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A SIMILARITY THEORY FOR TURBULENT BOUNDARY LAYER OVER A FLAT PLATE IN COMPRESSIBLE FLOW

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SUMMARY

As an application of the concepts in NACA TN 2541, the problem of turbulent boundary layer over a flat plate in compressible flow is treated. The dissipation term in the energy equation, often neglected, is first carefully studied and found to be of importance. In the differential equations governing the fluctuations, the lower-frequency components of all quantities are regarded to participate in an equilibrium in accordance with the similarity concept. After proper linearization, a set of differential equations containing only the lower-order-frequency fluctuations is obtained. In parallel with Von Kármán's theory in incompressible flow, the similarity scales for all the flow variables are derived. Two possible length scales are found, and the significance of this possibility is discussed.

INTRODUCTION

Prior to the development of high-speed aircraft, the aerodynamic phenomena were satisfactorily explained by theories based on an incompressible fluid. Such a situation certainly no longer remains true. An increasing abundance of research is now available, attempting to extend the theories into the compressible range for almost every aspect of practical importance. Yet not until recently has the problem of turbulent boundary layer in compressible flow attracted the attention of investigators. The reason is, of course, that any turbulence problem is a difficult one even without compressibility. Perhaps demanded by its practical importance in the flows over an airfoil or in a wind tunnel, or just to keep pace with the advancement of knowledge in other aerodynamic problems, papers dealing with the behavior of turbulent boundary layer in compressible flow finally begin to appear.

Thus one has, for instance, the works of Ferrari (reference 1), Wilson (reference 2), Van Driest (reference 3), Ladenburg and Bershader (reference 4), and Eckert (reference 5). Except for reference 4, which is a pure experimental measurement using interferometric technique, all of them present a theoretical analysis. On reviewing these, it appears that, in these analyses, the turbulent-dissipation terms were neglected by the argument that any molecular phenomenon is probably of no importance in comparison with the turbulent transfer. This argument, being without experimental backing, should be carefully investigated.

The first part of this report is concerned with such an investigation. The energy equation is studied in some detail, and the orders of magnitude of the various terms are estimated, with the help of available experiments. The turbulent dissipation is found to be not negligible at least for boundary-layer flow over an insulated wall. Von Kármán's formula for the mixture length has been used in this analysis; but the conclusions reached on the order of magnitude of the various terms are presumably correct, even though the distribution of scale of turbulence may be only roughly given by that formula.

The second part of this report is an attempt to extend Von Kármán's similarity theory to the turbulent flow of a gas in the boundary layer. This extension is based on the ideas developed in reference 6. It turns out (see reference 7) that the theory has a certain universal character with regard to the Mach number and the heat-transfer conditions at the wall, and the unavoidable correlation constants can be determined once for all. Owing to the lack of suitable experimental data, the numerical part of the theory cannot be pursued and is left to the future.

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ENERGY EQUATION AND DISSIPATION TERM

In problems of compressible flow, a relation between the velocity field and the temperature field is furnished by the equation of energy. In applying the energy equation to turbulent phenomena, it has been the customary practice to neglect both the heat conductivity and the viscosity terms. The reason implied, or sometimes stated, is that the molecular phenomena would make but very small contributions in comparison with the turbulent transfer terms, just like the situation in the

equations of momentum. Undoubtedly such vague arguments have definite limitations. The earlier applications were for the problem of temperature distribution in turbulent flow at low mean velocities. There is a predominant temperature field. The conduction terms are of the same nature as the viscous shear in the equations of momentum. The heat generated through viscous dissipation, being caused by the velocity fluctuations, is probably small because of the large mechanical equivalent of heat. Hence it seems reasonable that for low Mach numbers the neglect of molecular terms would be justified. Recently, in treating turbulent boundary layer in supersonic flow, various authors have retained such a procedure without questioning its validity (references 1 and 3). The following is an attempt to disclose the deciding factors in assessing the relative weight of the molecular terms and the turbulent terms. Most important of all, the dissipation term is found to be not negligible in the case of boundary-layer flow over an insulated wall, irrespective of Mach number.

One may first write down the complete energy equation (reference 8, p. 606)

$$\rho c_v \frac{DT}{Dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = k\nabla^2 T + \Phi \tag{1}$$

where c_v is the specific heat at constant volume, k is the heat conductivity, and Φ is the dissipation, defined by

$$\Phi = \mu\left[2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2 + \ldots\right] \tag{2}$$

The first term on the left-hand side represents the convective action. Equation (1) is the equivalent of the statement that following a particle the temperature change is due to three kinds of heat sources: The compressibility, the heat conduction, and the dissipation. Without essential difference, let the incompressible case be considered. Then equation (1) becomes,

$$\frac{1}{\gamma\nu}\frac{DT}{Dt} = \frac{1}{\sigma}\nabla^2 T + \frac{1}{c_p}\epsilon \tag{3}$$

obtained by omitting the compressibility terms and dividing through with $c_p\mu$. (See appendix for definitions of symbols.) Here $\gamma = c_p/c_v$, the ratio of specific heats, $\sigma = c_p\mu/k$, the Prandtl number, and $\epsilon = \Phi/\mu$. In turbulent flow, each quantity may be split into mean and fluctuating parts; that is,

$$\frac{1}{\gamma\nu}\left(\frac{D\overline{T}}{Dt} + \frac{DT'}{Dt}\right) = \frac{1}{\sigma}\nabla^2\overline{T} + \frac{1}{\sigma}\nabla^2 T' + \frac{1}{c_p}\left(\overline{\epsilon} + \epsilon'\right) \tag{3a}$$

It is now intended to compare the terms on the right-hand side with a typical term on the left-hand side, such as $v' \frac{d\overline{T}}{dy}$. The mean flow is assumed to have a characteristic velocity $\overline{U}$ in the x-direction, as in boundary-layer or wake flows.

(1) The molecular conduction term $\frac{1}{\sigma}\nabla^2\overline{T}$ due to mean temperature: Let

$$r_1 = \frac{1}{\sigma}\nabla^2\overline{T}\Big/\frac{1}{\gamma\nu}\, v' \frac{d\overline{T}}{dy}$$

Then

$$r_1 \approx \frac{1}{\sigma}\frac{d^2\overline{T}}{dy^2}\Big/\frac{1}{\gamma\nu}\, v' \frac{d\overline{T}}{dy}$$

$$= \gamma\,\frac{\nu}{\sigma}\,\frac{1}{v'}\left(\frac{d^2\overline{T}}{dy^2}\Big/\frac{d\overline{T}}{dy}\right)$$

(In the following discussions of the orders of magnitudes suitable root-mean-square values will be implied without explicit notation.) At least for a study of the order of magnitude, one may use Reynolds' analogy (reference 8, p. 649 ff.) to relate mean temperature with mean velocity; that is,

$$\frac{d^2\overline{T}}{dy^2}\Big/\frac{d\overline{T}}{dy} \approx \frac{d^2\overline{U}}{dy^2}\Big/\frac{d\overline{U}}{dy} \approx 1/l$$

where l may be interpreted as either the mixing length or Von Kármán's similarity scale. Thus,

$$r_1 = O\left(\frac{1}{\sigma}\frac{\nu}{v' l}\right) \tag{4a}$$

or

$$r_1 = O\left(\frac{1}{\sigma}\frac{\overline{U}}{v'}\frac{L}{l}\frac{1}{R_L}\right) \tag{4b}$$

where L is the characteristic length of the mean flow and R_L is the mean-flow Reynolds number, defined by

$$R_L = \overline{U}L/\nu$$

Two illustrative cases may be given. For jet or wake flow, L may be taken as the "breadth" of the wake, where $\overline{U} = \frac{1}{2}\overline{U}_{max}$. The well-known theories (see, e.g., reference 9) assume

$$L/l = \text{Constant}$$

the constant being of the order of 0.2 for two-dimensional jets. The ratio r_1 is indeed small throughout the section. In the case of boundary-layer flow, L may be taken as the thickness of the layer and

$$l \approx 0.4y$$

Then equation (4b) indicates that r_1 becomes important only when very close to the wall, where laminar sublayer comes in anyway.

