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A CRITICAL EXAMINATION OF SIMILARITY THEORY FOR INCOMPRESSIBLE FLOWS

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SUMMARY

The classical theories for turbulent shear flow are the momentum transfer theory of Prandtl, the vorticity transfer theory of Taylor, and the similarity theory of Von Kármán. The two transfer theories both involve a mixture length, which must be given by an additional assumption. On the other hand, the similarity theory is a more determinate scheme, because it makes a more definite hypothesis about the nature of the turbulent fluctuations. Goldstein, however, introduced an alternative form of the similarity theory. A great amount of work has been done to evaluate the relative merits of these three theories.

Further investigation into the nature of turbulent motion is, however, done largely in connection with the simpler case of isotropic turbulence. In this field, much recent progress has been made, particularly following the concept of Kolmogoroff. The concept of similarity also plays a dominant role. Since Kolmogoroff's theory is also applicable to shear flow, it is natural that one should reexamine the similarity theory of Von Kármán by using modern concepts. This is the main purpose of the present investigation. It is found that the original form of the theory is supported by modern concepts.

INTRODUCTION

The concept of similarity was first introduced by Von Kármán in 1930 (reference 1). Even at the very beginning, he realized that it was not possible to have complete similarity, including both the components of fluctuation essentially free from viscous forces and those largely influenced by viscous forces. It was Taylor (reference 2), however, who

first made a more penetrating analysis into this question in connection with isotropic turbulence. He introduced a macroscale of turbulence l and a microscale λ. He also fully discussed, both theoretically and experimentally, how two turbulent fields may be similar in all the large eddies, which contain practically all the energy, and yet differ completely in their rates of dissipation, which are governed chiefly by the small eddies having a negligible contribution to the total energy. In particular, he gave the relation

$$\epsilon \sim \frac{(u')^3}{l} \sim \nu\frac{(u')^2}{\lambda^2} \tag{1}$$

for the energy dissipation ϵ in terms of the scales of turbulence, its intensity u', and the kinematic-viscosity coefficient ν. This relation shows clearly that full similarity cannot be possible in general.

A further advance in this direction was made by Kolmogoroff (reference 3). According to his concept, for large Reynolds numbers of turbulence, as defined by $R_\lambda = u'\lambda/\nu$, the small eddies are independent of the behavior of the large eddies, except to the extent that they supply the energy to be dissipated. There are then only the two parameters ϵ and ν for the viscous range. From dimensional arguments, Kolmogoroff introduced the characteristic velocity v and the characteristic scale η defined, respectively, by

$$\left.\begin{aligned} v &= (\nu\epsilon)^{1/4} \\ \eta &= \left(\nu^3/\epsilon\right)^{1/4} \end{aligned}\right\} \tag{2}$$

In terms of the spectrum of turbulence, the high (spatial) frequency components are dependent only on v and η. By assuming that the lower end of this range is independent of ν explicitly, one arrives at the spectrum

$$F \sim \epsilon^{2/3}\kappa^{-5/3} \tag{3}$$

where κ is the wave number. This relation was first given by Obukhoff (reference 4) and was found independently later by Onsager (reference 5), Heisenberg (reference 6), and Von Weizsäcker (reference 7).

The idea that the low-frequency range takes care of almost all of the energy and that the high-frequency range takes care of almost all of the dissipation has been fully demonstrated by Von Kármán and Lin (references 8 and 9). It is found that, in either case, the unimportant part to the important part is of the order of R_λ^{-m} (where m is some positive number) and is consequently negligible for large Reynolds numbers of turbulence. The analysis was made by evaluating the integrals

$$\left.\begin{aligned} I_0 &= \int_0^\infty F(\kappa)\ d\kappa \\ I_2 &= \int_0^\infty \kappa^2 F(\kappa)\ d\kappa \end{aligned}\right\} \tag{4}$$

for energy and dissipation, assuming the characteristic quantities V, l for the low-frequency range and the characteristic quantities v, η for the high-frequency range and using Taylor's relation (equation (1)) and the Obukoff spectrum formula (equation (3)).

This kind of analysis is now used to remove a difficulty raised by Goldstein in his analysis of Von Kármán's similarity theory (reference 10) Goldstein showed that there are at least two ways of applying the similarity theory, one analogous to the momentum transfer theory (the τ-theory) as given originally by Von Kármán, and the other analogous to the vorticity transfer theory (the M-theory). By an analysis of the relative importance of the high-frequency and low-frequency components, it is possible to show that the τ-theory is a logical consequence of Von Kármán's similarity concept, while the M-theory does not follow directly. This kind of investigation also shows that the usual discussion of the similarity theory needs some modification, although the final conclusions are not altered.

