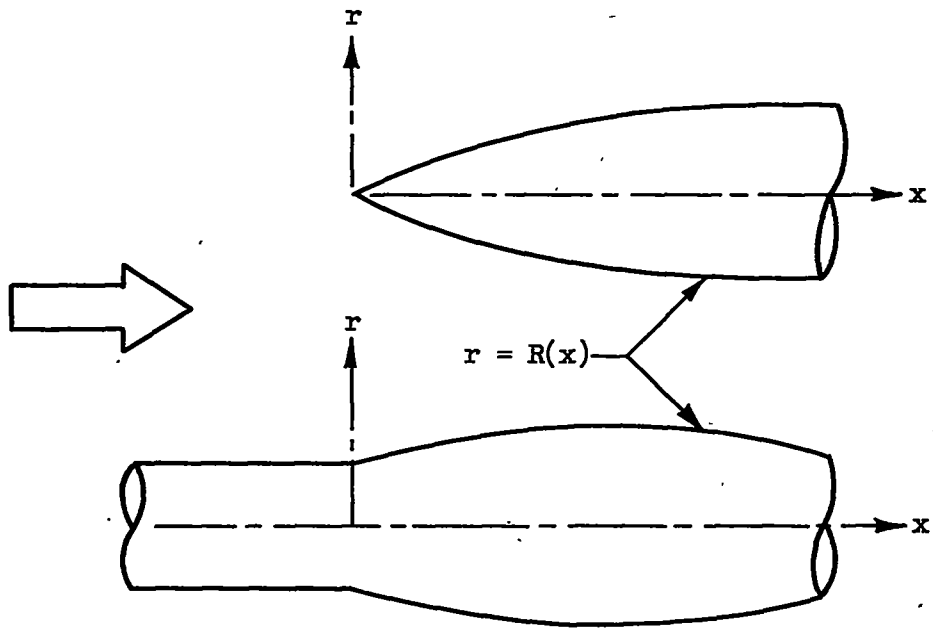
$$\tan\lambda - \frac{1}{\beta} = \frac{3}{8}\frac{(\gamma + 1)^2M^8}{\beta^5}\epsilon^4 \tag{78}$$

which is $1\frac{1}{2}$ times the above result. The discrepancy means that the

second-order solution fails near the Mach cone. It seems remarkable that the result is in error only to the extent of a constant factor.

The entropy increase through a weak oblique shock wave is proportional to the cube of its inclination away from the Mach lines. Consequently, the entropy rise through the shock wave from a cone is $O(\epsilon^{12})$, as noted by Lighthill (reference 21).

Particular solution for axially symmetric flow.- Consider flow past a body of revolution which is either a slender pointed body with nose at the origin or one which extends indefinitely upstream with constant radius a for $x \leqq 0$ (see diagram). The latter shape corresponds to



Flow past bodies of revolution.

the external flow past a sharp-edged, open-nosed body with supersonic internal flow. With slight modification the subsequent development can be applied to internal flow as well. The meridian curve can be represented in the first case by

$$r = R(x) = \epsilon g(x) \qquad x \geqq 0 \tag{79a}$$

and in the second by

$$r = R(x) = \begin{cases} a & x \leqq 0 \\ a + \epsilon g(x) & x \geqq 0 \end{cases} \tag{79b}$$

Here ϵ is again a parameter small compared with unity, and $g(x)$ is a continuous function of order unity which vanishes for $x \leqq 0$.

The first-order problem is

$$\phi_{rr} + \frac{\phi_r}{r} - \beta^2\phi_{xx} = 0 \tag{80a}$$

$$\phi_r(x,R) = R'(x)\left[1 + \phi_x(x,R)\right] \tag{80b}$$

$$\phi(0,r) = \phi_x(0,r) = 0 \tag{80c}$$

The solution is known to be (reference 23)

$$\phi(x,r) = -\int_b^{x-\beta r} \frac{F(\xi)\ d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} = -\int_0^{\cosh^{-1}\frac{x-b}{\beta r}} F(x - \beta r \cosh u)\ du \tag{81}$$

The second form is useful for carrying out differentiation, after which the first form can be restored. The derivatives which will be required are

$$\phi_x = -\int_0^{\cosh^{-1}\frac{x-b}{\beta r}} F'(x - \beta r \cosh u)\ du = -\int_b^{x-\beta r} \frac{F'(\xi)\ d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \tag{82a}$$

$$\phi_r = \beta\int_0^{\cosh^{-1}\frac{x-b}{\beta r}} F'(x - \beta r \cosh u) \cosh u\ du = \frac{1}{r}\int_b^{x-\beta r} \frac{(x-\xi)F'(\xi)\ d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} \tag{82b}$$

With coordinates as shown in the preceding diagram, the lower limit of integration b is 0 for the pointed body and $-\beta a$ for the semi-infinite body. The function $F(x)$ may be interpreted physically as the strength of a supersonic line source along the x-axis. It is determined by the tangency condition, equation (80b), which gives the following integral equation of Volterra type for F':

$$\int_b^{x-\beta R(x)} \frac{(x-\xi)F'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2(x)}} = R(x)R'(x)\left[1-\int_b^{x-\beta R(x)} \frac{F'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2(x)}}\right] \tag{83}$$

The second-order iteration equation is found from equation (13) to be

$$\varphi_{rr} + \frac{\varphi_r}{r} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} 2(N-1)\beta^2\phi_{xx}\phi_x + 2\phi_{xr}\phi_r + \\ \phi_{rr}\phi_r^2 + \\ O\left(\phi_{xx}\phi_x^2,\ \phi_{xx}\phi_r^2,\ \phi_{xr}\phi_x\phi_r\right) \end{bmatrix} \tag{84}$$

The solution for the cone suggests that the terms indicated in the last line are negligible.

It will now be shown that a particular solution of this equation is given by

$$\psi(x,r) = M^2\phi_x\left(\phi + Nr\phi_r\right) - \frac{1}{4}M^2r\phi_r^3 \tag{85}$$

The first group of terms contributes the first line in equation (84), as can be verified by direct substitution. The last term in equation (85) accounts for the term $\phi_{rr}\phi_r^2$ as follows:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\beta^2\frac{\partial^2}{\partial x^2}\right)\left(-\frac{1}{4}r\phi_r^3\right)=-\phi_r\left(\frac{9}{4}\phi_{rr}\phi_r+\frac{3}{2}r\phi_{rr}^2+\frac{3}{4}r\phi_{rrr}\phi_r+\right.$$
$$\left.\frac{1}{4}\frac{\phi_r^2}{r}-\frac{3}{2}\beta^2 r\phi_{xr}^2-\frac{3}{4}\beta^2 r\phi_{xxr}\phi_r\right)$$
$$=\phi_r\left[\phi_{rr}\phi_r-\frac{3}{4}r\phi_r\left(\phi_{rrr}+\frac{\phi_{rr}}{r}-\frac{\phi_r}{r^2}-\beta^2\phi_{xxr}\right)-\right.$$
$$\left(\phi_r+\frac{3}{2}r\phi_{rr}\right)\left(\phi_{rr}+\frac{\phi_r}{r}-\beta^2\phi_{xx}\right)-$$
$$\left.\beta^2\left(\frac{3}{2}r\phi_{xx}\phi_{rr}+\phi_{xx}\phi_r-\frac{3}{2}r\phi_{xr}^2\right)\right]$$
$$=\phi_{rr}\phi_r^2-\beta^2\phi_r\left(\frac{3}{2}r\phi_{xx}\phi_{rr}+\phi_{xx}\phi_r-\frac{3}{2}r\phi_{xr}^2\right) \quad (86)$$

where repeated use is made of the fact that ϕ satisfies equation (80a). The last group of terms consists of triple products involving x-derivatives, which have already been neglected in equation (84), so that the result is proved.

The correction potential χ' is a solution of equation (80) and can be written as

$$\chi(x,r)=-\int_b^{x-\beta r}\frac{f(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}}=-\int_0^{\cosh^{-1}\frac{x-b}{\beta r}}f(x-\beta r\cosh u)\,du \quad (87)$$

Using equation (82) the second-order tangency condition (equation (19b')) is found to be

$$\psi_r(x,R)+\frac{1}{R}\int_b^{x-\beta R}\frac{(x-\xi)f'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2}}=R'\left[\psi_x(x,R)-\int_b^{x-\beta R}\frac{f'(\xi)\,d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2}}\right] \quad (88)$$

which is again a Volterra integral equation.

Methods of solving integral equation.- Discovery of a particular integral for bodies of revolution reduces the second-order problem to

the same form as the first-order problem - namely, the solution of a Volterra integral equation. Various methods of attacking this problem are listed by Hayes (reference 15, p. 140).

An indirect method consists in assuming that the unknown source strengths in equations (81) and (87) can be represented by a few terms of a polynomial, for example, that in equation (81)

$$F(x) = C_1x + C_2x^2 + \ldots + C_nx^n \tag{89}$$

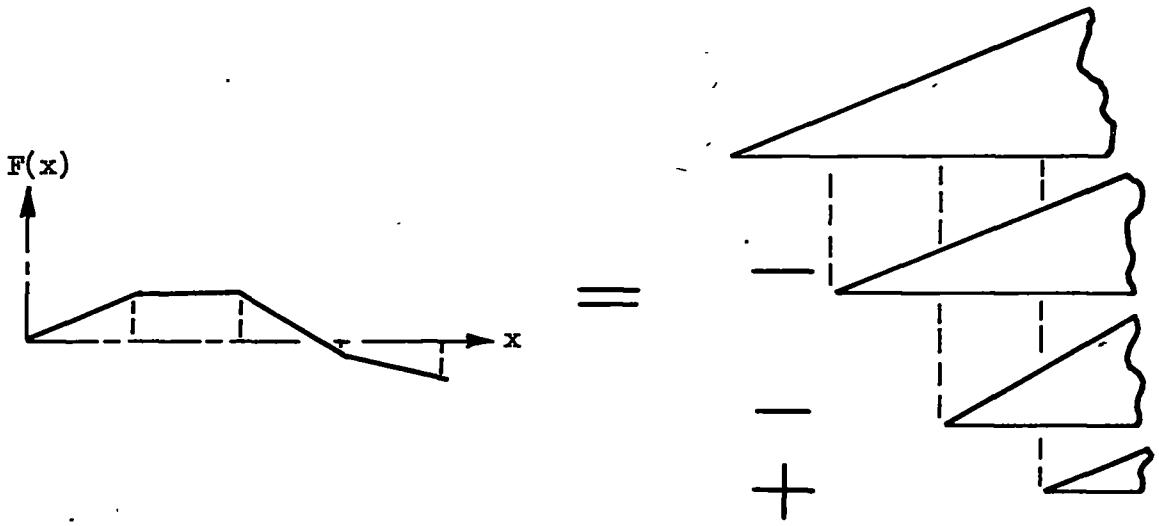
The resulting solutions were introduced in a more formal manner by Hayes (reference 15, p. 38), who has discussed their properties in detail. The first term alone gives the potential for the cone, equation (69). Additional terms give the solution for simple families of shapes. However, the method is not suitable for bodies having discontinuities in slope or curvature. Consequently, a more direct procedure is desirable.

Kármán first introduced an asymptotic solution of the integral equation (equation (83)) which has come to be known as the slender-body approximation. For slender bodies, the source strength $F(x)$ appearing in equation (81) is found to be approximately proportional to the rate of change of cross-sectional area. Thus

$$F(x) \approx \frac{1}{2\pi}\frac{dS(x)}{dx} = R(x)R'(x) \tag{90}$$

where $S(x) = \pi R^2(x)$ is the cross-sectional area of the body. Lighthill has shown (reference 16) that if $R(x)$ and its first two derivatives are of order ϵ, and R' is continuous, then this determination of $F(x)$ is correct to the order of terms retained in the first-order solution. For purposes of the second-order solution, it can be shown that $F(x)$ may be determined in this way only if R'' is also continuous. This means that the body must have continuous curvature, which is a severe limitation. Moreover, the slender-body approximation is found generally to cause unnecessary loss of accuracy even though the mathematical order estimate of the error is small. Consequently, this approximation is not to be recommended if it can be avoided.

The most satisfactory way of solving the integral equations is to use a step-by-step numerical procedure. In first-order theory the usual method, introduced by Kármán and Moore (reference 24), is to assume that the unknown source distribution can be approximated by a polygonal graph. This is equivalent to superimposing a number of conical source lines of different strengths, each shifted downstream with respect to its predecessor, as indicated in the following diagram:

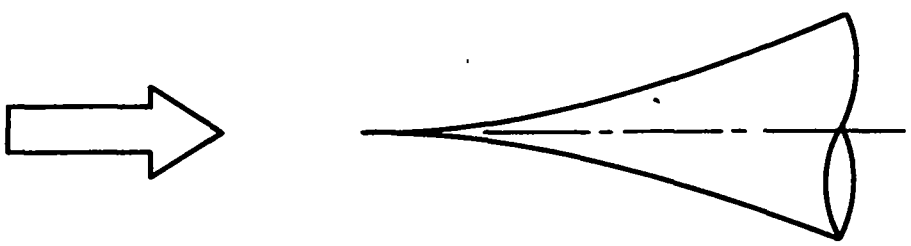


Equivalence of polygonal source and sum of conical sources.

The latter viewpoint is more convenient for computation. The strengths of the source lines are determined in succession by satisfying the tangency condition at a series of points on the surface of the body. The details of this procedure are clearly explained in reference 1, page 77.

For purposes of a second-order solution, this procedure must be modified in one respect. The source distribution $F(x)$ must not be approximated by a polygon unless it actually has corners. The reason for this is that a corner corresponds locally to adding a conical source line, which gives the solution for a cone. But it was found in the case of the cone that to second order the velocities are discontinuous across the Mach wave. This would cause false pressure jump in the flow field.

