REPORT 1081

A STUDY OF SECOND-ORDER SUPERSONIC FLOW THEORY

By Milton D. Van Dyke [1]

SUMMARY

An attempt is made to develop a second approximation to the solution of problems of supersonic flow which can be solved by existing first-order theory. The method of attack adopted is an iteration process using the linearized solution as the first step.

For plane flow it is found that a particular integral of the iteration equation can be written down at once in terms of the first-order solution. The second-order problem is thereby reduced to an equivalent first-order problem and can be readily solved. At the surface of an isolated body, the solution reduces to the well-known result of Busemann. The plane case is considered in some detail insofar as it gives insight into the nature of the iteration process.

Again, for axially symmetric flow the problem is reduced to a first-order problem by the discovery of a particular integral. For smooth bodies, the second-order solution can then be calculated by the method of von Kármán and Moore. Bodies with corners are also treated by a slight modification of the method. The second-order solution for cones represents a considerable improvement over the linearized result. Second-order theory also agrees well with several solutions for other bodies of revolution calculated by the numerical method of characteristics.

For full three-dimensional flow, only a partial particular integral has been found. As an example of a more general problem, the solution is derived for an inclined cone. The possibility of treating other inclined bodies of revolution and three-dimensional wings is discussed briefly.

INTRODUCTION

As the linearized theory of supersonic flow approaches full development, the question arises whether more exact approximations are practical. If viscous effects are large, refinement of the perfect-fluid solution is impractical. If viscosity is negligible, however, higher approximations are known to yield a closer approach to reality. In intermediate cases, an improved solution is desirable in order to assess the relative effects of viscosity and nonlinearity.

The prototype of a higher-order solution for supersonic flow is Busemann's series for the surface pressure in plane flow past an isolated body. This simple result is of considerable value in analyzing supersonic airfoil sections. Two terms of the series prove sufficient for almost all requirements; the extension to third and fourth order is chiefly of academic interest.

The aim of the present study is, therefore, to find a second approximation, analogous to Busemann's result, for supersonic flow past bodies which can be treated by existing first-order theory. The natural method of attack, and apparently the only practical one, is by means of an iteration process, taking the usual linearized result as the first step. Several writers have applied this procedure to two-dimensional subsonic flow. In supersonic flow, as usual, the solution is simpler, so that more general problems can be solved.

This paper is based upon a thesis for the degree of doctor of philosophy in aeronautics written at the California Institute of Technology under a National Research Council predoctoral fellowship and under the guidance of Prof. P. A. Lagerstrom (reference 1). It was published in revised form as NACA TN 2200, 1951 (reference 2). The present version has been further slightly revised, in particular to include references to the recent literature.

ITERATION PROCEDURE

BASIC ASSUMPTIONS

The problem to be considered is that of steady three-dimensional supersonic flow of a polytropic gas past one or more slender bodies. As indicated in figure 1, the bodies

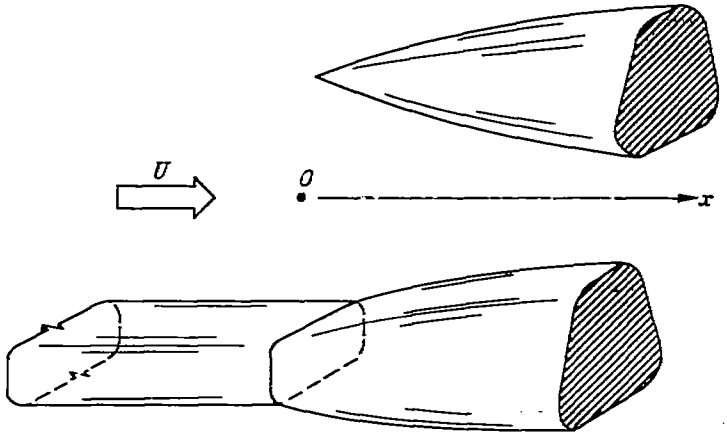


Figure 1.—The problem.

are assumed either to be pointed or to extend upstream indefinitely as cylinders parallel to the free-stream direction. In either case, the origin of coordinates can be chosen so that all variations in body shape are confined to the half-space $x>0$. Wind axes are introduced, so that for $x\leq 0$ the flow is uniform and parallel to the x axis, with the free-stream velocity U and Mach number M. (For definitions of all symbols, see the appendix.)

The bodies are slender, which means that at any point the component of U normal to the surface is small compared

[1] Supersedes NACA TN 2200, "A Study of Second-Order Supersonic-Flow Theory," by Milton D. Van Dyke, 1951.

with U itself. The symbol ϵ will be used throughout as a measure of this smallness. Thus the ordinates of a body will be written as ϵ times a function of order unity. Used in this way, ϵ serves to distinguish terms of various orders.

The aim of this investigation is to find a second approximation to the solution of problems for which the first-order solution is available. The first-order, or linearized, solution is defined as the result of keeping only linear perturbation terms in the equation of motion. Similarly, the second-order solution is the result of retaining second-degree terms in perturbation quantities. In addition, however, certain of the triple products are in some cases found to be as important as one or more double products and are therefore also retained in the equation. It may be noted that the second-order solution will not generally consist simply of terms of order ϵ and ϵ^2, though this is the case for plane flow. For example, the second-order solution for flow past a body of revolution will be found to contain terms as high as $\epsilon^4 ln\epsilon$.

The flow is assumed to be irrotational and isentropic. This assumption is justified in the first- and second-order solutions, since the resulting error is found to be at most of the order of terms neglected elsewhere.

EXACT PERTURBATION EQUATION

Under the assumption of irrotational flow, there exists a velocity potential Ω. In Cartesian coordinates, the equation of motion is (reference 3, equation (39)).

$$(c^2-\Omega_x^2)\Omega_{xx}+(c^2-\Omega_y^2)\Omega_{yy}+(c^2-\Omega_z^2)\Omega_{zz}-2\Omega_y\Omega_z\Omega_{yz}-2\Omega_z\Omega_x\Omega_{zx}-2\Omega_x\Omega_y\Omega_{xy}=0 \tag{1}$$

Here the local speed of sound c is related to c_∞, its value in the uniform stream, by

$$c^2=c_\infty^2-\frac{\gamma-1}{2}(\Omega_x^2+\Omega_y^2+\Omega_z^2-U^2) \tag{2}$$

where γ is the adiabatic exponent. The subscript notation is used to indicate differentiation.

A perturbation potential Φ is now introduced in the usual way. For convenience, however, Φ is normalized through division by the free-stream velocity so that

$$\Omega=U(x+\Phi) \tag{3}$$

The perturbation velocity at any point is then the gradient of Φ multiplied by U.

Introducing the perturbation potential into the equation of motion gives, after some manipulation,

$$\Phi_{yy}+\Phi_{zz}-\beta^2\Phi_{xx}=M^2\left[\frac{\gamma-1}{2}(2\Phi_x+\Phi_x^2+\Phi_y^2+\Phi_z^2)(\Phi_{xx}+\Phi_{yy}+\Phi_{zz})+2\Phi_x\Phi_{xx}+\Phi_x^2\Phi_{xx}+\Phi_y^2\Phi_{yy}+\Phi_z^2\Phi_{zz}+2\Phi_y\Phi_z\Phi_{yz}+2\Phi_z(1+\Phi_x)\Phi_{zz}+2(1+\Phi_x)\Phi_y\Phi_{xy}\right] \tag{4}$$

where $\beta=\sqrt{M^2-1}$.

SOLUTION BY ITERATION

The exact perturbation equation (equation (4)) is completely equivalent to the original nonlinear potential equation (equation (1)). Simplifying assumptions must therefore be introduced in order to solve it. If it is assumed that squares and products of the derivatives of Φ can be neglected, the right-hand side of equation (4) disappears, leaving the wave equation

$$\Phi_{yy}^{(1)}+\Phi_{zz}^{(1)}-\beta^2\Phi_{xx}^{(1)}=0 \tag{5}$$

This equation is the basis of the linearized or first-order perturbation theory, and its solution is designated by $\Phi^{(1)}$.

More exact solution of equation (4) by means of iteration was first suggested by Prandtl (reference 4). The procedure has been applied to plane subsonic flow by Görtler (reference 5), Hantsche and Wendt (references 6 and 7), Imai and Ōyama (references 8 to 10), and Kaplan (references 11 to 13). Schmieden and Kawalki (reference 14) applied the procedure to subsonic flow past an ellipsoid of revolution. Most of these writers have considered the stream function rather than the potential, which restricts the method to plane or axially symmetric flows. The procedure is clearly described by Sauer (reference 3, p. 140) for the case of plane flow.

The linearized solution $\Phi^{(1)}$, subject to proper boundary conditions, is taken as the first approximation. Substituting this known solution into the right-hand side of equation (4) gives

$$\Phi_{yy}^{(2)}+\Phi_{zz}^{(2)}-\beta^2\Phi_{xx}^{(2)}=F_1(x,y,z) \tag{6}$$

where F_1 is a known function of the independent variables. This is again a linear equation, the nonhomogeneous wave equation. A second-order solution $\Phi^{(2)}$, subject to proper boundary conditions, can be sought by standard methods. The procedure can be repeated by substituting $\Phi^{(2)}$ into the right-hand side of equation (4) and solving again. Continuing this process yields a sequence of solutions $\Phi^{(n)}$ which, under proper conditions, perhaps converges to the exact solution.

A significant feature of this procedure is that in each step the left-hand side of the iteration equation is the same. As a consequence, the characteristic curves of each iteration equation are just the Mach lines of the undisturbed flow. However, in actuality the local Mach lines are usually neither straight nor parallel; that is, the characteristics of the original nonlinear equation are curved, in a manner which is initially unknown and which depends upon the shape of the body. Because of the fundamental role played by the characteristics in the theory of hyperbolic equations (see, e. g., reference 15, ch. 5; reference 16, ch. 2), it might be anticipated that an iteration procedure should be chosen such that in each step the approximate characteristics would be successively revised so as to approach the actual characteristics. For purely subsonic flow, the counterpart of such a procedure is known to converge under proper conditions (reference 15, p. 288–289). Convergence might reasonably be anticipated also in the case of purely supersonic flow.

Unfortunately, an iteration procedure in which the approximate characteristics are successively revised would, except in the first step, involve equations with nonconstant coefficients. This would greatly complicate the procedure. Fortunately, it will be found that the scheme adopted here, which makes no provision for such revision, nevertheless

gives an improved solution nearly everywhere in the flow field.

SECOND-ORDER ITERATION EQUATION

Henceforth, only the first two steps of the iteration process will be considered in detail. In order to eliminate cumbersome superscripts, it is therefore convenient to introduce a simplified notation for the first two approximations. The first-order perturbation potential will hereafter be denoted by φ, and the second-order potential[2] by ϕ

$$\left.\begin{aligned}\varphi&\equiv\Phi^{(1)}\\ \phi&\equiv\Phi^{(2)}\end{aligned}\right\} \tag{7}$$

Introducing these quantities into the exact perturbation equation (equation (4)) gives the following second-order iteration equation for ϕ:

$$\begin{aligned}\phi_{yy}+\phi_{zz}-\beta^2\phi_{xx}=M^2\Big[&\frac{\gamma-1}{2}(2\varphi_x+\varphi_x^2+\varphi_y^2+\varphi_z^2)(\varphi_{xx}+\varphi_{yy}+\\&\varphi_{zz})+2\varphi_x\varphi_{xx}+\varphi_x^2\varphi_{xx}+\varphi_y^2\varphi_{yy}+\varphi_z^2\varphi_{zz}+\\&2\varphi_y\varphi_z\varphi_{yz}+2\varphi_z(1+\varphi_x)\varphi_{zx}+2(1+\varphi_x)\varphi_y\varphi_{xy}\Big]\end{aligned} \tag{8}$$

Since φ satisfies equation (5), the term $(\varphi_{xx}+\varphi_{yy}+\varphi_{zz})$ in the right-hand side of equation (8) can be replaced by $M^2\varphi_{xx}$, and the equation for ϕ becomes

$$\begin{aligned}\phi_{yy}+\phi_{zz}-\beta^2\phi_{xx}=M^2\Big[&\frac{\gamma-1}{2}M^2(2\varphi_x+\varphi_x^2+\varphi_y^2+\varphi_z^2)\varphi_{xx}+\\&2\varphi_x\varphi_{xx}+\varphi_x^2\varphi_{xx}+\varphi_y^2\varphi_{yy}+\varphi_z^2\varphi_{zz}+\\&2\varphi_y\varphi_z\varphi_{yz}+2\varphi_z(1+\varphi_x)\varphi_{zx}+2(1+\varphi_x)\varphi_y\varphi_{xy}\Big]\end{aligned} \tag{9}$$

Here the right-hand side contains not only double products of perturbation quantities but also triple products. The latter can be omitted for plane flow, since they contribute terms of smaller order (equal to those found in the next iteration). Otherwise, certain of the triple products should be retained because their contribution is as great as that of one or more of the double products and greater than any contribution from a third approximation. It will be seen later that triple products should be retained if they involve only derivatives normal to the free stream. Triple products which involve x derivitives can be neglected, so that the iteration equation becomes

$$\begin{aligned}\phi_{yy}+\phi_{zz}-\beta^2\phi_{xx}=M^2\{&[2+(\gamma-1)M^2]\varphi_x\varphi_{xx}+2\varphi_y\varphi_{xy}+\\&2\varphi_z\varphi_{zx}+\varphi_y^2\varphi_{yy}+2\varphi_y\varphi_z\varphi_{yz}+\varphi_z^2\varphi_{zz}\}\end{aligned} \tag{10}$$

Here the triple products which may be important are the last three terms on the right-hand size.

The adiabatic exponent γ will be found to occur always in the combination

$$N=\frac{(\gamma+1)M^2}{2\beta^2} \tag{11}$$

[2] Here ϕ is regarded as being the complete second-order potential rather than (as in references 1 and 2) a small correction to be added to the first-order solution.

Introducing this expression in place of γ gives the final form of the second-order iteration equation:

$$\begin{aligned}\phi_{yy}+\phi_{zz}-\beta^2\phi_{xx}=M^2[&2(N-1)\beta^2\varphi_x\varphi_{xx}+2\varphi_y\varphi_{xy}+2\varphi_z\varphi_{zx}+\\&\varphi_y^2\varphi_{yy}+2\varphi_y\varphi_z\varphi_{yz}+\varphi_z^2\varphi_{zz}\end{aligned} \tag{12}$$

ITERATION EQUATION IN OTHER COORDINATES

In cylindrical coordinates, equation (9) becomes

$$\begin{aligned}\phi_{rr}+\frac{\phi_r}{r}+\frac{\phi_{\theta\theta}}{r^2}-\beta^2\phi_{xx}=M^2\Big[&2(N-1)\beta^2\varphi_x\varphi_{xx}+2\varphi_r\varphi_{xr}+2\frac{\varphi_\theta}{r^2}\varphi_{x\theta}+\\&\varphi_r^2\varphi_{rr}+2\frac{\varphi_r\varphi_\theta}{r^2}\varphi_{r\theta}-\frac{\varphi_r\varphi_\theta^2}{r^3}+\frac{\varphi_\theta^2}{r^4}\varphi_{\theta\theta}+\\&0\Big(\varphi_x^2\varphi_{xx},\ \varphi_r^2\varphi_{xx},\ \frac{\varphi_\theta^2}{r^2}\varphi_{xx},\ \frac{\varphi_x\varphi_\theta}{r^2}\varphi_{x\theta},\ \varphi_x\varphi_r\varphi_{xr}\Big)\Big]\end{aligned} \tag{13}$$

The terms whose form is indicated at the end of the equation are the triple products which will be found to be negligible.