It is the purpose of the present paper to reexamine critically Von Kármán's similarity theory for incompressible flows by using modern concepts in order to provide a basis for extension and application of the theory to compressible flows. In references 11 and 12 turbulent boundary layer over a flat plate in compressible flow is treated in the same spirit.

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The authors wish to thank Professor S. Goldstein for stimulating discussions during his visit here in September 1950.

THE CLASSICAL SIMILARITY THEORY OF VON KÁRMÁN

It seems convenient to begin with a brief sketch of the original similarity theory of Von Kármán, together with Goldstein's discussion of its difficulties and his alternative suggestion (the M-theory). Remarks will then be made from the present point of view, leading to the fuller discussions of the next section.

Von Kármán considered a steady two-dimensional mean flow and made the following hypotheses (see reference 10): (1) The turbulence mechanism is essentially independent of the viscosity of the fluid (except in the viscous layer near the walls); (2) In comparing the turbulence mechanisms at two different points, consideration of the fields of turbulent flow may be restricted to the immediate neighborhoods of these points; (3) The turbulence flow patterns at different points are similar (relative to frames of reference moving with the mean velocities at the points) and differ only in scales of length and time (or velocity). On the basis of these hypotheses (the validity of which will be examined later), the following development of the theory may be made. (See appendix A for definitions of important symbols.)

Consider a two-dimensional parallel mean motion with velocity $U(y)$ in the direction of the x-axis. If (u,v,w) are the turbulent velocity components and p is the pressure, the Navier-Stokes equations are

$$\left.\begin{aligned}
\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(uu) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw) &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\,\Delta u \\
\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + \frac{\partial}{\partial x}(vu) + \frac{\partial}{\partial y}(vv) + \frac{\partial}{\partial z}(vw) &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\,\Delta v \\
\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + \frac{\partial}{\partial x}(wu) + \frac{\partial}{\partial y}(wv) + \frac{\partial}{\partial z}(ww) &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\,\Delta w
\end{aligned}\right\} \quad (5)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{6}$$

According to the first assumption the term in ν may be neglected. One may now differentiate the set of equations (5) and combine the resultant equations to yield the equations for the turbulence vorticity components (ξ,η,ζ). The usual derivation is based on these vorticity equations.

From the present point of view, the neglect of viscosity means that the "low-frequency" components of the turbulent fluctuations are being considered. It is known, however, that the "high-frequency" components of the vorticity fluctuations are more important. It is consequently difficult to justify the neglect of viscosity from the vorticity equation, as it is usually done. Even the neglect of viscosity from the original Navier-Stokes equations (5) requires a careful examination. These points will be discussed more fully in the next section.

To proceed with the derivation, take the origin at the point P under consideration and axes moving with the mean velocity at P, so that $U = 0$ at the origin. The assumption is introduced that only the immediate neighborhood of P may be considered: For dU/dy, d^2U/dy^2, their values are taken at P, while for U the first term only of the Taylor expansion is taken, namely, $y(dU/dy)_P$. The scales of length and velocity are introduced by writing

$$x = l\tilde{x} \qquad t = l\tilde{t}/A$$

$$y = l\tilde{y} \qquad u = A\tilde{u}$$

$$z = l\tilde{z} \qquad v = A\tilde{v}$$

$$w = l\tilde{w}$$

so that

$$\xi = A\tilde{\xi}/l$$

$$\eta = A\tilde{\eta}/l$$

$$\zeta = A\tilde{\zeta}/l$$

where $\tilde{\xi} = \frac{\partial\tilde{w}}{\partial\tilde{y}} - \frac{\partial\tilde{v}}{\partial\tilde{z}}$, and so forth. Substituting these into the equations

for vorticity and requiring them to be independent of the position of the origin and of the values of dU/dy, d^2U/dy^2 there, one obtains

$$A = \text{Constant } l \frac{\partial U}{dy}$$

$$l = \text{Constant } \frac{dU}{dy}\bigg/\frac{d^2U}{dy^2}$$

The Reynolds shearing stress at different points of the fluid is $\tau = -\rho\overline{uv} = -\rho A^2\overline{\tilde{u}\tilde{v}}$ and is therefore proportional to $\rho l^2(dU/dy)^2$; and, if l is multiplied by a suitable constant, there may, when signs are regarded, be written

