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TECHNICAL NOTE 2647

IMPLICATION OF THE TRANSPORT EQUATION FOR THE SEMIEMPIRICAL TREATMENT OF SHIELDS

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SUMMARY

The semiempirical method of dealing with shields by treating them as composed of layers is revised and extended by taking the angular distribution of the radiation into account and by making use of the transport equation. It is shown that breaking up the ranges of direction and energy of radiation into finite intervals is appropriate where multiple scattering with angular deviation does not dominate, and a procedure is given for using data referring to very thin layers to calculate the parameters for finite layers in this case. In the contrary case where multiple elastic scattering is dominant, it is shown that the number of unknowns to be considered is decreased if the angular distribution is represented by an appropriate expansion in terms of Legendre polynomials and the Laplace transformation of the coefficients is taken with respect to energy. The method is illustrated by consideration of the effect upon neutrons of a thick nonabsorbing shield of high atomic weight. The physical significance of the new variables thus introduced is determined.

INTRODUCTION

Shielding is accomplished by interposing a material barrier between a source of radiation and the region to be protected. In passing through the shield, the particles making up the radiation (which may contain particles of zero mass such as photons) undergo collisions with the atoms of the shield. In a given collision, a particle may lose some portion of its energy while simultaneously changing direction and may also generate new particles. The probability of a particular type of collision depends upon the energy of the particle before the collision and, hence, upon its past history; the number of collisions a particle undergoes within the shield depends upon its length of path in the shield and, hence, on its original energy and the geometry of its trajectory. In general, therefore, the particles leaving the shield will be of several types and will cover a large range of energies and directions. The complexity of the problem makes the solution of the mathematical equation governing the travel of the particles through the

shield (the transport equation) quite difficult. However, the large number of possible processes makes the use of composite shields seem particularly promising. For example, the first section of a shield might slow the particles down sufficiently that the second section could absorb them. This means that to find the most efficient shield for a given source many combinations of materials must be considered. Because of the difficulties of calculation, direct measurement seems the proper procedure for comparing these combinations, but to try all the possibilities would require a large volume of experimentation.

In order to avoid both the need for elaborate computation and for large-scale experimentation, Bobrowsky (reference 1) proposed a method (herein designated as semiempirical) for predicting the effectiveness of composite shields on the basis of actual measurements made upon a relatively small number of homogeneous shields. The basic idea presented in reference 1 is to treat the shield the effect of which is to be computed as being built up of a number of layers or elements upon each of which the required measurements of shielding effectiveness have been made. The results of such measurements serve to relate the distribution in direction, energy, and surface position of the particles leaving any element to the distribution in direction, energy, and surface position of those entering the element. If the individual elements are combined to form a composite shield, the same relations as before hold between the distribution of particles entering a particular element and the distribution of those leaving that element. Except for the front face of the first element and the back face of the last element, however, the incident fluxes are unknown. Thus, in the case of n elements and p species of particles, there are $2np$ unknown functions describing the emergent fluxes, one for each species of particle at each of the two faces of each element. When these fluxes are connected with one another and with the incident flux, there result $2np$ integral equations which are, however, of a relatively simple structure.

Briefly, the procedure which has been used by Bobrowsky is to approximate the integrals in question by appropriate sums, usually with the additional assumption that the incident flux is independent of surface position. The intensity is thus no longer expressed in terms of a function of direction and energy but instead is expressed in terms of a set of numbers, each of which is the average value of the intensity within the appropriate range of energy and of angle. (Actually, Bobrowsky suppressed the angular dependence by assuming all particles to travel in a single direction.) Thus, the set of integral equations in the unknown intensity functions is replaced by a set of ordinary algebraic equations in the intensity numbers. The solution of these can be accomplished by ordinary methods. In practice, however, the computation may be quite tedious because of the large number of unknowns involved; in addition, the amount of experimental data required

may still be excessive. This is especially the case if considerable accuracy is desired and, consequently, the intervals of angle and energy are narrow.