For conical flows it is convenient to introduce nonorthogonal conical coordinates (x, t, θ) where

$$t=\frac{\beta r}{x} \tag{14}$$

If the body itself is conical, the perturbation potentials are reduced to functions of two variables by introducing conical perturbation potentials (reference 17) so that

$$\varphi(x, t, \theta)=x\bar{\varphi}(t, \theta) \tag{15}$$

with a corresponding definition for $\bar{\phi}$. The derivatives are given by

$$\left.\begin{aligned}&\varphi_x=\bar{\varphi}-t\bar{\varphi}_t &&\varphi_{xx}=\frac{t^2}{x}\bar{\varphi}_{tt} &&\varphi_{r\theta}=\beta\bar{\varphi}_{t\theta}\\&\varphi_r=\beta\bar{\varphi}_t &&\varphi_{rr}=\frac{\beta^2}{x}\bar{\varphi}_{tt} &&\varphi_{\theta x}=\bar{\varphi}_\theta-t\bar{\varphi}_{t\theta}\\&\varphi_\theta=x\bar{\varphi}_\theta &&\varphi_{\theta\theta}=x\bar{\varphi}_{\theta\theta} &&\varphi_{xr}=-\frac{\beta t}{x}\bar{\varphi}_{tt}\end{aligned}\right\} \tag{16}$$

with the same relations connecting ϕ and $\bar{\phi}$. The iteration equation (equation (13)) becomes

$$\begin{aligned}(1-t^2)\bar{\phi}_{tt}+\frac{\bar{\phi}_t}{t}+\frac{\bar{\phi}_{\theta\theta}}{t^2}=M^2\Big\{&2(N-1)t^2(\bar{\varphi}-t\bar{\varphi}_t)\bar{\varphi}_{tt}-2t\bar{\varphi}_t\bar{\varphi}_{tt}+\\&\frac{2}{t_2}\bar{\varphi}_\theta(\bar{\varphi}_\theta-t\bar{\varphi}_{t\theta})+\beta^2\bar{\varphi}_t^2\bar{\varphi}_{tt}+2\frac{\beta^2}{t^2}\bar{\varphi}_t\bar{\varphi}_\theta\bar{\varphi}_{t\theta}-\\&\frac{\beta^2}{t^3}\bar{\varphi}_t\varphi_\theta^2+\frac{\beta^4}{t^4}\bar{\varphi}_\theta^2\bar{\varphi}_{\theta\theta}+0\Big[t^2(\bar{\varphi}-t\bar{\varphi}_t)^2\bar{\varphi}_{tt},\\&t^2\bar{\varphi}_t^2\bar{\varphi}_{tt},\ \bar{\varphi}_\theta^2\bar{\varphi}_{tt},\ \frac{1}{t^2}(\bar{\varphi}-t\bar{\varphi}_t)\bar{\varphi}_\theta(\bar{\varphi}_\theta-t\bar{\varphi}_{t\theta}),\\&t\bar{\varphi}_t(\bar{\varphi}-t\bar{\varphi}_t)\bar{\varphi}_{tt}\Big]\Big\}\end{aligned} \tag{17}$$

Here the grouping of terms corresponds to that in equation (13).

BOUNDARY CONDITIONS

Physical considerations suggest that the flow should satisfy the following conditions:

Tangency condition: The resultant flow is tangent to the surface of the body.

Upstream condition: All flow perturbations vanish everywhere upstream of the body.

These two requirements are sufficient to determine the solution. The first imposes one condition along the time-like surface of the body, and the second may be regarded as imposing two conditions on a spacelike surface. This is the case of mixed boundary conditions (reference 15, p. 172) and leads to a determinate solution (see reference 16, p. 85).

The tangency condition may be written formally as

$$\text{grad } \Omega \cdot \text{grad } S=0 \qquad \text{at } S=0 \tag{18}$$

where $S=0$ is the equation of the surface of the body. In a more useful form it becomes, for the first- and second-order problems,

$$\varphi_c=(\text{slope})\ (1+\varphi_x) \qquad \text{(on the surface)} \tag{19a}$$

$$\phi_c=(\text{slope})\ (1+\phi_x) \qquad \text{(on the surface)} \tag{19b}$$

Here ϕ_c means the cross-wind component of the normal derivative of ϕ at the surface of the body. In plane flow ϕ_c is ϕ_y, and in axially symmetric flow ϕ_c is ϕ_r. The slope of the body is measured with respect to the free-stream direction.

In first-order theory, the tangency condition is frequently approximated by neglecting φ_x compared with 1 in equation (19a), which causes only a second-order error. Correspondingly, in the second-order tangency condition (equation (19b)), ϕ_x can be replaced by its first-order counterpart φ_x with only a third-order error. Thus the tangency condition simplifies to

$$\varphi_c=\text{slope} \qquad \text{(on the surface)} \tag{20a}$$

$$\phi_c=(\text{slope})\ (1+\varphi_x) \qquad \text{(on the surface)} \tag{20b}$$

This approximation will not be made except for plane flow.

Another approximation in the tangency condition can be made for planar bodies. A planar body is one whose entire surface lies near to a plane parallel to the free stream, say the plane $y=0$ (reference 17, p. 52). Thin flat wings are planar bodies, whereas slender pointed bodies of revolution are not. For a planar body the first-order tangency condition can be imposed at the plane rather than on the surface of the body. Correspondingly, the second-order tangency condition can also be imposed at the plane provided that the difference in ϕ_y between the plane and the surface of the body is accounted for approximately by retaining the second term in its Taylor series expansion about $y=0$. Furthermore, to first order φ_c is φ_y for a planar body. Thus if the surface of a planar body is given by $y=Y(x,z)$, the simplified tangency condition is

$$\varphi_y=Y_x(1+\varphi_x) \qquad \text{at } y=0 \tag{21a}$$

$$\phi_y+Y\phi_{yy}-Y_z\phi_z=Y_x(1+\phi_x) \qquad \text{at } y=0 \tag{21b}$$

The term containing ϕ_z in equation (21b) accounts for the fact that to second order the cross-wind component ϕ_c may not be vertical. Corresponding simplifications can be made for any quasi-cylindrical body, which is a body whose entire surface lies near to a cylinder parallel to the free stream.

Finally, the approximation which led to equations (20) may be adopted in addition to that just discussed for planar systems, in which case the tangency condition simplifies further to

$$\varphi_y=Y_x \qquad \text{at } y=0 \tag{22a}$$

$$\phi_y=Y_x(1+\varphi_x)-Y\varphi_{yy}+Y_z\varphi_z \qquad \text{at } y=0 \tag{22b}$$

The upstream condition implies that both Φ and Φ_x vanish at the plane $x=0$, which completes the boundary conditions required. For the first- and second-order problems the upstream condition is therefore

$$\varphi=\varphi_x=0 \qquad \text{at } x=0 \tag{23a}$$

$$\phi=\phi_x=0 \qquad \text{at } x=0 \tag{23b}$$

DETERMINATION OF PRESSURE

When the potential field has been determined, the net velocity q at any point is given by

$$q^2=(U+u)^2+v^2+w^2 \tag{24}$$

where

$$\left.\begin{aligned} \frac{u}{U}&=\Phi_x \\ \frac{v}{U}&=\begin{cases}\Phi_y \\ \Phi_r\end{cases} \\ \frac{w}{U}&=\begin{cases}\Phi_z \\ \frac{1}{r}\Phi_\theta\end{cases}\end{aligned}\right\} \tag{25}$$

in Cartesian and cylindrical coordinates, respectively. Because the flow is assumed to be isentropic, the pressure coefficient is given by

$$C_p\equiv\frac{p-p_\infty}{\frac{1}{2}\rho_\infty U^2}=\frac{2}{\gamma M^2}\left\{\left[1+\frac{\gamma-1}{2}M^2\left(1-\frac{q^2}{U^2}\right)\right]^{\frac{\gamma}{\gamma-1}}-1\right\} \tag{26}$$

where p_∞ and ρ_∞ are the free-stream pressure and density.

It is the practice in linearized theory also to simplify the pressure relation. Substituting equation (24) into equation (26) and expanding gives

$$C_p=-2\frac{u}{U}-\frac{v^2+w^2}{U^2}+\beta^2\frac{u^2}{U^2}+M^2\frac{u}{U}\frac{v^2+w^2}{U^2}+\frac{M^2}{4}\left(\frac{v^2+w^2}{U^2}\right)^2+ 0\left[\frac{u^3}{U^3},\frac{u^2}{U^2}\frac{v^2+w^2}{U^2},\frac{u}{U}\left(\frac{v^2+w^2}{U^2}\right)^2,\left(\frac{v^2+w^2}{U^2}\right)^3\right] \tag{27}$$

All the terms shown here explicitly may give contributions of second order; the remaining terms whose form is merely indicated are always of higher order. In linearized theory only the first term is ordinarily retained. This is satisfactory for plane flow or flow past planar systems, since the contribution of the remaining terms is definitely of higher order.

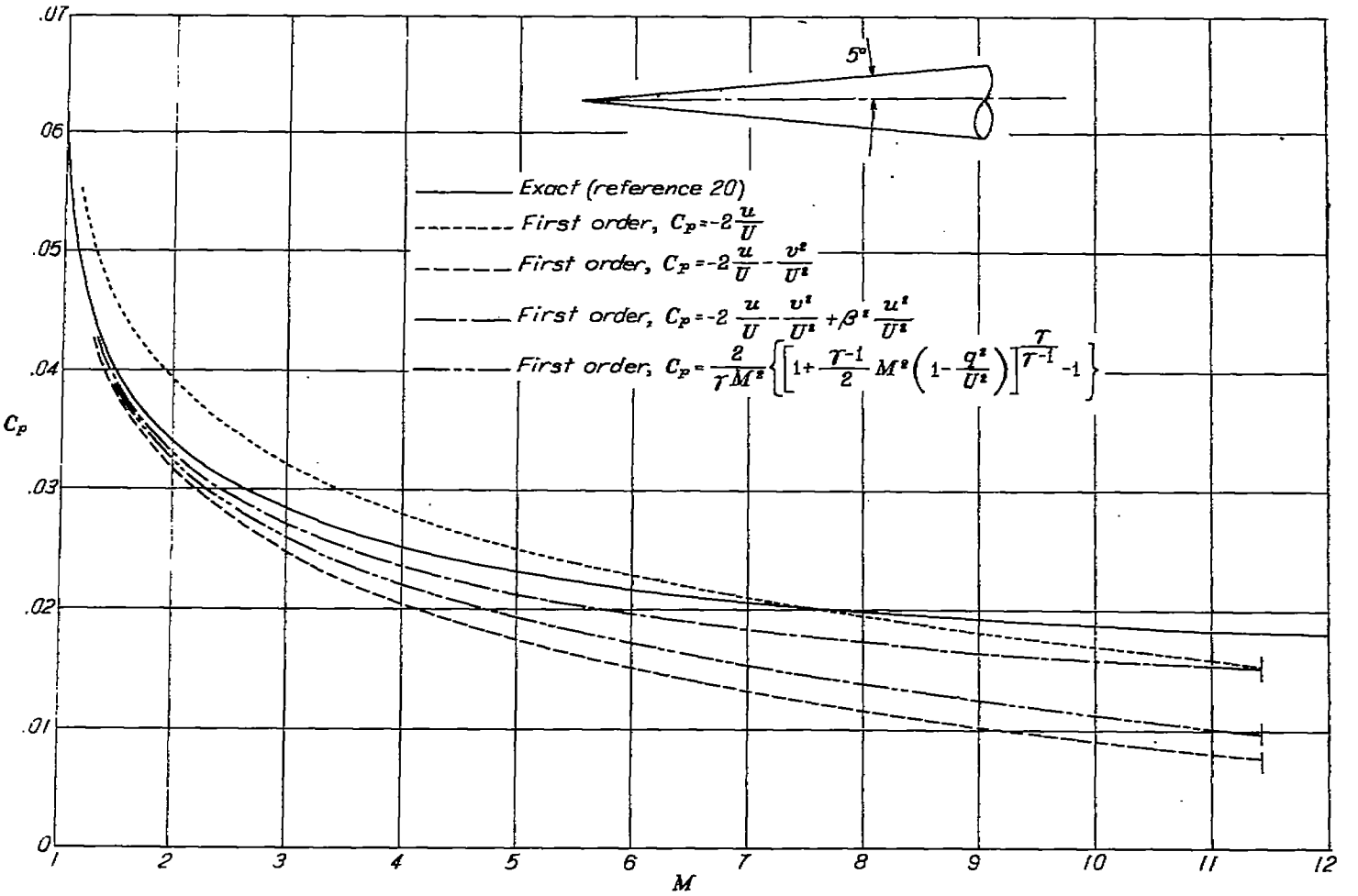


FIGURE 2.—Effect of using various pressure relations in first-order solution for 5° cone.

In fact, for plane flow past an isolated body it happens that the next two terms cancel identically.

For slender bodies such as a cone, however, orders of magnitude are not so clearly distinguished. Busemann suggests (reference 18) that the second term, $(v^2+w^2)/U^2$, is then sufficiently large compared with the first that it should be retained also. This view is supported by Lighthill (reference 19), who shows that the resulting solution is correct up to the order of the quantities contributed by the second term. Again, the third term, β^2u^2/U^2, which is also the square of a perturbation quantity, is comparable with the second at high Mach numbers (where Lighthill's order estimates become invalid) and might logically be retained. Having gone this far, it may be simpler to use the exact relation.

Each of these four possibilities for the first-order flow past a 5° cone is compared with the exact solution (reference 20) in figure 2. The series (equation (27)) is seen to alternate in this case. It converges so slowly, however, that at moderate Mach numbers (say, near $M=\sqrt{2}$) where first-order theory is most accurate, linearizing the pressure relation introduces much greater errors than linearizing only the equation of motion. Adding each of the quadratic terms in turn causes changes nearly as great as the error due directly to nonlinearity in the equation.

The point of view to be adopted here is that calculating the velocities and calculating the pressure are two essentially distinct operations. A certain degree of approximation may be necessary in order to solve for the velocities, but the pressure relation need not then be approximated to the same extent simply for the sake of consistency. For it may happen that the resulting errors (though of the same mathematical order) are greater than those due to the original approximation. Indeed, this is evidently the case at moderate Mach numbers in the first-order solution for a cone and will be found true to a greater extent in the second-order solution. Moreover, in the second-order solution so many terms of equation (27) must be retained that it is usually simpler to use the exact relation. For these reasons, the exact pressure equation (equation (26)) will be used throughout except in the case of plane flow.

ROLE OF A PARTICULAR INTEGRAL

The second-order iteration equation can be attacked by standard methods, and in the case of plane flow a solution can be found directly. For plane and axially symmetric flows, however, it will be found that a particular integral of the iteration equation can be written down at once in terms of the first-order solution. This solves the problem because the complete solution consists of a particular integral plus a solution of the homogeneous equation, and the latter can be obtained by existing methods. That is, the second-order potential may be written as

$$\phi=\psi+\chi \tag{28}$$

where

ψ is any particular integral of the nonhomogeneous iteration equation

χ is the complementary function which serves to correct the tangency condition

The problem for χ is the usual first-order problem for which methods of solution are assumed to be known.

The role of the particular integral is to transfer the nonhomogeneity in the problem from the equation, where it is troublesome, to the boundary conditions, where it can be handled by existing theory. For linear partial differential equations it is always possible in principle to transfer nonhomogeneities in this way from the equation to the boundary conditions, and vice versa, by adding a suitable function to the dependent variable (see reference 21, p. 236).

Since the particular solution ψ will be found in terms of the first-order solution, it will vanish upstream of the plane $x=0$. Then the complementary function must also vanish there, so that the upstream condition for χ is

$$\chi(0,y,z)=\chi_x(0,y,z)=0 \tag{29}$$

From equation (19b), the tangency condition for χ is found to be

$$\psi_c+\chi_c=(\text{slope})(1+\psi_x+\chi_x) \qquad \text{(on the surface)} \tag{30}$$

or, in the case of a planar body given by $y=Y(x,z)$, from equation (22b),

$$\psi_y+\chi_y=Y_x(1+\varphi_x)-Y\varphi_{yy}+Y_z\varphi_z \qquad \text{at } y=0 \tag{31}$$

It should be noted that, although ϕ is of the same magnitude as φ, this is not necessarily true of either ψ or χ alone.

PLANE FLOW

The second-order solution for conditions at an isolated surface in plane supersonic flow was given by Busemann (references 22 and 23). By using the iteration procedure, the solution will now be found throughout the flow field, including the case when several bodies interact.

The solution for plane flow is of interest chiefly insofar as it serves as a guide in more complicated problems. In particular, it provides insight into such details of the iteration process as the question of its success and the effect of sharp corners.

PARTICULAR INTEGRAL FOR PLANE FLOW

The first-order equation for plane flow is

$$\varphi_{yy}-\beta^2\varphi_{xx}=0 \tag{32}$$

The general solution is

$$\varphi(x,y)=h(x-\beta y)+j(x+\beta y) \tag{33}$$

where h and j are functions determined by the first-order boundary conditions.