A different form of formula (4a) will also show that r_1 is generally small. Introducing the turbulence Reynolds number $R_\lambda = u'\lambda/\nu$, where λ is the microscale, equation (4a) becomes

$$r_1 \approx O\left(\frac{1}{\sigma}\frac{u'}{v'}\frac{1}{R_\lambda}\right) \tag{4c}$$

Except when near a solid wall, $u'/v' \approx O(1)$, so $r_1 \approx O\left(\frac{1}{R_\lambda}\right)$.

(2) The molecular conduction term $\frac{1}{\sigma}\nabla^2 T'$ due to fluctuating temperature: The term $\nabla^2 T'$ involves second derivatives of a fluctuating quantity, therefore greatly influenced by the high-frequency components of the fluctuation. It will be compared with a term $\frac{1}{\gamma\nu} v' \frac{\partial T'}{\partial y}$. Before the comparison, however, one may observe that the neglect of viscous shear in the equation of momentum implies the following relation:

$$r_2 = \nu \frac{\partial^2 u'}{\partial y^2} \Big/ v' \frac{\partial u'}{\partial y}, \ll 1 \tag{5}$$

in spite of the high-frequency components in $\partial^2 u'/\partial y^2$. Now form the ratio

$$r_3 = \frac{1}{\sigma}\nabla^2 T' \Big/ \frac{1}{\gamma\nu} v' \frac{\partial T'}{\partial y}$$

$$\approx \gamma \frac{\nu}{\sigma} \frac{1}{v'} \left(\frac{\partial^2 T'}{\partial y^2} \Big/ \frac{\partial T'}{\partial y} \right) \tag{6a}$$

Rewriting,

$$r_3 \approx 0 \ \frac{1}{\sigma} \left(\frac{\partial^2 T'/\partial y^2}{\partial T'/\partial y} \frac{\partial u'/\partial y}{\partial^2 u'/\partial y^2} \right) r_2 \tag{6b}$$

Reynolds' analogy assumed that the temperature and velocity fluctuations are proportional. Although such an approximate picture cannot be expected to hold for the derivatives largely controlled by the high-frequency components, it is likely that, at least,

$$\frac{1}{\Delta\overline{T}} \frac{\partial^2 T'}{\partial y^2} \Big/ \frac{1}{\overline{U}} \frac{\partial^2 u'}{\partial y^2} \approx O(1)$$

$$\frac{1}{\Delta\overline{T}} \frac{\partial T'}{\partial y} \Big/ \frac{1}{\overline{U}} \frac{\partial u'}{\partial y} \approx O(1)$$

where $\Delta\overline{T}$ is the mean temperature difference across the boundary layer and $\overline{U}$ is the mean free-stream velocity. Then equation (6b) reduces to

$$r_3 \approx 0\left(\frac{r_2}{\sigma}\right), \ll 1$$

(3) The dissipation terms $\overline{\epsilon}$ and ϵ': The order of magnitude of the dissipation term is to be compared with $v' \, d\overline{T}/dy$ on the left-hand side. To begin with, one discards the contribution of the mean flow velocity gradients to the dissipation, as a consequence of the discussion in reference 6. Then,

$$\left.\begin{aligned} \overline{\epsilon} &\approx 2\overline{\left(\frac{\partial u'}{\partial x}\right)^2} + 2\overline{\left(\frac{\partial v'}{\partial y}\right)^2} + \ldots \\ \epsilon' &\approx 2\left[\left(\frac{\partial u'}{\partial x}\right)^2 - \overline{\left(\frac{\partial u'}{\partial x}\right)^2}\right] + 2\left[\left(\frac{\partial v'}{\partial y}\right)^2 - \overline{\left(\frac{\partial v'}{\partial y}\right)^2}\right] + \ldots \end{aligned}\right\} \tag{7}$$

The relative magnitude of ϵ' and $\overline{\epsilon}$ may be seen by studying the following equation:

$$\begin{aligned} r_4 = \frac{|\epsilon'|}{\overline{\epsilon}} &\approx \left|\left(\frac{\partial u'}{\partial x}\right)^2 - \overline{\left(\frac{\partial u'}{\partial x}\right)^2}\right| \Bigg/ \overline{\left(\frac{\partial u'}{\partial x}\right)^2} \\ &\approx \left\{\overline{\left[\left(\frac{\partial u'}{\partial x}\right)^2 - \overline{\left(\frac{\partial u'}{\partial x}\right)^2}\right]^2}\right\}^{1/2} \Bigg/ \overline{\left(\frac{\partial u'}{\partial x}\right)^2} \\ &= \left\{\frac{\overline{\left(\frac{\partial u'}{\partial x}\right)^4}}{\left[\overline{\left(\frac{\partial u'}{\partial x}\right)^2}\right]^2} - 1\right\}^{1/2} \end{aligned} \tag{8}$$

Batchelor and Townsend (reference 10) measured the quantity $\overline{\left(\frac{\partial u'}{\partial x}\right)^4} \Big/ \left[\overline{\left(\frac{\partial u'}{\partial x}\right)^2}\right]^2$ for both isotropic turbulence and in the wake of a cylinder. Since the

high-frequency components are responsible for ϵ, one expects the ratios in the two cases to be close to each other. It turns out that the wake case has a higher ratio but both were within 10 percent of the value 4.0.[1] It may therefore be concluded that

$$r_4 \approx O(1)$$

For the order of magnitude of the mean dissipation $\overline{\epsilon}$ Taylor's work may first be cited (reference 12). For a channel flow and omitting compressibility, the work done on an elementary volume is $-\tau \frac{d\overline{u}}{dy}$, where τ is the total shearing stress, viscous plus turbulent. Based on experimental measurements, the microscale was evaluated and the dissipation estimated by the formula

$$\overline{\epsilon} \approx 15\mu \frac{\overline{(u')^2}}{\lambda^2}$$

which was originally derived for isotropic turbulence. Plotted across the section, his results indicate that these two terms are essentially of the same order of magnitude over a greater part of the section except near the center line and the wall. The dissipation further does not change its order of magnitude in these regions. The transfer of turbulent energy, being the difference of the two, is therefore, in general, not sufficiently important to alter their orders of magnitude. It seems now justifiable, based on Taylor's experience, to estimate the order of magnitude of $\overline{\epsilon}$ by evaluating $\tau \frac{d\overline{u}}{dy}$ at points near half the thickness in the case of a boundary-layer flow, where the velocity profile is rather similar to a channel flow. In a jet flow, the region of maximum viscous shear might be chosen for the estimation in order to avoid the nearly isotropic turbulence in regions of weak shear. The correspondence there is undoubtedly less satisfactory but perhaps still close enough for the present purpose.