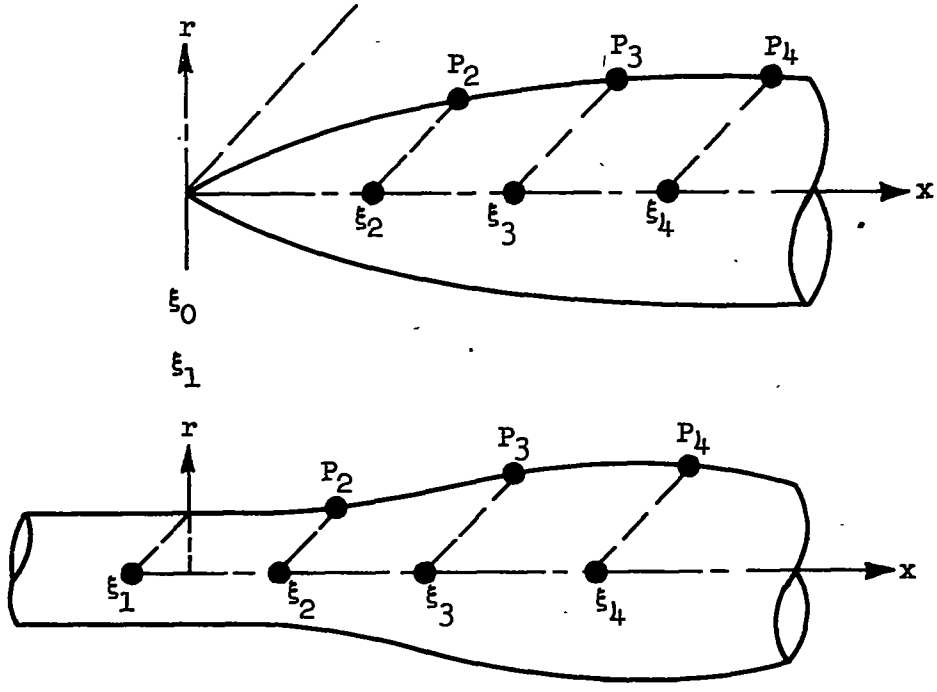
Instead, the procedure must be carried out using source lines of quadratic strength. The source strength $F(x)$ can then be approximated smoothly, so that false pressure jumps do not occur. A single source line of this type represents the flow past a slender pointed body with a cusped nose (see sketch), as is clear from the slender-body approximation, equation (90).



Body formed by source line of quadratic strength.

Method of solution for smooth bodies.- The second-order solution will be described first for bodies having continuous slope. Modifications for treating sharp corners will be discussed in the next section.

The procedure is indicated in the following diagram. The axis is



Method of solution for smooth bodies.

divided into intervals by choosing points with abscissas ξ_n, at each of which a source line is to begin. Good accuracy is usually obtained if the interval length is not greater than β times the local radius. The tangency condition will be imposed on the surface of the body at the points P_n, which lie on the Mach lines from the points at ξ_n.

For pointed bodies, the first-order solution is started with a conical source from the origin which gives the proper conical tip. This potential and the derivatives which are required are

$$\left.\begin{aligned}
\phi_o &= -C_o x\left(\operatorname{sech}^{-1}t - \sqrt{1 - t^2}\right) & (\phi_o)_{xx} &= -\frac{C_o}{x}\frac{1}{\sqrt{1 - t^2}} \\
(\phi_o)_x &= -C_o \operatorname{sech}^{-1}t & (\phi_o)_{xr} &= \frac{\beta C_o}{x}\frac{1}{t\sqrt{1 - t^2}} \\
(\phi_o)_r &= \beta C_o \frac{\sqrt{1 - t^2}}{t} & (\phi_o)_{rr} &= -\frac{\beta^2 C_o}{x}\frac{1}{t^2\sqrt{1 - t^2}}
\end{aligned}\right\} \tag{91}$$

where

$$C_o = \frac{\epsilon^2}{\sqrt{1 - \beta^2\epsilon^2} + \epsilon^2 \operatorname{sech}^{-1}\beta\epsilon}$$

and the semivertex angle of the conical tip is $\tan^{-1}\epsilon$. No such term is required for the semi-infinite body.

The subsequent procedure is the same for either body. Quadratic source lines are started from each of the points ξ_1, ξ_2, and so forth. For the pointed body ξ_1 is also at the origin, while for the semi-infinite body it is at $-\beta a$. For the nth such source line, the potential and its derivatives are given by

$$\left.\begin{aligned}
\phi_n &= -C_n(x - \xi_n)^2\left[\left(1 + \frac{1}{2}\tau_n^2\right)\operatorname{sech}^{-1}\tau_n - \frac{3}{2}\sqrt{1 - \tau_n^2}\right] \\
(\phi_n)_x &= -2C_n(x - \xi_n)\left(\operatorname{sech}^{-1}\tau_n - \sqrt{1 - \tau_n^2}\right) \\
(\phi_n)_r &= \beta C_n(x - \xi_n)\left(\frac{\sqrt{1 - \tau_n^2}}{\tau_n} - \tau_n \operatorname{sech}^{-1}\tau_n\right) \\
(\phi_n)_{xx} &= -2C_n \operatorname{sech}^{-1}\tau_n \\
(\phi_n)_{xr} &= 2\beta C_n \frac{\sqrt{1 - \tau_n^2}}{\tau_n} \\
(\phi_n)_{rr} &= -\beta^2 C_n\left(\frac{\sqrt{1 - \tau_n^2}}{\tau_n^2} + \operatorname{sech}^{-1}\tau_n\right)
\end{aligned}\right\} \tag{92}$$

where

$$\tau_n = \frac{\beta r}{x - \xi_n}$$

The constants C_n are determined successively by imposing the first-order tangency condition in turn at each of the points P_n. From equation (80b), the condition is that

$$\frac{\sum^{n-1} (\phi_n)_r}{1 + \sum^{n-1} (\phi_n)_x} = \text{Slope} \tag{93}$$

where the summation begins with $n = 0$ for the pointed body and $n = 1$ for the semi-infinite body. In this way, values of the complete first-order potential ϕ and its first and second derivatives are calculated at each of the points P_n.

The velocities due to the particular second-order solution ψ can then be calculated at the same points. Differentiating equation (85) gives

$$\left.\begin{aligned}
\psi_x &= M^2\left[\phi_{xx}\left(\phi + Nr\phi_r\right) + \phi_x\left(\phi_x + Nr\phi_{xr}\right) - \frac{3}{4} r\phi_{xr}\phi_r{}^2\right] \\
\psi_r &= M^2\left\{\phi_{xr}\left(\phi + Nr\phi_r\right) + \phi_x\left[(N + 1)\phi_r + Nr\phi_{rr}\right] - \frac{1}{4} \phi_r{}^2\left(\phi_r + 3r\phi_{rr}\right)\right\}
\end{aligned}\right\} \tag{94}$$

Finally, the second-order correction potential χ is determined by repeating the procedure used for ϕ, finding new constants such that the second-order tangency condition is satisfied. From equation (19b'), the condition is that

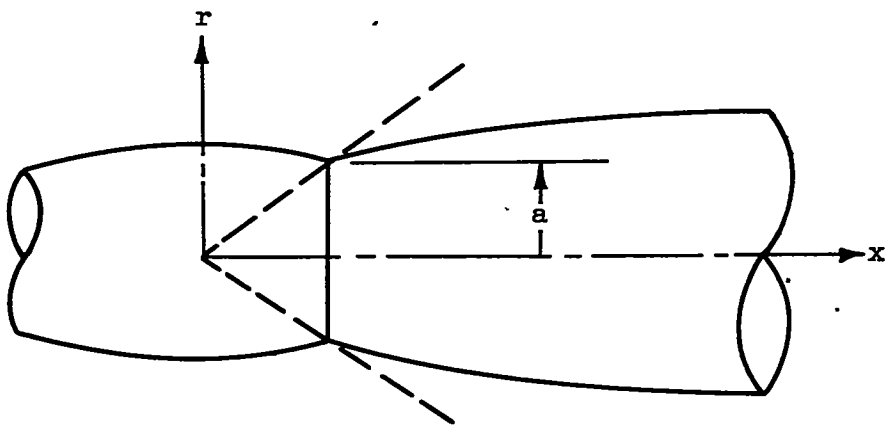
$$\frac{\psi_r + \sum^{n-1} (\chi_n)_r}{\psi_x + \sum^{n-1} (\chi_n)_x} = \text{Slope} \tag{95}$$

The second derivatives of χ need not be calculated.

The complete second-order perturbation velocities are the sums of the contributions from ϕ, ψ, and χ. Then the pressure coefficient can be calculated at each point P_n from equation (26).

The computing time required is several times that for a careful first-order solution.

Treatment of bodies with corners.- Suppose that the meridian curve of the body has a sharp corner, which for convenience may be assumed to lie on the Mach cone from the origin (see fig.). Then the method of



Body with corner.

solution must be modified for two reasons.

In the first place, the intervals between source lines would have to be chosen extremely small in order to obtain an accurate first-order solution behind the corner. This difficulty can be eliminated, however, by adding a new solution which causes a sharp deflection of the streamlines. In this way the corner is effectively removed.

Such a solution can be found by approximating to equation (83) in the vicinity of the Mach cone. The resulting Abel integral equation can be solved to show that, in general, a potential having discontinuous nth derivatives results from a source distribution along the axis which is initially proportional to $x^{n-\frac{1}{2}}$. Setting $F(x) = x^{n-\frac{1}{2}}$ in equation (81) gives

$$\phi(x,r) = -\int_0^{x-\beta r} \frac{\xi^{n-\frac{1}{2}}\,d\xi}{\sqrt{(x-\xi)^2 - \beta^2 r^2}} = -\frac{(x-\beta r)^n}{\sqrt{2\beta r}} \int_0^1 \frac{(1-\xi)^{n-\frac{1}{2}}\,d\xi}{\sqrt{\xi\left(1 + \frac{x-\beta r}{2\beta r}\,\xi\right)}} \tag{96}$$

This integral represents the analytical continuation of the hypergeometric function, so that, except for a constant factor,

$$\phi(x,r) = (x - \beta r)^n \sqrt{\frac{a}{r}}\, F\left(\frac{1}{2}, \frac{1}{2}, n + 1, -\frac{x - \beta r}{2\beta r}\right) \tag{97}$$

where a is the radius at the corner. The potential is understood to vanish except within the downstream Mach cone from the origin. The hypergeometric functions occurring here can all be expressed in terms of complete elliptic integrals with real modulus.

The solution for a corner is obtained by taking $n = 1$. Then

$$\left.\begin{aligned}
\phi &= -\frac{4}{\pi} x \sqrt{\frac{a}{r}} (1 + t) \sqrt{\frac{2t}{1 + t}} \left[K\left(\frac{1 - t}{1 + t}\right) - E\left(\frac{1 - t}{1 + t}\right)\right] \\
\phi_x &= -\frac{2}{\pi} \sqrt{\frac{a}{r}} \sqrt{\frac{2t}{1 + t}}\, K\left(\frac{1 - t}{1 + t}\right) \\
\phi_r &= \frac{2\beta}{\pi} \sqrt{\frac{a}{r}} \sqrt{\frac{2t}{1 + t}} \left[\frac{1 + t}{t} E\left(\frac{1 - t}{1 + t}\right) - K\left(\frac{1 - t}{1 + t}\right)\right] \\
\phi_{xx} &= \frac{1}{\pi} \frac{1}{x} \sqrt{\frac{a}{r}} \frac{1}{1 - t} \sqrt{\frac{2t}{1 + t}} \left[K\left(\frac{1 - t}{1 + t}\right) - E\left(\frac{1 - t}{1 + t}\right)\right] \\
\phi_{xr} &= \frac{\beta}{\pi} \frac{1}{x} \sqrt{\frac{a}{r}} \frac{1}{1 - t} \sqrt{\frac{2t}{1 + t}} \left[\frac{1}{t} E\left(\frac{1 - t}{1 + t}\right) - K\left(\frac{1 - t}{1 + t}\right)\right] \\
\phi_{rr} &= -\frac{\beta^2}{\pi} \frac{1}{x} \sqrt{\frac{a}{r}} \cdot \frac{1}{1 - t} \sqrt{\frac{2t}{1 + t}} \left[\frac{2 - t^2}{t^2} E\left(\frac{1 - t}{1 + t}\right) - \frac{2 - t}{t} K\left(\frac{1 - t}{1 + t}\right)\right]
\end{aligned}\right\} \tag{98}$$

Here t is the conical variable introduced in equation (14), and $K\left(\frac{1 - t}{1 + t}\right)$ and $E\left(\frac{1 - t}{1 + t}\right)$ are the complete elliptic integrals of the first and second kinds with modulus k such that $k^2 = \frac{1 - t}{1 + t}$.

From the tangency condition, equation (80b), it can be shown that the above solution should be multiplied by

$$\frac{(R_2' - R_1')\left[1 + (\Phi_x)_1\right]}{\beta + R_2'} \tag{99}$$

in order to cancel the corner. Here R_1' and R_2' are the slopes of the meridian curve just before and after the corner and $(\Phi_x)_1$ is the value before the corner. The first-order solution can then be continued as described in the previous section.

The second difficulty which arises is that, even though the first-order solution may be exact, the second-order solution described previously is incorrect behind the corner. It is clear that the local pressure jump at a corner should have the value corresponding to plane flow past a corner of the same angle. However, the second-order solution described above yields the first-order, rather than the second-order, value of the pressure jump. The method of solution must be modified in order to obtain the correct result.

The proper procedure would be to find the solution for the case when the corner has been slightly rounded off and then pass to the limit of a sharp corner. However, the following simpler procedure is found to give exactly the same result.