In the iteration equation, all triple products can be neglected, and equation (12) becomes

$$\phi_{yy}-\beta^2\phi_{xx}=2M^2[(N-1)\beta^2\varphi_x\varphi_{xx}+\varphi_y\varphi_{xy}] \tag{34}$$

It can be verified directly that a particular integral of this equation is given in terms of the first-order solution by

$$\psi=M^2\varphi_x\left[\left(1-\frac{N}{2}\right)\varphi+\frac{N}{2}y\varphi_y\right] \tag{35}$$

To this must be added a solution χ of the homogeneous equation (equation (32)), which has the form

$$\chi=H(x-\beta y)+J(x+\beta y) \tag{36}$$

where H and J are functions determined by the second-order boundary condition.

For flow past a single boundary (such as one surface of an airfoil) the first-order potential (equation (33)) contains only one or the other of the functions h and j. In this case $\varphi_y\varphi_{xy}=\beta^2\varphi_x\varphi_{xx}$ so that the iteration equation reduces to

$$\phi_{yy}-\beta^2\phi_{xx}=2M^2\beta^2N\varphi_x\varphi_{xx} \tag{37}$$

The particular integral may then be simplified to

$$\psi=M^2\frac{N}{2}y\varphi_x\varphi_y \tag{38}$$

and the complementary function contains only H or J, according as the first-order solution contains only h or j.

FLOW PAST A CURVED WALL

As an example of the application of the particular solution, consider flow past a wall which at some point begins to deviate slightly from a plane (see fig. 3). The wall can be represented by

$$y=Y(x)=\epsilon g(x) \tag{39}$$

where ϵ is a parameter small compared with unity and $g(x)$ is a continuous function of order unity which vanishes for $x\leqq 0$.

This is a planar body, so that the tangency condition is given by equation (22a). Consequently, the first-order problem is

$$\left.\begin{aligned}\varphi_{yy}-\beta^2\varphi_{xx}&=0\\ \varphi_y(x,0)&=\epsilon g'(x)\\ \varphi(0,y)=\varphi_x(0,y)&=0\end{aligned}\right\} \tag{40}$$

where the prime indicates differentiation of g with respect to its argument. The solution is

$$\varphi=-\frac{\epsilon}{\beta}g(x-\beta y) \tag{41}$$

Substituting this first-order solution into the right-hand side of equation (37) gives the iteration equation

$$\phi_{yy}-\beta^2\phi_{xx}=2M^2N\epsilon^2g'(x-\beta y)g''(x-\beta y) \tag{42}$$

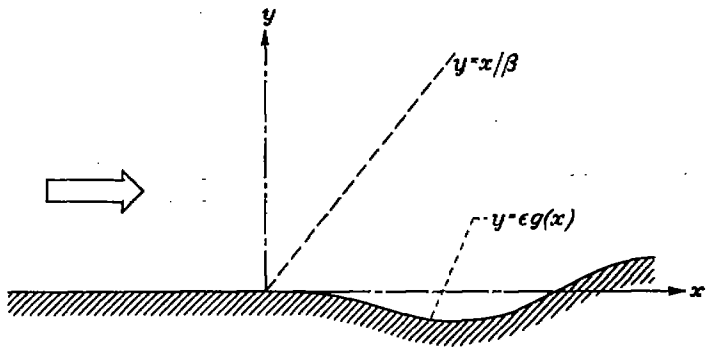


FIGURE 3.—Flow past a curved wall.

According to equations (38) and (36), the solution is

$$\phi=\psi+\chi=-\frac{M^2N}{2\beta}\epsilon^2y[g'(x-\beta y)]^2+H(x-\beta y) \tag{43}$$

Imposing the approximate second-order planar tangency condition (equation (22b)) gives

$$H'(x)=\epsilon^2\left\{\frac{2-M^2N}{2\beta^2}[g'(x)]^2-g(x)g''(x)\right\} \tag{44}$$

so that

$$H(x)=-\epsilon^2\left\{g(x)g'(x)+\frac{M^2(N-2)}{2\beta^2}\int_0^x[g'(\xi)]^2d\xi\right\} \tag{45}$$

The complete second-order perturbation potential is therefore

$$\phi=-\frac{\epsilon}{\beta}g(x-\beta y)-\epsilon^2\left\{g(x-\beta y)g'(x-\beta y)+\frac{M^2N}{2\beta}y[g'(x-\beta y)]^2+\frac{M^2(N-2)}{2\beta^2}\int_0^{x-\beta y}[g'(\xi)]^2d\xi\right\} \tag{46}$$

The same result can be found by solving equation (42) directly, using the impulse method (reference 15, p. 164).

Flow quantities at the surface of the wall can be related to their values at the plane $y=0$ by expanding in Taylor series and discarding terms in ϵ^3. In this way the streamwise velocity perturbation at the wall is found to be given by

$$\frac{u}{U}=-\frac{\epsilon}{\beta}g'(x)-\frac{M^2N-2}{2\beta^2}\epsilon^2[g'(x)]^2 \tag{47}$$

The pressure coefficient at the wall can now be calculated from equation (27) which, upon replacing N by its value from equation (11), gives

$$C_p=\frac{2}{\beta}\epsilon g'(x)+\frac{(\gamma+1)M^4-4\beta^2}{2\beta^4}[\epsilon g'(x)]^2 \tag{48}$$

This is the well-known result of Busemann (references 22 and 23). To second order, the surface pressure coefficient depends only upon the local slope.

ROLE OF CHARACTERISTICS

It was pointed out previously that because of the underlying significance of the characteristics for solutions of hyperbolic equations, it might be expected that the approximate characteristics of the iteration equation would have to be revised successively at each stage. However, an iteration procedure was adopted which involves no such revision. It is therefore pertinent to inquire in this simple example what roles have been played by the actual and approximate characteristics.

The flow past a single curved wall is given (until shock waves form) by a simple wave or distributed Prandtl-Meyer expansion. Of the two families of characteristics, those of primary importance in a simple wave run downstream away from the wall. We therefore confine attention to that family.

For the first-order equation, equation (32), the conventional theory (e. g., reference 15, ch. 5; reference 16, ch. 2) shows that the characteristics of the downstream family are the lines of slope

$$\frac{dy}{dx}=\frac{1}{\beta} \tag{49}$$

This means that to first order the actual characteristics are approximated by the Mach lines of the undisturbed flow.

For the second-order solution, a closer approximation to the characteristics could easily be found. It can be shown that if the first-order streamwise perturbation velocity at any point in the flow is $u^{(1)}$, then the revised local values of Mach number and β are approximately [3]

$$M^{(1)}=M\left[1+\beta^2(N-1)\frac{u^{(1)}}{U}\right] \tag{50a}$$

$$\beta^{(1)}\equiv\sqrt{M^{(1)2}-1}=\beta\left[1+M^2(N-1)\frac{u^{(1)}}{U}\right] \tag{50b}$$

Combining this result with the first-order solution (equation (41)) shows that the revised Mach lines have the slope

$$\frac{dy}{dx}=\frac{1}{\beta}\left[1+\frac{M^2N}{\beta}\epsilon g'(x-\beta y)\right] \tag{51}$$

However, because the iteration scheme adopted does not allow for such revision, these are not actually the characteristics of the second-order equation. Instead, the characteristics of equation (42) are still the original Mach lines of the undisturbed flow.

Physically, the characteristics are lines along which discontinuities in velocity derivatives are propagated, and this definition is completely equivalent to the mathematical one (reference 15, p. 297). Therefore, in the second-order solution given above, discontinuities in acceleration will occur along the original characteristics rather than, as they more properly should, along the revised characteristics.

Suppose, however, that no such discontinuities occur. For flow past an isolated body the downstream characteristics are also lines along which the velocity is constant, provided that shock waves do not appear. Setting

$$\left.\begin{aligned}d\phi_x&=\phi_{xx}dx+\phi_{xy}dy=0\\ d\phi_y&=\phi_{xy}dx+\phi_{yy}dy=0\end{aligned}\right\} \tag{52}$$

it is seen that the velocity is constant if

$$\frac{dy}{dx}=-\frac{\phi_{xx}}{\phi_{xy}}=-\frac{\phi_{xy}}{\phi_{yy}} \tag{53}$$

For the second approximation (equation (46)) the velocity is constant along lines of slope

$$\frac{dy}{dx}=\frac{1}{\beta}\left[1+\frac{M^2N}{\beta}\epsilon g'(x-\beta y)\right] \tag{54}$$

which, according to equation (51), are the revised characteristics. Consequently, although the characteristics have not been revised in the mathematical sense, the solution behaves physically as if they had, so long as discontinuities do not occur. The question of discontinuities will be considered in the next section.

The connection between the original and revised characteristics can be interpreted physically. The right-hand side of the iteration equation may be regarded as representing the

[3] As implied by the superscript, these are regarded as first-order values, because they are determined from the first-order solution. From this point of view, M and β are zero-order quantities.

effect of a known distribution of supersonic sources throughout the flow field. The influence of this source distribution spreads downstream along both families of original characteristics. The resulting velocity changes are just such that to second order the velocities become constant along the revised rather than the original characteristics.

Finally, it is interesting to note that the second-order potential is constant on lines which bisect the original and revised characteristics. For, setting

$$d\phi=\phi_x dx+\phi_y dy=0 \tag{55}$$

ϕ is found to be constant along lines of slope

$$\frac{dy}{dx}=\frac{1}{\beta}\left[1+\frac{M^2N}{2\beta}\epsilon g'(x-\beta y)\right] \tag{56}$$

FLOW PAST A CORNER AND A PARABOLIC BEND

A simple case in which discontinuities may occur is that of flow past a sharp corner. The exact solution is known to involve an oblique shock wave with attendant velocity disdiscontinuities for compression and a continuous Prandtl-Meyer fan for expansion.

Denoting the tangent of the deflection angle by ϵ, positive for compression (see fig. 4), the function $g(x)$ appearing in equation (39) is

$$g(x)=\begin{cases}0 & x\leq 0\\ x & x\geq 0\end{cases} \tag{57}$$

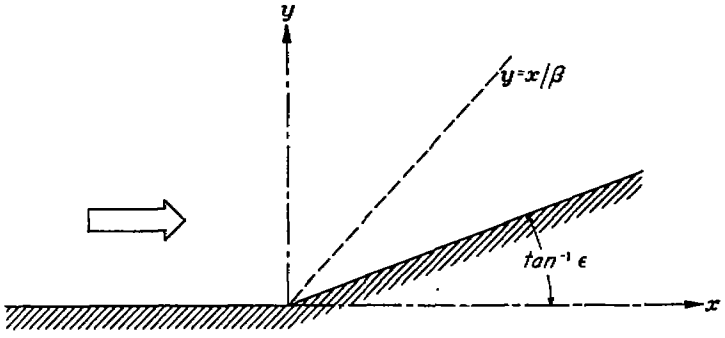


FIGURE 4.—Flow past a corner.

From equation (46) the second-order perturbation potential is found to be

$$\phi=-\frac{\epsilon}{\beta}(x-\beta y)+\frac{\epsilon^2}{\beta^2}(x-\beta y)-\frac{M^2N}{2\beta^2}\epsilon^2 x \tag{58}$$

to the right of the line $x=\beta y$ and zero to the left. Consequently, in either compression ($\epsilon>0$) or expansion ($\epsilon<0$) the second-order potential suffers a discontinuous drop along the Mach line from the corner, of strength proportional to the distance from the corner. Such a discontinuity cannot be admitted, which indicates that the iteration process fails in this region.

In the case of compression, the solution can be corrected by analytically continuing the perturbation potential upstream until it can be joined continuously to the free-stream potential. (This is permissible since the line of discontinuity is not actually a characteristic.) From the result of equation (56) the juncture is seen to occur along the line from the corner which bisects the upstream and downstream Mach directions, as indicated in figure 5. The adjusted juncture corresponds to a shock wave, for it is known that an oblique shock bisects the Mach lines to a first approximation (reference 16, p. 354). In the case of expansion, this type of correction cannot be justified since it would involve continuation of the free-stream potential across a true characteristic. Instead, a Prandtl-Meyer fan must be inserted.

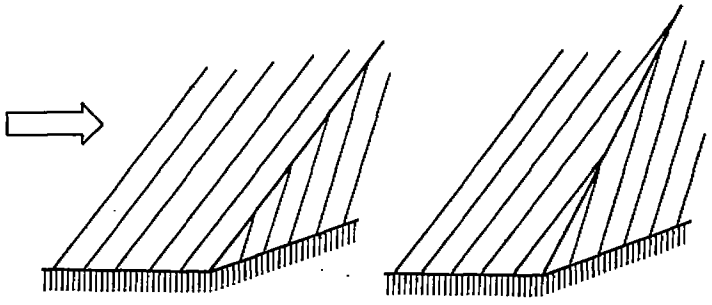


FIGURE 5.—Mach lines before and after adjustment of potential discontinuity.

Evidently the iteration process is successful except within an angular region of order ϵ lying near the Mach line from the corner. In particular, the pressure is given correctly everywhere on the surface of the wall.

It is enlightening to observe that the alternative method of iteration, in which the characteristics are successively revised, fails in the same region. The potential is double-valued over a fan-shaped region in the case of compression and is left undefined over a similar region in the case of expansion (see fig. 6). The same artificial corrections are necessary to complete the solution.

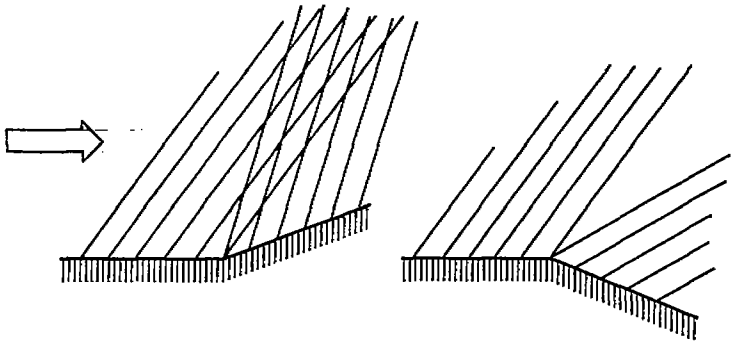


FIGURE 6.—Second-order flow past a corner using revised characteristics.

Consider next flow past a parabolic bend which is represented by

$$y=\frac{1}{2}\epsilon x^2 \qquad x\geq 0 \tag{59}$$

From equation (46) the second-order perturbation potential is found to be

$$\phi=\frac{\epsilon}{2\beta}(x-\beta y)^2-\frac{M^2(N+1)-3}{6\beta^2}\epsilon^2(x-\beta y)^3-\frac{M^2N}{2\beta}\epsilon^2 y(x-\beta y)^2 \tag{60}$$

The potential and also the velocities are continuous, so that the previous difficulties do not occur. The acceleration is discontinuous across the original characteristic $x=\beta y$, which in this case happens to be also a revised characteristic. However, a new complication arises. It is known that, in the exact solution for the compressive case, the characteristics form an envelope, as shown in figure 7. Inside the

cusp the potential is triple-valued (reference 16, p. 111), so that a shock wave must be inserted. This envelope must also arise in the second approximation since the characteristics are no longer parallel. However, the second-order potential given by equation (60) is single-valued, so that it cannot predict the formation of an envelope. Again the iteration process fails in a part of the flow field.

It can be seen that the alternative iteration process, using revised characteristics, will produce an envelope.

CONVERGENCE FOR PLANE FLOW

The examples just considered indicate that the success of the iteration procedure should be carefully investigated. A step of an iteration process may be considered successful if, in some sense, it significantly improves the solution. In particular, one is interested in the success of the second-order solution.

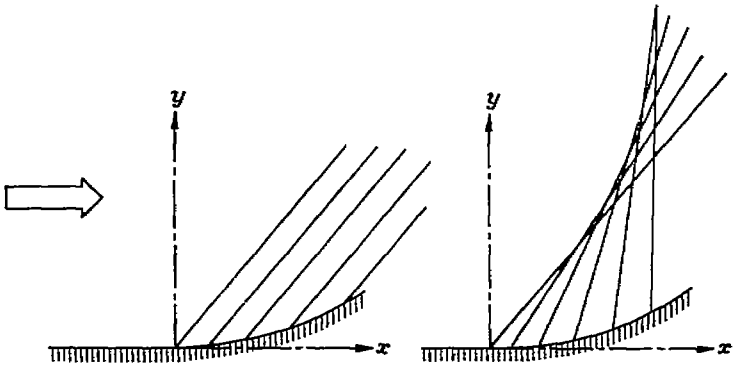


FIGURE 7.—First- and second-order flow past a parabolic bend.