$$\tau = \rho l^2\left|\frac{dU}{dy}\right|\frac{dU}{dy}$$

$$l = K_1\left|\frac{dU}{dy}\bigg/\frac{d^2U}{dy^2}\right|$$

The average state of affairs under consideration is essentially independent of x and z, and the rate M at which x-momentum is communicated to unit volume is

$$M = d(-\rho\overline{uv})/dy$$

$$= -(\rho A^2/l)\ d\left(\overline{\tilde{u}\tilde{v}}\right) d\tilde{y}$$

and is therefore proportional to $\rho l(dU/dy)^2$ or to $\rho l^2(dU/dy)(d^2U/dy^2)$. If l is multiplied by a suitable constant, there may therefore be written

$$M = \rho l^2 \frac{dU}{dy}\frac{d^2U}{dy^2}$$

$$l = K_2\left|\frac{dU}{dy}\bigg/\frac{d^2U}{dy^2}\right|$$

Notice also that $M = \rho\bar{v}\zeta$, if the average state of affairs is independent of x. Thus, the first alternative, the τ-theory given by Von Kármán, is formally related to the momentum transfer theory, while the second alternative, the M-theory given by Goldstein, is formally related to the vorticity transfer theory, although the basic concept of the similarity theory is essentially different from a transfer theory.

The above derivation is essentially the one used by Goldstein, the only difference being that he started with the vorticity equations directly. The original derivation of Von Kármán is made by a consideration of two-dimensional fluctuations, which is sufficient to provide the essential steps. In Goldstein's paper, there are also an extension of the theory to the flow through a pipe and a critical discussion of the validity of the theory. His investigations show that in the axially symmetrical case, it is impossible to have a simple similarity as that in the two-dimensional case if terms of the same order of magnitude are all kept. This becomes obvious when the radius of curvature of the surface of constant velocity is recognized as an additional parameter affecting the length scale for a local similarity. He also found that the velocity distribution from the τ-theory agrees better with the experiments for flow between parallel planes, while the M-theory appears to be more satisfactory for pipe flow.

Another generally recognized difficulty with the similarity theory is the following. As a consequence of the theory, the ratios $\overline{u^2}:\overline{v^2}:\overline{w^2}:\overline{uv}:\overline{uw}:\overline{vw}$ should remain constants. In a channel, this is found to be a good approximation only for points not too close to either the wall or the center line.

To summarize, the following unsatisfactory points in the similarity theory have been discussed:

(a) There is nothing in the original theory to decide between the τ-theory and the M-theory and any other alternative obtained, say, by an application of the theory to the calculation of $\partial^2\tau/\partial y^2$.

(b) It cannot be extended to cases other than the original case of two-dimensional parallel flows.

(c) The ratios $\overline{u^2} : \overline{v^2}$ and so forth are not constants near the center of the channel or near the boundary.

(d) In addition, there is the difficulty in applying the theory to flows with a point of inflection in the velocity profile.

(e) Again, the scale l turns out to be only moderately small in most cases.

It must be noted that most of the difficulties are limitations which do not bear on the basic concept of the theory. The final form of the theory is applicable only when the scale may be expected to be determined by dU/dy and d^2U/dy^2. This is obviously not the case in wakes or in the central part of the channel. This is also the reason why the axially symmetrical case cannot be treated. The presence of another length scale makes similarity unlikely. Near the boundary, similarity might also break down as soon as the scale is comparable with the distance from the boundary.

Thus, the essential difficulties are (a) and (e) discussed above if one limits oneself to flows through channels and in the boundary layer.

It is to be recognized that the whole boundary layer or the channel should be regarded as an "organic" field of motion. One might even describe the turbulent motion as a nonlinear oscillation superposed over a field of flow. Thus, any attempt to "localize" the theory, such as is done in the similarity theory and to a certain extent in the transfer theories, is at best a rough approximation. One is faced with the dilemma of either treating each individual case separately with proper emphasis on the influence of the boundary conditions or being satisfied with an approximate theory having a fairly general applicability. The latter course has often been taken.

The only remaining difficulty (a) is to be settled by a critical examination of the concept of similarity from the standpoint of recent developments of the statistical theory of turbulence. It will be seen that the basic concept of Von Kármán's similarity prefers the original form of the theory (τ-theory) to any theory based on higher-order derivatives (M-theory, etc.).