The particular solution ψ calculated from equation (85) is discontinuous along the Mach wave springing from the corner. If the discontinuity vanished at the corner, the solution could subsequently be revised as in the case of plane flow (see diagrams of Mach lines in section "Flow past a corner and a parabolic bend"). However, there is a finite jump in ψ directly at the corner, which cannot be allowed. Consequently, the correction potential χ must involve an equal and opposite jump. A potential having such a discontinuity is obtained by setting $n = 0$ in equation (97). Then

$$\left.\begin{aligned}
\chi &= \frac{2}{\pi}\sqrt{\frac{a}{r}}\sqrt{\frac{2t}{1+t}}\,K\left(\frac{1-t}{1+t}\right) \\
\chi_x &= -\frac{1}{\pi}\frac{1}{x}\sqrt{\frac{a}{r}}\frac{1}{1-t}\sqrt{\frac{2t}{1+t}}\left[K\left(\frac{1-t}{1+t}\right) - E\left(\frac{1-t}{1+t}\right)\right] \\
\chi_r &= -\frac{\beta}{\pi}\frac{1}{x}\sqrt{\frac{a}{r}}\frac{1}{1-t}\sqrt{\frac{2t}{1+t}}\left[\frac{1}{t}E\left(\frac{1-t}{1+t}\right) - K\left(\frac{1-t}{1+t}\right)\right]
\end{aligned}\right\} \tag{100}$$

Adding a suitable multiple of this potential cancels the discontinuity in ψ. The second-order solution can then be continued as described in the preceding section. It can be verified that the pressure jump at the corner has then the correct second-order value.

It is instructive to analyze the behavior of a corner from another viewpoint. It was pointed out before that the right-hand side of the

iteration equation may be considered to represent the effects of a known distribution of sources throughout the flow field. In the case of a slightly rounded corner, this source distribution will be weak except between the Mach lines from the corner. As the corner shrinks to a point, the source intensity will increase in that region in such a way that the total strength remains constant. In the limit, the source distribution will behave like a Dirac δ-function along the Mach line from the corner. The particular solution for plane flow (equation (35)) takes account of this impulsive function, so that the correct solution is automatically obtained. In the case of axially symmetric flow, however, it is clear that the particular solution given by equation (85) misses the contribution of the impulse. It is therefore necessary to correct this shortcoming by adding the step-function potential given by equation (100).

Comparison with numerical solutions.- The accuracy of the second-order solution for bodies of revolution can be evaluated by comparison with examples calculated using the numerical method of characteristics.

The first body to be considered is a 12.5-caliber ogive, which has a semivertex angle of 16.26° at the tip. The second-order solution was calculated for this body at a Mach number of 3.24. This represents a severe test of the method, because the Mach angle is then only 10 percent greater than the tip cone angle. Intervals were chosen such that the points P_n lay at 0.1, 0.25, 0.5, 1, 2, and 3.5 calibers (fig. 3). The pressure distributions calculated by first- and second-order theory are compared in figure 3 with the results of various characteristics solutions summarized in reference 25. The second-order solution apparently coincides with the characteristics solutions to within the accuracy of the characteristics method.

The second body to be considered consists of a cone of 10° semivertex angle followed by a circular cylinder. The characteristics solution for this body at a Mach number of 2.075 has been given by Liepmann and Lapin in reference 26. The first- and second-order solutions were calculated beyond the corner using the modifications discussed in the preceding section. Figure 4 shows the shape of the body, the location of source lines, and the velocity distributions calculated by first-order theory, second-order theory, and the method of characteristics. Again, the second-order results agree with the characteristics solution to within the accuracy of the latter method.

Series expansion with respect to thickness.- An alternative method of solving the exact perturbation equation (equation (4)) by successive approximations is to assume that the solution can be expanded in powers of the thickness parameter ϵ. Thus the exact perturbation potential is written as

$$\Phi = \overset{(1)}{\Phi}\epsilon + \overset{(2)}{\Phi}\epsilon^2 + \overset{(3)}{\Phi}\epsilon^3 + \overset{(4)}{\Phi}\epsilon^4 + \ldots \tag{102}$$

Substituting into equation (4) and equating like powers of ϵ yield a sequence of equations

$$\left.\begin{aligned}
&\overset{(1)}{\Phi}_{yy} + \overset{(1)}{\Phi}_{zz} - \beta^2 \overset{(1)}{\Phi}_{xx} = 0 \\
&\overset{(2)}{\Phi}_{yy} + \overset{(2)}{\Phi}_{zz} - \beta^2 \overset{(2)}{\Phi}_{xx} = 2M^2\left[(N - 1)\beta^2 \overset{(1)}{\Phi}_{xx}\overset{(1)}{\Phi}_x + \overset{(1)}{\Phi}_{xy}\overset{(1)}{\Phi}_y + \overset{(1)}{\Phi}_{zx}\overset{(1)}{\Phi}_z\right] \\
&\qquad\qquad \ldots\ldots\ldots
\end{aligned}\right\} \tag{103}$$

which can be solved in succession. The first is again the usual linearized equation. This method was applied to plane subsonic flow in references 4 and 9.

Schmieden and Kawalki first pointed out (reference 11) that the power series assumed here does not always exist. In general, terms of the form $\epsilon^m(\log \epsilon)^n$ begin to appear in the third-order solution for plane flow and in the second-order solution for three-dimensional flow. This difficulty can be met by assuming a more general series of the form

$$\Phi = \overset{(1)}{\Phi}\epsilon + \overset{(2)}{\Phi}\epsilon^2 + \overset{(3)}{\Phi}\epsilon^3 + \overset{(4)}{\Phi}\epsilon^4\log \epsilon + \overset{(5)}{\Phi}\epsilon^4 + \ldots \tag{104}$$

On the basis of this assumption, Broderick has developed a second-order solution for supersonic flow past slender pointed bodies of revolution (reference 27). The analysis is rather lengthy, since the simplification resulting from the discovery of a particular solution does not appear. The results are limited to shapes for which the cross-sectional area is given by an analytic function, or at any rate possesses continuous derivatives up to the fourth order. This is a severe limitation since, for example, the two bodies discussed in the previous section are not admissible.

Broderick's result can be obtained by expanding the present second-order solution in powers of t and $\log t$ for small values of t. The logarithmic terms arise from the series

$$\operatorname{sech}^{-1} t = \log \frac{2}{t} - \frac{1}{4} t^2 - \frac{3}{32} t^4 - \ldots \qquad (0 < t < 1) \tag{105}$$

The expansion will now be carried out for the case of flow past a cone.

It is clear from equation (70) that the constant A in the first-order solution (equation (69)) is given approximately by

$$A = \epsilon^2 + \ldots \tag{106}$$

Substituting this value into equations (73a) and (73b), expanding in powers of t and $\log \frac{2}{t}$, and imposing the tangency condition, equation (71b), shows that

$$B = -\frac{2M^2 - 1}{M^2} \log \frac{2}{\beta\epsilon} + \left(N + 1 + \frac{1}{2M^2}\right) + \ldots \tag{107}$$

Then according to equation (73), the velocity perturbations on the surface of the cone are

$$\frac{u}{U} = -\epsilon^2 \log \frac{2}{\beta\epsilon} - \epsilon^4 \left[\beta^2 \left(\log \frac{2}{\beta\epsilon}\right)^2 - \frac{4M^2 + 1}{2} \log \frac{2}{\beta\epsilon} + M^2 N + \frac{6M^2 + 1}{4}\right] + \ldots \tag{108a}$$

$$\frac{v}{U} = \epsilon - \epsilon^3 \log \frac{2}{\beta\epsilon} + \ldots \tag{108b}$$

Replacing N by its value from equation (11), equation (27) gives for the pressure coefficient on the surface of the cone

$$\begin{aligned} C_p = {} & \epsilon^2 \left(2 \log \frac{2}{\beta\epsilon} - 1\right) + \\ & \epsilon^4 \left[3\beta^2 \left(\log \frac{2}{\beta\epsilon}\right)^2 - \left(5M^2 - 1\right) \log \frac{2}{\beta\epsilon} + (\gamma + 1) \frac{M^4}{\beta^2} + \frac{13}{4} M^2 + \frac{1}{2}\right] + \\ & O\left[\epsilon^6 \left(\log \frac{2}{\beta\epsilon}\right)^3\right] \end{aligned} \tag{109}$$

This is Broderick's result (reference 27, equation (81)).

This series is compared with the original form of the second-order solution in figure 5. For the most slender cone, the expansion in series causes only a moderate loss in accuracy. For more practical thicknesses, however, the expansion reduces the accuracy to such an extent that for the cone of 20° semivertex angle, Broderick's solution is inferior to the first-order result. The reason must be that the iteration process itself converges more rapidly than do the subsequent expansions which are required to reduce it to series form. Hence, terminating all expansions at terms of the order of those retained in the iteration process results in an unnecessary loss of accuracy.

THREE-DIMENSIONAL FLOW

Partial particular solution.- It might be hoped that a particular solution, which so greatly simplifies the iteration for plane and axially symmetric flows, could be found for the general three-dimensional case. The various methods of existing first-order theory could then be applied immediately to the problems of second-order flow past such shapes as inclined bodies of revolution and three-dimensional wings.

A part of such a particular solution is found at once, being common to the two special cases. Consider the three-dimensional iteration equation (equation (12)):

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} 2(N - 1)\beta^2\phi_{xx}\phi_x + 2\phi_{xy}\phi_y + 2\phi_{zx}\phi_z + \\ \phi_{yy}\phi_y{}^2 + 2\phi_{yz}\phi_y\phi_z + \phi_{zz}\phi_z{}^2 \end{bmatrix} \tag{110}$$

It can be readily verified that taking $N = 0$ and neglecting the triple products in the last line, a particular solution is given by

$$\psi' = M^2\phi\phi_x \tag{111}$$

which appears in both equations (35) and (85).

The iteration equation is thereby reduced to

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = M^2\begin{bmatrix} 2\beta^2N\phi_{xx}\phi_x + \\ \phi_{yy}\phi_y{}^2 + 2\phi_{yz}\phi_y\phi_z + \phi_{zz}\phi_z{}^2 \end{bmatrix} \tag{112}$$

It has not been possible to find a particular solution of this equation in terms of the first-order potential. The solutions for plane and axially symmetric flow do not appear to suggest a generalization. On the other hand, there is no assurance that such a solution cannot be found, so that one is tempted to search further. If the triple products are neglected, the right-hand side of equation (112) vanishes for $\gamma = -1$ $(N = 0)$. However, investigation of the previous solutions indicates that the idea of here taking $\gamma = -1$ is not legitimate.

In the absence of a complete particular integral, the reduced iteration equation (equation (112)) must be attacked by more conventional methods. In principle, it is always possible to find a particular solution of a linear nonhomogeneous equation with the aid of the fundamental solution associated with the differential operator. For the three-dimensional wave operator which occurs here, the fundamental solution is

$$\frac{1}{\sqrt{(x - \xi)^2 - \beta^2\left[(y - \eta)^2 + (z - \zeta)^2\right]}} \tag{113}$$

which can be interpreted as the potential at any point (x,y,z) lying inside the downstream Mach cone from a unit supersonic source at (ξ,η,ζ). With the aid of Green's formula, it can be shown that a particular solution of

$$\varphi_{yy} + \varphi_{zz} - \beta^2\varphi_{xx} = F(x,y,z) \tag{114}$$

is given by

$$\psi(x,y,z) = -\frac{1}{2\pi}\iiint \frac{F(\xi,\eta,\zeta)\ d\xi\ d\eta\ d\zeta}{\sqrt{(x - \xi)^2 - \beta^2\left[(y - \eta)^2 + (z - \zeta)^2\right]}} \tag{115}$$

where the integration extends throughout that portion of the forward Mach cone from the point (x,y,z) within which F is defined.

In practice, the integration indicated in equation (115) is generally not feasible. For example, even the simplification of axial symmetry reduces equation (115) only to a double integral of $F(x,r)$ multiplied by a complete elliptic integral of complicated argument. Avoiding such integrals by discovery of the particular solution clearly represents a great simplification in this case.[1]

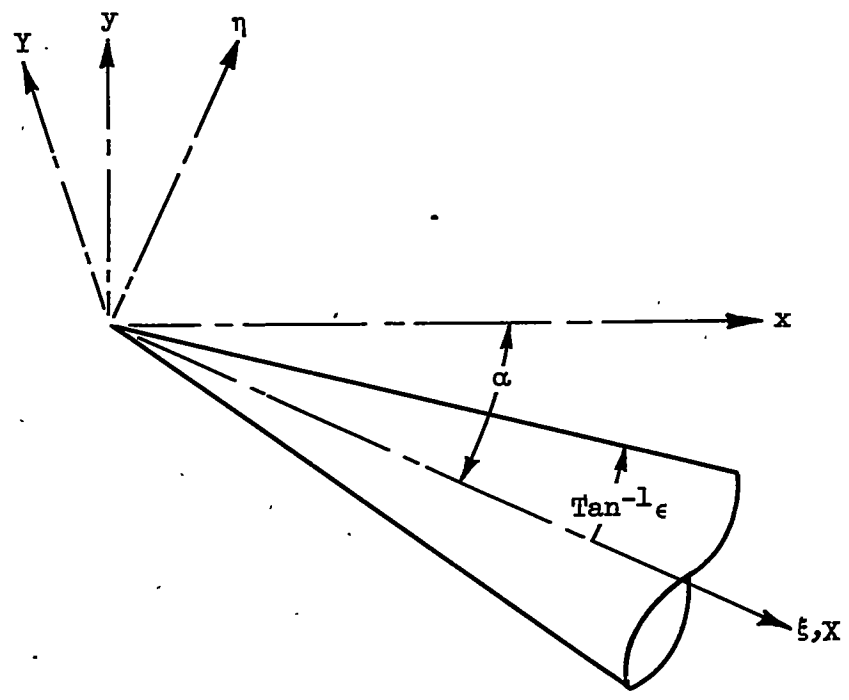
[1]Comparing the two methods would lead to the evaluation of definite integrals involving complete elliptic integrals, which might be of some interest.

In the following sections, one example of a three-dimensional solution will be given, and the possibility of treating other shapes will be discussed thereafter.

Inclined cone.- The problem of a cone at an angle of attack illustrates the use of separation of variables to reduce the three-dimensional iteration equation to tractable form.

Two alternative coordinate systems are suitable for bodies of revolution at an angle of attack. In wind axes the body is inclined, while in body axes the stream impinges on the body obliquely. The latter system is simpler for first-order problems and is probably better for the second-order solution also. However, wind axes will be used here, since otherwise the iteration equations must be rederived.