The present work, which was conducted at the NACA Lewis laboratory, is directed toward extending and simplifying the semiempirical method. The first step toward accomplishing these objectives is the determination of approximate solutions of the transport equation for two special conditions: first, that in which multiple elastic scattering dominates; and second, that in which the role of multiple scattering with change in direction is small compared with other effects. The results obtained serve three purposes: (1) They demonstrate the importance of taking into account the direction of motion of the particles; (2) they provide a means for calculating the effect of any element which is not too thick and in which multiple elastic scattering is not too important an effect, from the results of measurements on an infinitesimal element of the same material; and (3) they suggest a way of simplifying the computations necessary in the application of the semiempirical method.

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In order to generalize this method to include the effects of angular dependence and to achieve the simplification mentioned, the integral equations displaying the angular dependence of the intensity are first written down. These equations are to be treated by approximating the integrals by sums and solving the resulting algebraic equations. This replacement may be accomplished by writing each unknown intensity function as the sum of a series of terms, each term being the product of an unknown constant coefficient and a known function of the appropriate variables. If this representation is substituted into the original equations, the integrations may be performed at once, and a set of equations in the unknown coefficients results. Knowledge of the approximate solutions of the transport equation makes it possible to choose the known functions in such a way that all but a few of the terms in each of the equations so obtained are small; the resulting equations are therefore amenable to solution by successive approximation. After the coefficients are obtained, the intensities may be found by summing the appropriate series. It is shown, however, that the coefficients themselves contain information of considerable physical significance; for some purposes, it may be sufficient to know the coefficients in order to estimate the usefulness of the shield.

ANALYSIS

Assumptions

In each case it is assumed that the following conditions hold:

(1) The elements are homogeneous slabs of finite thickness with infinite plane parallel surfaces.

(2) No new particles are produced by collisions.

(3) The system is in a steady state.

(4) The incident radiation is uniform over the surface and independent of the angle measured around the normal but not necessarily independent of the angle it makes with this normal.

(5) All sources are external to the elements.

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Conditions (1) and (4) are essential to the type of analysis employed here because they represent restrictions on the geometrical aspects of the problem which make the resulting system of equations relatively tractable. The requirement with regard to angular distribution contained in condition (4) is fulfilled if the sources are disposed in a uniform manner with respect to the shield. The assumptions contained in conditions (2), (3), and (5) are not essential but are introduced primarily to avoid complications in the analysis which would obscure the ideas presented herein. In particular, reference 1 deals with the case in which condition (2) does not hold and reference 2 deals with the case in which conditions (2), (3), and (5) do not hold.

The Individual Element

The procedure to be followed in simplifying the application of the semiempirical method is to determine the way in which the particles traverse a single element and to deduce from the results obtained an appropriate representation for use in this method. More explicitly, the representation sought is in each case to be obtained from an approximate solution of the transport equation valid under certain general assumptions about the nature of the element. The two cases to be considered are that in which scattering dominates and an expansion in terms of Legendre polynomials is appropriate (approach 1) and that in which absorption dominates and iteration is appropriate (approach 2). It is to be emphasized that the expansions thus obtained are applicable in general but are most convenient under the circumstances stated.

Approach 1: Expansion in Legendre polynomials. - In this section the transport equation is transformed by expressing the dependence on energy in terms of a new variable η by a Laplace transformation and by representing the angular dependence in terms of an expansion in Legendre polynomials. It will be shown that for elastic scattering the introduction of η simplifies the equations considerably, that the use of Legendre polynomials is well adapted to the boundary conditions, and that in the case of multiple scattering only a few terms of the expansion need be retained. The need for including angular dependence also becomes obvious.

The starting point of this discussion is the form of the transport equation given by Marshak (reference 3, equation (63); see also appendix B of this report) here specialized to the time-independent and surface-position-independent case with no internal sources:

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$$\lambda(u)\mu \frac{\partial\psi(u,\mu,z)}{\partial z} + \psi(u,\mu,z) =$$

$$\int_0^u du' \int d\Omega'\psi(u',\mu',z)\ f(\mu_0,u-u')\ h(u') \tag{1}$$

All symbols are defined in appendix A.