It should be noted that a divergent process may be successful for many steps and that, on the other hand, convergence does not necessarily imply success. In practice, however, one would expect a convergent process to be successful. As used here, success is a subjective notion, not amenable to analysis. Consequently, only the convergence of the iteration procedure can be considered in any detail.

Unfortunately, proofs of sufficient conditions for convergence have not been obtained, even in the case of plane flow. However, the preceding examples suggest certain conjectures regarding convergence. These will be stated and some arguments for their plausibility will be advanced.

For flow past a slightly curved plane wall represented by $y=\epsilon g(x)$, the solution obtained by iteration using the revised characteristics is conjectured to converge in any bounded region adjacent to the wall, provided that

(a) ϵ is sufficiently small,

(b) $g(x)$ is continuously differentiable.

If $g(x)$ has only a piecewise continuous derivative, the convergence holds except possibly in fan-shaped regions springing from each corner, which lie near the original Mach line and subtend an angle of order ϵ.

For the iteration process actually adopted, in which the characteristics are not revised, the first n steps are conjectured to form part of a convergent process, provided that

(a) ϵ is sufficiently small,

(b′) $g(x)$ has continuous derivatives up to $(n-1)$ st order if the potential is required, nth order if the velocities are required.

If condition (b′) is satisfied only piecewise, the result holds except possibly in fan-shaped regions springing from each corner.

In the first case, condition (a) is necessary in order to insure that the solution be unique, as is clear from the example of the parabolic wall. The preceding examples also show that condition (b) is necessary.

If the sufficiency of these two conditions is assumed, their connection with condition (b′) in the second case can be illustrated by analogy with a mathematical model which retains the essential difference between the two iteration processes; namely, that the correct characteristics are not used in the method actually adopted.[4] Consider the first-order problem given by equation (40):

$$\left.\begin{aligned} &\varphi_{yy}-\varphi_{xx}=0\\ &\varphi_y(x,0)=\epsilon g'(x)\\ &\varphi(0,y)=\varphi_x(0,y)=0 \end{aligned}\right\} \tag{61}$$

where we have taken $\beta=1$ for convenience. The solution (equation (41)) is

$$\varphi=-\epsilon g(x-y) \tag{62}$$

Now we attempt to solve this problem using characteristics which differ from the true characteristics by $0(\epsilon)$. Thus we consider the equivalent problem

$$\left.\begin{aligned} &\varphi_{yy}-(1-\epsilon)\varphi_{xx}=\epsilon\varphi_{xx}\\ &\varphi_y(x,0)=\epsilon g'(x)\\ &\varphi(0,y)=\varphi_x(0,y)=0 \end{aligned}\right\} \tag{63}$$

and solve by iteration. In the first approximation the right-hand side is neglected, so that the differential equation becomes

$$\varphi_{yy}^{(1)}-(1-\epsilon)\varphi_{xx}^{(1)}=0 \tag{64}$$

which has the solution, subject to the boundary conditions,

$$\varphi^{(1)}=-\epsilon g(x-\sqrt{1-\epsilon}\,y) \tag{65}$$

Substituting this into the right-hand side of equation (63) gives the iteration equation for the second approximation:

$$\varphi_{yy}^{(2)}-(1-\epsilon)\varphi_{xx}^{(2)}=-\epsilon^2 g''(x-\sqrt{1-\epsilon}\,y) \tag{66}$$

Using the impulse method (reference 15, p. 164) gives the solution, subject to the boundary conditions, as

$$\varphi^{(2)}=-\epsilon g(x-\sqrt{1-\epsilon}\,y)+\frac{1}{2}\epsilon^2 y g'(x-\sqrt{1-\epsilon}\,y) \tag{67}$$

But this is just the Taylor series expansion, correct to $0(\epsilon^2)$, of the true solution (equation (62)). Subsequent iterations add additional terms to the Taylor series expansion. Hence, despite the use of slightly incorrect characteristics, the iteration process converges to the correct solution. The connection between conditions (b) and (b′) is thus seen to be that the existence of sufficiently many continuous derivatives compensates for the fact that the wrong characteristics are used.

[4] This model was suggested by Prof. C. R. DePrima of the California Institute of Technology.

AXIALLY SYMMETRIC FLOW

Before discussing the general solution for bodies of revolution, it is convenient to consider the simple problem of a cone. In this case the second-order solution can be found directly. The results will be useful in indicating which triple products should be retained in the general case.

FLOW PAST A CONE

Consider flow past a slender cone of semivertex angle $\tan^{-1}\epsilon$, as shown in figure 8. The flow is conical and axially symmetric, so that the iteration equation is given by equation

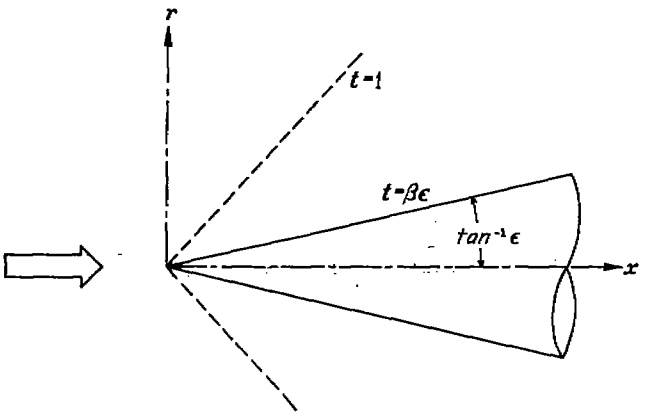


FIGURE 8.—Flow past a cone.

(17) with θ derivatives omitted. Including the boundary conditions from equations (19a) and (23a), the first-order problem is

$$(1-t^2)\bar{\varphi}''(t)+\frac{\bar{\varphi}'(t)}{t}=0 \tag{68a}$$

$$\beta\bar{\varphi}'=\epsilon(1+\bar{\varphi}-t\bar{\varphi}') \qquad \text{at } t=\beta\epsilon \tag{68b}$$

$$\bar{\varphi}=\bar{\varphi}'=0 \qquad \text{at } t=\infty \tag{68c}$$

The differential equation can be solved by using the integrating factor $t/\sqrt{1-t^2}$. Imposing the upstream condition (equation (68c)) gives the well-known result

$$\bar{\varphi}=-A(\operatorname{sech}^{-1} t-\sqrt{1-t^2}) \tag{69}$$

which is understood to vanish except within the downstream Mach cone, where $t\leqq 1$. The tangency condition (equation (68b)) is satisfied by putting

$$A=\frac{\epsilon^2}{\sqrt{1-\beta^2\epsilon^2}+\epsilon^2\operatorname{sech}^{-1}(\beta\epsilon)} \tag{70}$$

At the Mach cone ($t=1$), all velocity perturbations vanish, so that the first-order solution predicts no deflection of the shock wave from the Mach cone (see reference 16, p. 403).

Substituting the first-order solution into the iteration (equation (17)) gives

$$(1-t^2)\bar{\phi}''(t)+\frac{\bar{\phi}'(t)}{t}=A^2M^2\left[2(N-1)\frac{\operatorname{sech}^{-1} t}{\sqrt{1-t^2}}+2\frac{1}{t^2}-\beta^2A\frac{\sqrt{1-t^2}}{t^4}\right] \tag{71a}$$

and from equations (19b) and (23b) the corresponding boundary conditions are

$$\beta\bar{\phi}'=\epsilon(1+\bar{\phi}-t\bar{\phi}') \qquad \text{at } t=\beta\epsilon \tag{71b}$$

$$\bar{\phi}=\bar{\phi}'=0 \qquad \text{at } t=\infty \tag{71c}$$

Equation (71a) can again be solved using the integrating factor $t/\sqrt{1-t^2}$. The various integrals encountered can be treated by integrating by parts one or more times. Imposing the upstream condition (equation (71c)), the complete conical second-order perturbation potential is found to be

$$\bar{\phi}=-B(\operatorname{sech}^{-1} t-\sqrt{1-t^2})+A^2M^2\left[(\operatorname{sech}^{-1} t)^2-(N+1)\sqrt{1-t^2}\operatorname{sech}^{-1}t-\frac{1}{4}\beta^2A\frac{(1-t^2)^{3/2}}{t^2}\right]+0\left[\epsilon^6(\operatorname{sech}^{-1} t)^3\right] \tag{72}$$

From equation (16), the streamwise and radial velocity perturbations are

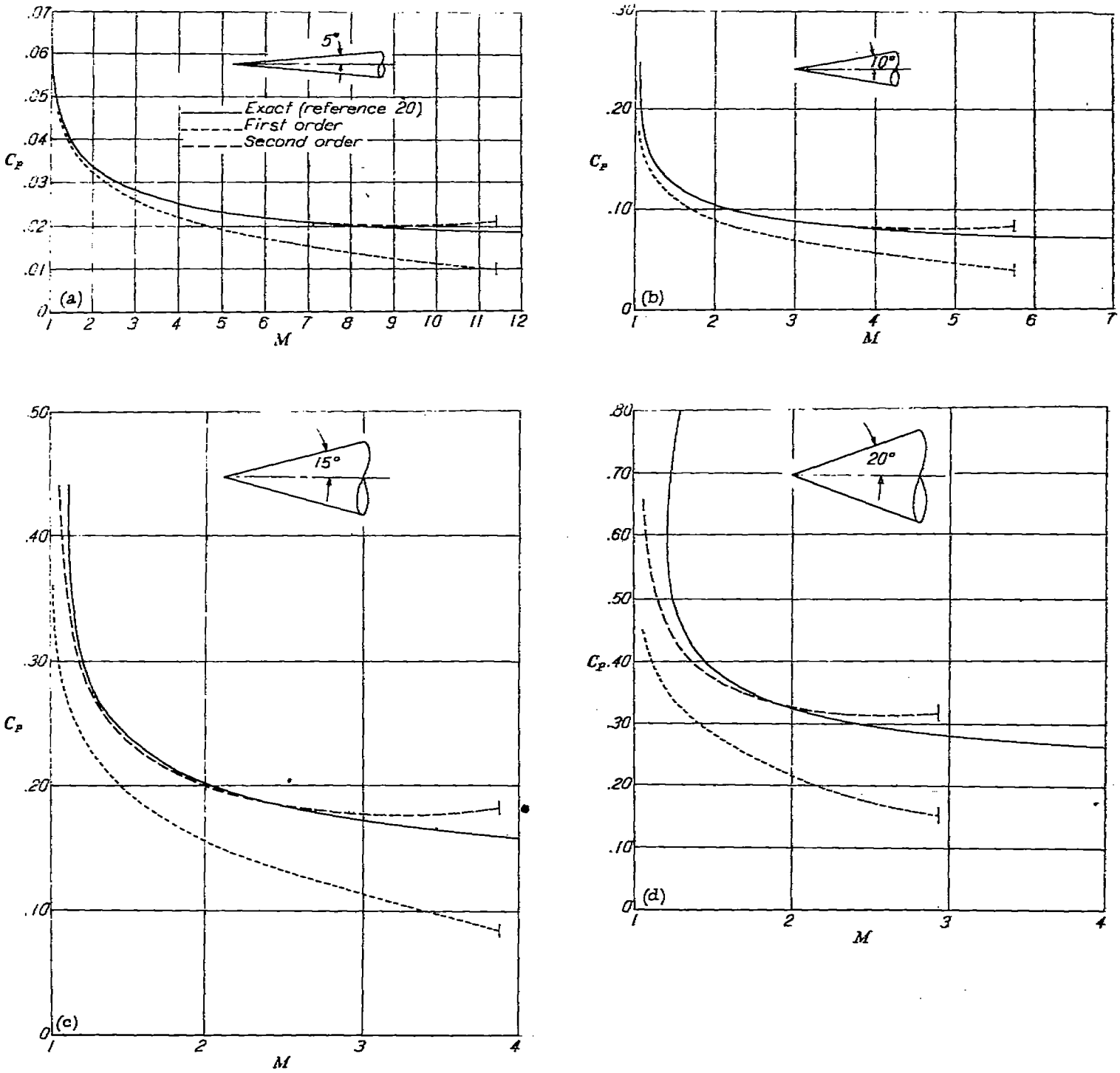
$$\frac{u}{U}=-B\operatorname{sech}^{-1} t+A^2M^2\left[(\operatorname{sech}^{-1}t)^2-(N-1)\frac{\operatorname{sech}^{-1}t}{\sqrt{1-t^2}}-(N+1)-\frac{3}{4}\beta^2A\frac{\sqrt{1-t^2}}{t^2}\right] \tag{73a}$$

$$\frac{1}{\beta}\frac{v}{U}=B\frac{\sqrt{1-t^2}}{t}+A^2M^2\left[-2\frac{\sqrt{1-t^2}\operatorname{sech}^{-1} t}{t}+(N+1)\frac{1}{t}+(N-1)\frac{t\operatorname{sech}^{-1} t}{\sqrt{1-t^2}}+\frac{1}{4}\beta^2A\frac{(2+t^2)\sqrt{1-t^2}}{t^3}\right] \tag{73b}$$

The constant B must be adjusted so as to satisfy the tangency condition (equation (71b)). In actual computation it is easier to adjust B numerically in exactly this fashion rather than to calculate it from the cumbersome expression which could be written down. The pressure coefficient at any point can then be calculated from equation (26).

The last term in the bracket in equation (71a) is the triple product $\beta^2\bar{\varphi}_t^2\bar{\varphi}_{tt}$ which is retained in the iteration equation (equation (17)). Its retention is now justified by noting that its contribution—the last term in equation (72)—is of the same order as other terms near the surface of the cone ($t=\beta\epsilon$). Actually, it contributes a second term, which has been neglected since it is at most of order $\epsilon^6\operatorname{sech}^{-1}\beta\epsilon$. It can also be verified that the other triple products, whose form is indicated at the end of equation (17), are in fact negligible since they contribute at most terms of order $\epsilon^6(\operatorname{sech}^{-1}\beta\epsilon)^2$. Consideration of a further iteration indicates that a third approximation would add terms of order $\epsilon^6(\operatorname{sech}^{-1}\beta\epsilon)^3$, which is greater than the terms just neglected.

The second-order result for surface pressure coefficient is compared in figure 9 with the exact solution (reference 20) for cones of 5°, 10°, 15°, and 20° semivertex angles. Also shown for comparison are the first-order results based upon the exact isentropic expression (equation (26)) for the pres-



(a) Semi-vertex angle 5°.
(b) Semi-vetex angle 10°.
(c) Semi-vertex angle 15°.
(d) Semi-vertex angle 20°.

FIGURE 9.—Comparison of various solutions for pressure on a cone

sure coefficient. The second-order solution is seen to provide a much better approximation throughout the range of Mach numbers up to the point at which the Mach angle equals the cone angle, beyond which the perturbation solutions have no physical meaning.

SHOCK-WAVE POSITION FOR CONE

The solution for plane flow fails near the Mach wave from a corner, which suggests that the second-order solution for the cone may likewise fail near the Mach cone from the vertex. In the plane case, nevertheless, a first approximation to the shock-wave position (and hence to the entropy change) can be calculated from the velocity perturbations near the Mach wave. We now consider whether this is true also for the cone.