To facilitate imposing the tangency condition, it is convenient to apply an oblique transformation (see, for example, reference 28, p. 18). This effectively unyaws the axis of the body (but distorts the surface) while leaving the wave operator unchanged. Thus three different



Coordinate systems for inclined cones.

coordinate systems are required, as shown in the above sketch:

Wind axes:	x,y,z		
Body axes:	ξ,η,ζ	ξ,ρ,ϑ	ξ,τ,ϑ
Oblique axes:	X,Y,Z	X,R,Θ	X,T,Θ

the latter two being used also in cylindrical and conical form.

To simplify the solution, it will be assumed that the angle of attack α is so small that its square can be neglected. This will give a solution nonlinear in the body thickness but linear in α and will therefore yield the correct initial slope of the lift curve. Then the three systems of coordinates are related according to the following table:

$$z = \zeta = Z$$

$\xi = x - \alpha y$	$x = \xi + \alpha\eta$	$X = x + \alpha\beta^2 y$	$x = X - \alpha\beta^2 Y$	$\xi = X - \alpha M^2 Y$	$X = \xi + \alpha M^2\eta$
$\eta = y + \alpha x$	$y = \eta - \alpha\xi$	$Y = y + \alpha x$	$y = Y - \alpha X$	$\eta = Y$	$Y = \eta$

To this approximation

$$R = \rho \tag{116a}$$

$$T = \tau\left(1 - \frac{M^2}{\beta}\tau\alpha\cos\vartheta\right) \tag{116b}$$

$$\Theta = \vartheta \tag{116c}$$

the surface of the cone is

$$\tau_s = \beta\epsilon \tag{117a}$$

$$T_s = \beta\epsilon - \beta M^2\epsilon^2\alpha\cos\vartheta \tag{117b}$$

and the velocity components are related by

$$\Phi_\xi = \overline{\Phi} - T\overline{\Phi}_T \tag{118a}$$

$$\Phi_\rho = \beta\overline{\Phi}_T + M^2\left(\overline{\Phi} - T\overline{\Phi}_T\right)\alpha\cos\Theta \tag{118b}$$

$$\frac{\Phi_\vartheta}{\rho} = \frac{\overline{\Phi}_\Theta}{T} - \beta M^2\left(\overline{\Phi} - T\overline{\Phi}_T\right)\alpha \sin \Theta \tag{118c}$$

where, as in equation (15), the conical potential is introduced by

$$\Phi(X,R,\Theta) = X\overline{\Phi}(T,\Theta) \tag{119a}$$

where

$$T = \frac{\beta R}{X} \tag{119b}$$

The first-order problem, referred to oblique coordinates, is found to be

$$\left(1 - T^2\right)\overline{\phi}_{TT} + \frac{\overline{\phi}_T}{T} + \frac{\overline{\phi}_{\Theta\Theta}}{T^2} = 0 \tag{120a}$$

$$\beta\overline{\phi}_T + M^2\left(\overline{\phi} - T\overline{\phi}_T\right)\alpha \cos \Theta + \alpha \cos \Theta = \epsilon\left(1 + \overline{\phi} - T\overline{\phi}_T\right) \quad \text{at} \quad T = T_s \tag{120b}$$

$$\overline{\phi}(\infty,\Theta) = \overline{\phi}_T(\infty,\Theta) = 0 \tag{120c}$$

The solution is the sum of potentials for a conical line source and dipole (reference 1, p. 74) and has the form

$$\overline{\phi}(T,\Theta) = -A\left(\operatorname{sech}^{-1}T - \sqrt{1 - T^2}\right) + C\left(\frac{\sqrt{1 - T^2}}{T} - T \operatorname{sech}^{-1}T\right)\alpha \cos \Theta \tag{121}$$

Substituting into the tangency condition (equation (120b)) and expressing values of functions on the cone in terms of their values at $T = \beta\epsilon$ by means of Taylor expansions, it is found that

$$A = \frac{\epsilon^2}{\sqrt{1 - \beta^2\epsilon^2} + \epsilon^2 \operatorname{sech}^{-1}\beta\epsilon} \tag{122a}$$

$$C = \beta\epsilon^2 \frac{1 + M^2A\left(\frac{1 + \epsilon^2}{\sqrt{1 - \beta^2\epsilon^2}} - \operatorname{sech}^{-1}\beta\epsilon\right)}{\left(1 + 2\epsilon^2\right)\sqrt{1 - \beta^2\epsilon^2} + \beta^2\epsilon^2 \operatorname{sech}^{-1}\beta\epsilon} \tag{122b}$$

The streamwise perturbation velocity is

$$\phi_z = \left(\bar{\phi} - T\bar{\phi}_T\right) + \beta\bar{\phi}_T\alpha \cos\Theta$$

$$= -A \operatorname{sech}^{-1}T + (2C + \beta A)\frac{\sqrt{1 - T^2}}{T}\alpha \cos\Theta \tag{123}$$

Then according to equation (111), the partial particular solution is, in conical form,

$$\bar{\psi}' = \frac{1}{Z}M^2\phi\phi_z = M^2A\left\{A \operatorname{sech}^{-1}T\left(\operatorname{sech}^{-1}T - \sqrt{1 - T^2}\right) - \right.$$

$$\left[(3C + \beta A)\frac{\sqrt{1 - T^2}\operatorname{sech}^{-1}T}{T} - (2C + \beta A)\frac{1 - T^2}{T} - \right.$$

$$\left.\left.CT\left(\operatorname{sech}^{-1}T\right)^2\right]\alpha \cos\Theta\right\}$$

$$= \bar{\psi}^{(I)} + \bar{\psi}^{(II)}\alpha \cos\Theta \tag{124}$$

There remains to solve the reduced iteration equation given by equation (112) which, after transformation of coordinates, becomes

$$(1 - T^2)\bar{\varphi}_{TT} + \frac{\bar{\varphi}_T}{T} + \frac{\bar{\varphi}_{\Theta\Theta}}{T^2} = M^2\left\{2N\left[T^2\bar{\phi}_{TT}\left(\bar{\phi} - T\bar{\phi}_T\right) + \right.\right.$$

$$\left.\left.\beta T\bar{\phi}_{TT}\left(3T\bar{\phi}_T - 2\bar{\phi}\right)\alpha \cos\Theta\right] + \beta^2\bar{\phi}_{TT}\bar{\phi}_T{}^2\right\} \tag{125}$$

Substituting equation (121) into the right-hand side gives

$$(1 - T^2)\bar{\varphi}_{TT} + \frac{\bar{\varphi}_T}{T} + \frac{\bar{\varphi}_{\Theta\Theta}}{T^2} = AM^2\left\{A\left(2N\frac{\operatorname{sech}^{-1}T}{\sqrt{1 - T^2}} - \beta^2A\frac{\sqrt{1 - T^2}}{T^4}\right) - \right.$$

$$\left.2\left[2N(C + \beta A)\frac{\operatorname{sech}^{-1}T}{T\sqrt{1 - T^2}} + N(2C + \beta A)\frac{1}{T} - 2\beta^2AC\frac{\sqrt{1 - T^2}}{T^5}\right]\alpha \cos\Theta\right\} \tag{126}$$

This is reduced to two total differential equations by setting

$$\bar{\varphi}(T,\Theta) = \bar{\varphi}^{(I)}(T) + \bar{\varphi}^{(II)}(T)\alpha \cos \Theta \tag{127}$$

Therefore the complete second-order potential consists of a term independent of α plus one proportional to $\alpha \cos \Theta$. The first of these must be the previous solution for the symmetrical cone (equation (72)), so that $\bar{\varphi}^{(I)}$ is known. The equation for $\bar{\varphi}^{(II)}$ is

$$\left(1 - T^2\right)\bar{\varphi}_{TT}^{(II)} + \frac{\bar{\varphi}_T^{(II)}}{T} - \frac{\bar{\varphi}^{(II)}}{T^2} = -2AM^2\left[2N(C + \beta A)\frac{\operatorname{sech}^{-1}T}{T\sqrt{1 - T^2}} + N(2C + \beta A)\frac{1}{T} - 2\beta^2 AC\frac{\sqrt{1 - T^2}}{T^5}\right] \tag{128}$$

Setting

$$\bar{\varphi}^{(II)}(T) = T\omega(T) \tag{129}$$

reduces this to a linear first-order equation in ω which can be integrated to find that

$$\bar{\varphi}^{(II)}(T) = AM^2\left[D\left(\frac{\sqrt{1 - T^2}}{T} - T \operatorname{sech}^{-1}T\right) + (3C + 2\beta A)N\frac{1 - T^2}{T} + (C + \beta A)NT\left(\operatorname{sech}^{-1}T\right)^2 + \frac{1}{2}\beta^2 AC\frac{\sqrt{1 - T^2}}{T^3}\right] \tag{130}$$

The tangency condition (equation (19b')) separates into the two conditions

$$\beta\left[\bar{\psi}_T^{(I)} + \bar{\varphi}_T^{(I)}\right] = \epsilon\left\{\left[\bar{\psi}^{(I)} + \bar{\varphi}^{(I)}\right] - T\left[\bar{\psi}_T^{(I)} + \bar{\varphi}_T^{(I)}\right]\right\} \quad \text{at} \quad T = \beta\sigma \tag{131a}$$

$$\beta\left[\bar{\Psi}_T(II) + \bar{\varphi}_T(II)\right] + M^2\left\{\left[\bar{\Psi}(I) + \bar{\varphi}(I)\right] - T(1 + T)\left[\bar{\Psi}_T(I) + \bar{\varphi}_T(I)\right]\right\} =$$

$$\epsilon\left\{\left[\bar{\Psi}(II) + \bar{\varphi}(II)\right] - T\left[\bar{\Psi}_T(II) + \bar{\varphi}_T(II)\right] + \right.$$

$$\left.\epsilon M^2T^2\left[\bar{\Psi}_{TT}(I) + \bar{\varphi}_{TT}(I)\right]\right\} \qquad \text{at} \quad T = \beta\epsilon \qquad (131b)$$

The first of these is the previous relation (equation (71b)) which determined the constant B in equation (72). Similarly, the second of these determines the constant D in equation (130).

Series expansions for pressure and normal force.- Numerical results have been calculated only for the case in which the solution is expanded in powers of T and $\log \frac{2}{T}$. Carrying out the expansion, the constant D is found to be

$$D = \left[\frac{2M^2 + 1}{M^2} \log \frac{2}{\beta\epsilon} - \left(5N - 1 + \frac{5}{2M^2}\right) + \ldots\right]\beta\epsilon^2 \qquad (132)$$

Then calculating the velocity components from equation (118) and the pressure from equation (27) gives, on the surface of the cone,

$$C_p = (C_p)_o - 4\epsilon\left[1 - \epsilon^2\left(M^2\log \frac{2}{\beta\epsilon} - \frac{3}{2} M^2 + 1\right)\right]\alpha \cos \vartheta + \ldots \qquad (133)$$

Here $(C_p)_o$ is the value for zero angle of attack, given by equation (109). Integrating gives the normal-force coefficient, based on cross-sectional area:

$$C_N = \frac{\text{Normal force}}{\frac{1}{2} \rho_o U^2(\text{Area})} = 2\alpha\left[1 - \epsilon^2\left(M^2\log \frac{2}{\beta\epsilon} - \frac{3}{2} M^2 + 1\right)\right] + \ldots \qquad (134)$$

This result has been obtained also by Lighthill (reference 29), who has calculated the lift on bodies of revolution having analytic meridian curves by assuming a series expansion for the velocity potential.

Stone (reference 30) has developed a solution for inclined cones which is linearized with respect to α, but otherwise exact. Kopal (reference 31) has published tables of the numerical results of Stone's theory. A comparison of equation (134) with this exact theory and with

Tsien's first-order solution (reference 32) is shown in figure 6 for 5° and 10° cones. The earlier discussion of series expansions suggests that the agreement might improve if the solution were not expanded in series.

Shock-wave position.- If the solution were valid at the Mach cone, the velocity components there would be, from equations (72), (118), (124), and (130):

$$-\left(\frac{u}{U}\right) = \frac{1}{\beta}\left(\frac{v}{U}\right) \overset{?}{=} \epsilon^4 M^2 \left[2N - 3(3N - 1)\beta\alpha \cos \vartheta\right] \tag{135}$$

For simplicity, using equation (122), A and C have here been approximated by ϵ^2 and $\beta\epsilon^2$. Comparing equations (74) and (77) it is seen that the difference between the shock-wave angle and the Mach angle would be

$$\lambda - \sin^{-1}\frac{1}{M} \overset{?}{=} \beta M^2 N^2 \epsilon^4 - \frac{3}{2}\beta^2 M^2 N(3N - 1)\epsilon^4 \alpha \cos \vartheta \tag{136}$$

Hence the ratio of the angular rotation of the shock wave to that of the cone would be

$$\frac{\delta}{\alpha} \overset{?}{=} \frac{3}{2}\beta^2 M^2 N(3N - 1)\epsilon^4 \tag{137a}$$

It was seen previously that although the solution does not in fact converge at the Mach cone, the shock-wave deflection calculated in this way is correct for the unpitched cone except for a factor of $1\frac{1}{2}$. It might be supposed that the same correction factor would apply here. Kopal (reference 31) tabulates values of δ/α calculated from Stone's theory, and from these it appears that a factor of 3 rather than $1\frac{1}{2}$ is required, so that actually

$$\frac{\delta}{\alpha} = \frac{9}{2}\beta^2 M^2 N(3N - 1)\epsilon^4 \tag{137b}$$

Figure 7 shows a comparison of this modified result with the exact values for a 5° cone.

It must be emphasized that equation (137b) represents nothing more than a conjecture. It could probably be verified, however, by extending the solution of Lighthill (reference 21) or Broderick (reference 22) to the case of angle of attack.