It also may be easily shown that, in the case of axial symmetry, the work done per unit volume becomes $\tau \frac{d\overline{u}}{dr}$. Taking this to represent the order of magnitude of $\overline{\epsilon}$, one may compare with the term $\rho c_v u_r' \frac{\partial \overline{T}}{\partial r}$ to determine the relative importance of $\overline{\epsilon}$ in the energy equation.

---

[1]For a Gaussian distribution of $\partial u'/\partial x$, the ratio would have been 3.0. Experiments by Simmons and Salter (reference 11) indicated that u' follows very closely a Gaussian law and indeed gave $\overline{(u')^4} \Big/ \left[\overline{(u')^2}\right]^2 \approx 3.0$.

To estimate τ, when the mean motion depends essentially on y and the normal velocity $\tilde{v} \approx 0$,

$$\tau \approx \mu \frac{d\bar{u}}{dy} + \overline{\rho u v}$$

$$= \mu \frac{d\bar{u}}{dy} + \bar{\rho}\, \overline{u'v'}$$

for incompressible flow. Except very close to the wall, $\tau \approx \bar{\rho}\, \overline{u'v'}$. This formula may be regarded as valid even in compressible flow if only the order of magnitude is desired. Rewriting,

$$\tau \approx \bar{\rho} R_{uv} |u'||v'|$$

where R_{uv} is the correlation function. To compare the dissipation with transfer terms in the equation of motion, form the ratio

$$r_5 = \tau \frac{d\bar{u}}{dy} \Big/ \bar{\rho} c_v v' \frac{d\bar{T}}{dy}$$

Then,

$$r_5 \approx \frac{R_{uv} |u'| \frac{d\bar{u}}{dy}}{c_v \frac{d\bar{T}}{dy}}$$

$$\approx \frac{R_{uv} \left|\frac{u'}{\bar{U}_m}\right| \bar{U}_m^{\,2}}{\frac{\Delta\bar{T}_m}{\bar{T}_m} c_v \bar{T}_m} \tag{9a}$$

where $\bar{U}_m$ is the characteristic mean velocity, $\bar{T}_m$ is the characteristic mean temperature, and $\Delta\bar{T}_m$ is the characteristic mean temperature difference. The replacement of $\frac{d\bar{u}}{dy} \Big/ \frac{d\bar{T}}{dy}$ by $\bar{U}_m/\Delta\bar{T}_m$ is again based on Reynolds'

analogy, identifying the mean velocity and temperature distributions. With the introduction of a characteristic Mach number M_m equation (9a) becomes

$$r_5 \approx O\left(R_{uv}\left|\frac{u'}{\overline{U}_m}\right|\frac{M_m^2}{\frac{\Delta\overline{T}_m}{\overline{T}_m}}\right) \tag{9b}$$

Thus it is seen that, with a similar turbulence pattern, the importance of the dissipation term is proportional to $M_m^2\Big/\frac{\Delta\overline{T}_m}{\overline{T}_m}$. The similarity of the turbulence pattern, incidentally, needs only to cover the lower part of the frequency spectrum, as would prevail when mean streams of different velocities are passed through the same grid.

The criterion (9b) will now be applied to two cases of experiments on jet flow made by Corrsin (references 13 and 14). In reference 13, the data are as follows:

Jet, 1 inch

u_o = Nozzle velocity = 10 meters per second

ΔT_o = Nozzle temperature difference $\approx 10^\circ$ C

At $x/d = 20$, $r = 5$ centimeters from axis

$$\overline{U}_m/u_o = 0.26$$

$$\Delta\overline{T}_m/\Delta\overline{T}_o = 0.22$$

$$\left|u'/\overline{U}_m\right| \approx 0.18$$

$$\left|v'/\overline{U}_m\right| \approx 0.14$$

$$\overline{u'v'}\Big/\overline{U}_m^2 = 0.010$$

$$\frac{d}{dr}\left(\overline{u}/\overline{U}_m\right) \approx 0.132 \text{ per centimeter}$$

$$\frac{d}{dr}\left(\Delta\overline{T}/\Delta\overline{T}_m\right) \approx 0.120 \text{ per centimeter}$$

(The slopes are almost identical, indicating the approximate validity of the Reynolds' analogy in this case.) Hence,

$$r_5 \approx 0\left(0.07 \frac{M_m^2}{\Delta\overline{T}_m/\overline{T}_m}\right)$$

But

$$M_m^2 \approx 64 \times 10^{-6}$$

$$\Delta\overline{T}_m/\overline{T}_m \approx 7 \times 10^{-3}$$

Thus

$$r_5 \approx 0(0.001)$$

In reference 14 by Corrsin and Uberoi there are the following data:

Jet, 1 inch

$$u_o = 100 \text{ feet per second}$$

$$\Delta\overline{T}_o = 170^\circ \text{ C}$$

At $x/d = 15$, $\Delta\overline{T}_m \approx 0.3$

$$\Delta\overline{T}_o = 50^\circ \text{ C}$$

$$\overline{U}_m \approx 0.4$$

$$u_o = 40 \text{ feet per second}$$

At $r/r_2 = 1.0$ of the section, where r_2 is the radius at which $\Delta\overline{T}/\Delta\overline{T}_m = 1/2$,

$$\frac{d}{dr}\left(\overline{u}/\overline{U}_m\right) \approx 0.6 \text{ per centimeter}$$

$$\frac{d}{dr}\left(\Delta\overline{T}/\Delta\overline{T}_m\right) \approx 0.55 \text{ per centimeter}$$

$$\left|v'/\overline{U}_m\right| \approx 0.1$$

$$\overline{u'v'}\Big/\overline{U}_m^2 \approx 0.006$$

Hence

$$r_5 \approx 0\left(0.06 \frac{M_m^2}{\Delta\overline{T}_m/\overline{T}_m}\right)$$

With $M_m^2 \approx 0.0016$ and $\Delta\overline{T}_m/\overline{T}_m \approx 0.14$, one has finally, again, $r_5 \approx 0(0.001)$. The experience in these two cases seems to suggest that in jet flow the dissipation is of no importance in the energy equation.

For qualitative results, equation (9b) may be put into another form. Introducing the mixing length,

$$\overline{u'v'} = l|v'|\frac{d\overline{u}}{dy}$$

Then

$$r_5 \approx 0\left[l \frac{d}{dy}\left(\frac{\overline{u}}{\overline{U}_m}\right)\frac{M_m^2}{\Delta\overline{T}_m/\overline{T}_m}\right] \tag{10}$$

This formula may be checked by the jet-flow results. If the "breadth" b is taken to characterize the jet spreading, similarity of the mean velocity profile will lead to $\frac{d}{dy}\left(\frac{\overline{u}}{\overline{U}_m}\right) = A/b$ at any given y/b, say $y/b = 1$. Using Prandtl's assumption of l in jet flow,

$$l = Bb$$

Consequently, equation (10) becomes

$$r_5 \approx A \times B \times 0\left(\frac{M_m^2}{\Delta\overline{T}_m/\overline{T}_m}\right) \tag{11}$$

From previous experiments, $A \times B \approx 0.06$; its constancy is thus verified.