It was noted before that first-order theory predicts no disturbance at the Mach cone and hence no shock wave. According to second-order theory the velocity perturbations just behind the Mach cone are (equations (73))

$$-\left(\frac{u}{U}\right)_{t=1}=\frac{1}{\beta}\left(\frac{v}{U}\right)_{t=1}=-2M^2N\epsilon^4 \tag{74}$$

so that the perturbation is normal to the Mach cone. Here A (equation (70)) has been approximated by ϵ^2. From equation (50b) the cotangent of the revised Mach angle just behind the Mach cone is found to be

$$\beta^{(1)}=\beta[1-2M^4N(N-1)\epsilon^4] \tag{75}$$

The upward stream inclination there is approximately $(v/U)_{t=1}$, so that the Mach lines have the slope

$$\frac{dr}{dx}=\frac{1}{\beta}(1+2M^4N^2\epsilon^4) \tag{76}$$

Now if this can be taken to be the slope also of the revised characteristics just behind the shock wave, then the slope of the shock wave differs from that of the original Mach cone by

$$\tan\lambda-\frac{1}{\beta}\overset{?}{=}\frac{M^4N^2}{\beta}\epsilon^4=\frac{(\gamma+1)^2M^8}{4\beta^5}\epsilon^4 \tag{77}$$

This problem has been treated rigorously by Lighthill (references 24 and 25) and by Broderick (reference 26), who find that actually

$$\tan\lambda-\frac{1}{\beta}=\frac{3}{8}\frac{(\gamma+1)^2M^8}{\beta^5}\epsilon^4 \tag{78}$$

which is one and one-half times the preceding result. The discrepancy means that the second-order solution fails near the Mach cone. The nature of this failure and the proper method of remedying it have recently been studied by Lighthill (reference 25).

The entropy increase through a weak oblique shock wave is proportional to the cube of its inclination away from the Mach lines. Consequently, the entropy rise through the shock wave from a cone is $0(\epsilon^{12})$, as noted by Lighthill (reference 24).

PARTICULAR INTEGRAL FOR AXIALLY SYMMETRIC FLOW

Consider flow past a body of revolution which is either a slender pointed body with nose at the origin or one which extends indefinitely upstream with constant radius a for $x\leqq 0$ (see fig. 10). The latter shape corresponds to the external flow past a sharp-edged, open-nosed body with supersonic flow at the lip. With slight modification the subsequent development can be applied to internal flow as

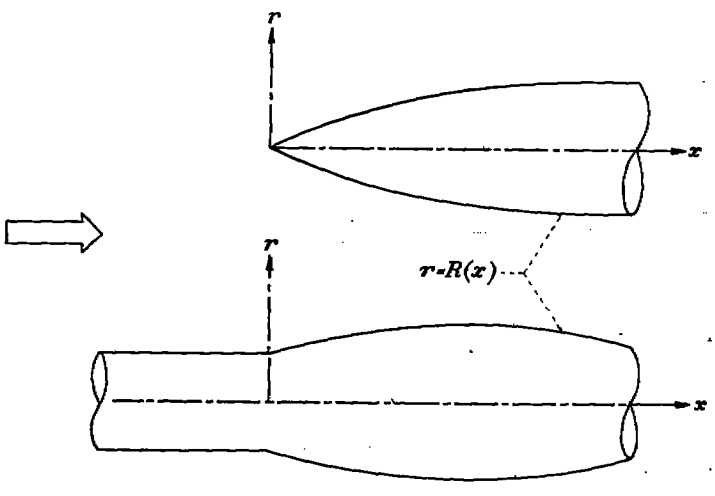


FIGURE 10.—Flow past bodies of revolution.

well. The meridian curve can be represented in the first case by

$$r=R(x)=\epsilon g(x) \qquad x\geqq 0 \tag{79a}$$

and in the second by

$$r=R(x)=\begin{cases}a & x\leqq 0\\ a+\epsilon g(x) & x\geqq 0\end{cases} \tag{79b}$$

Here ϵ is again a parameter small compared with unity, and $g(x)$ is a continuous function of order unity.

The first-order problem is

$$\varphi_{rr}+\frac{\varphi_r}{r}-\beta^2\varphi_{xx}=0 \tag{80a}$$

$$\varphi_r(x,R)=R'(x)[1+\varphi_x(x,R)] \tag{80b}$$

$$\varphi(0,r)=\varphi_x(0,r)=0 \tag{80c}$$

The solution is known to be (reference 27)

$$\varphi(x,r)=-\int_b^{x-\beta r}\frac{f(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}}$$
$$-\int_0^{\cosh^{-1}\frac{x-b}{\beta r}}f(x-\beta r\cosh u)du \tag{81}$$

The second form is useful for carrying out differentiation, after which the first form can be restored. The derivatives which will be required are

$$\varphi_x=-\int_0^{\cosh^{-1}\frac{x-b}{\beta r}}f'(x-\beta r\cosh u)du=-\int_b^{x-\beta r}\frac{f'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \tag{82a}$$

$$\varphi_r=\beta\int_0^{\cosh^{-1}\frac{x-b}{\beta r}}f'(x-\beta r\cosh u)\cosh u\,du=$$
$$\frac{1}{r}\int_b^{x-\beta r}\frac{(x-\xi)f'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}} \tag{82b}$$

In carrying out the differentiation the fact has been used that $f(b)=0$ for a body with finite slope. With coordinates as shown in figure 10, the lower limit of integration b is 0 for the pointed body and $-\beta a$ for the semi-infinite body. The function $f(x)$ may be interpreted physically (aside from a numerical factor) as the strength of a supersonic line source along the x axis. It is determined by the tangency condition (equation (80b)) which gives the following integral equation of the Volterra type for f':

$$\int_0^{x-\beta R(x)}\frac{(x-\xi)f'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2(x)}}=$$
$$R(x)R'(x)\left[1-\int^{x-\beta R(x)}\frac{f(\xi')d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2(x)}}\right] \tag{83}$$

The second-order iteration equation is found from equation (13) to be

$$\phi_{rr}+\frac{\phi_r}{r}-\beta^2\phi_{xx}=M^2[2(N-1)\beta^2\varphi_x\varphi_{xx}+2\varphi_r\varphi_{xr}+\varphi_r^2\varphi_{rr}+$$
$$0(\varphi_x^2\varphi_{xx},\ \varphi_r^2\varphi_{xx},\ \varphi_x\varphi_r\varphi_{xr})] \tag{84}$$

The solution for the cone suggests that the terms indicated at the end of the equation are negligible.

It will now be shown that a particular integral of this equation is given in terms of the first-order solution by

$$\psi(x,r)=M^2\left[\varphi_x(\varphi+Nr\varphi_r)-\frac{1}{4}r\varphi_r^3\right] \tag{85}$$

The first group of terms contributes the first two terms on the right-hand side of equation (84), as can be verified by direct substitution. The last term in equation (85) accounts for the term $\varphi_r^2\varphi_{rr}$ as follows:

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\beta^2\frac{\partial^2}{\partial x^2}\right)\left(-\frac{1}{4}r\varphi_r^3\right)=$$
$$\varphi_r^2\varphi_{rr}-\beta^2\varphi_r\left(\frac{3}{2}r\varphi_{xx}\varphi_{rr}+\varphi_r\varphi_{xx}-\frac{3}{2}r\varphi_{xr}^2\right) \tag{86}$$

where repeated use has been made of the fact that φ satisfies equation (80a). The last group of terms consists of triple products involving x derivatives, which have already been neglected in equation (84), so that the result is proved.

The complementary function χ is a solution of equation (80) and can be written as

$$\chi(x,r)=-\int_b^{x-\beta r}\frac{F(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}}=-\int_0^{\cosh^{-1}\frac{x-b}{\beta r}}F(x-\beta r\cosh u)du \tag{87}$$

Using equations (82) the second-order tangency condition (equation (19b)) is found to be

$$\psi_r(x,R)+\frac{1}{R}\int_b^{x-\beta R}\frac{(x-\xi)F'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2}}=$$
$$R'\left[1+\psi_x(x,R)-\int_b^{x-\beta R}\frac{F'(\xi)d\xi}{\sqrt{(x-\xi)^2-\beta^2R^2}}\right] \tag{88}$$

which is again a Volterra integral equation.

METHODS OF SOLVING INTEGRAL EQUATION

Discovery of a particular integral for bodies of revolution reduces the second-order problem to the same form as the first-order problem; namely, the solution of a Volterra integral equation. Various methods of attacking this problem are listed by Hayes (reference 17, p. 140).

An indirect method consists in assuming that the unknown source strengths in equations (81) and (87) can be represented by a few terms of a polynomial, for example,

$$f(x)=C_1x+C_2x^2+\ .\ .\ .\ +C_nx^n \tag{89}$$

The resulting solutions were introduced in a more formal manner by Hayes (reference 17, p. 38), who has discussed their properties in detail. The first term alone gives the potential for the cone, (equation (69)). Additional terms give the solution for simple families of shapes. However, the method is not suitable for bodies having discontinuities in slope or curvature. Consequently, a more direct procedure is desirable.

von Kármán first introduced an asymptotic solution of the integral equation (equation (83)) which has come to be known as the slender-body approximation (reference 27). For slender bodies, the source strength $f(x)$ appearing in equation (81) is found to be approximately proportional to the rate of change of cross-sectional area. Thus

$$f(x)\approx\frac{1}{2\pi}\frac{d}{dx}[\pi R^2(x)]=R(x)R'(x) \tag{90}$$

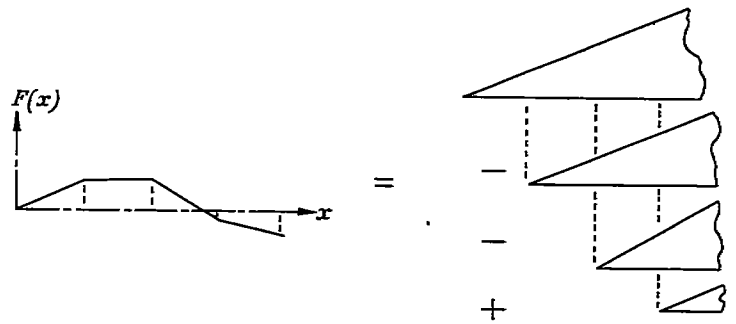


FIGURE 11.—Equivalence of polygonal source and sum of conical sources.

Lighthill has shown (reference 19) that if $R(x)$ and its first two derivatives are of order ϵ and R' is continuous, this determination of $f(x)$ is correct to the order of terms retained in the first-order solution. For purposes of the second-order solution, it can be shown that $f(x)$ may be determined in this way only if the first four derivatives of R are of order ϵ and R'' is continuous. This means that the body must have continuous curvature, which is a severe limitation. Moreover, the slender-body approximation is found generally to cause unnecessary loss of accuracy even though the mathematical order estimate of the error is small. Consequently this approximation should be avoided if possible.

The most satisfactory way of solving the integral equations is to use a step-by-step numerical procedure. In first-order theory the usual method, introduced by von Kármán and Moore (reference 28), is to assume that the unknown source distribution can be approximated by a polygonal graph. This is equivalent to superimposing a number of conical source lines of different strengths, each shifted downstream with respect to its predecessor, as indicated in figure 11. The latter viewpoint is more convenient for computation. The strengths of the source lines are determined in succession by satisfying the tangency condition at a series of points on the surface of the body. The details of this prodedure are clearly explained in reference 3, page 77.

For purposes of a second-order solution, this procedure must be modified in one respect. Unless the source distribution $f(x)$ actually has corners, it must not be approximated by a polygon. The reason is that a corner corresponds locally to adding a conical source line which would, according to the solution for the cone, produce false second-order discontinuities in velocity and pressure across the Mach cone from the corner. Instead, the procedure must be carried out using source lines of quadratic strength. The source strength $f(x)$ can then be approximated smoothly so that false pressure jumps do not occur. A single source line of this type represents the flow past a slender pointed body with a cusped nose (see fig. 12), as is clear from the slender-body approximation (equation (90)).

METHOD OF SOLUTION FOR SMOOTH BODIES

The second-order solution will be described first for bodies having continuous slope. Modifications for treating sharp corners will be discussed in the next section.

The procedure is indicated in figure 13. The axis is divided into intervals by choosing points with abscissas ξ_n, at each of which a source line is to begin. Good accuracy is usually obtained if the interval length is not greater than β times the local radius. The tangency condition will be imposed on the surface of the body at the points P_n, which lie on the Mach lines from the points ξ_n.

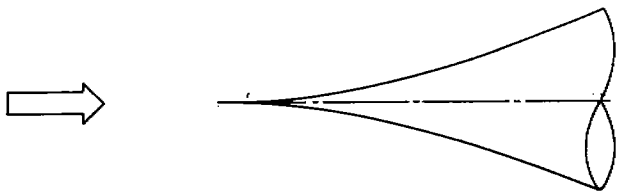


FIGURE 12.—Body formed by source line of quadratic strength.

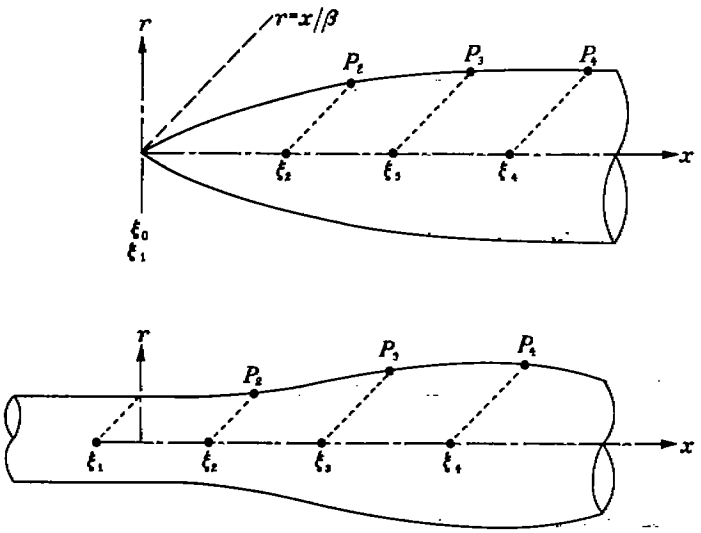


FIGURE 13.—Method of solution for smooth bodies.

For pointed bodies, the first-order solution is started with a conical source from the origin which gives the proper conical tip. This potential and the derivatives which are required are

$$\left.\begin{aligned}
&\varphi_0=-C_0x\left(\operatorname{sech}^{-1}t-\sqrt{1-t^2}\right) && \varphi_{0_{xx}}=-\frac{C_0}{x}\frac{1}{\sqrt{1-t^2}}\\
&\varphi_{0_x}=-C_0\operatorname{sech}^{-1}t && \varphi_{0_{xr}}=\frac{\beta C_0}{x}\frac{1}{t\sqrt{1-t^2}}\\
&\varphi_{0_r}=\beta C_0\frac{\sqrt{1-t^2}}{t} && \varphi_{0_{rr}}=\frac{\beta^2C_0}{x}\frac{1}{t^2\sqrt{1-t^2}}\\
&\text{where}\\
&C_0=\frac{\epsilon^2}{\sqrt{1-\beta^2\epsilon^2}+\epsilon^2\operatorname{sech}^{-1}\beta\epsilon}
\end{aligned}\right\}\tag{91}$$

Here C_0 is the same as the A of equations (69) to (73), ϵ being the tangent of the semivertex angle of the conical tip. No such term is required for the semi-infinite body.

The subsequent procedure is the same for either body. Quadratic source lines are started from each of the points ξ_1, ξ_2, and so forth. For the pointed body ξ_1 also lies at the origin, while for the semi-infinite body it is at $-\beta a$. For the nth such source line, the potential and its derivatives are given by

$$\left.\begin{aligned}
&\varphi_n=-C_n(x-\xi_n)^2\left[\left(1+\frac{1}{2}\tau_n^2\right)\operatorname{sech}^{-1}\tau_n-\frac{3}{2}\sqrt{1-\tau_n^2}\right]\\
&\varphi_{n_x}=-2C_n(x-\xi_n)\left(\operatorname{sech}^{-1}\tau_n-\sqrt{1-\tau_n^2}\right)\\
&\varphi_{n_r}=\beta C_n(x-\xi_n)\left(\frac{\sqrt{1-\tau_n^2}}{\tau_n}-\tau_n\operatorname{sech}^{-1}\tau_n\right)\\
&\varphi_{n_{xx}}=-2C_n\operatorname{sech}^{-1}\tau_n\\
&\varphi_{n_{xr}}=2\beta C_n\frac{\sqrt{1-\tau_n^2}}{\tau_n}\\
&\varphi_{n_{rr}}=-\beta^2C_n\left(\frac{\sqrt{1-\tau_n^2}}{\tau_n^2}+\operatorname{sech}^{-1}\tau_n\right)
\end{aligned}\right\}\tag{92}$$

where

$$\tau_n=\frac{\beta_r}{x-\xi_n}$$

The constants C_n are determined successively by imposing the first-order tangency condition in turn at each of the points P_{n+1}. From equation (80b), the condition is

$$\sum^n\varphi_{n_r}=R'\left(1+\sum^n\varphi_{n_x}\right)\quad\text{at } P_{n+1}\tag{93}$$

where the summation begins with $n=0$ for the pointed body and $n=1$ for the semi-infinite body. In this way, values of the complete first-order potential φ and its first and second derivatives are calculated at each of the points on the body.