CRITICAL DISCUSSION OF SIMILARITY CONCEPT FROM STANDPOINT OF MODERN STATISTICAL THEORIES

The concept of the similarity theory in shear flow will now be formulated and it will be shown how it is related to the concept of similarity developed in the statistical theory of isotropic turbulence. The concepts developed from the statistical theory will then be applied to the present problem to show that the M-theory does not follow from Von Kármán's concept of nonviscous similarity in shear flow.

General concept of similarity.- As discussed above, the similarity of the large-scale eddies (which are responsible for transfer) is at best a rough approximation. The similarity of the small eddies (which

are responsible for the energy dissipation) is however a much closer approximation at very large Reynolds numbers of turbulence. According to the concept of Kolmogoroff, this part of the turbulent fluctuation will depend only on the kinematic viscosity coefficient ν and the rate of energy transferred to these scales from larger scales. Furthermore, this part of turbulent motion is isotropic and has a universal character independent, say, of the amount of shear in the main flow. The rate of energy transfer is approximately the same as the total rate of dissipation ϵ. The order of accuracy of such approximations has been estimated by Von Kármán and Lin (references 8 and 9). In other words, in the case of shear flow, there is a production of turbulent energy from main motion by the usual transfer mechanism (at a scale of $l = KU'/U''$, say). This large-scale turbulence breaks down into motions at smaller scales and eventually passes into the viscous range to be dissipated into heat. The rate of transfer may be estimated, say, by Taylor's formula. These large-scale motions, being produced directly from the mean motion, must depend on their characteristics. According to the concept of Von Kármán, the turbulent fluctuations in this range have a universal structure, with a length scale $l = KU'/U''$ and a velocity scale lU'. Thus, two regimes of similarity are visualized: (a) The large-scale similarity of Von Kármán and (b) the small-scale similarity of Kolmogoroff. The first range is anisotropic and contributes to the shear; the second range is isotropic and contributes only to the dissipation. The transition range probably has an energy spectrum $\epsilon^{2/3}\kappa^{-5/3}$ with decreasing amount of shear.

This picture of similarity is analogous to the one visualized by Von Kármán and Lin (references 8 and 9) for the intermediate stage of decay of isotropic turbulence. There, the large-scale eddies are also isotropic, having a scale determined by the Loitsiansky invariant. It must be remarked that the establishment of an equilibrium or quasi-equilibrium state for large-scale motions takes a long time. Thus, it may be subjected to doubt whether the idealized picture thus visualized can actually occur for decaying turbulence. On the other hand, in the case of shear flow, a stationary system being considered, the condition is more conducive to the establishment of Von Kármán's similarity.

With this general concept of similarity in mind, the classical theory of similarity for shear flow may now be examined with the help of results developed for isotropic turbulence. In the first place, one may examine the orders of magnitude of length, velocity, and vorticity for the eddies of large scales and small scales.

For large-scale motions,

$$\left.\begin{array}{ll} \text{Length scale} & l \sim U'/U'' \\ \text{Velocity scale} & V \sim lU' \\ \text{Vorticity scale} & V/l \sim U' \end{array}\right\} \quad (7)$$

For small-scale motions,

$$\left.\begin{array}{ll} \text{Length scale} & \eta = \left(\nu^3/\epsilon\right)^{1/4} = \text{Number} \times lR_\lambda^{-3/2} \\ \text{Velocity scale} & v = (\nu\epsilon)^{1/4} = \text{Number} \times VR_\lambda^{-1/2} \\ \text{Vorticity scale} & \frac{v}{\eta} = (\nu/\epsilon)^{1/2} = \text{Number} \times (V/l)R_\lambda \end{array}\right\} \quad (8)$$

where R_λ is the Reynolds number of turbulence and is usually very large.

These order-of-magnitude relations will now be applied to the development of the similarity theory. In particular, the following two points will be considered:

(a) The neglect of the viscous components from the equations for turbulent fluctuations

(b) The relative plausibility of the τ-theory and the M-theory

In making these considerations, the low-frequency and the high-frequency components will be considered as behaving independently in a linear equation. For nonlinear terms, low-frequency components can be obtained by the product of two terms both of the high frequency or both of the low frequency, while high-frequency components can be obtained by the product of one term of the high frequency and one term of the low frequency. Thus, if one rites

$$u = u_l + u_h \quad (9)$$

where u_l denotes the low-frequency components, and u_h, the high-frequency components, then in

$$u^2 = u_l^2 + 2u_l u_h + u_h^2 \tag{10}$$

the low-frequency components come from u_l^2 and u_h^2, while the high-frequency components come from $u_l u_h$ and u_h^2.