Let equation (10) be now applied to boundary-layer flow. Here one may take

M_m = Free-stream Mach number

$\overline{T}_m$ = Free-stream temperature

Consider the case of an insulated wall,

$$\Delta\overline{T}_m = \text{Stagnation temperature} - \overline{T}_m$$

$$\approx \frac{\gamma - 1}{2} \overline{T}_m M_m{}^2$$

as usually accepted. It follows that

$$\frac{M_m{}^2}{\Delta\overline{T}_m/\overline{T}_m} \approx \frac{2}{\gamma - 1}$$

a constant, and

$$r_5 \approx 0\left[\frac{2}{\gamma - 1}\, l\, \frac{d}{dy}\left(\frac{\overline{u}}{\overline{U}_m}\right)\right] \tag{12a}$$

Now in incompressible flow, a popular approximate form of the velocity distribution is

$$\frac{\overline{u}}{\overline{U}_m} = \left(\frac{y}{\delta}\right)^{1/7}$$

where δ is the boundary-layer thickness. Ladenburg and Bershader measured for a supersonic stream over a flat plate (reference 4), concluding

$$\frac{\overline{u}}{\overline{U}_m} \approx \left(\frac{y}{\delta}\right)^{1/9}$$

Putting $\overline{u}/\overline{U}_m = (y/\delta)^n$, and taking $l = Ky$ as for incompressible flow, one gets for order-of-magnitude purposes

$$r_5 \approx 0\left[\frac{2}{\gamma - 1}\, Kn\left(\frac{y}{\delta}\right)^n\right] \tag{12b}$$

As previously stated, the dissipation may be estimated at the station $y/\delta \approx 0.5$. With $K \approx 0.4$, $\gamma = 1.4$ for air,

$$
\begin{aligned}
r_5 &\approx O\left[2n(0.5)^n\right] \\
&\approx O(0.3) \quad \text{for} \quad n = 1/7 \\
&\approx O(0.25) \quad \text{for} \quad n = 1/9
\end{aligned}
\tag{13}
$$

The result of equation (13) shows that for an insulated wall, the dissipation is of the same order of magnitude as the terms kept in the energy equation. The usual practice of omitting the dissipation term is therefore incorrect. It is also interesting to see that, in boundary-layer flow over an insulated wall, the weight of the dissipation term is sensibly independent of the Mach number. Such a situation naturally will be greatly modified when excessive cooling or heating of the wall is introduced by auxiliary means.

EXTENSION OF SIMILARITY THEORY TO COMPRESSIBLE TURBULENT BOUNDARY LAYER

After the exploration in reference 6 of the foundations and limitations of the similarity concept in the case of incompressible turbulent boundary-layer flow, it is natural to attempt to formulate a theory of the compressible turbulent boundary layer on a somewhat similar basis. One recognizes that the additional density and temperature fields, the mean distribution as well as the fluctuations, must bring in many more difficulties. The similarity theory was shown to involve many approximations for the incompressible problem. The extension to compressible flow must therefore be made to an even more approximate degree. Nevertheless, it may again be stressed that the similarity concept leads to a simple model of the intrinsic turbulence mechanism, which is essentially correct at least in incompressible flow. By using this model a unified theory is possible for the mean velocity and temperature phenomena, without the necessity of separate assumptions whose consistency with each other cannot be ascertained. Moreover, the mean distributions are not very sensitive to the assumed mechanism of the turbulence, as the experience in compressible flow has indicated. Any discrepancy of the similarity model from the true one may be expected to influence but little the usefulness of the theory in predicting mean distributions.

A review of prevailing theories of compressible turbulent boundary layer discloses that a better fundamental concept can be claimed by none. The earliest attempt by Von Kármán (reference 15) was to substitute the wall conditions for the stream conditions in the formulas derived for incompressible flow. More recent ones include the comparatively simple theories of Wilson (reference 2) and Van Driest (reference 3), both starting from the incompressible expression

$$\tau = \bar{\rho} l^2 \left|\frac{d\bar{u}}{dy}\right| \frac{d\bar{u}}{dy}$$

Van Driest took l to be Prandtl's mixing length $l = Ky$, while Wilson used Von Kármán's form

$$l = \kappa \frac{d\bar{u}}{dy} \Big/ \frac{d^2\bar{u}}{dy^2}$$

The extension to the compressible case was formally carried out by allowing $\bar{\rho}$ to vary as a consequence to the temperature distribution. By assuming the effective turbulent Prandtl number equal to unity, the mean temperature and the mean velocity, in the case of an insulated wall, satisfy the isoenergetic law,

$$c_p\bar{T} + \frac{1}{2}\bar{u}^2 = \text{Constant}$$

in an analogous manner as for laminar flow. A recovery factor (references 16 and 17) was sometimes employed as a refinement of the isoenergetic law. For empirical data on the effective turbulent Prandtl number and the validity of the isoenergetic law in turbulent flow, discussions can be found in literature. In jet flow, for instance, the effective Prandtl number for air lies between 0.7 and 0.8 (reference 18). But if dissipation terms are not negligible, the isoenergetic law cannot follow in any case. Though useful, it must be regarded as largely empirical.

The work of Ferrari (reference 1) started on a more solid basis by including the energy equation to furnish a relation between the mean temperature and velocity. The assumption of a special relation, such as the isoenergetic law for the insulated wall, is thus avoided. He used Prandtl's mixture-length concept to deal with momentum and enthalpy transfers. When the two mixture lengths are taken to be identical, a temperature-velocity relation is obtained and appears to be formally the same one which occurs in a laminar boundary layer for a Prandtl number of unity. This coincidence perhaps may be used to explain the fact that

the isoenergetic law has been found to lead to no serious error. However, Ferrari, too, neglected the viscosity terms in both the equations of momentum and that of energy. His procedure cannot be easily applied if the dissipation terms, which are shown to be not negligible, are kept. In addition, his argument for identifying the two mixture lengths (for momentum and for enthalpy) is by no means conclusive.

It is thought that especially in the compressible case an intuitive guess to the proper form of the mixture length is rather difficult because of the complicated interactions of the velocity, density, and temperature. A systematic approach based on a clearly defined mechanism becomes necessary in order to achieve self-consistency.

As was previously shown in the discussion of the incompressible case, a theory can be established on an assumed similarity among the lower-frequency components of the turbulent fluctuations. In extending to the compressible case, the behaviors of the density and temperature fluctuations require certain additional simplifications based on physical concepts. The differential equations are to be satisfied in a more approximate manner. Let the complete set of equations be first written down:

$$\frac{Du_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i}, \quad i = 1,2,3 \tag{14}$$

$$\frac{1}{\rho}\frac{D\rho}{Dt} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{15}$$

$$\rho c_v \frac{DT}{Dt} + p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \mu\epsilon \tag{16}$$

$$p = \rho RT \tag{17}$$

One may note that in the equations of momentum the viscosity terms are neglected but not in the equation of energy as a consequence of the discussion in the section "Energy Equation and Dissipation Term." Prior to a mathematical derivation, the procedure may be summarized to consist of the following considerations:

(1) Partial linearization of the differential equations by neglecting the fluctuations of density and temperature against the mean quantities themselves but not the derivatives of the fluctuations.

(2) The separation of the density and temperature fluctuations into lower- and higher-frequency components in a manner analogous to the velocity fluctuations. A similarity of the turbulence is then assumed to include all these lower-frequency components.

(3) The dilatation of fluid element, that is,

$$e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

is regarded as a separate entity, not deducible from similarity of the lower-frequency components.