Because ψ does not depend upon φ, the right side of equation (1) can be integrated over φ' immediately so that it becomes

$$\int_0^u du' \int_{-1}^{1} d\mu' f_1(\mu,\mu',u-u')\psi(u',\mu',z)h(u') \tag{2}$$

where

$$f_1(\mu,\mu',u-u') = \int_0^{2\pi} f(\mu_0,u-u')\ d\varphi' \tag{3}$$

Equation (1) may be further simplified by taking its Laplace transform with respect to u. If, for purposes of simplicity, it is assumed that $\lambda(u)$ and $h(u)$ are constant, then it is possible to put $\lambda(u)=1$ (this amounts to a change in the choice of units for λ constant) and $h(u)=1$ (this amounts to a change in the way f is normalized for constant h); with these changes the equation becomes (see reference 3, equation (17))

$$\mu \frac{\partial\Phi(z,\eta,\mu)}{\partial z} + \Phi\ (z,\eta,\mu) = \int_{-1}^{1} d\mu' F(\mu,\mu',\eta)\ \Phi\ (z,\eta,\mu') \tag{4}$$

where Φ and F are defined by

$$\Phi(z,\eta,\mu) = \int_0^\infty du\ e^{-\eta u}\psi(u,\mu,z) \tag{5}$$

and

$$F(\mu,\mu',\eta) = \int_0^\infty du\ e^{-\eta u}\ f_1(\mu,\mu',u') \tag{6}$$

respectively. In equation (4) the integration over u' has been performed by making use of the fact that u' enters f_1 only through $u-u'$. The extension to the case where λ and h are not constant is immediate (reference 3, equation (80)). It is only necessary to expand each of these quantities in a power series in E: 2386

$$\lambda = \sum_j \lambda_j E^j = \sum_j \Lambda_j\ e^{-ju} \tag{7}$$

$$h = \sum_j h_j\ E^j = \sum_j H_j\ e^{-ju} \tag{8}$$

where the quantities Λ_j and H_j are constants and the j's represent some set of numbers of either sign which may be, but are not necessarily, integers. These expansions need contain only a few terms if λ and h do not vary too fast with u. The representation given is not very adequate if resonance is marked; but the error is not serious for a narrow resonance band because the chance of a particle attaining an energy within such a band is small. If equations (7) and (8) are substituted in equation (1) and the Laplace transform is obtained, equation (1) becomes

$$\mu \sum_j \Lambda_j \frac{\partial\Phi(z,\eta+j,\mu)}{\partial z} + \Phi\ (z,\eta,\mu) =$$

$$\sum_j H_j \int_{-1}^{1} d\mu'\ F(\mu,\mu',\eta)\ \Phi\ (z,\eta+j,\mu') \tag{9}$$

It is to be emphasized that the validity of equation (9) depends upon the assumption that u and u' enter f_1 only through the combination $u-u'$. This restriction on f_1, while not necessarily always holding, is correct in many cases of physical significance (reference 3, p. 186), for example, whenever the scattering can be represented as a classical collision between the incident particle and a free particle at rest. The same result holds in quantum mechanics for the case of

so-called S-wave scattering, that is, the case when the "wavelength" of the incident particle is great compared with the range of the scattering force. In the general case, the function f_1 can be written in the form

$$f_1 = f_1(u-u',u') \tag{10}$$

so that the treatment of this case can be extended by expanding f_1 in a series in $e^{-ju'}$ just as was done in the case where $h(u)$ was not constant.

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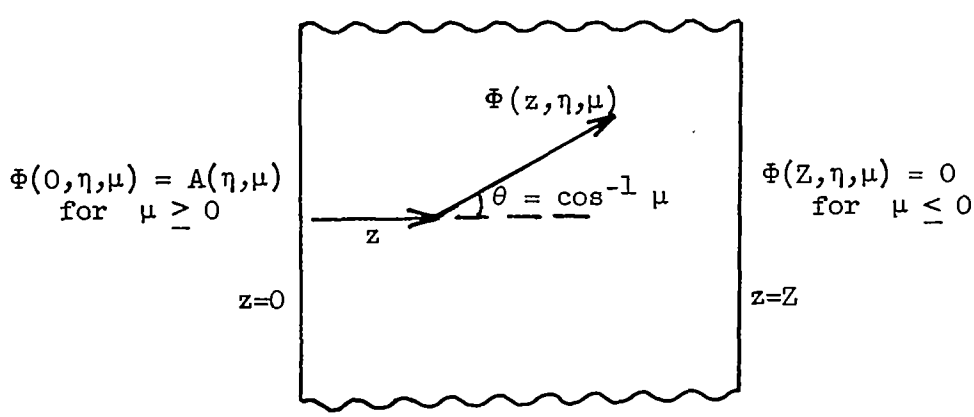
In order to complete the formulation of the problem, it is necessary to specify the boundary conditions. This may be done quite simply as follows. Let $z=0$ represent the face upon which the radiation is incident and let $z=Z$ represent the opposite face. Then the conditions in question are