The velocities due to the particular second-order integral ψ can then be calculated at the same points. Differentiating equation (85) gives

$$\left.\begin{aligned}
&\psi_x=M^2\left[(\varphi+Nr\varphi_r)\varphi_{xx}+\varphi_x(\varphi_x+Nr\varphi_{xr})-\frac{3}{4}r\varphi_r^2\varphi_{xr}\right]\\
&\psi_r=M^2\left\{(\varphi+Nr\varphi_r)\varphi_{xr}+\varphi_x[(N+1)\varphi_r+Nr\varphi_{rr}]-\frac{1}{4}\varphi_r^2(\varphi_r+3r\varphi_{rr})\right\}
\end{aligned}\right\}\tag{94}$$

Finally, the second-order complementary function χ is determined by repeating the procedure used for φ, finding new constants such that the second-order tangency condition is satisfied. According to equation (19b), the condition is

$$\psi_r+\sum^n\chi_{n_r}=R'\left(1+\psi_x+\sum^n\chi_{n_x}\right)\quad\text{at } P_{n+1}\tag{95}$$

The second derivatives of χ need not be calculated.

The complete second-order perturbation velocities are found as the sums of the contributions from ψ and χ. Then the pressure coefficient can be calculated at each point P_n from equation (26).

TREATMENT OF BODIES WITH CORNERS

Suppose that the meridian curve of the body has a sharp corner, which for convenience may be assumed to lie on the Mach cone from the origin, as indicated in figure 14. Then the method of solution must be modified for two reasons. In the first place, the intervals between source lines would

have to be chosen extremely small in order to obtain an accurate first-order solution behind the corner. This difficulty can be overcome by adding a new solution which causes a sharp deflection of the streamlines. In the second place, even if the first-order solution is determined exactly, the second-order solution does not yield the Busemann result just behind the corner, as it should since the flow is locally plane. This defect is remedied by properly canceling a

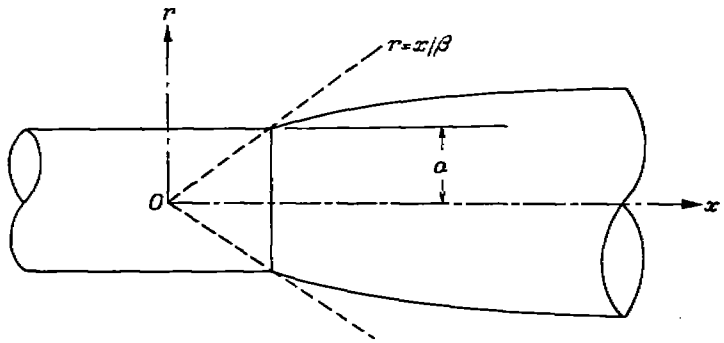


FIGURE 14.– Body with corner.

discontinuity which arises in the second-order solution at the corner.

These two modifications require special solutions of the first-order equation which along the Mach cone from the origin have discontinuities in velocity in the first case, and a discontinuity in potential in the second case. Such solutions can be found by approximating to equation (83) in the vicinity of the Mach cone. Imposing the condition that there x is only slightly less than βR and keeping only leading terms in the difference $x-\beta R$ leads to an Abel integral equation for the source strength. Inverting the integral equation shows that a potential having discontinuous nth derivatives results from a source distribution along the axis which is proportional to $x^{n-\frac{1}{2}}$ Setting $f(x)=x^{n-\frac{1}{2}}$ in equation (81) gives

$$\varphi(x,r)=-\int_0^{x-\beta r}\frac{\xi^{n-\frac{1}{2}}d\xi}{\sqrt{(x-\xi)^2-\beta^2r^2}}=$$

$$-\frac{(x-\beta r)^n}{\sqrt{2\beta r}}\int_0^1\frac{(1-\xi)^{n-\frac{1}{2}}d\xi}{\sqrt{\xi\left(1+\frac{x-\beta r}{2\beta r}\xi\right)}} \tag{96}$$

This integral represents the analytical continuation of the hypergeometric function, so that, except for a constant factor,

$$\varphi=(x-\beta r)^n\sqrt{\frac{a}{r}}\,F\left(\frac{1}{2},\frac{1}{2};n+1;-\frac{x-\beta r}{2\beta r}\right) \tag{97}$$

where a is the radius at the corner. The potential is understood to vanish except within the downstream Mach cone from the origin. The hypergeometric functions occurring here can all be expressed in terms of complete elliptic integrals with real moduli.

In the first-order solution, a sharp deflection of the streamlines at a corner is produced by adding a multiple of the potential which has discontinuous first derivatives. This is found by setting $n=1$, which gives

$$\left.\begin{aligned}
\varphi&=-\frac{4}{\pi}x\sqrt{\frac{a}{r}}(1+t)\sqrt{\frac{2t}{1+t}}(K-E)\\
\varphi_x&=-\frac{2}{\pi}\sqrt{\frac{a}{r}}\sqrt{\frac{2t}{1+t}}K\\
\varphi_r&=\frac{2\beta}{\pi}\sqrt{\frac{a}{r}}\sqrt{\frac{2t}{1+t}}\left(\frac{1+t}{t}E-K\right)\\
\varphi_{xx}&=\frac{1}{\pi}\frac{1}{x}\sqrt{\frac{a}{r}}\frac{1}{1-t}\sqrt{\frac{2t}{1+t}}(K-E)\\
\varphi_{xr}&=\frac{\beta}{\pi}\frac{1}{x}\sqrt{\frac{a}{r}}\frac{1}{1-t}\sqrt{\frac{2t}{1+t}}\left(\frac{1}{t}E-K\right)\\
\varphi_{rr}&=-\frac{\beta^2}{\pi}\frac{1}{x}\sqrt{\frac{a}{r}}\frac{1}{1-t}\sqrt{\frac{2t}{1+t}}\left(\frac{2-t^2}{t^2}E-\frac{2-t}{t}K\right)
\end{aligned}\right\} \tag{98}$$

Here t is the conical variable introduced in equation (14), and K and E are the complete elliptic integrals of the first and second kinds with modulus $k=[(1-t)/(1+t)]^{1/2}$.

From the tangency condition (equation (80b)), it can be shown that in order to account for the corner the solution given by equation (98) should be multiplied by

$$\frac{(R_b'-R_a')[1+(\varphi_x)_a]}{\beta+R_b'} \tag{99}$$

Here R_a' and R_b' are the slopes of the meridian curve just ahead of and behind the corner and $(\varphi_x)_a$ is the value of φ_x ahead of the corner. The first-order solution can thereafter be continued as described in the previous section.

The second difficulty noted before was that the second-order solution is found to be incorrect just behind the corner. The proper method of treating this difficulty is to solve the case when the corner has been slightly rounded and then pass to the limit of a sharp corner. However, the following simpler procedure is found to give exactly the same result.

The particular solution ψ calculated from equation (85) is discontinuous along the Mach wave springing from the corner. If the discontinuity vanished at the corner, the solution could subsequently be revised as in the case of plane flow (see fig. 5). However, there is a finite jump in ψ directly at the corner, which cannot be allowed. Consequently, the correction potential χ must involve an equal and opposite jump. A potential having such a discontinuity is obtained by setting $n=0$ in equation (97). Then

$$\left.\begin{aligned}
\chi&=\frac{2}{\pi}\sqrt{\frac{a}{r}}\sqrt{\frac{2t}{1+t}}K\\
\chi_x&=-\frac{1}{\pi}\frac{1}{x}\sqrt{\frac{a}{r}}\frac{1}{1-t}\sqrt{\frac{2t}{1+t}}(K-E)\\
\chi_r&=-\frac{\beta}{\pi}\frac{1}{x}\sqrt{\frac{a}{r}}\frac{1}{1-t}\sqrt{\frac{2t}{1+t}}\left(\frac{1}{t}E-K\right)
\end{aligned}\right\} \tag{100}$$

Adding a suitable multiple of this potential cancels the discontinuity in ψ. The second-order solution can then be continued as described in the preceding section. It can be verified that the pressure jump at the corner has then the correct second-order value.

It is instructive to analyze the behavior of a corner from another viewpoint. It was pointed out before that the right-

hand side of the iteration equation can be considered to represent the effects of a known distribution of sources throughout the flow field. In the case of a slightly rounded corner, this source distribution will be weak except between the Mach lines from the corner. As the corner shrinks to a point, the source intensity will increase in that region in such a way that the total strength remains constant. In the limit, the source distribution will behave like a Dirac delta function along the Mach line from the corner. The particular integral for plane flow (equation (35)) takes account of this impulsive function so that the correct solution is automatically obtained. In the case of axially symmetric flow, however, it is clear that the particular integral given by equation (85) misses the contribution of the impulse. It is therefore necessary to correct this shortcoming by adding the step-function potential given by equation (100).

COMPARISON WITH NUMERICAL SOLUTIONS

The accuracy of the second-order solution for bodies of revolution can be evaluated by comparison with examples calculated using the numerical method of characteristics.

The first body to be considered is a circular-arc ogive of 12½-caliber radius of curvature followed by a cylinder, which has a half angle of 16.26° at the tip. The second-order solution has been calculated for this body at a Mach number of 3.24. This represents a severe test of the method because the Mach angle is then only 10 percent greater than the tip cone angle. Intervals were chosen such that the points P_n lay at 0.1, 0.25, 0.5, 1, 2, and 3.5 calibers. The pressure distributions calculated by first- and second-order theory are compared in figure 15 with the results of various computations by the numerical method of characteristics. Of the latter, the result obtained from the interpolation chart given by Rossow in reference 29 is believed to be more accurate than the earlier German computations which were taken from the summary report of reference 30. Except near the tip, the second-order solution agrees very closely with the numerical results.

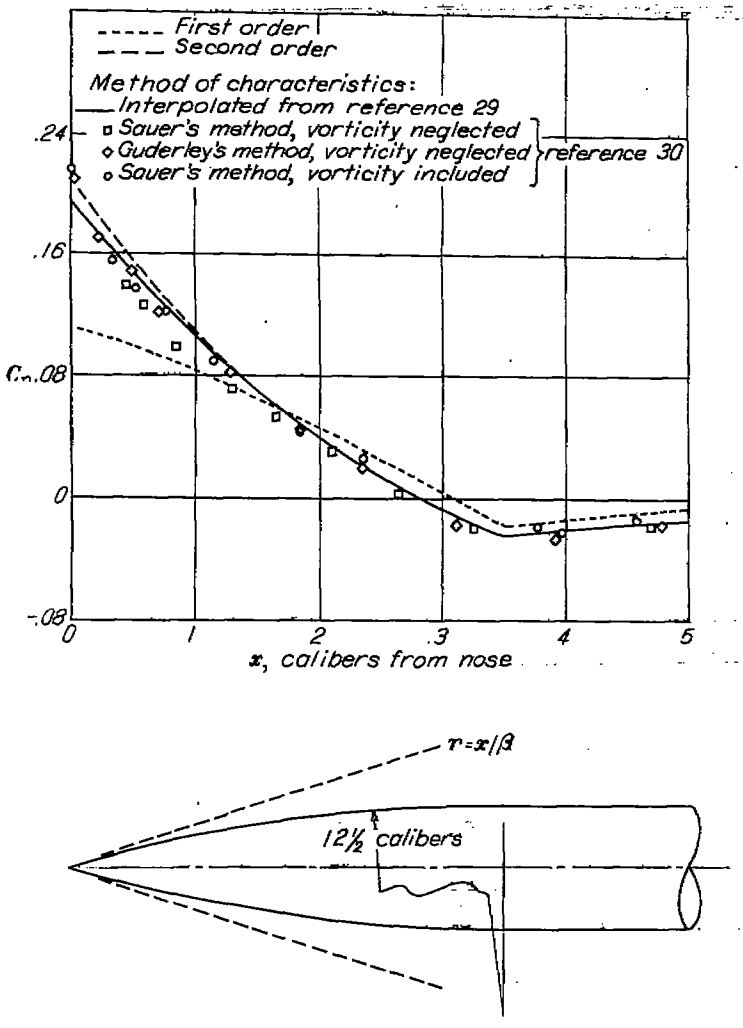


FIGURE 15.—Pressure distribution on smooth body of revolution at M=3.24.

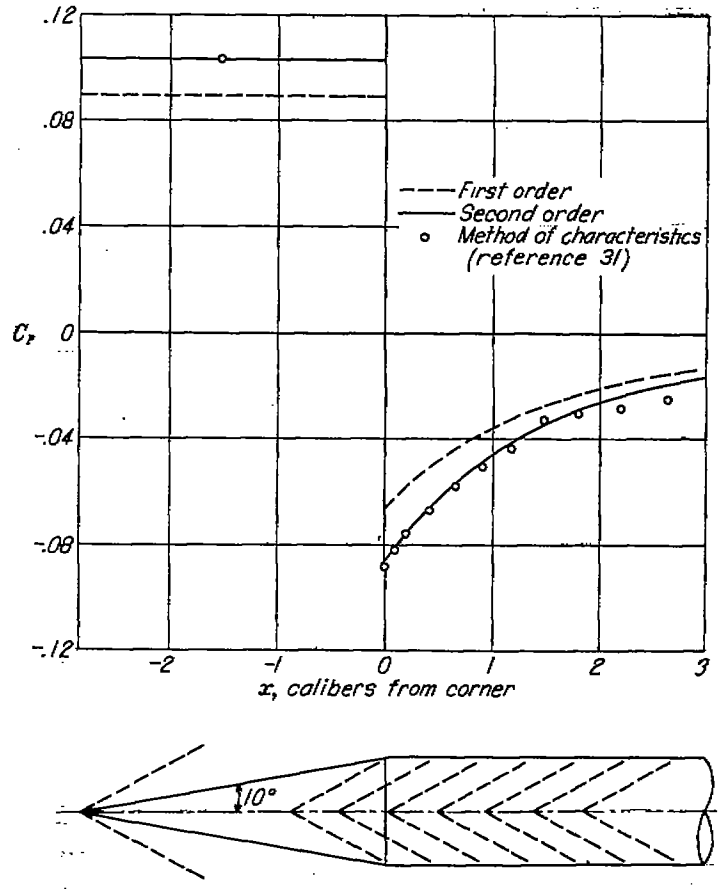


FIGURE 16.—Pressure distribution on body of revolution with corner at M=2.075.

The second body to be considered consists of a cone of 10° semi-vertex angle followed by a cylinder. The characteristics solution for this body at a Mach number of 2.075 has been given by Liepmann and Lapin in reference 31. The first- and second-order solutions were calculated beyond the corner using the modifications discussed in the preceding section. Figure 16 shows the shape of the body, the location of source lines, and the pressure distributions calculated by first-order theory, second-order theory, and the method of characteristics. Again, the second-order results agree well with the characteristics solution.

SERIES EXPANSION WITH RESPECT TO THICKNESS

An alternative method of solving the exact perturbation equation (equation (4)) by successive approximations is to

assume that the solution can be expanded in powers of the thickness parameter ϵ. Thus the exact perturbation potential is written as

$$\Phi=\overset{(1)}{\Phi}\epsilon+\overset{(2)}{\Phi}\epsilon^2+\overset{(3)}{\Phi}\epsilon^3+\overset{(4)}{\Phi}\epsilon^4+\ldots \tag{101}$$

Substituting into equation (4) and equating like powers of ϵ yields a sequence of equations

$$\left.\begin{aligned}&\overset{(1)}{\Phi}_{yy}+\overset{(1)}{\Phi}_{zz}-\beta^2\overset{(1)}{\Phi}_{xx}=0\\&\overset{(2)}{\Phi}_{yy}+\overset{(2)}{\Phi}_{zz}-\beta^2\overset{(2)}{\Phi}_{xx}=2M^2\left[(N-1)\beta^2\overset{(1)}{\Phi}_x\overset{(1)}{\Phi}_{xx}+\overset{(1)}{\Phi}_y\overset{(1)}{\Phi}_{xy}+\overset{(1)}{\Phi}_z\overset{(1)}{\Phi}_{xz}\right]\\&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\end{aligned}\right\} \tag{102}$$

which can be solved in succession. The first is again the usual linearized equation. This method was applied to plane subsonic flow in references 6 and 12.