Since $u_h \ll u_l$, it is seen that u^2 may be approximated by u_l^2. Thus, in the left-hand side of any one of equations (5), one is justified in putting a subscript l to every fluctuating quantity, if the low-frequency components of the whole left side are desired. The right-hand side is linear, and hence one can again consider the low-frequency components separately. For these, the viscous dissipation is known to be negligible $\left(\text{as may be verified by noting that } \nu\,\Delta u \approx \nu V/l^2 \text{ is small compared with } \frac{\partial}{\partial x}(uu) \text{ when } Vl/\nu \text{ is large}\right)$. Consequently, Von Kármán's original assumption of neglecting ν is justified.

The equation of continuity (6) is linear and can therefore be considered separately for components of low and high frequencies.

It is remarked that the above discussion can be carried through only when equations (5) are put into the form given. If the left sides of equations (5) were written in the form

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{11}$$

the neglect of the influence of the high-frequency components on the low-frequency components would have been dubious. Consider, for example, the term

$$v\frac{\partial u}{\partial y} = v_l\frac{\partial u_l}{\partial y} + v_h\frac{\partial u_l}{\partial y} + v_l\frac{\partial u_h}{\partial y} + v_h\frac{\partial u_h}{\partial y} \tag{12}$$

The low-frequency components are given by $v_l \frac{\partial u_l}{\partial y}$ as well as by $v_h \frac{\partial u_h}{\partial y}$. Now, by equations (7) and (8),

$$\left.\begin{aligned} v_l \frac{\partial u_l}{\partial y} &\approx \frac{V^2}{l} \\ v_h \frac{\partial u_h}{\partial y} &\approx \frac{v^2}{\eta} \\ &\approx \frac{V^2}{l} R_\lambda^{1/2} \end{aligned}\right\} \tag{13}$$

and it would not be legitimate to replace $v \frac{\partial u}{\partial y}$ by $v_l\left(\partial u_l/\partial y\right)$. The mathematical argument can be carried through only when Du/Dt, and so forth are put into the form given in equations (5), with the help of the equation of continuity.

<u>τ-theory and M-theory</u>.- The above arguments also suggest that in the evaluation of $\tau = -\rho\overline{uv}$, the contribution of the low-frequency components predominates, while that of the high-frequency components is negligible. This would justify the application of the similarity concept to the calculation of τ. On the other hand, in the product $v\zeta$, the high-frequency components predominate. This suggests that $\overline{v\zeta}$ would be dependent more on the high-frequency components than on the low-frequency components. Thus, the similarity concept at low frequencies, as developed above, cannot be used for obtaining a formula for $\overline{v\zeta}$. The M-theory is therefore not a direct consequence of Von Kármán's concept of similarity.

These arguments should be somewhat modified by the fact that turbulence tends to be isotropic at high frequencies. This fact strengthens the argument for τ but weakens the argument against $\overline{v\zeta}$. It is possible that the contribution to $\overline{v\zeta}$ is equally important from high-frequency and low-frequency components. But there is no convincing argument to show that the high-frequency components can be entirely neglected. To make the ideas more precise, use will now be made of the correlation tensor and the spectral tensor in the discussion.

For convenience, first consider homogeneous anisotropic turbulence. Then, the correlation tensor $R_{ik} = \overline{u_i(P)u_k(P')}$ is a function of the relative positional vector ξ_l. The spectral tensor $F_{ik}(\kappa_l)$ stands in Fourier transform relation to R_{ik}:

$$\left.\begin{aligned} F_{ik}(\kappa) &= \frac{1}{(2\pi)^3}\iiint R_{ik}(\overset{o}{\xi})e^{i(\kappa\xi)}\ d\tau(\xi) \\ R_{ik}(\xi) &= \iiint F_{ik}(\kappa)e^{-i(\kappa\xi)}\ d\tau(\kappa) \end{aligned}\right\} \tag{14}$$

From these, it is seen that

$$\overline{u_i(P)\frac{\partial u_k}{\partial x_l}(P')} = \iiint F_{ik}(\kappa)\kappa_l e^{-i(\xi\kappa)}\ d\tau(\kappa) \tag{15}$$

If one lets $\xi \longrightarrow 0$ in the formula for R_{ik}, it is seen that

$$\overline{u_i u_k} = \iiint F_{ik}(\kappa)\ d\tau(\kappa) \tag{16}$$

If a spherical coordinate system is considered in the κ-space, integration with respect to the angular variables leads to

$$\overline{u_i u_k} = \int_0^\infty F(\kappa)\varphi_{ik}(\kappa)\ d\kappa \tag{17}$$

where φ_{ik} is a "correlation coefficient," being equal to unity when $i = k$. Isotropy at high frequencies requires that, for $i \neq k$, $\varphi_{ik}(\kappa)$ approach zero rapidly with $\kappa \longrightarrow \infty$, being substantially zero in the viscous range.