GENERAL REMARKS

Future investigation.- Two large classes of problems which have only been touched upon deserve further study. One of these is wings; the other, bodies of revolution at an angle of attack. The example of the inclined cone was undoubtedly made awkward by the use of wind coordinates. The iteration equation should be rederived in body coordinates and the solution extended to general bodies of revolution. It is possible that in this form a particular integral could be discovered. That there is good possibility of success with this problem is suggested by the fact that Lighthill was able to obtain a general solution by assuming a series expansion (reference 29).

The possibility of discovering particular integrals of the iteration equation might be investigated more systematically. If none can be found for general three-dimensional flow, special cases such as conical flow should be studied.

Possible treatment of wings.- Possibly the most useful application of first-order theory is to thin flat wings. No attempt has so far been made to find the second-order solution for a wing. It seems likely, however, that solutions can be found at least for conical problems. In this case the iteration equation can be reduced, by the standard conical theory (references 14 and 28), to the problem of solving Poisson's equation inside a circle.

Two difficulties can be anticipated. First, if the wing has subsonic edges, infinite velocities arise there, so that the assumption of small perturbations is violated. It is known that in first-order theory this is no essential objection, since the pressure is found correctly except in the immediate neighborhood of the singularity, and the integrated values of lift and moment are correct to first order. Kaplan (reference 10) and Schmieden and Kawalki (reference 11) have indicated that this result extends to the second approximation for subsonic flow, so that probably no real difficulty exists.

Secondly, if the wing has supersonic edges, the failure of the iteration process along Mach lines from the apex can be expected to affect the surface pressures. Again it is possible that integrated values will be correct to second order. Otherwise, it may be possible to adjust the solution in those regions, in a manner similar to that shown in the diagram of Mach lines in the section "Flow past a corner and a parabolic bend."

Higher approximations.- It seems unlikely that a third or higher approximation would ever be justified. Other neglected factors, chiefly viscosity and heat conduction, should certainly be considered first.

However, the Busemann second-order result has been extended to third and even fourth order (reference 33), and various writers have considered the third approximation for plane subsonic flow (references 5, 7, and 8). If a third approximation should be considered worth while, the iteration could be repeated. Again the cases of flow past a curved wall and a cone would serve as helpful examples.

Application to subsonic flow.- The iteration equation and the particular integrals are in no way restricted to supersonic flow. The particular solution for plane flow might profitably be compared with the subsonic solutions of references 4 to 10.

The particular solution for axially symmetric flow makes possible a second-order solution for bodies of revolution at subsonic speed. In this case, the integral equation can be treated by the methods used for the airship problem.

California Institute of Technology
Pasadena, Calif., December 9, 1949

APPENDIX

SYMBOLS

a	constant reference radius for body of revolution
b	abscissa at which source distribution for body of revolution begins
A,B,C,D	constants determined by boundary conditions
c	local speed of sound
C_n	constant coefficients of series
C_p	pressure coefficient
$E(k^2)$	complete elliptic integral of the second kind with modulus k
$f(x),F(x)$	source-distribution functions for body of revolution
$F_n(x,y,z)$	known right-hand side of nth-order iteration equation
$g(x)$	continuous function of order unity which vanishes for $x \leqq 0$
h,j,H,J	arbitrary functions of one variable
$K(k^2)$	complete elliptic integral of the first kind with modulus k
M	free-stream Mach number
$N = \frac{(\gamma + 1)M^2}{2\beta^2}$	
p	local static pressure
P_n	points on body of revolution at which tangency condition is imposed
q	local speed of flow
r	radius in cylindrical coordinates

$R(x)$	radius of meridian curve of body of revolution
$S(x)$	cross-sectional area of body of revolution
t	conical variable $\left(\frac{\beta r}{x}\right)$
u,v,w	perturbation velocity components in Cartesian or cylindrical coordinates
U	free-stream velocity
x,y,z	Cartesian coordinates with x in free-stream direction
$\left.\begin{matrix} X,Y,Z \\ X,R,\theta \\ X,T,\theta \end{matrix}\right\}$	oblique axes (see diagram of coordinate systems in section entitled "Inclined cone")
α	angle of attack
$\beta = \sqrt{M^2 - 1}$	
γ	adiabatic exponent
δ	angular rotation of shock wave on cone due to angle of attack
ϵ	parameter small compared with unity
θ	azimuthal variable in cylindrical coordinates
λ	angle of shock wave from free-stream direction
ξ,η,ζ	Cartesian coordinates of variable point
$\left.\begin{matrix} \xi,\eta,\zeta \\ \xi,\rho,\vartheta \\ \xi,\tau,\vartheta \end{matrix}\right\}$	body coordinates (see diagram of coordinate systems in section entitled "Inclined cone")
ρ	local density
τ	conical variable referred to $x = \xi$ rather than $x = 0$
φ	additional second-order perturbation potential
ϕ	first-order (linearized) perturbation potential, same as $\phi^{(1)}$

Φ	exact perturbation potential
$\phi^{(n)}$	nth-order perturbation potential
$\overset{(n)}{\Phi}$	nth term in series expansion of perturbation potential
χ	second-order correction potential
ψ	particular solution of second-order iteration equation
ψ'	partial particular solution for three-dimensional flow
Ω	complete velocity potential
ω	auxiliary variable (see equation (129))
$(\ )^{(n)}$	result of nth iteration
$(\ )^{(I)}$	independent of Θ (see equation (127))
$(\ )^{(II)}$	proportional to $\alpha \cos \Theta$ (see equation (127))
$\overline{(\ )}$	conical potential; for example, $\Phi = x\overline{\Phi}$

Subscripts:

c	differentiation in cross-stream direction - component of normal direction which is perpendicular to free stream
o	free-stream conditions
s	surface of cone
1, 2	values ahead of and behind a corner

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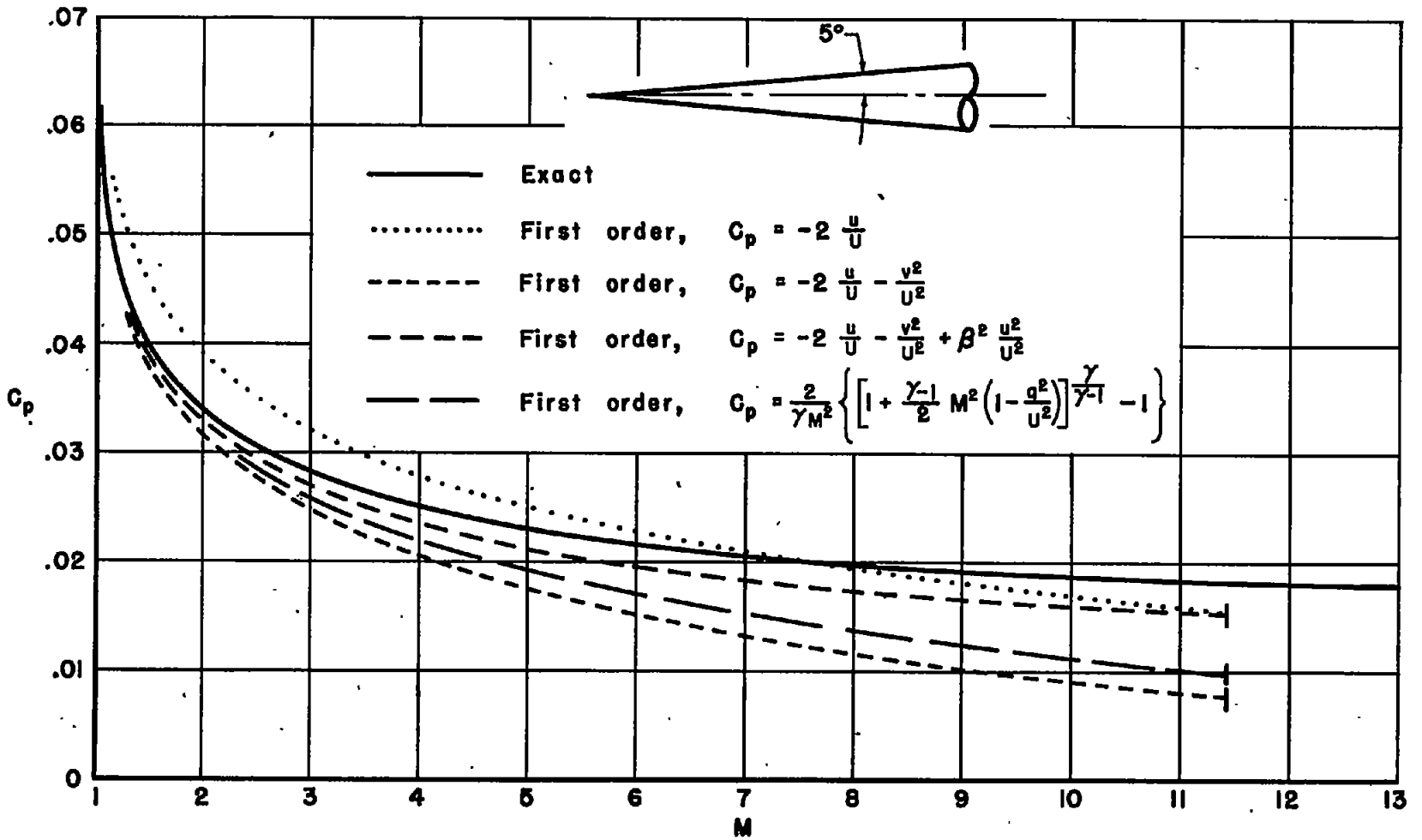
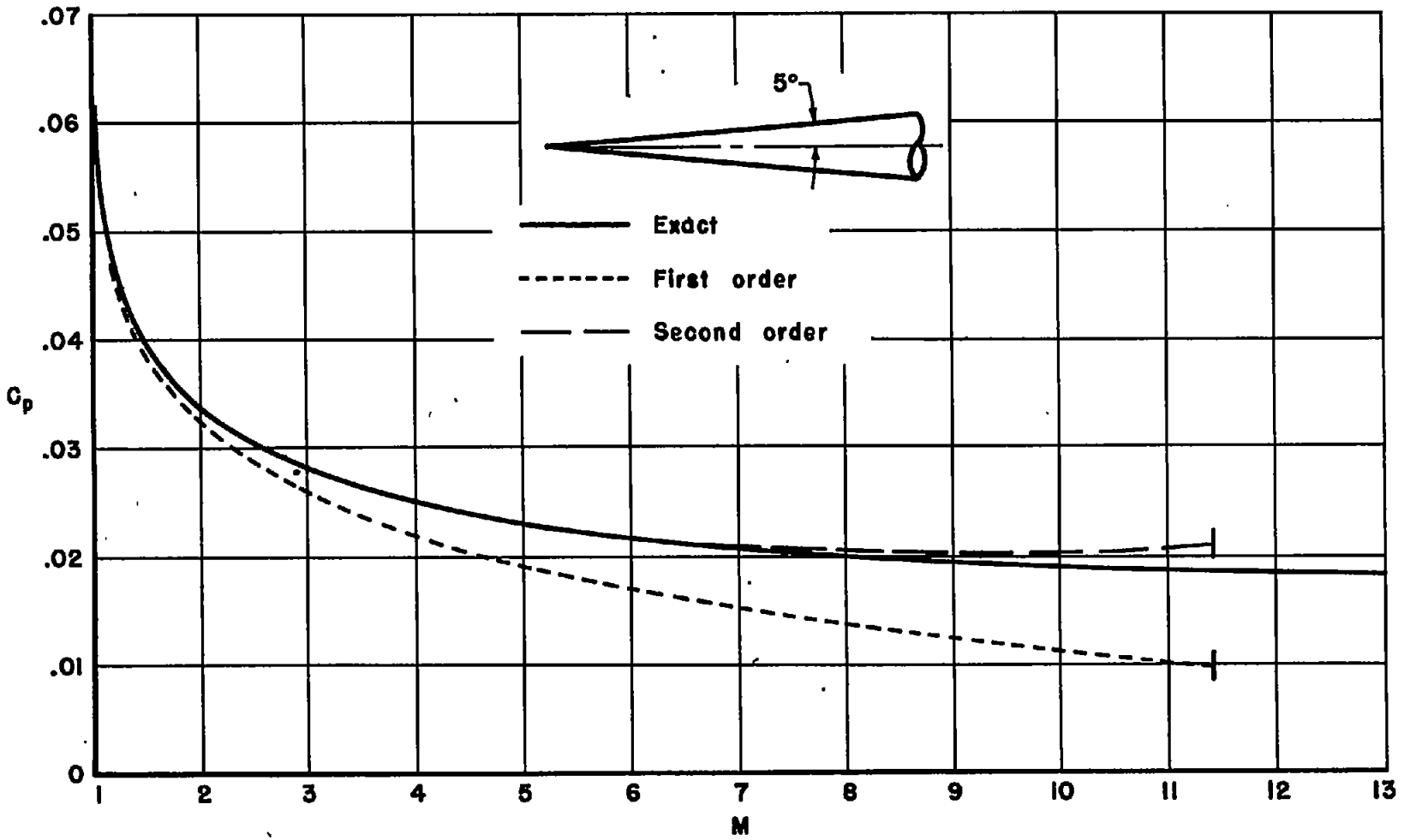
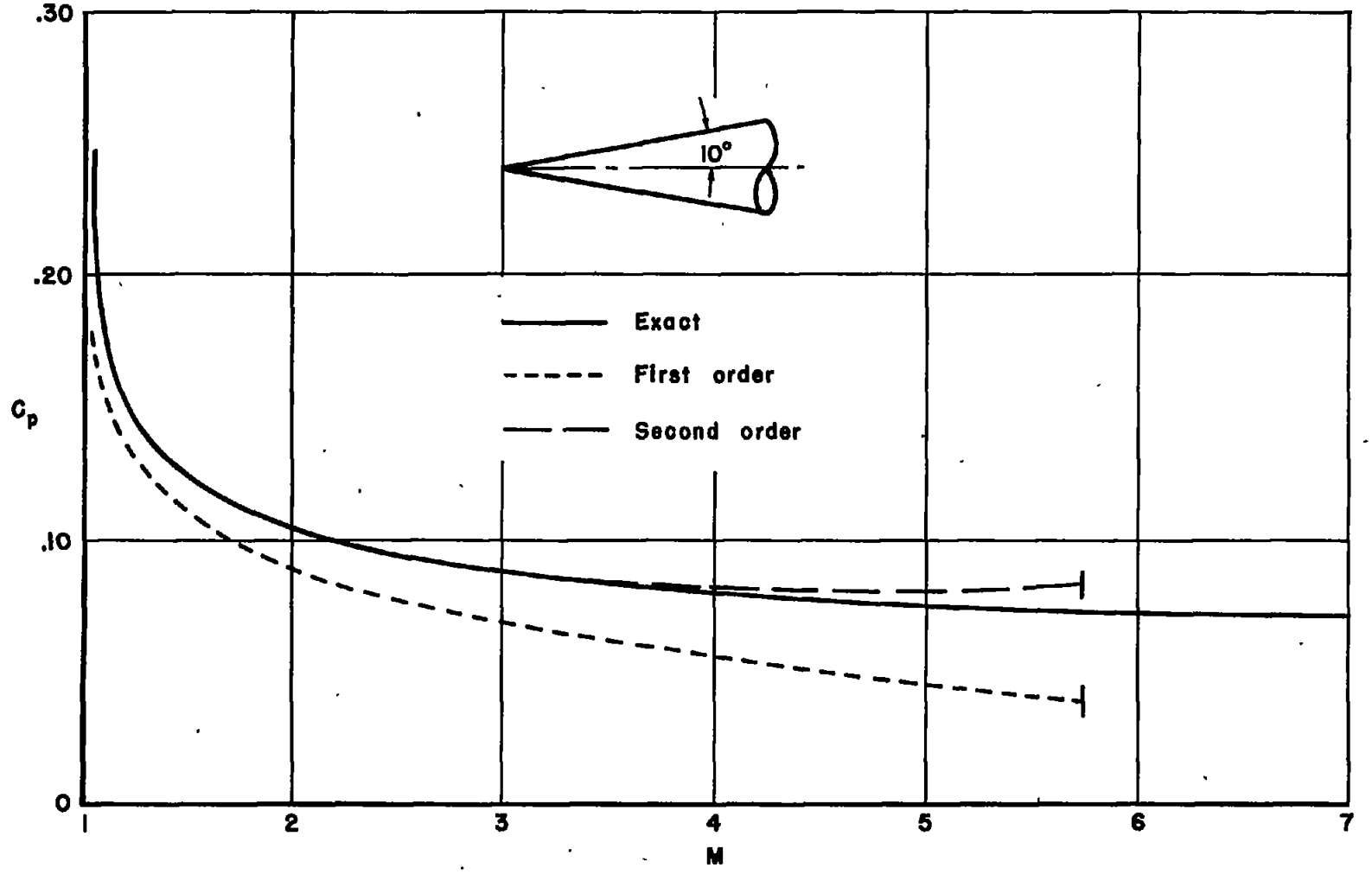