The following discussions will substantiate these points:

The partial linearization process is based on the fact that, ordinarily, the magnitude of turbulent fluctuations is only a small fraction of the mean quantity. For a first-order approximation, one may take

$$\left.\begin{aligned} \frac{\rho'}{\bar{\rho}} &\approx 0\left[\sqrt{\overline{(\rho')^2}}/\bar{\rho}\right], \quad \ll 1 \\ \frac{T'}{\bar{T}} &\approx 0\left[\sqrt{\overline{(T')^2}}/\bar{T}\right], \quad \ll 1 \end{aligned}\right\} \tag{18}$$

In taking derivatives, however, the fluctuation terms must be kept as was previously done for the velocity fluctuations and with the same arguments. By so doing it is seen that the differential equations are satisfied to a lesser degree in comparison with the incompressible case. The difference between velocity and other quantities lies in that the mean density and temperature cannot be dismissed like the mean velocity by the introduction of relative motion. The mean density distribution alone essentially provides the coupling between temperature and velocities. A consequence of equations (18) is that the equations of momentum will look exactly like those in incompressible flow, only with a variable mean density distribution. Thus, as very rough approximations, the starting point of the theories of Wilson (reference 2) and Van Driest (reference 3) is not without justification.

For a closer examination, one separates the larger eddies from the small ones,

$$\left.\begin{aligned} \rho' &= \rho_l' + \rho_h' \\ T' &= T_l' + T_h' \end{aligned}\right\} \tag{19}$$

where subscripts l and h denote the lower- and the higher-frequency components, respectively. The fluctuations are then seen to be due to two sources, convective actions of the larger eddies and heating from viscous dissipation. Loosely speaking, the convective actions, being carried by eddies of the larger sizes, would show up only in the lower-frequency components. The viscous heating comes from the very small eddies and would contribute to both the lower- and the higher-frequency components. The demarcation between the "larger" and "small" eddies therefore becomes even less clean-cut than that in incompressible flow, where the coupling through a density variation does not exist. It is nevertheless still conceivable that for eddies small enough, their equilibrium is controlled by no other than Kolmogoroff's (reference 19) parameters $\bar{\mu}$ and ϵ, $\bar{\mu}$ being now the local mean viscosity. For points across the boundary-layer thickness, there is merely a shift of the Kolmogoroff range in the wave-number space, the ratio of the wave numbers separating the Kolmogoroff range from the rest being of order unity since the ratio of $\bar{\mu}$ is only of order unity. The higher-frequency components ρ_h' and T_h' therefore must participate in a different equilibrium from ρ_l' and T_l'. The attempt now is to establish differential equations governing the approximate behavior of the lower-frequency component only.

It is found that, after partial linearization, a typical term to be examined in the process of setting up the differential equations for the lower-frequency fluctuations is $\frac{\partial}{\partial x} T'u'$. Reasoning in the same way as before, the lower-frequency part of $T'u'$ consists of $T_l'u_l'$ plus some contribution from $T_h'u_h'$. Now the magnitude of T_h' is probably of the same order as T_l'. For, one could visualize that T_l' for order of magnitude is associated with the mean distribution $d\bar{T}/dy$, while T_h' is essentially related to the dissipation ϵ'. The quantity ϵ' was shown in the section "Energy Equation and Dissipation Term" to be of the same order as $\bar{\epsilon}$ and also of the same order as the contribution from $d\bar{T}/dy$. However, previous discussions indicated that $\left(u_h'\right)^2 \ll \left(u_l'\right)^2$; hence

$$T_h'u_h' = T_l'u_l' \approx u_h' = u_l' \approx \sqrt{\overline{\left(u_h'\right)^2}} = \sqrt{\overline{\left(u_l'\right)^2}}, \quad \ll 1 \tag{20}$$

Therefore, the high-frequency parts again may be omitted in terms like $T'u'$ without seriously affecting the equilibrium of the lower-frequency components.

There is another way of looking upon such terms as $\frac{\partial}{\partial x} T'u'$ occurring in the energy equation, relying more upon a physical interpretation. One could regard the operator D/Dt as representing the change following the fluid element. The higher-frequency velocity fluctuations tend to average out without contributing appreciably to the movement of the element.[2] The operator D/Dt should therefore be associated with only u_l', v_l', and w_l'. In equation (14), the quantity after D/Dt is the velocity. Being now linear, the equation can be split into one for the higher frequency u_h' and another for u_l'. The previous results for the velocity fluctuation are reproduced. In equation (16), the temperature appears also in a linear fashion, the lower- and higher-frequency parts can again be simply separated.

The dilatation $e = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$, deceptively simple in appearance, requires a different handling when a theory of the lower-frequency components is desired. Firstly, the magnitude of each term in e is mainly due to the high-frequency component rather than the low, since

$$\frac{\partial u'}{\partial x} \approx O\left[\sqrt{\overline{\left(\frac{\partial u'}{\partial x}\right)^2}}\right]$$

and $\overline{\left(\frac{\partial u'}{\partial x}\right)^2}$ has been shown in reference 6 to depend on the very small eddies in the study of the dissipation ϵ. In the second place, the

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[2]One may get an idea of the relative importance to diffusion of the low-frequency and the high-frequency components by examining Taylor's formula (reference 20)

$$\frac{1}{2}\overline{y^2} = \overline{(v')^2} \int_0^T \int_0^t R_\xi \, d\xi \, dt$$

for diffusion by continuous movements where R_ξ is the correlation of the velocities at time interval ξ apart. Being proportional to $\overline{(v')^2}$, the movement is largely due to the lower-frequency components.

equations for compressible flow must be reconciled with the incompressible case when the density remains constant. If one applies a similarity consideration in a very naive manner, the apparent conclusion is that

$$e \approx \frac{\partial u'}{\partial x} \approx O(V/L)$$

where V and L are the similarity scales. It would then be difficult to produce a consistent theory aside from making the postulate that

$$e \equiv 0 \quad \text{if} \quad \bar{\rho} = \text{Constant}$$

$$e = O(V/L) \quad \text{if} \quad \bar{\rho} \neq \text{Constant}$$

To say the least, such a postulate would be too arbitrary. A more satisfactory concept is to regard e as an entity by itself. Although composed of terms like $\partial u'/\partial x$, the net effect of all three terms combined together cause e to behave in its own way. The equation of continuity (15) tells how this entity e is related to the density variation.

One is now in a position to write down the differential equations for the lower-frequency fluctuating components. From equations (14) to (16), with partial linearization and separation of lower- and higher-frequency components, the following are obtained:

$$\left.\begin{aligned}
\frac{D}{Dt}\left(\bar{u} + u_l'\right) &= -\frac{1}{\bar{\rho}}\frac{\partial}{\partial x}\left(\bar{p} + p_l'\right) \\
\frac{D}{Dt}\, v_l' &= -\frac{1}{\bar{\rho}}\frac{\partial}{\partial y}\left(\bar{p} + p_l'\right) \\
\frac{D}{Dt}\, w_l' &= -\frac{1}{\bar{\rho}}\frac{\partial}{\partial z}\left(\bar{p} + p_l'\right)
\end{aligned}\right\} \tag{21}$$

$$\frac{D}{Dt}\left(\bar{\rho} + \rho_l'\right) + e = 0 \tag{22}$$

$$\frac{D}{Dt}\left(\overline{T} + T_l'\right) + \left(\overline{p} + p_l'\right) e = \frac{\overline{\mu\epsilon} + (\mu\epsilon)_l'}{\overline{\rho} c_v} \tag{23}$$

Here $\frac{D}{Dt}$ is to be understood as

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\overline{u} + u_l'\right)\frac{\partial}{\partial x} + v_l' \frac{\partial}{\partial y} + w_l' \frac{\partial}{\partial z} \tag{24}$$

From the equation of state (17), one gets

$$\left.\begin{aligned} \overline{p} &= R\overline{\rho}\overline{T} \\ \frac{p_l'}{\overline{p}} &= \frac{\rho_l'}{\overline{\rho}} + \frac{T_l'}{\overline{T}} \end{aligned}\right\} \tag{25}$$

where the double correlation $\overline{\rho'T'}$ is omitted since

$$\frac{\overline{\rho'T'}}{\overline{\rho}\overline{T}} \approx O\left[\frac{\sqrt{\overline{(\rho')^2}}}{\overline{\rho}} \frac{\sqrt{\overline{(T')^2}}}{\overline{T}}\right], \quad \ll 1$$

In the next section, similarity scales will be derived from the above differential equation in parallel with the incompressible theory. The mean velocity and temperature distributions will then follow.