Thus, the contribution of the low-frequency components to $\overline{u_i u_k}$ dominates that of the high-frequency components even to a greater extent when $i \neq k$ than when $i = k$. In the latter case, it is well-known that the high-frequency components are negligible (cf. appendix B), and this is therefore still more so in the case $i \neq k$ where the factor φ_{ik} reduces the influence of the high-frequency components still further.

It is not possible to make a similar discussion for $\overline{v\zeta}$ by using homogeneous and anisotropic turbulence, because this term is zero. However, one may write a formula analogous to equation (17) in the form

$$\overline{u_i \frac{\partial u_k}{\partial x_l}} = \int_0^\infty F(\kappa)\psi_{ikl}(\kappa)\kappa \, d\kappa \tag{18}$$

where ψ_{ikl} is a dimensionless "correlation coefficient," approaching zero rapidly as $\kappa \longrightarrow \infty$, and the factor κ is suggested by formula (15). It will be shown in appendix B that, without the factor ψ_{ikl}, the contribution to the above integral would practically all come from the high-frequency components. However, since ψ_{ikl} must approach zero rapidly as $\kappa \longrightarrow \infty$, this argument is not certain. But, at any rate, equation (18) does not show that $\overline{v\zeta}$ will be essentially determined by the low-frequency components. One may also argue as follows. The factor ψ_{ikl} probably begins to become insignificant only in the $\kappa^{-5/3}$ range. For various Reynolds numbers of turbulence, corresponding to identical low-frequency components, the extent of the $\kappa^{-5/3}$ range differs. Since this range is now obviously important for the determination of integral (18), the amount of vorticity transfer would depend on the local Reynolds number of turbulence. This is another way of stating that there is no similarity in this sense between the various points of the flow field.

The above discussion shows that the τ-theory should be used in connection with Von Kármán's similarity concept.

DISCUSSION

It is perhaps in order now to make a general survey of the present theories of shear flow. In contrast with the similarity theory, there are the theories of transfer of momentum and vorticity. In comparing the similarity theory with transfer theories, it should be noted that, although

there is formal similarity in the final formulas, there is a definite difference in basic concept. The similarity theory does not directly use the physical picture of the mechanism of transfer. It, however, asserts more definitely on the nature of the turbulent fluctuations. On the one hand, this has the advantage of leading to a formula for the scale. On the other hand, one can apply the concepts evolved from modern statistical theories to the theory of shear flow by following the concept of similarity. The present work seems to be one of the few attempts in this direction, and it tends to bear out the classical form of the theory of Von Kármán (τ-theory). The use of Von Kármán's concept to calculate average quantities involving velocity derivatives (such as in the M-theory) cannot be justified by current concepts of the statistical theory of turbulence.

The conclusion that the τ-theory (analogous to the momentum transfer theory) is preferable to the M-theory (analogous to the vorticity transfer theory) perhaps requires further clarification. A great deal of work has been done, following Taylor, which shows that the vorticity transfer theory is better than the momentum transfer theory. The strongest case is perhaps the one involving joint velocity and temperature distributions in a wake and in a jet. Whereas the momentum transfer theory predicts the same distribution for velocity and temperature, the vorticity transfer theory can account for the difference in distribution which is actually observed experimentally. In fact, if θ is the fluctuation of temperature, the transfer of heat is

$$-\overline{\theta v} = l^2 \left|\frac{dU}{dy}\right| \frac{d\overline{\theta}}{dy} \tag{19}$$

in either theory. The momentum transfer theory gives further

$$\frac{\tau}{\rho} = -\overline{uv} = l^2 \left|\frac{dU}{dy}\right| \frac{dU}{dy} \tag{20}$$

while the vorticity transfer theory gives

$$\frac{1}{\rho}\frac{\partial\tau}{\partial y} = \overline{\zeta v} = l^2 \frac{dU}{\partial y}\frac{d^2U}{\partial y^2} \tag{21}$$

If l is assumed constant, this leads to

$$\frac{\tau}{\rho} = \frac{1}{2} l^2 \left|\frac{dU}{dy}\right| \frac{dU}{dy} \tag{22}$$

The difference in the numerical factor 1/2 is the essential point in question.