Figure 1.- Comparison of first-order solutions for a 5° cone using various pressure relations.



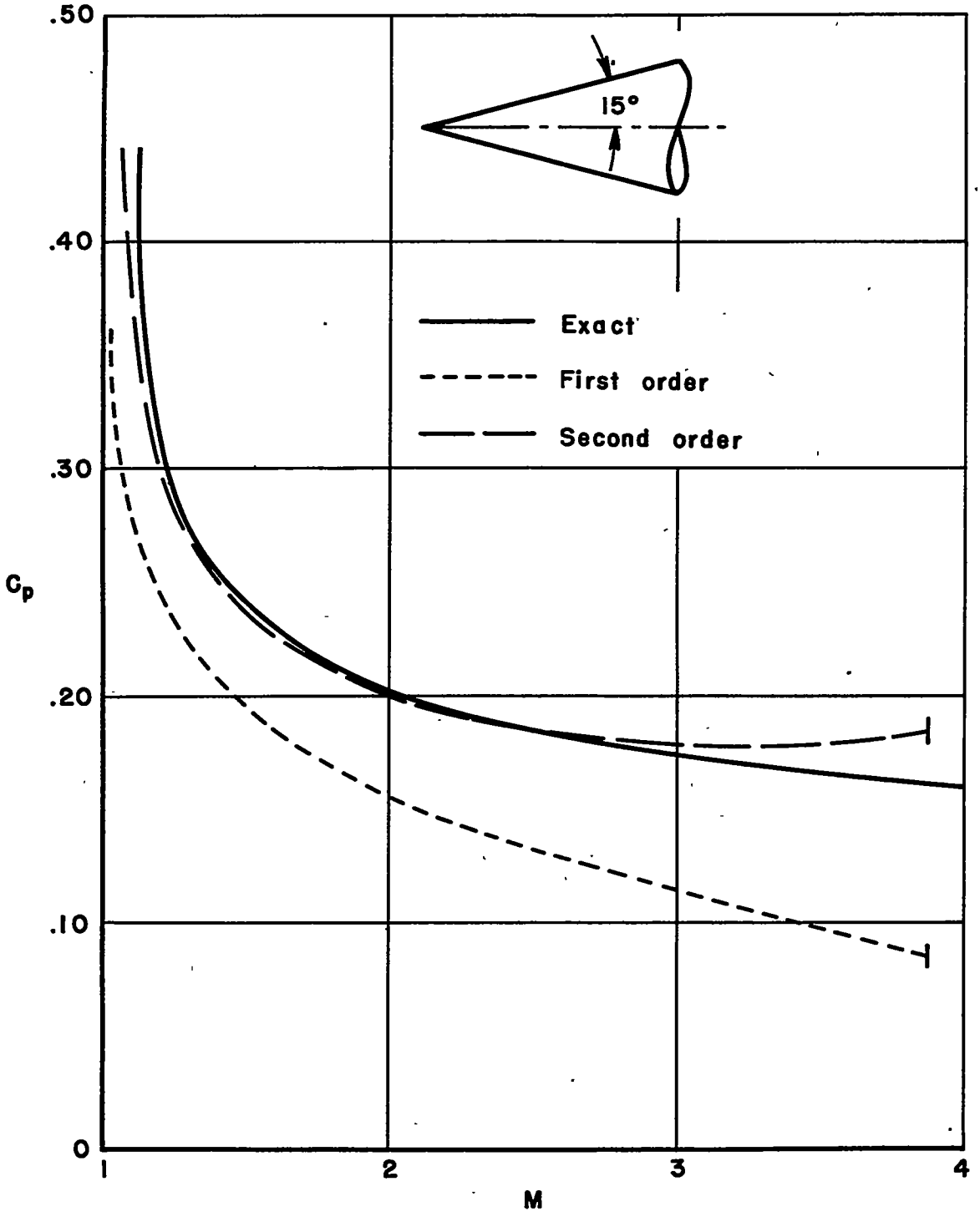
(a) 5° cone.

Figure 2.- Comparison of various solutions for pressure on cones of 5°, 10°, 15°, and 20° semivertex angles.



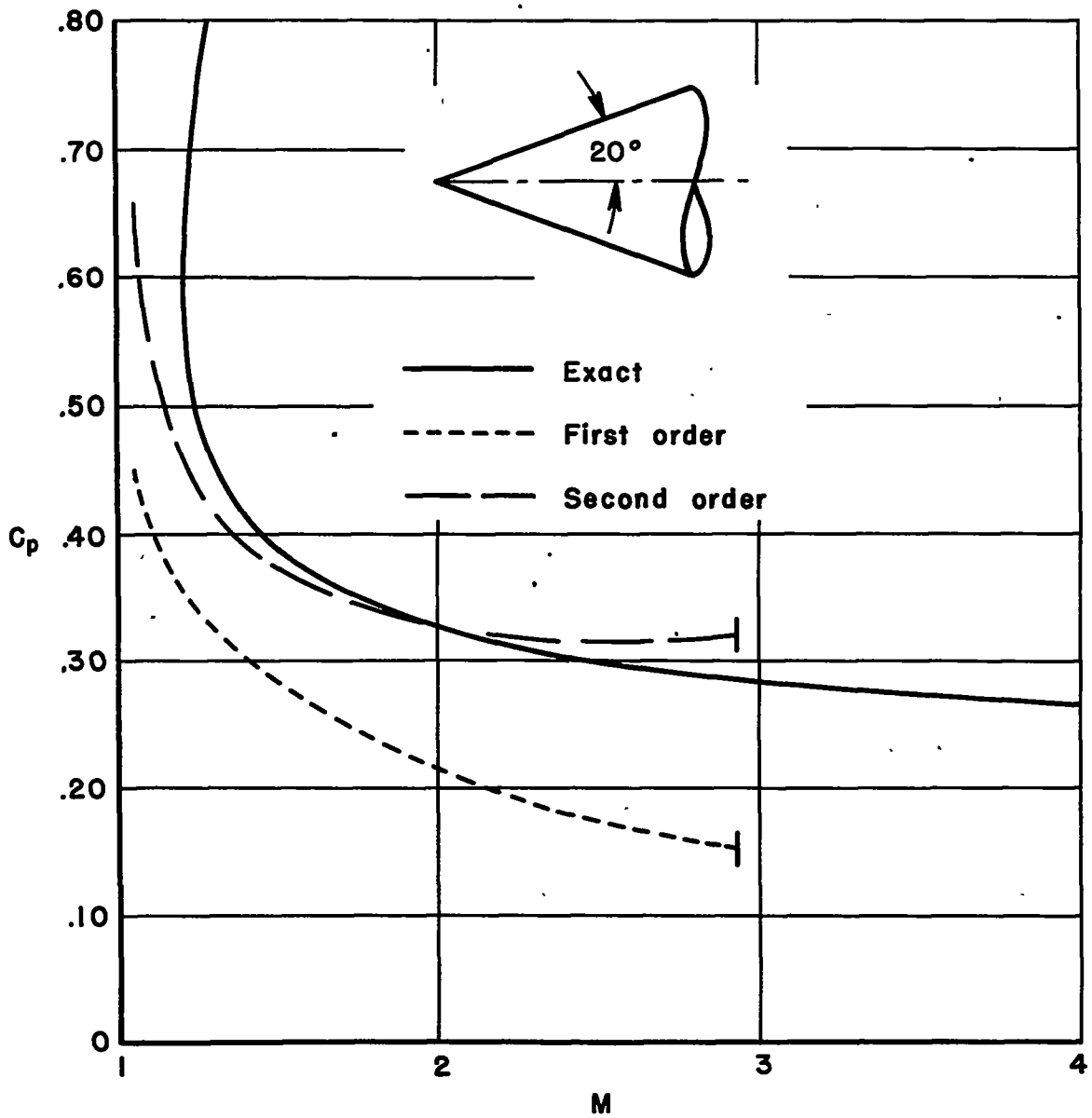
(b) 10° cone.

Figure 2.- Continued.



(c) 15° cone.

Figure 2.- Continued.



(d) 20° cone.

Figure 2.- Concluded.

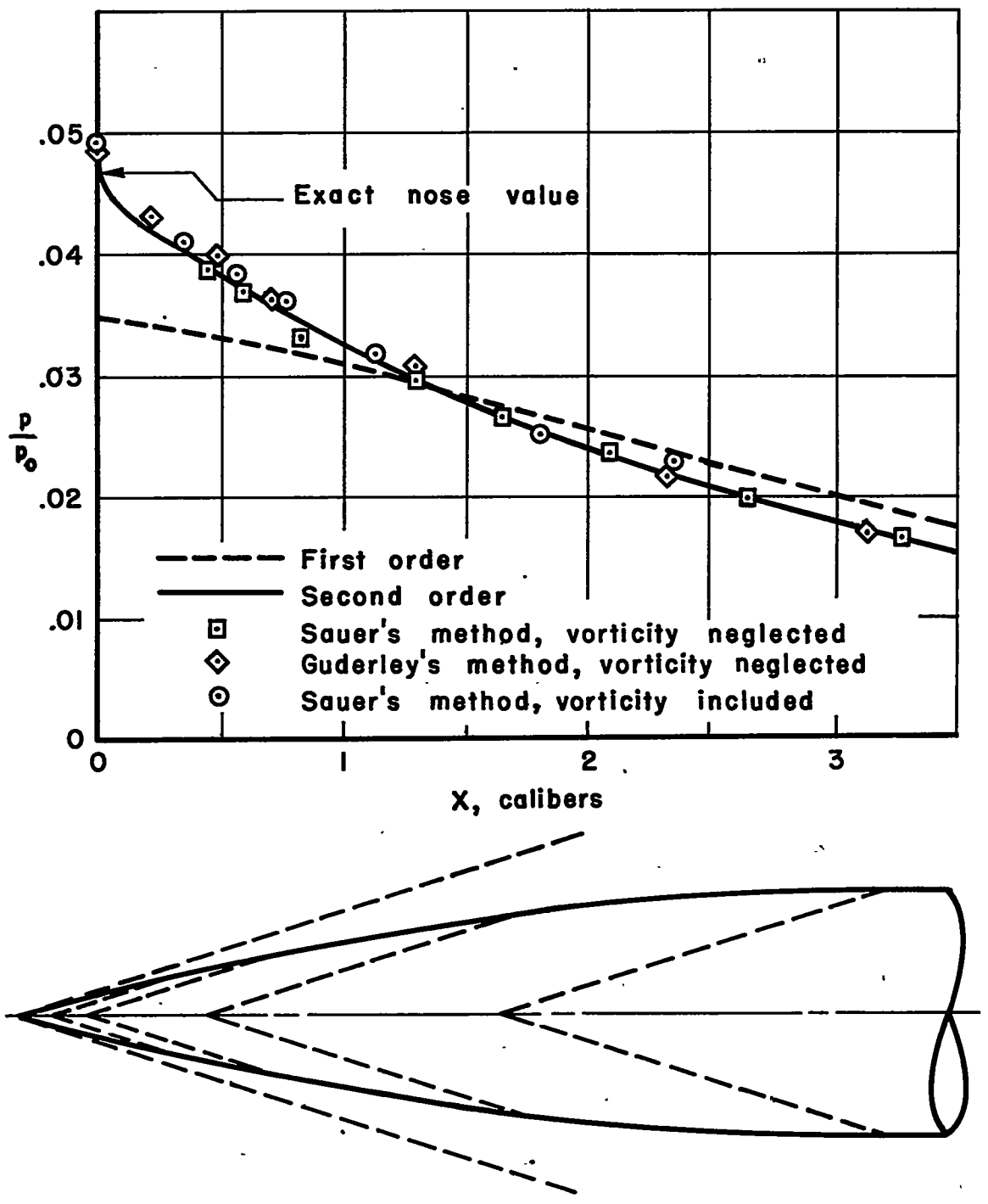


Figure 3.- Pressure distribution on a smooth body of revolution.