SIMILARITY SCALES

Based on the system of differential equations (21) to (23) for the lower-frequency components of the fluctuations, it is now possible to deduce the similarity scales for these lower-frequency components and build upon them a theory of the compressible boundary layer in parallel with the incompressible case. For the sake of simplicity, let the subscript l be dropped and rewrite the equations as:

$$\frac{D}{Dt}\left(\overline{u}_i + u_i'\right) = -\frac{1}{\overline{\rho}} \frac{\partial}{\partial x_i}\left(\overline{p} + p'\right), \quad i = 1, 2, 3 \tag{26}$$

$$-\frac{1}{\overline{\rho}}\frac{D}{Dt}\left(\overline{\rho} + \rho'\right) = e \tag{27}$$

$$c_p \frac{D}{Dt}\left(\overline{T} + T'\right) = \frac{R}{\overline{\rho}}\frac{D}{Dt}p'$$

$$= \overline{\nu\epsilon} + (\nu\epsilon)' \tag{28}$$

$$\left.\begin{aligned} \overline{p} &= R\overline{\rho}\overline{T} \\ \frac{p'}{\overline{p}} &= \frac{\rho'}{\overline{\rho}} + \frac{T'}{\overline{T}} \end{aligned}\right\} \tag{29}$$

where

$$e = \frac{\partial}{\partial x_i}\left(\overline{u}_i + u_i'\right)$$

and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \left(\overline{u} + u'\right)\frac{\partial}{\partial x} + v'\frac{\partial}{\partial y} + w'\frac{\partial}{\partial z}$$

Equation (28) is obtained by eliminating e in the energy equation by means of the continuity relation.

For the case of boundary layer over a flat plate, consider the state of affairs at large distances from the leading edge. There, if the slope of the outer edge of the boundary layer be assumed small as is usually done for the incompressible case,[3] a first approximation will be to regard

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[3]In the incompressible case, a popular formula, though not too well substantiated, for the growth of the boundary-layer thickness is

$$\delta \propto x^{4/5}$$

hence $\frac{d\delta}{dx} \longrightarrow 0$ as $x \longrightarrow \infty$. See, e.g., reference 21.

the mean flow within the boundary layer as a parallel flow varying only in the normal direction, namely

$$\bar{u} = \bar{u}(y)$$

$$\bar{v} = \bar{w} = 0$$

The mean pressure is, of course, constant and equal to the value in the main stream. In considering local similarity, the observer may be taken to move with the local mean motion, again as in the incompressible case. Thus,

$$\left.\begin{aligned} \bar{u} &= y\,\frac{d\bar{u}}{dy} + \frac{1}{2}\,y^2\,\frac{d^2\bar{u}}{dy^2} + \ldots \\ \frac{D}{Dt} &= \frac{\partial}{\partial t} + \left(y\,\frac{d\bar{u}}{dy} + \frac{1}{2}\,y^2\,\frac{d^2\bar{u}}{dy^2} + \ldots + u'\right) + v'\,\frac{\partial}{\partial y} + w'\,\frac{\partial}{\partial z} \end{aligned}\right\} \tag{30}$$

and so on; the right-hand side of equation (26) becomes

$$-\frac{1}{\bar{\rho}}\,\frac{\partial p'}{\partial x_i}$$

and the dilatation reduces to

$$e = \frac{\partial u_i'}{\partial x_i}$$

One may recall that, at this point, for the incompressible case the procedure was to use the vorticity equation by eliminating the pressure fluctuation terms on the right-hand side of equation (26) from each other. Such a step is not readily useful here because the mean density now varies with y. Instead, by cross-multiplication, there follows

$$\frac{\partial p'}{\partial x}\,\frac{Dv'}{Dt} = \frac{\partial p'}{\partial y}\,\frac{D}{Dt}\,(\bar{u} + u') \tag{31}$$

and so forth. Hence, if a scale can be found for the pressure fluctuation, equation (31) indicates that the velocity and time scales are

determined by a relation identical with that for the incompressible case. In other words, these scales should come from a consideration of the operator D/Dt. Expanding $\frac{D}{Dt}(\bar{u} + u')$,

$$\frac{D}{Dt}(\bar{u} + u') = v'\left(\frac{d\bar{u}}{dy} + y\frac{d^2\bar{u}}{dy^2} + \ldots\right) + \frac{\partial u'}{\partial t} +$$

$$\left(y\frac{d\bar{u}}{dy} + \frac{1}{2}y^2\frac{d^2\bar{u}}{dy^2} + \ldots + u'\right)\frac{\partial u'}{\partial x} + v'\frac{\partial u'}{\partial y} + w'\frac{\partial u'}{\partial z}$$

Let the fluctuating motion have a length scale l_o, a time scale t_o, and a velocity scale v_o, where

$$v_o \propto l_o/t_o$$

After substitution,

$$\frac{D}{Dt}(\bar{u} + u') = v_o\left(\frac{d\bar{u}}{dy} + \frac{y}{l_o}l_o\frac{d^2\bar{u}}{dy^2} + \ldots\right)v_1' + \frac{v_o}{l_o}\frac{\partial u_1'}{\partial t_1} + \left(\frac{y}{l_o}l_o\frac{d\bar{u}}{dy} +\right.$$

$$\left.\frac{y^2}{l_o^2}\frac{l_o^2}{2}\frac{d^2\bar{u}}{dy^2} + \ldots + v_o v_1'\right)\frac{v_o}{l_o}\frac{\partial u_1'}{\partial x_1} + \frac{v_o}{l_o}\left(\frac{\partial u_1'}{\partial y_1} + \frac{\partial u_1'}{\partial z_1}\right)$$

where subscript 1 denotes the normalized quantity:

$$(u_1', v_1') = (u', v')/v_o$$

$$(x_1, y_1, z_1) = (x, y, z)/l_o$$

The similarity solutions require

$$u_1' = u_1'(x_1, y_1, z_1, t_1)$$

and so forth. Therefore, the mean flow quantities must be separable from the normalized fluctuating quantities in the above expression. Exactly as in the incompressible case, it follows that

$$\left.\begin{aligned} \frac{1}{t_o} &\propto \frac{d\bar{u}}{dy} \\ v_o &\propto l_o \frac{d\bar{u}}{dy} \end{aligned}\right\} \tag{32}$$

and, if the mean velocity distribution is of predominant importance, so that the term involving $\frac{d^2\bar{u}}{dy^2}$ must be retained,

$$l_o \propto \frac{d\bar{u}}{dy}\bigg/\frac{d^2\bar{u}}{dy^2} \tag{33}$$

Applying the same procedure to Dv'/Dt, the same set of scales is obtained. Equation (31) is seen to be satisfied by such a choice of scales, provided a scale for pressure fluctuation exists.