Schmieden and Kawalki first pointed out (reference 14) that the power series assumed here does not always exist, even for plane flow. In general, terms of the form $\epsilon^m ln^n\epsilon$ appear, beginning with $\epsilon^4 ln\epsilon$ in the third-order solution for plane flow and in the second-order solution for axially symmetric flow. Furthermore, for a body of revolution only even powers of ϵ arise. Consequently, it is necessary to assume a more general series of the form

$$\Phi=\overset{(1)}{\Phi}\epsilon+\overset{(2)}{\Phi}\epsilon^2+\overset{(3)}{\Phi}\epsilon^3+\overset{(4)}{\Phi}\epsilon^4\, ln\,\epsilon+\overset{(5)}{\Phi}\epsilon^4+\ldots \tag{103a}$$

for plane flow and of the form

$$\Phi=\overset{(1)}{\Phi}\epsilon^2+\overset{(2)}{\Phi}\epsilon^4\, ln\,\epsilon+\overset{(3)}{\Phi}\epsilon^4+\overset{(4)}{\Phi}\epsilon^6\, ln^2\,\epsilon+\overset{(5)}{\Phi}\epsilon^6\, ln\,\epsilon+\overset{(6)}{\Phi}\epsilon^6+\ldots \tag{103b}$$

for axially symmetric flow.

On the basis of this assumption, Broderick has developed a second-order solution for supersonic flow past slender pointed bodies of revolution (reference 32). The analysis is rather lengthy, since the simplification resulting from the discovery of a particular integral does not appear. The results are limited to shapes for which the cross-sectional area is given by an analytic function, or at any rate has its first four derivatives small of order ϵ, and the first two continuous. This is a severe limitation since, for example, the two bodies discussed in the previous section are not admissible.

Broderick's method yields the slender-body counterpart of the present second-order theory. Just as the usual first-order slender-body results can be derived as asymptotic forms of linearized solutions, so Broderick's second-order slender-body results can be obtained by expanding the present second-order solution in powers of t and $ln\ t$ for small t and retaining secondary as well as leading terms. The logarithmic terms arise from the series

$$\operatorname{sech}^{-1}t=ln\frac{2}{t}-\frac{1}{4}t^2-\frac{3}{32}t^4-\ldots \qquad (0<t<1) \tag{104}$$

The expansion will now be carried out for the case of flow past a cone.

It is clear from equation (70) that the constant A in the first-order solution (equation (69)) is given approximately by

$$A=\epsilon^2+\ldots \tag{105}$$

Substituting this value into equations (73a) and (73b), expanding in powers of t and $ln\ t$, and imposing the tangency condition (equation (71b)) shows that

$$B=\epsilon^2+\epsilon^4\left[(2M^2-1)ln\frac{2}{\beta\epsilon}-M^2(N+1)-\frac{1}{2}\right]+\cdots \tag{106}$$

Then according to equation (73), the velocity perturbations on the surface of the cone are

$$\frac{u}{U}=-\epsilon^2 ln\frac{2}{\beta\epsilon}-\epsilon^4\left[\beta^2\left(ln\frac{2}{\beta\epsilon}\right)^2-\frac{4M^2+1}{2}ln\frac{2}{\beta\epsilon}+M^2N+\frac{6M^2+1}{4}\right]+\cdots \tag{107a}$$

$$\frac{v}{U}=\epsilon-\epsilon^3 ln\frac{2}{\beta\epsilon}+\cdots \tag{107b}$$

Replacing N by its value from equation (11), the approximate pressure relation of equation (27) gives for the pressure coefficient on the surface of the cone

$$C_p=\epsilon^2\left(2ln\frac{2}{\beta\epsilon}-1\right)+\epsilon^4\left[3\beta^2\left(ln\frac{2}{\beta\epsilon}\right)^2-(5M^2-1)ln\frac{2}{\beta\epsilon}+(\gamma+1)\frac{M^4}{\beta^2}+\frac{13}{4}M^2+\frac{1}{2}\right]+0\left[\epsilon^6\left(ln\frac{2}{\beta\epsilon}\right)^3\right] \tag{108}$$

This is Broderick's result (reference 32, equation (81)).

This series is compared in figure 17 with the original form of the second-order solution which uses the exact pressure relation. For the most slender cone, the expansion in series causes only a moderate loss in accuracy. For more practical thicknesses, however, the expansion reduces the accuracy to such an extent that for the cone of 20° semivertex angle Broderick's solution is inferior to the first-order result (with the exact pressure relation). The reason must be that the iteration process itself converges more rapidly than do the subsequent expansions which are required to reduce it to the slender-body series form. Hence terminating all expansions at terms of the order of those retained in the iteration process results in an unnecessary loss of accuracy.

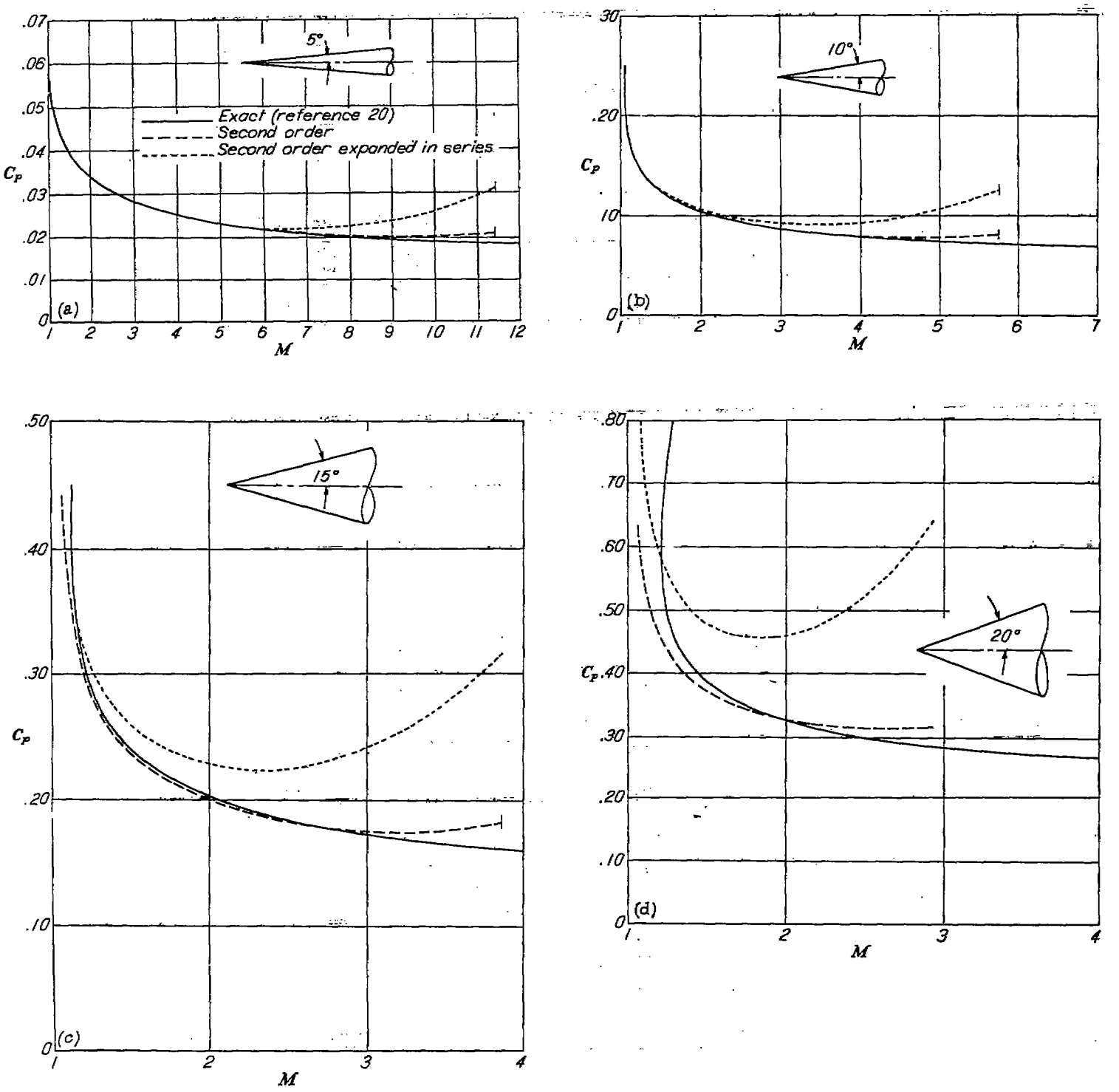
THREE-DIMENSIONAL FLOW

PARTIAL PARTICULAR INTEGRAL

It might be hoped that a particular integral, which so greatly simplifies the iteration for plane and axially symmetric flows, could be found for the general three-dimensional case. The various methods of existing first-order theory could then be applied immediately to the problems of second-order flow past such shapes as inclined bodies of revolution and three-dimensional wings.

A part of such a particular integral is found at once, being common to the two special cases. Consider the three-dimensional iteration equation (equation (12)), which may be written

$$\phi_{yy}+\phi_{zz}-\beta^2\phi_{xx}=M^2(2N\beta^2\varphi_x\varphi_{xx}-2\beta^2\varphi_x\varphi_{xx}+2\varphi_y\varphi_{xy}+2\varphi_z\varphi_{xz}+\varphi_y^2\varphi_{yy}+2\varphi_y\varphi_z\varphi_{yz}+\varphi_z^2\varphi_{zz}) \tag{109}$$



(a) Semi-vertex angle 5°.
(b) Semi-vertex angle 10°.
(c) Semi-vertex angle 15°.
(d) Semi-vertex angle 20°.

FIGURE 17.—Effect of expanding in series upon second-order pressure on a cone.

It can be readily verified that except for the term in N and the triple products (the last three terms) a particular integral is given by

$$\psi_* = M^2\varphi\varphi_x \tag{110}$$

which appears in both equations (35) and (85).

The iteration equation is thereby reduced to

$$\phi_{yy}+\phi_{zz}-\beta^2\phi_{xx}=M^2(2\beta^2N\varphi_x\varphi_{xx}+\varphi_y^2\varphi_{yy}+2\varphi_y\varphi_z\varphi_{yz}+\varphi_z^2\varphi_{zz}) \tag{111}$$

It has not been possible to find a particular integral of this equation in terms of the first-order potential. The solutions for plane and axially symmetric flow do not appear to suggest a generalization. On the other hand, there is no assurance that such an integral cannot be found. When the triple products are negligible, the right-hand side of equation (111) vanishes for $\gamma=-1$ ($N=0$). However, investigation of the previous solutions indicates that the idea of here taking $\gamma=-1$ is not legitimate.

In the absence of a complete particular integral, the reduced iteration equation (equation (111)) must be attacked by more conventional methods. In principle, it is always possible to find a particular integral of a linear nonhomogeneous equation with the aid of the fundamental solution associated with the differential operator. For the three-dimensional wave operator which occurs here, the fundamental solution is

$$\frac{1}{\sqrt{(x-\xi)^2-\beta^2[(y-\eta)^2+(z-\zeta)^2]}} \tag{112}$$

which can be interpreted as the potential at any point (x, y, z) lying inside the downstream Mach cone from a unit supersonic source at (ξ, η, ζ). With the aid of Green's formula, it can be shown that a particular integral of

$$\phi_{yy}+\phi_{zz}-\beta^2\phi_{xx}=F(x,y,z) \tag{113}$$

is given by

$$\psi(x,y,z)=-\frac{1}{2\pi}\int\int\int\frac{F(\xi,\eta,\zeta)d\xi d\eta d\zeta}{\sqrt{(x-\xi)^2-\beta^2[(y-\eta)^2+(z-\zeta)^2]}} \tag{114}$$

where the integration extends throughout that portion of the forward Mach cone from the point (x, y, z) within which F is defined.

In practice, the integration indicated in equation (114) is generally not feasible. For example, even the simplification of axial symmetry reduces equation (114) only to a double integral of $F(x,r)$ multiplied by a complete elliptic integral of complicated argument. Avoiding such integrals by discovery of the particular solution clearly represents a great simplification in this case.[5]

In the following sections, one example of a three-dimensional solution will be given, and the possibility of treating other shapes will be discussed thereafter.

FLOW PAST AN INCLINED CONE

The problem of a cone at an angle of attack illustrates the use of separation of variables to reduce the three-dimensional iteration equation to tractable form.

Two alternative coordinate systems are suitable for bodies of revolution at an angle of attack. In wind axes the body is inclined, while in body axes the stream impinges on the body obliquely. The latter system is simpler for first-order problems and is probably better for the second-order solution also. However, wind axes will be used here, since otherwise the iteration equations must be rederived.

To simplify the solution, it will be assumed that the angle of attack α is so small that its square can be neglected. This will give a solution nonlinear in the body thickness but linear in α, and will therefore yield the initial slope of the lift curve correct to second order. The coordinate system is indicated in figure 18.[6]

[5] Comparing the two methods would lead to the evaluation of definite integrals involving complete elliptic integrals, which might be of some interest.

[6] In previous versions of this work (references 1 and 2) an oblique transformation was applied which effectively unyaws the axis of the body. However, this additional transformation has been found to complicate rather than simplify the analysis, and has been omitted here.

To this approximation the surface of the cone is given by

$$r=R(x,\theta)=[\epsilon-\alpha(1+\epsilon^2)\cos\theta]\,x \tag{115a}$$

$$t=T(\theta)=\beta\,[\epsilon-\alpha(1+\epsilon^2)\cos\theta] \tag{115b}$$

The first-order equation for the conical perturbation potential is given by the left-hand side of equation (17):

$$(1-t^2)\bar{\varphi}_{tt}+\frac{\bar{\varphi}_t}{t}+\frac{\bar{\varphi}_{\theta\theta}}{t^2}=0 \tag{116}$$

The solution required here is the sum of potentials for a conical line source and dipole (reference 3, p. 74) and has the form

$$\bar{\varphi}=-A\left(\operatorname{sech}^{-1}t-\sqrt{1-t^2}\right)+C\left(\frac{\sqrt{1-t^2}}{t}-t\operatorname{sech}^{-1}t\right)\alpha\cos\theta \tag{117}$$

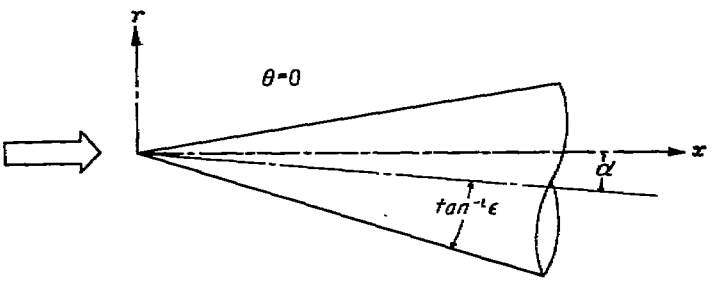


FIGURE 18.—Inclined cone in wind axes.