$$\left.\begin{aligned} \Phi(z,\eta,\mu)\Big|_{z=0} &= A(\eta,\mu) \qquad \mu \geq 0 \\ \Phi(z,\eta,\mu)\Big|_{z=Z} &= 0 \qquad \mu \leq 0 \end{aligned}\right\} \tag{11}$$

The quantity $A(\eta,\mu)$ is here an arbitrary known function of the appropriate variables. Physically, these equations mean that the flux incident upon the front face of the element is known while the flux incident upon the back face is zero. An individual element of the shield is shown in the following diagram:



Equations (4) as they stand are still integral equations with respect to μ as well as differential equations with respect to z. It is desirable to transform from this set of integro-differential equations to a (infinite) set of ordinary linear differential equations. This can be simply done if Φ and F are expanded in terms of complete set of orthogonal functions of μ. Marshak (reference 3, equation (65a)) used for this purpose the set of Legendre polynomials of μ. For the present problem this is inconvenient because the boundary conditions (equation (11)) impose a sharp distinction between forward $(\mu > 0)$ and backward $(\mu < 0)$ directions. It is therefore natural to make use rather of an expansion in which this dichotomy is reflected in the expansion of the function Φ in terms of the variable μ. A procedure in which this is accomplished and in which some of the advantages of the use of the Legendre polynomials are retained is based on the use of the representation:

$$\Phi(z,\eta,\mu) = \begin{cases} \sum_{n=0}^{\infty} (2n+1)\varphi_n^+(\eta,z)P_n(2\mu-1) & \mu > 0 \\ \sum_{n=0}^{\infty} (2n+1)\varphi_n^-(\eta,z)P(-2\mu-1) & \mu < 0 \end{cases} \tag{12}$$

The + and - superscripts on the expansion coefficients φ indicate, of course, that they are associated, respectively, with positive and negative values of μ and hence with forward and backward directions. The coefficients have thus a significance different from that to be associated with the usual expansion in terms of Legendre polynomials (reference 4). It is to be emphasized, however, that the number of nodes of the Legendre polynomial of degree n increases as n increases; and, hence; the larger is n, the greater is the degree of angular assymetry to be associated with φ_n. It will be advantageous for future work to define

$$P_n^+(2\mu-1) = \begin{cases} P_n(2\mu-1) & 0 \leq \mu \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{13}$$

and

$$P_n^-(-2\mu-1) = \begin{cases} P_n(-2\mu-1) & -1 \leq \mu \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

With this definition Φ may be written

$$\Phi = \sum_{n=0}^{\infty} \left\{(2n+1)\left[\varphi_n^+(\eta,z)P_n^+(2\mu-1) + \varphi_n^-(\eta,z)P_n^-(-2\mu-1)\right]\right\} \tag{14}$$

It is to be noted that two P's for which the n's and the superscripts are not both identical are orthogonal. This is one of the chief advantages of the expansion. Because of the isotropy of the element, it is assumed that f depends on θ and θ' only through $|\mu_0|$ and, therefore, F may be written in terms of these functions as

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$$F = \sum_{n,n'} (2n+1) \left\{f_{nn'}^{++}\left[P_n^+(2\mu-1)P_{n'}^+(2\mu'-1) + P_n^-(-2\mu-1)P_{n'}^-(-2\mu-1)\right] + f_{nn'}^{+-}\left[P_n^+(2\mu-1)P_{n'}^-(-2\mu'-1) + P_n^-(-2\mu-1)P_{n'}^+(2\mu'-1)\right]\right\} \tag{15}$$