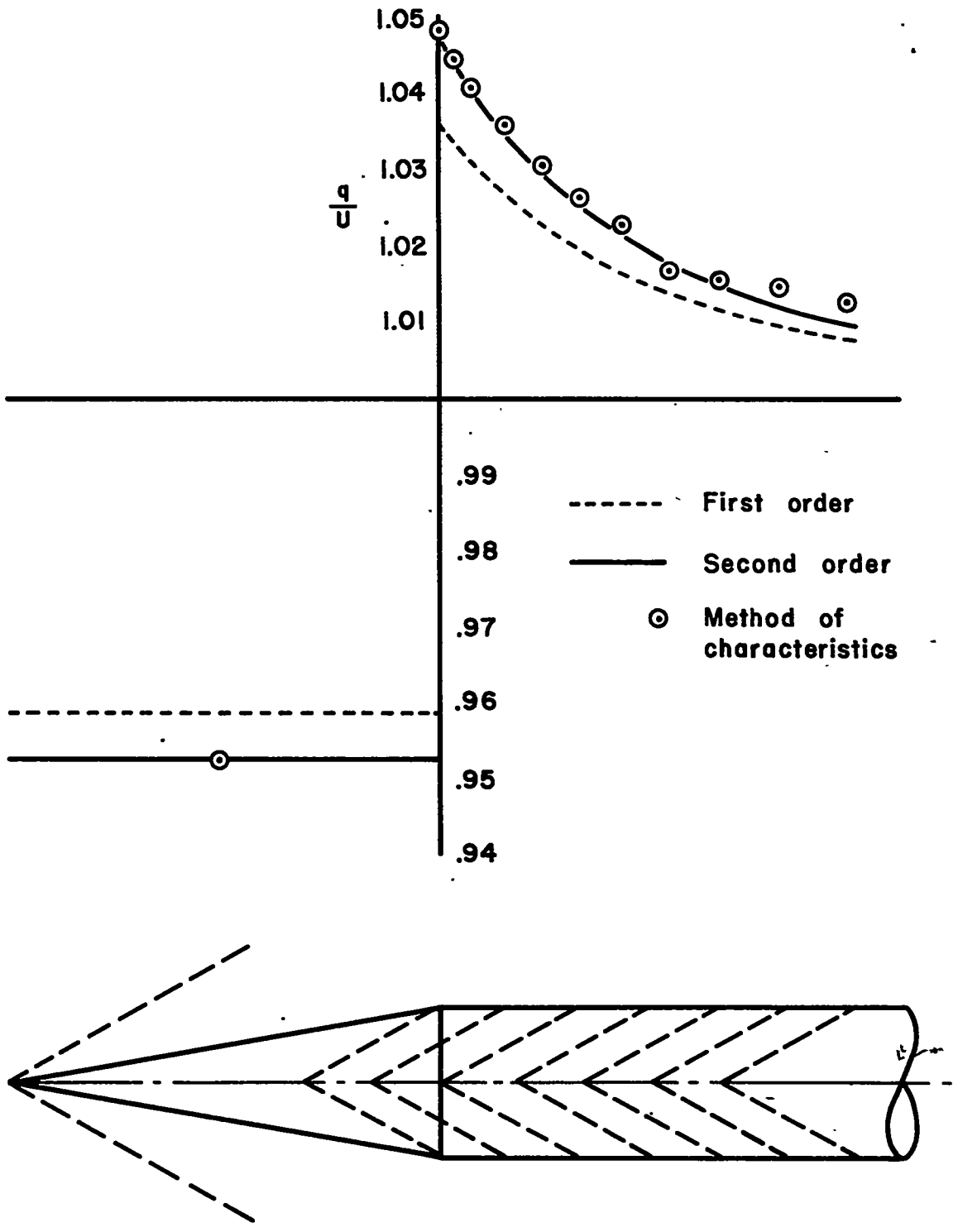
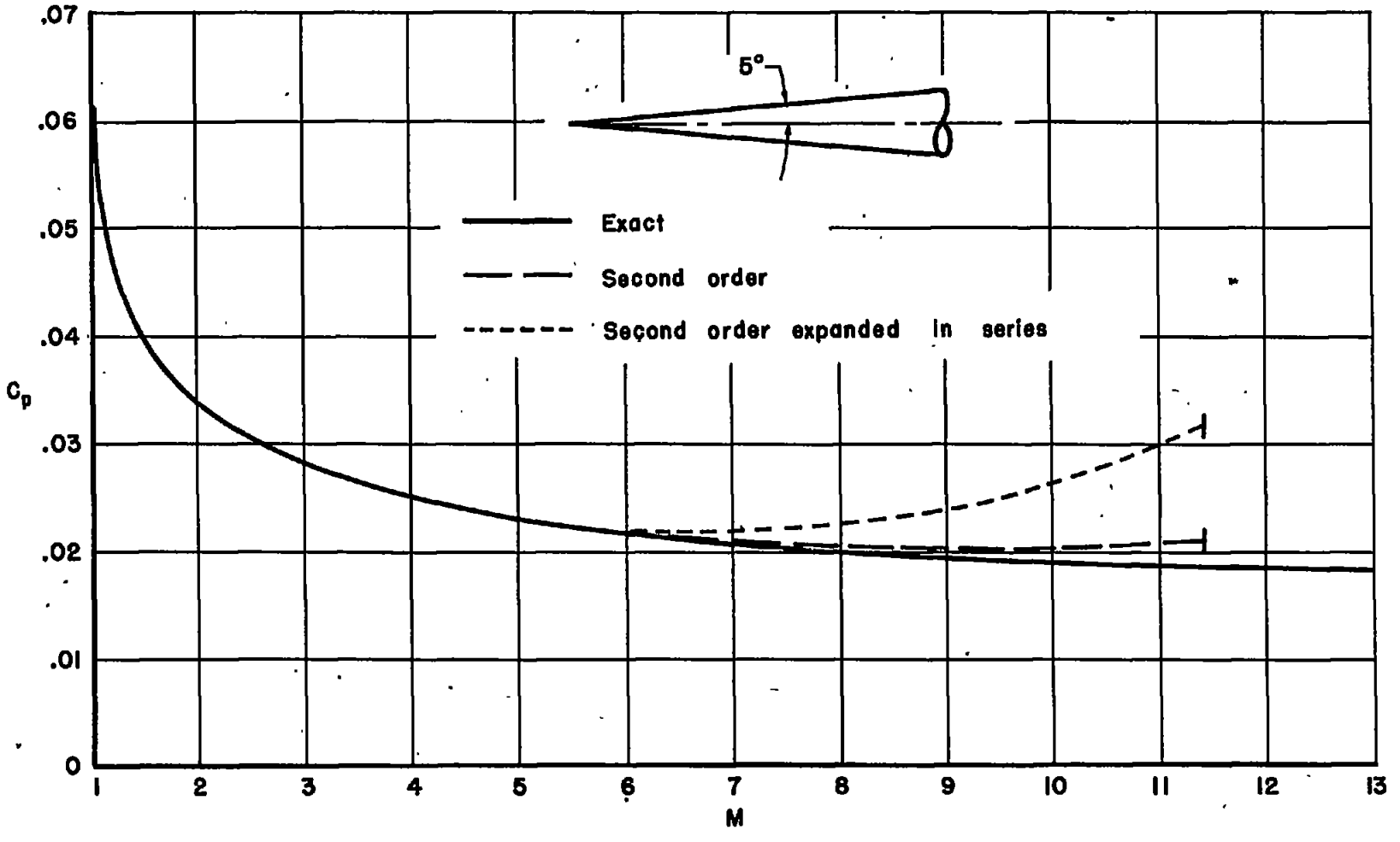
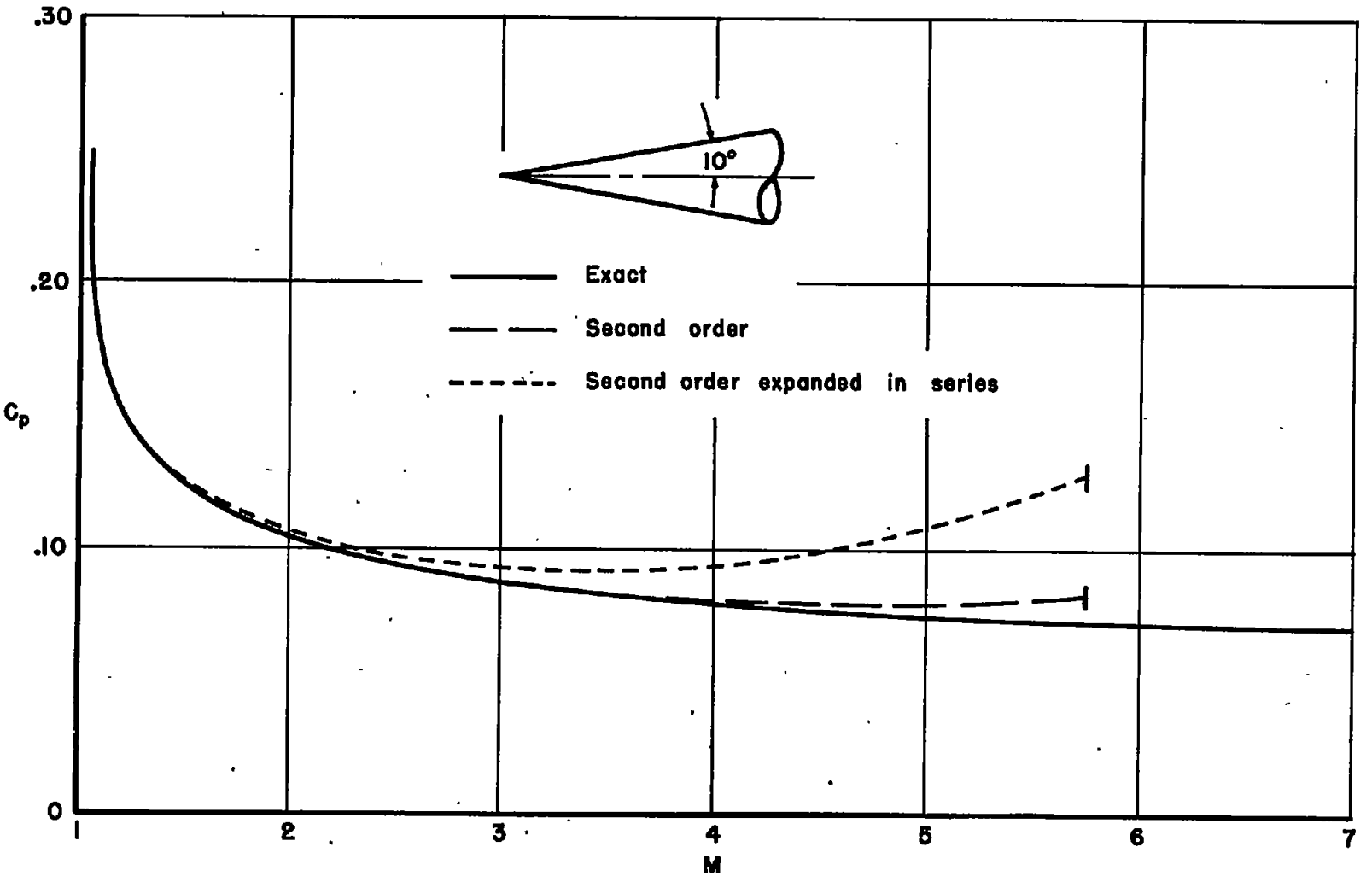


Figure 4.- Pressure distribution on a body of revolution with discontinuity in slope.