In applying this kind of argument to the similarity theory, one must recognize, as mentioned above, that the mechanism of transfer is not directly used in the similarity theory. In the transfer theories, it is natural to use the same transfer coefficient $l^2|dU/dy|$ in all the formulas, such as equations (20) to (22). On the basis of the similarity concept,

$$u \sim l \frac{dU}{dy}$$

$$v \sim l \frac{dU}{dy}$$

$$\theta \sim l \frac{d\overline{\theta}}{dy}$$

According to the τ-theory, then

$$\frac{\tau}{\rho} \sim l^2 \left|\frac{dU}{dy}\right| \frac{dU}{dy}$$

$$\overline{v\theta} \sim l \left|\frac{dU}{dy}\right| \frac{d\overline{\theta}}{dy}$$

but nothing can be said of the constants of proportionality. Thus, the above method of testing cannot distinguish between the τ-theory and the M-theory.

It must be added that the similarity theory with the formula $l \sim U'/U''$ does not apply to the case of wakes and jets, because the velocity-distribution curve has a point of inflection. The general concept of similarity of low-frequency components, however, might still apply even though the scale of similarity in such cases is probably not determined by the local velocity distribution. In fact, even in other cases of shear flow, the validity of the general considerations of similarity does not depend on the correctness of Von Kármán's formula.

Massachusetts Institute of Technology
Cambridge, Mass., December 27, 1950

APPENDIX A

SYMBOLS

l	similarity scale of length
p	pressure
t	time
u,v,w	fluctuating velocity components in x-, y-, and z-direction, respectively
x,y,z	Cartesian coordinates; x-axis in direction of mean flow
L	macroscale of turbulence
R_λ	Reynolds number of turbulence
U	velocity in x-direction
V	velocity scale
ϵ	rate of dissipation
θ	fluctuation of temperature
κ	wave number
λ	Taylor's microscale of turbulence
ν	coefficient of kinematic viscosity
ξ,η,ζ	turbulence vorticity components
ρ	density of fluid
τ	shearing stress

Subscripts:

l	low-frequency part of fluctuations
h	high-frequency part of fluctuations

Barred quantities always represent mean values; primed quantities represent fluctuations.

APPENDIX B

MOMENTS OF THE SPECTRUM OF TURBULENCE

As indicated in the section "Critical Dissussion of Similarity Concept from Standpoint of Modern Statistical Theories," the relative contribution of the low-frequency and high-frequency components to the moments of the spectral function plays an important role in deciding the physical concept of the mechanism of turbulence. In this section, it will be shown that in the case of large Reynolds numbers, the low-frequency contribution to the integral

$$I_n = \int_0^\infty F(\kappa)\kappa^n \, d\kappa \tag{A1}$$

is much more important than the high-frequency contribution if $n < 2/3$, while the reverse is true if $n > 2/3$ (equation (A15)). The low frequencies and the high frequencies are separated at a fairly arbitrary point in the $\kappa^{-5/3}$ range. In the borderline case of $n = 2/3$, the relative contribution depends on where the separation of the two ranges is made (equation (A16)).

The special cases of $n = 0$ (giving the energy) and $n = 2$ (giving the dissipation) are well-known (cf. "Introduction"). The case $n = 1$ has been used in the section "Critical Discussion of Similarity Concept from Standpoint of Modern Statistical Theories" to bring out the difficulties associated with the M-theory. The case $n = -1$ would be of importance in determining the macroscale. It is clear that the macroscale would be proportional to

$$L = \int_0^\infty F(\kappa)\kappa^{-1} \, d\kappa \Bigg/ \int_0^\infty F(\kappa) \, d\kappa \tag{A2}$$

and is consequently a low-frequency property. This is in contrast with the microscale λ of Taylor. Since λ is proportional to I_0/I_2, it depends on the low-frequency components as well as on the high-frequency components.

The method of investigation is the one used by Von Kármán and Lin (references 8 and 9). Let κ^* be some frequency (as yet unspecified) in

the range where $F \sim \kappa^{-5/3}$. The spectrum below this frequency κ^* is assumed similar with a length scale L and a velocity scale V. Above it, the characteristic quantities are v and η. Thus, the spectrum may be described as follows. For low frequencies,

$$F(\kappa) = V^2 L f(\kappa L) \tag{A3}$$

where the function $f(X)$ has the behavior

$$f(X) \approx CX^{-5/3} \tag{A4}$$

for large values of X, for example, of the order of $X^* = \kappa^* L$. For high frequencies,

$$F(\kappa) = v^2 \eta g(\kappa\eta) \tag{A5}$$

where the function $g(x)$ has the behavior

$$g(x) \approx cx^{-5/3} \tag{A6}$$

for small values of x, for example, of the order of $x^* = \kappa^*\eta$.