The pressure scale is easily deduced from equation (26). Let the scale be π. One has

$$\frac{1}{\bar{\rho}} \frac{\pi}{l_o} \frac{\partial p_1}{\partial x_1} = \frac{D}{Dt} (\bar{u} + u')$$

$$\approx \frac{v_o}{t_o} \frac{Du_1'}{Dt_1}$$

with Du_1'/Dt_1 representing the normalized expression of $\frac{D}{Dt} (\bar{u} + u')$. Thus,

$$\pi \propto \bar{\rho} v_o^2 \tag{34}$$

As the next step, the energy equation (28) may be considered to establish a possible scale for the temperature fluctuation T'. A naive approach of again assuming a single scale for T' and substituting the rest by the results (32) to (34) will lead to additional restrictions on the mean flow and rule out a self-consistent description. This approach

must therefore be abandoned. In view of the linearity of the variable T' in equation (28), however, one can at least formally split the temperature fluctuation T' into three parts and rewrite equation (28) into a system of equations:

$$c_p \frac{D}{Dt}\left(\overline{T} + T_1'\right) = 0 \tag{35}$$

$$c_p \frac{D}{Dt} T_2' - \frac{R}{\overline{\rho}} \frac{D}{Dt} p' = 0 \tag{36}$$

$$c_p \frac{D}{Dt} T_3' = \overline{\nu\epsilon} + (\nu\epsilon)' \tag{37}$$

Equation (35) is the same as the usual incompressible heat-transfer equation in turbulent flow. Consequently, T_1' may be interpreted as the temperature fluctuation caused by the mixing of fluid elements from strata of different temperatures, due to the turbulent velocity fluctuations. Alternatively, one may regard equation (35) as describing that a certain portion of the temperature is attached to each fluid element during the process of mixing. Equation (36) describes the contribution to temperature fluctuation by the pressure field acting on the volume change of the fluid elements - the compressibility effect. Such an interpretation is most clearly seen when the pressure term is converted back to the form of equation (35) involving explicitly the dilatation e. The third part T_3' in equation (37) obviously represents the heating effect of the viscous dissipation.

One may now introduce different scales θ_1, θ_2, and θ_3 for the fluctuations T_1', T_2', and T_3'. Expansion of equation (35) leads to

$$v_o v_1'\left(\frac{d\overline{T}}{dy} + \frac{y}{l_o} l_o \frac{d^2\overline{T}}{dy^2} + \ldots\right) + \frac{\theta_1}{t_o}\left[\frac{\partial}{\partial t}\left(\frac{T_1'}{\theta_1}\right) + \ldots\right] = 0$$

Hence,

$$\theta_1 \propto l_o \frac{d\overline{T}}{dy} \tag{38}$$

and, if the higher derivatives of the mean temperature distribution are of predominant importance,

$$l_o \propto \frac{d\overline{T}}{dy}\bigg/\frac{d^2\overline{T}}{dy^2} \tag{39}$$

In a similar way, by expanding equations (36) and (37),

$$\theta_2 \propto v_o^2 \tag{40}$$

$$\theta_3 \propto t_o\left[\overline{\nu\epsilon} + (\nu\epsilon)'\right] \tag{41}$$

It was previously shown that, for the dissipation, both the mean and the fluctuating parts are related to terms like $\left(\frac{\partial u_h'}{\partial x}\right)^2$ and are of the same order of magnitude. Taylor's expression of balancing the energy yields now

$$\overline{\nu\epsilon} \propto l_o^2/t_o^3$$

Substitution into equation (41) shows that θ_3 is of the same form as θ_2. There are but two scales for the temperature fluctuation given by equations (38) and (40).

The scales of the density fluctuation must satisfy the second relation of equations (29). Again one has a linear form and may split ρ' into several parts:

$$\rho_1' + \rho_2' + \ldots = \frac{\overline{\rho}}{\overline{T}}\left(T_1' + T_2' + \ldots\right) + \frac{\overline{\rho}}{\overline{p}}\,p'$$

Denoting the scales for ρ_1', ρ_2', . . . by r_1, r_2, . . ., one recasts the above into a system of equations:

$$r_1 \propto \frac{\overline{\rho}}{\overline{T}}\,\theta_1$$

$$\propto \frac{\overline{\rho}}{\overline{T}}\,l_o\,\frac{d\overline{T}}{dy} \tag{42}$$

$$r_2 \propto \frac{\bar{\rho}}{\bar{T}} \theta_2$$

$$\propto \frac{\bar{\rho}}{\bar{T}} v_o^2 \tag{43}$$

$$r_3 \propto \frac{\bar{\rho}}{\bar{p}} \pi$$

$$\propto \frac{\bar{\rho}}{\bar{T}} v_o^2 \tag{44}$$

Thus r_2 and r_3 also are proportional. With the help of the first relation of equations (29), equation (42) may be rewritten as

$$r_1 \propto l_o \frac{d\bar{\rho}}{dy} \tag{45}$$

Equations (33) and (45) are the expressions for the two scales of ρ'.

It is desirable at this moment to bring attention to the entity "dilatation." The equation of continuity (27) has not been used in the previous derivation of the scales. One could formally write $e \propto v_o/l_o$ and find that the above scales satisfy equation (27) as well. A procedure like this, however, is not justifiable on the ground of the arguments in the section "Extension of Similarity Theory to Compressible Turbulent Boundary Layer."

All the scales of the lower-frequency fluctuations have now been related to the mean flow. There remains one point which needs some clarification. The question is: For the length scale l_o, which of the expressions, (33) or (39), should be adopted? Evidently, if they are compatible, there must be

$$\frac{d^2\bar{u}}{dy^2}\bigg/\frac{d\bar{u}}{dy} : \frac{d^2\bar{T}}{dy^2}\bigg/\frac{d\bar{T}}{dy} = a_1 : a_2 \tag{46}$$

Integrating,

$$\frac{d\bar{u}}{dy} = A\left(\frac{d\bar{T}}{dy}\right)^{a_1/a_2} \tag{47}$$

On the other hand, it will be seen in reference 7 that a relation between $\bar{u}$ and $\bar{T}$ may be obtained from averaging the equations of motion and energy. The relation is further independent of the choice of l_o. If the expressions of l_o from equations (33) and (39) are equivalent, the relation must also be reconcilable with equation (47). Such is the case for incompressible flow when Reynolds' analogy is valid (i.e., $a_1/a_2 = 1$). But, in general, one can hardly expect the compatibility relation (47) to be fulfilled, except perhaps approximately. It will be seen later that the choice of the form of l_o determines the distribution in the space coordinate (y-direction). At present, one will be contented with the notion that the choice of equation (33), say, means more emphasis on the satisfaction of the momentum relations at the expense of the energy relations, and vice versa for the choice of equation (39). Some additional discussions are included in the following section.

DISCUSSION

Before a detailed treatment of the theory of turbulent boundary layer, it may be worth while to remind oneself of all the approximations involved and, hence, the limitations to be introduced in the investigation.