From equations (16) and (19a) the tangency condition is found to be

$$\beta\bar{\varphi}_t=[\epsilon-\alpha(1+\epsilon^2)\cos\theta](1+\bar{\varphi}-t\bar{\varphi}_t) \qquad \text{at } t=T \tag{118}$$

Substituting from equation (117) and expressing values on the cone in terms of their values at $t=\beta\epsilon$ by means of Taylor expansions, it is found that

$$A=\frac{\epsilon^2}{\sqrt{1-\beta^2\epsilon^2}+\epsilon^2\operatorname{sech}^{-1}\beta\epsilon} \tag{119a}$$

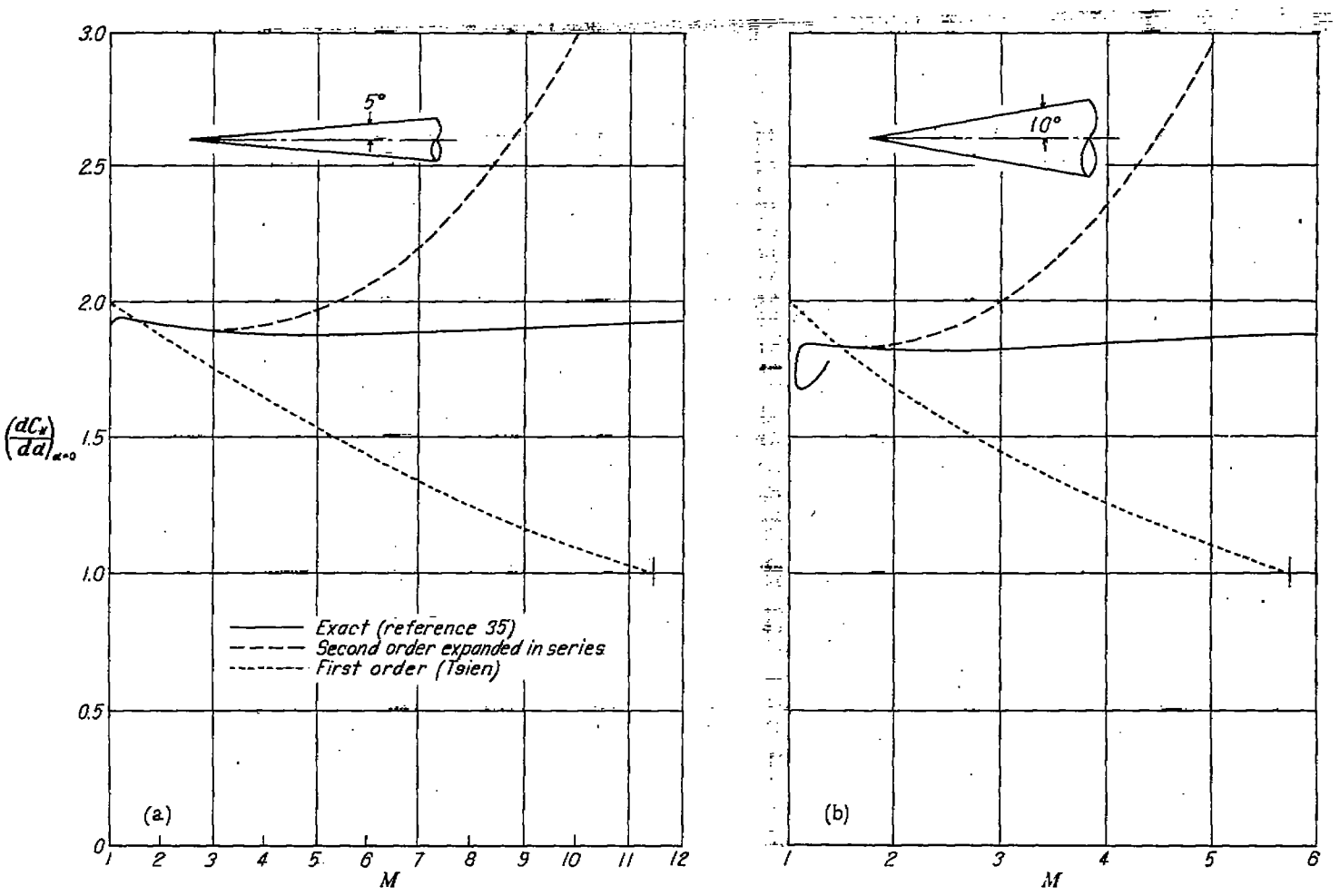
$$C=\beta\epsilon^2(1+\epsilon^2)\frac{1+A\left(\frac{1+\epsilon^2}{\epsilon^2\sqrt{1-\beta^2\epsilon^2}}-\operatorname{sech}^{-1}\beta\epsilon\right)}{(1+2\epsilon^2)\sqrt{1-\beta^2\epsilon^2}+\beta^2\epsilon^2\operatorname{sech}^{-1}\beta\epsilon} \tag{119b}$$

According to equation (110), a partial particular integral of the second-order iteration equation is, in conical form,

$$\bar{\psi}_*=M^2\bar{\varphi}(\bar{\varphi}-t\bar{\varphi}_t)=AM^2\{A\operatorname{sech}^{-1}t(\operatorname{sech}^{-1}t-\sqrt{1-t^2})-C\left[3\frac{\sqrt{1-t^2}\operatorname{sech}^{-1}t}{t}-2\frac{1-t^2}{t}-t(\operatorname{sech}^{-1}t)^2\right]\alpha\cos\theta\}=\bar{\psi}_0+\bar{\psi}_1\alpha\cos\theta \tag{120}$$

There remains to solve the reduced iteration equation given by equation (111), which becomes

$$(1-t^2)\bar{\phi}_{tt}+\frac{\bar{\phi}_t}{t}+\frac{\bar{\phi}_{\theta\theta}}{t^2}=AM^2\left[A\left(2N\frac{\operatorname{sech}^{-1}t}{\sqrt{1-t^2}}-\beta^2A\frac{\sqrt{1-t^2}}{t^4}\right)-4C\left(N\frac{\operatorname{sech}^{-1}t}{t\sqrt{1-t^2}}+N\frac{1}{t}-\beta^2A\frac{\sqrt{1-t^2}}{t^5}\right)\alpha\cos\theta\right] \tag{121}$$



(a) Semi-vertex angle 5°.

(b) Semi-vertex angle 10°.

FIGURE 19.—Comparison of various solutions for normal force on a cone.

This is reduced to two total differential equations by setting

$$\bar{\phi}(t, \theta)=\bar{\phi}_0(t)+\bar{\phi}_1(t)\alpha \cos \theta \tag{122}$$

Here $\bar{\phi}_0$ is associated with the axial component of the free stream, and $\bar{\phi}_1$ with the cross flow. The first of these is known from the previous solution for the symmetrical cone (equation (72)). The equation for the cross-flow term $\bar{\phi}_1$ is

$$(1-t^2)\bar{\phi}_1''+\frac{\bar{\phi}_1'}{t}-\frac{\bar{\phi}_1}{t^2}=-4ACM^2\left(N\frac{\operatorname{sech}^{-1}t}{t\sqrt{1-t^2}}+N\frac{1}{t}-\beta^2A\frac{\sqrt{1-t^2}}{t^5}\right) \tag{123}$$

Setting

$$\bar{\phi}_1(t)=t\omega(t) \tag{124}$$

reduces this to a linear first-order equation in ω which can be integrated to find that

$$\bar{\phi}_1=D\left(\frac{\sqrt{1-t^2}}{t}-t\operatorname{sech}^{-1}t\right)+ACM^2\left[3N\frac{1-t^2}{t}+Nt(\operatorname{sech}^{-1}t)^2+\frac{1}{2}\beta^2A\frac{(1-t^2)^{3/2}}{t^3}\right] \tag{125}$$

The tangency condition separates into the two conditions

$$\beta(\bar{\psi}_0'+\bar{\phi}_0')=\epsilon[1+(\bar{\psi}_0-t\bar{\psi}_0')+(\bar{\phi}_0-t\bar{\phi}_0')] \quad \text{at } t=\beta\epsilon \tag{126a}$$

$$\beta(\bar{\psi}_1'+\bar{\phi}_1')-\frac{\epsilon}{1+\epsilon^2}(\bar{\psi}_1+\bar{\phi}_1)=\beta^2(1+\epsilon^2)(\bar{\psi}_0''+\bar{\phi}_0'')-[1+(\bar{\psi}_0+\bar{\phi}_0)-\beta\epsilon(\bar{\psi}_0'+\bar{\phi}_0')] \tag{126b}$$

The first of these is the previous relation (equation (71b)) which determined the constant B in equation (72). Similarly, the second of these determines the constant D in equation (125).

SERIES EXPANSIONS FOR PRESSURE AND NORMAL FORCE ON INCLINED CONE

Numerical results have been calculated only for the case in which the solution is expanded in powers of t and $ln\ t$ (the slender-body approximation). Carrying out the expansion, the constant D is found to be

$$D=2\beta\epsilon^2\left[1+2M^2\epsilon^2 ln\frac{2}{\beta\epsilon}-\left(3M^2N+\frac{3}{2}\right)\epsilon^2+\ldots\right] \tag{127}$$

Then calculating the velocity components from equation (117) and the pressure from equation (27) gives, on the surface of the cone,

$$C_p=(C_p)_0-4\epsilon\left[1-\epsilon^2\left(M^2 ln\frac{2}{\beta\epsilon}-\frac{3}{2}M^2+1\right)\right]\alpha\cos\theta+\ldots \tag{128}$$

Here $(C_p)_0$ is the value for zero angle of attack, given by equation (108). Integrating gives the normal-force coefficient, based on cross-sectional area:

$$C_N=\frac{\text{normal force}}{\frac{1}{2}\rho_\infty U^2(\text{area})}=2\alpha\left[1-\epsilon^2\left(M^2 ln\,\frac{2}{\beta\epsilon}-\frac{3}{2}M^2+1\right)+\ \ldots\right] \tag{129}$$

This result has been obtained also by Lighthill (reference 33), who has calculated the lift on bodies of revolution having analytic meridian curves by assuming a series expansion for the velocity potential.

Stone (reference 34) has developed a solution for inclined cones which is linearized with respect to α, but otherwise exact. Kopal (reference 35) has published tables of the numerical results of Stone's theory. A comparison of equation (129) with this exact theory and with Tsien's first-order solution (reference 36) is shown in figure 19 for 5° and 10° cones. The earlier discussion of series expansions suggests that the agreement might improve if the solution were not expanded in series.

SHOCK-WAVE POSITION FOR INCLINED CONE[7]

Just behind the Mach cone the velocity components are

$$-\left(\frac{u}{U}\right)_{t=1}=\frac{1}{\beta}\left(\frac{v}{U}\right)_{t=1}=2M^2N\epsilon^4(1-8\beta\alpha\cos\theta) \tag{130}$$

For simplicity, using equations (119), A and C have here been approximated by ϵ^2 and $2\beta\epsilon^2$. Comparing equations (74) and (77), it is seen that if these were the velocities just behind the shock wave, then the difference between the shock-wave angle and the Mach angle would be

$$\lambda-\sin^{-1}\frac{1}{M}\doteq\beta M^2N^2\epsilon^4(1-8\beta\alpha\cos\theta) \tag{131}$$

Hence the ratio of the angular rotation of the shock wave to that of the cone would be

$$\frac{\delta}{\alpha}\doteq 8\beta^2M^2N^2\epsilon^4 \tag{132a}$$

Recently Lighthill has derived a simple expression for the shock-wave position for any conical body lying inside the Mach cone (reference 25). From his results it is found that the preceding relation is incorrect, the correct expression being again one and one-half times as large, so that

$$\frac{\delta}{\alpha}=3(\gamma+1)^2\frac{M^6}{\beta^2}\epsilon^4=12M^2\beta^2N^2\epsilon^4 \tag{132b}$$

For moderate cone angles this latter result agrees well with the exact values calculated by Kopal (reference 35) from Stone's theory. Again the discrepancy indicates failure of the second-order solution near the Mach cone.

EXTENSIONS OF THE THEORY

Two important classes of problems have only been touched upon here. One is wings; the other, inclined bodies of revolution. The problem of inclined bodies has recently been studied further in reference 37. The iteration equations were there rederived in body rather than wind axes, which simplifies the tangency condition. Again only a partial particular integral could be found.

The possibility of discovering particular integrals of the iteration equation might be investigated more systematically. If none can be found for general three-dimensional flow, special cases such as conical flow should be studied.

TREATMENT OF WINGS

One of the most useful applications of first-order theory is to thin flat wings. For conical wings, the reduced iteration problem can be transformed, by the standard conical theory (reference 18), into the problem of solving Poisson's equation inside a circle. This case has recently been studied by Moore (reference 38), who has calculated results for the nonlifting wing of triangular plan form lying inside the Mach cone and of symmetrical wedge section. Although the two extremes of the plane airfoil and the slender cone show that compressive pressures are reinforced in the second approximation, Moore calculates a reduction for an intermediate sweptback case.

Two difficulties can be anticipated. First, if the wing has subsonic edges, infinite velocities arise there, so that the assumption of small perturbations is violated. It is known that in first-order theory this is no essential objection, since the pressure is found correctly except in the immediate neighborhood of the singularity, and the integrated values of lift and moment are correct to first order. Schmieden and Kawalki (reference 14) and Kaplan (reference 13) have indicated that this result extends to the second approximation for subsonic flow, so that probably no real difficulty exists.

Secondly, if the wing has supersonic edges, the failure of the iteration process along Mach lines from the apex can be expected to affect the surface pressures. Again it is possible that integrated values will be correct to second order. Otherwise, it may be possible to adjust the solution in those regions, in a manner similar to that shown in figure 5.

HIGHER APPROXIMATIONS

It seems unlikely that third or higher approximation would ever be justified. Other neglected factors, chiefly viscosity and heat conduction, should ordinarily be considered first. However, the Busemann second-order result has been extended to third and even fourth order (reference 39), and various writers have considered the third approximation for plane subsonic flow (references 7, 9, and 11). If a third approximation should be considered worthwhile, the iteration could be repeated. Again the cases of flow past a curved wall and a cone would serve as helpful examples.

APPLICATION TO SUBSONIC FLOW

The iteration equation and the particular integrals are in no way restricted to supersonic flow. The particular solution for plane flow might profitably be compared with the subsonic solutions of references 5 to 4.

Recently Kaplan has shown (reference 40) that the particular integral for plane flow (equation (35)) can be derived formally, using complex variable theory, and has similarly obtained the third-order particular integral. The latter is

[7]These relations were given incorrectly in references 1 and 2.

not applicable to supersonic flow except in special cases which are free of shock waves.

The particular integral for plane flow has recently been applied to several subsonic problems by Harder and Klunker (references 41 and 42).

The particular solution for axially symmetric flow makes possible a second-order solution for bodies of revolution at subsonic speed. In this case, the integral equation can be treated by the methods used for the airship problem.

AMES AERONAUTICAL LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
MOFFETT FIELD, CALIFORNIA, *April 2, 1952*

APPENDIX

SYMBOLS

a	constant reference radius for body of revolution
b	abscissa at which source distribution for body of revolution begins
A, B, C, D	constants determined by boundary conditions
c	local speed of sound
C_n	constant coefficients of series
C_N	normal-force coefficient
C_p	pressure coefficient
E	complete elliptic integral of the second kind with modulus $k=[(1-t)/(1+t)]^{1/2}$
$f(x), F(x)$	source-distribution functions for body of revolution
$F_n(x, y, z)$	known right-hand side of $(n+1)$ st-order iteration equation
$g(x)$	continuous function of order unity which vanishes for $x=0$
h, j, H, J	arbitrary functions of one variable
K	complete elliptic integral of the first kind with modulus $k=[(1-t)/(1+t)]^{1/2}$
M	free-stream Mach number $\left(\frac{U}{c_\infty}\right)$
N	$\frac{\gamma+1}{2}\frac{M^2}{\beta^2}$
p	local static pressure
P_n	points on body of revolution at which tangency condition is imposed
q	local speed of flow
r	radius in cylindrical coordinates
R	radius of meridian curve of body of revolution
S	function defining surface of body
t	conical variable $\left(\frac{\beta r}{x}\right)$
T	value of t on surface of body
u, v, w	perturbation velocity components in Cartesian or cylindrical coordinates
U	free-stream velocity
x, y, z	Cartesian coordinates with x in free-stream direction
Y	value of y on surface of planar body
α	angle of attack
β	$\sqrt{M^2-1}$
γ	adiabatic exponent
δ	angular rotation of shock wave on cone due to angle of attack
ϵ	parameter small compared with unity
θ	azimuthal variable in cylindrical coordinates
λ	angle of shock wave from free-stream direction
ξ_n	abscissa of origin of nth source line
ρ	local density
τ_n	conical variable referred to $x=\xi_n$ as origin rather than $x=0$
φ	first-order (linearized) perturbation potential, same as $\Phi^{(1)}$
ϕ	second-order perturbation potential, same as $\Phi^{(2)}$
Φ	exact pertubation potential
$\Phi^{(n)}$	nth-order perturbation potential
$\overset{(n)}{\Phi}$	nth term in series expansion of perturbation potential
χ	complementary function for second-order iteration equation
ψ	particular integral of second-order iteration equation
ψ_*	partial particular solution for three-dimensional flow (See equation (110).)
Ω	complete velocity potential
ω	auxiliary variable (See equation (124).)

SUPERSCRIPTS

(n)	result of nth-order solution
—	value in conical form (See equation (15).)

SUBSCRIPTS

0	associated with axial flow
1	associated with cross flow
∞	free-stream conditions
a	value just ahead of corner
b	value just behind corner
c	differentiation in direction of cross-wind component of normal to body surface

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