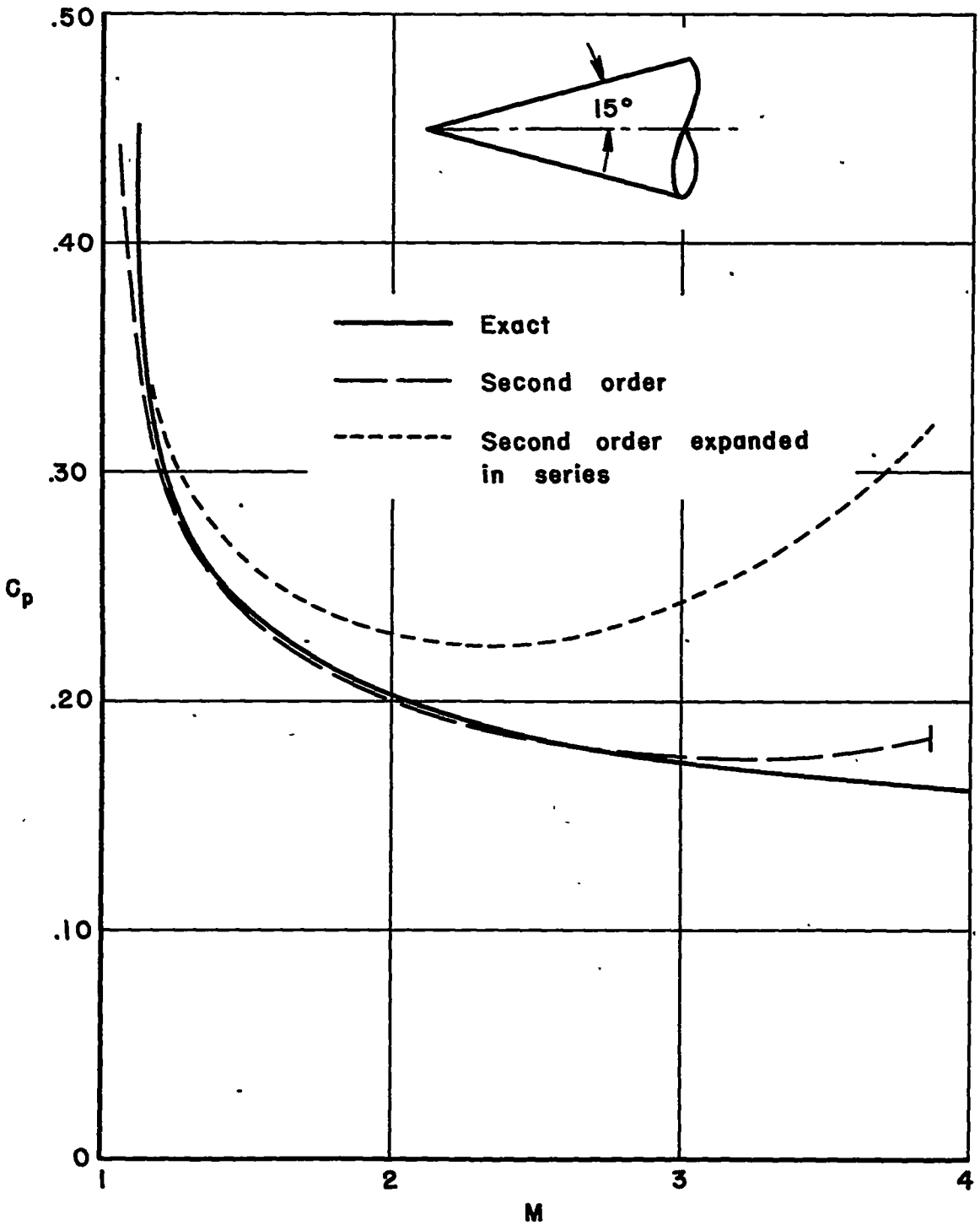
(a) 5° cone.

Figure 5.- Effect of expanding in series upon second-order pressure on cones of 5°, 10°, 15°, and 20° semivertex angles.



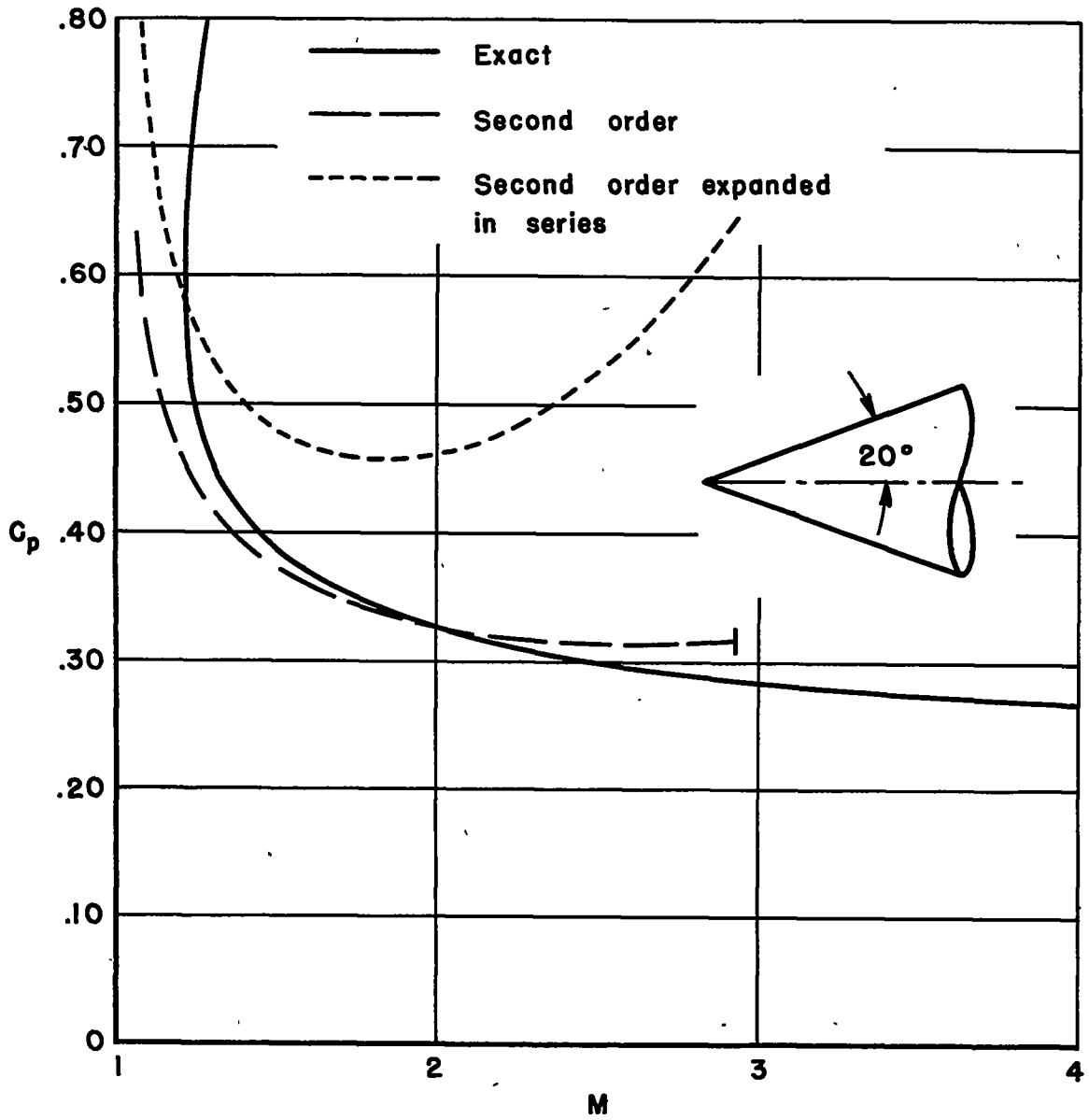
(b) 10° cone.

Figure 5.- Continued.



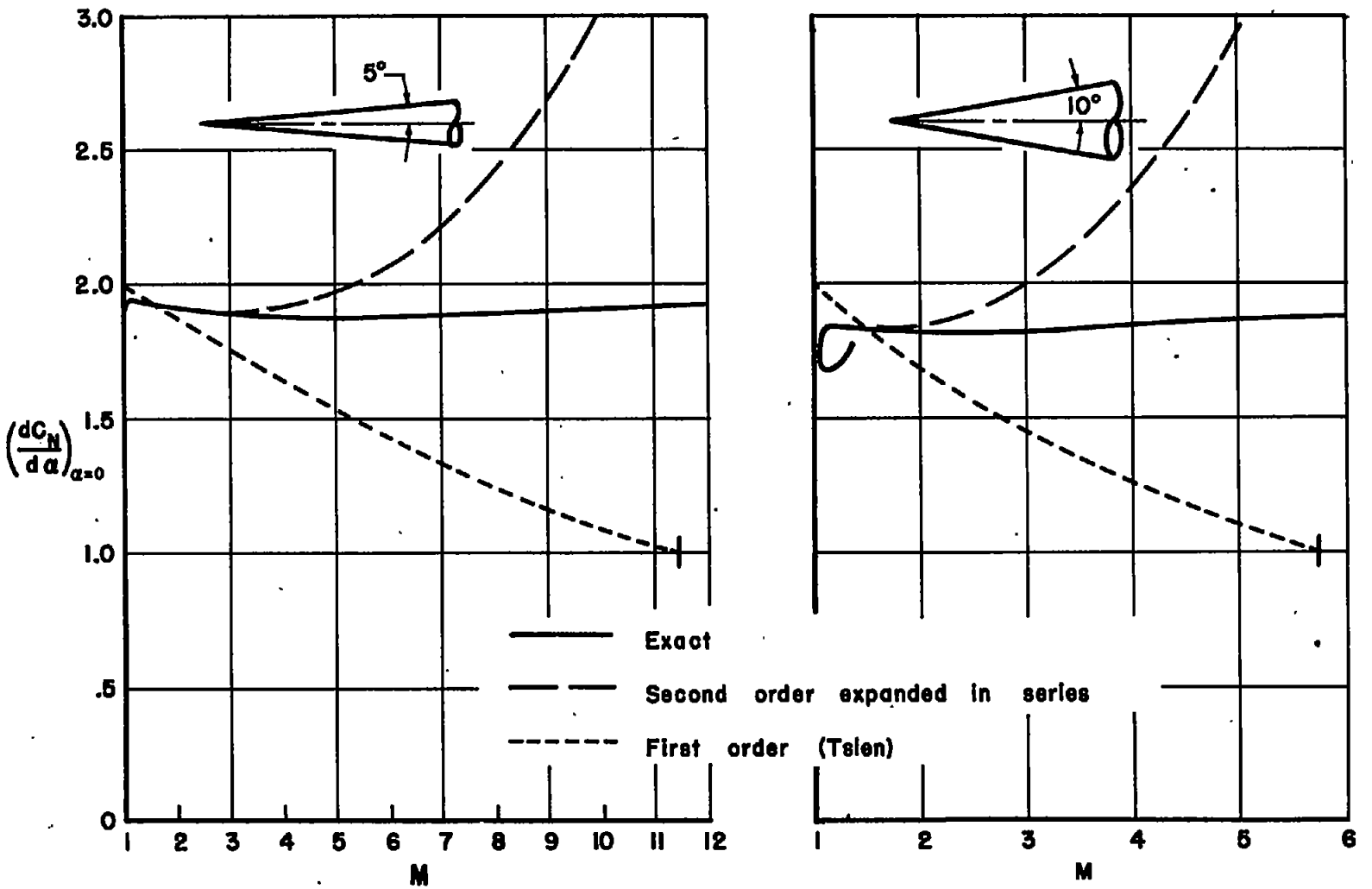
(c) 15° cone.

Figure 5.- Continued.



(d) 20° cone.

Figure 5.- Concluded.



(a) 5° semivertex angle.

(b) 10° semivertex angle.

Figure 6.- Comparison of various solutions for normal force on a cone.

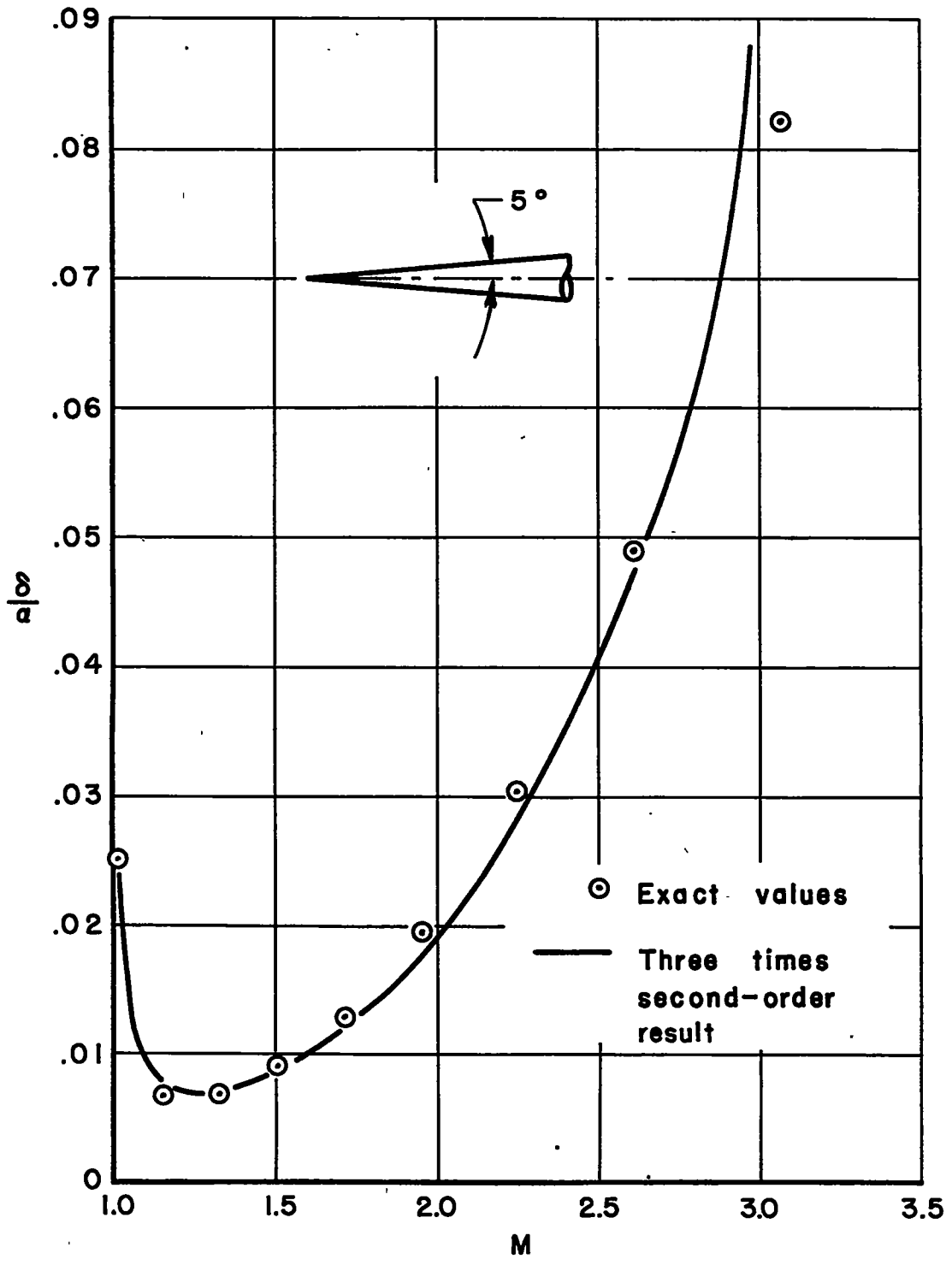


Figure 7.- Ratio of shock-wave rotation to angle of attack for a 5° cone.

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