The preceding theory for the turbulent boundary layer involves mainly the following approximations:

(1) The turbulence pattern: The turbulence pattern is idealized to be such that the small (high-frequency) dissipative eddies are separable from the larger (lower-frequency) energy-carrying eddies. In accordance with Kolmogoroff (reference 19), the small eddies are isotropic and participate in an equilibrium governed by the kinematic viscosity ν and the rate of dissipation ϵ. The larger eddies are assumed also to participate in an equilibrium, governed by the local conditions. Here one sees that three approximations are involved:

(a) The separability of small and larger eddies

(b) The possibility of similarity for the larger eddies

(c) The "local" character of the similarity for the larger eddies

In order that the small dissipative eddies may be separated, an essential condition is that the turbulence Reynolds number be high. This, in general, will follow if the mean-flow Reynolds number is high, for it is a commonly accepted notion that a higher mean-flow Reynolds number creates more small eddies. In this respect, a higher main-stream velocity outside the boundary layer should check better with the theory. The similarity for the larger eddies is mainly an intuitive hypothesis and certainly needs experimental verification. The arguments for its plausibility are given in reference 6. In the very simple configuration of a flat-plate boundary layer without pressure gradient, the chances for the realization of similarity are certainly on the favorable side. One may even take a heuristic point of view and regard the whole theory as being an attempt to seek a similarity solution, if at all possible. The behavior of the small eddies is decidedly different, but the order-of-magnitude study in reference 6 indicates that the original formulation by Von Kármán in 1930 (reference 22) remains valid if the fluctuations are interpreted as the lower-frequency ones only. The scales for length and time are determined from the local state of the mean flow by restricting attention to the immediate neighborhood of the point in question. It is well-known that the length scale from such considerations is only moderately small in comparison with the boundary-layer thickness. Experiments in the National Bureau of Standards for boundary layer with a pressure gradient further showed the existence of significant large eddies (reference 23). However, there is the consolation that for a class of the distribution profile, pointed out by Prandtl (reference 21, p. 132),

$$\bar{u} = A(y + B)^n + C \tag{48}$$

where A, B, and C are constants, the magnitude of the length scale is of no consequence. This is true because the higher derivatives, which bring in the conditions away from the point, will form length scales proportional to the one determined by $d\bar{u}/dy$ and $d^2\bar{u}/dy^2$. The logarithmic distribution belongs to the class (48) and so do the usual power (1/7, 1/9, etc.) laws.

(2) The boundary-layer growth: The growth of the thickness of the boundary layer has been neglected in the present theory, as everything is assumed to depend on the coordinate y only. Such a neglect is

equivalent to the stipulation that $d\delta/dx \ll 1$, δ being the boundary-layer thickness. In the incompressible case, semiempirically the generally accepted formula is

$$\delta \propto x^{4/5}$$

which shows that, although the rate of decrease of the slope is rather slow, $d\delta/dx$ could be as small as one wishes at distances very far from the leading edge. In applying the theory to moderate distances from the leading edge, one must be aware of this limitation when compared with experiments.

(3) The two choices of the length scale: With the extension to the compressible case, two possible choices of the length scale are found, depending on whether the equations of momentum or the equation of energy is to be better satisfied. Unless the two scales are essentially identical, in the manner discussed at the end of the section "Similarity Scales," an added degree of approximation is involved, because one of the expansions has to be cut down to the order of the first derivative only. In this situation, the validity therefore hinges on whether the second derivative is sufficiently small in comparison with the first for either $\bar{u}$ or $\bar{T}$. In other words, the approximation will still be good if one of the two scales is much larger than the other, for example,

$$\frac{d\bar{u}}{dy}\bigg/\frac{d^2\bar{u}}{dy^2} \ll \frac{d\bar{T}}{dy}\bigg/\frac{d^2\bar{T}}{dy^2}$$

so that, for local behavior, only the smaller scale $\left(\text{here } \frac{d\bar{u}}{dy}\bigg/\frac{d^2\bar{u}}{dy^2}\right)$ controls the motion. As an extreme case for the purpose of illustration, suppose that a large amount of heat were taken away at the flat plate to reduce the wall temperature to that in the free stream. The temperature profile in the boundary layer would be approximately uniform. The term $d\bar{T}/dy$, and so forth enters the determination of the length scale through the convective action and is obviously unimportant in this case. One must then choose the length scale from the velocity profile. In the general case, it can only be hoped that the actual distributions do not depart too much from the relation (46), which holds for approximately identical scales.

(4) The effect of pressure gradient: It is interesting to observe that the introduction of a pressure gradient renders impossible the extension to the compressible case. This is so because the scale for the pressure fluctuation can no longer be determined as related only to the velocity scale (cf. equation (34)). Without a similarity among the

pressure fluctuations, there follows no similarity for the temperature and the density fluctuations. The entire scheme breaks down. Difficulty like this does not influence the incompressible case, where the pressure fluctuation is of no consequence and of no interest. A remark may be in order about Von Kármán's initial derivation in considering a channel flow. His derivation started from the vorticity equation. Such a procedure is still permissible in the present theory when $\overline{\rho}$ remains constant, so that pressure terms can be eliminated by cross-differentiation of equation (26). The same similarity theory would then be obtained.

When the results of the present theory for a flat-plate boundary layer are to be applied to the case of an airfoil, the additional restriction to the possible degree of agreement should be kept in mind.

Massachusetts Institute of Technology
Cambridge, Mass., December 27, 1950

APPENDIX

SYMBOLS

a_1, a_2	constants
c_p, c_v	specific heat at constant pressure and constant volume, respectively
e	dilatation $\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$
k	coefficient of heat conductivity of fluid
l	mixture length or similarity scale of length
l_o	similarity scale of length
p	pressure
$\overline{p}, \overline{p}_1$	mean pressure and mean pressure in free stream, respectively
r_1, r_2, r_3 r_4, r_5	ratios of various heat conduction and viscous dissipation terms to turbulent transfer term in equation of energy, discussed in section "Energy Equation and Dissipation Term"
r_1, r_2, r_3	similarity scales of three components of density fluctuation, defined by equations (42) to (44)
t	time
t_o	similarity scale of time for fluctuations
u, v, w	velocity components in x-, y-, and z-directions, respectively
x, y, z	Cartesian coordinates; x, axis in direction of plate and free stream; y, axis normal to plate; and z, axis parallel to leading edge of plate
L	characteristic length
M_m	characteristic Mach number

R	gas constant in equation of state
R_L	characteristic Reynolds number
R_{uv}	correlation constant between u' and v'
R_λ	turbulence Reynolds number
T	temperature
$\overline{T}_m$	characteristic mean temperature
$\Delta\overline{T}_m$	characteristic mean temperature difference
$\overline{T}_o$	mean temperature at wall
$\overline{U}$	mean velocity component in x-direction
$\overline{U}_m$	characteristic mean velocity
γ	ratio of specific heats
δ	thickness of boundary layer
ϵ,ϵ'	rate of dissipation and its fluctuating component, respectively
$\theta_1,\theta_2,\theta_3$	similarity scales of temperature fluctuation defined by equations (38), (40), and (41), respectively
λ	Taylor's microscale of turbulence
μ	coefficient of viscosity
ν	coefficient of kinematic viscosity
π	similarity scale of pressure fluctuation
ρ	density of fluid
τ	shearing stress
Φ	dissipation function in equation of energy

Subscripts:

l	low-frequency part of fluctuations
h	high-frequency part of fluctuations
1	normalized fluctuating components, for example, $\mu_1' = \mu'/v_o$, $x' = x/l_o$, and so forth; also, quantities in free stream

Barred quantities always represent mean values; primed quantities represent fluctuations.

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