In evaluating the integral (A1), split it up into two parts:

$$I_n = \int_0^{\kappa^*} F(\kappa)\kappa^n \, d\kappa + \int_{\kappa^*}^{\infty} F(\kappa)\kappa^n \, d\kappa \tag{A7}$$

The low-frequency part is

$$\begin{aligned} I_{n,l} &= \int_0^{\kappa^*} F(\kappa)\kappa^n \, d\kappa \\ &= \frac{v^2}{L^n} \int_0^{X^*} X^n f(X) \, dX \end{aligned} \tag{A8}$$

where X^* is large. The high-frequency part is

$$I_{n,h} = \int_{\kappa^*}^{\infty} F(\kappa)\kappa^n \, d\kappa$$

$$= \frac{v^2}{\eta^n} \int_{x^*}^{\infty} x^n g(x) \, dx \tag{A9}$$

where x^* is small. To get the order of magnitude of the integrals, use is made of equations (A4) and (A6). Thus, for $n > 2/3$,

$$J_{n,l} = \int_0^{X^*} X^n f(X) \, dX$$

$$= \int_0^{X^*} X^n \left[f(X) - CX^{-5/3}\right] dX + CX^{*\,n-\frac{2}{3}} \Big/ \left(n - \frac{2}{3}\right) \tag{A10}$$

The first integral may be approximated by letting the upper limit go to infinity, and then

$$J_{n,l} = \int_0^{X^*} X^n f(X) \, dX$$

$$= \frac{C}{n - \frac{2}{3}} (X^*)^{n-\frac{2}{3}} + O(1) \tag{A11}$$

In the case $n < 2/3$, integral (A10) is convergent as $X^* \longrightarrow \infty$. Thus, write

$$\int_0^{X^*} X^n f(X) \, dX = \int_0^{\infty} X^n f(X) \, dX - \int_{X^*}^{\infty} X^n \left[f(X) - CX^{-5/3}\right] dX + \frac{C}{n - \frac{2}{3}} (X^*)^{n-\frac{2}{3}}$$

which leads to the same answer (A11). Similarly,

$$J_{n,h} = \int_{x^*}^{\infty} x^n g(x)\ dx = \frac{C}{n - \frac{2}{3}}(x^*)^{n-\frac{2}{3}} + O(1) \quad \text{for} \quad n \gtrless 2/3 \tag{A12}$$

The ratio of the two integrals $J_{n,l}$ and $J_{n,h}$ is therefore

$$\left.\begin{aligned} J_{n,l}/J_{n,h} &= O\left[(X^*)^{n-\frac{2}{3}}\right] && n > \frac{2}{3} \\ J_{n,l}/J_{n,h} &= O\left[(x^*)^{-\left(n-\frac{2}{3}\right)}\right] && n < \frac{2}{3} \end{aligned}\right\} \tag{A13}$$

On the other hand, from equations (A3) to (A6),

$$\begin{aligned} V^2L &= C(\kappa L)^{-5/3} \\ &= v^2\eta c(\kappa\eta)^{-5/3} \end{aligned}$$

so that

$$\begin{aligned} \frac{V}{v} &= \left(\frac{L}{\eta}\right)^{1/3} \\ &= \left(\frac{X^*}{x^*}\right)^{1/3} \end{aligned} \tag{A14}$$

Thus,

$$\left.\begin{aligned} I_{n,l}/I_{n,h} &= O\left[(x^*)^{\left(n-\frac{2}{3}\right)}\right] && n > \frac{2}{3} \\ I_{n,l}/I_{n,h} &= O\left[(X^*)^{-\left(n-\frac{2}{3}\right)}\right] && n < \frac{2}{3} \end{aligned}\right\} \tag{A15}$$

These are the statements made at the beginning of this section.

In the case $n = 2/3$, the above type of arguments give

$$I_{n,l} \approx \frac{V^2}{L^{2/3}} \log X^*$$

$$I_{n,h} \approx \frac{V^2}{\eta^{2/3}} \log x^*$$

and the ratio is

$$I_{n,l}/I_{n,h} = O\left(\frac{\log X^*}{|\log x^*|}\right) \tag{A16}$$

which depends on the choice of the frequency κ^*.

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