This equation serves to define the expansion coefficients f. (These coefficients have the property that $f_{n,n'} = f_{n',n}$.) When equations (14) and (15) are substituted into the right side of equation (4), there results

$$\mu \frac{\partial\Phi}{\partial z} + \Phi = \sum_{n,n'} (2n+1) \left\{f_{nn'}^{++}\left[\varphi_{n'}^+P_n^+(2\mu-1) + \varphi_{n'}^-P_n^-(-2\mu-1)\right] + f_{nn'}^{+-}\left[\varphi_{n'}^+P_n^-(-2\mu-1) + \varphi_{n'}^-P_n^+(2\mu-1)\right]\right\} \tag{16}$$

where use is made again of the properties of the Legendre polynomials (see reference 4). If equation (16) is multiplied by $P_n^{\pm}(\pm 2\mu-1)$ and the resulting equation integrated over the range $-1 \leq \mu \leq 1$, the following set of differential equations is obtained:

$$\left.\begin{aligned}
&\pm \frac{1}{2}\frac{\partial\varphi_0^{\pm}}{\partial z} \pm \frac{1}{2}\frac{\partial\varphi_1^{\pm}}{\partial z} + \varphi_0^{\pm} = \sum_m \left[f_{0,m}^{++}\varphi_m^{\pm} + f_{0,m}^{+-}\varphi_m^{\mp}\right] \\
&\pm \frac{1}{2}\frac{\partial\varphi_n^{\pm}}{\partial z} \pm \frac{1}{2}\frac{n+1}{2n+1}\frac{\partial\varphi_{n+1}^{\pm}}{\partial z} \pm \frac{1}{2}\frac{n}{2n+1}\frac{\partial\varphi_{n-1}^{\pm}}{\partial z} + \varphi_n^{\pm} \\
&\qquad = \sum_m \left[f_{nm}^{++}\varphi_m^{\pm} + f_{nm}^{+-}\varphi_m^{\mp}\right] \qquad n > 0
\end{aligned}\right\} \tag{17}$$

To this set are to be appended the boundary conditions which here take the simple form

$$\left.\begin{aligned} \varphi_n^+\Big|_{z=0} &= A_n(\eta) \\ \varphi_n^-\Big|_{z=Z} &= 0 \end{aligned}\right\} \tag{18}$$

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where Z is the thickness of the element and

$$A_n(\eta) = \int_0^1 P_n^+(2\mu-1)\ \Phi\ (z,\eta,\mu)\Big|_{z=0}\ d\mu$$

From the viewpoint of shielding, the problem is to find $\varphi_n^-\Big|_{z=0}$ and $\varphi_n^+\Big|_{z=Z}$ in terms of the $A_n(\eta)$. Generally speaking, for large Z one may expect these quantities to decrease as n increases in the case where there is little absorption because multiple scattering enhances the symmetry. A simple illustration of this is given by the following example:

Example of approach 1: Scattering of neutrons by a thick "heavy shield" without absorption. - When a neutron is scattered by an element of relatively high atomic weight, the neutron loses very little energy and the angular distribution of scattered neutrons is nearly uniform. As an approximation, the change in energy may be taken as zero and the angular distribution as uniform so that F is constant. As it is assumed that there is no absorption, F is normalized in such a way that

$$\int_{-1}^{1} F\ d\mu = 1 \tag{19}$$

(see equations (3) and (6) and equation (A1) in appendix A). Thus it follows that, to the approximation taken here, $F = \frac{1}{2}$. With the consistent neglect of all φ_n's for which n exceeds 1, the differential equations (17) for φ become

$$\left.\begin{aligned} \pm\frac{1}{2}\frac{\partial\varphi_1^\pm}{\partial z} \pm \frac{1}{2}\frac{\partial\varphi_0^\pm}{\partial z} + \varphi_0^\pm &= \frac{1}{2}\varphi_0^\pm + \frac{1}{2}\varphi_0^\mp \\ \pm\frac{1}{3}\frac{\partial\varphi_0^\pm}{\partial z} \pm \frac{\partial\varphi_1^\pm}{\partial z} + 2\varphi_1^\pm &= 0 \end{aligned}\right\} \tag{20}$$

The boundary conditions are

$$\left.\begin{aligned} \varphi_0^-\Big|_{z=Z} &= \varphi_1^-\Big|_{z=Z} = 0 \\ \varphi_0^+\Big|_{z=0} &= A_0 \\ \varphi_1^+\Big|_{z=0} &= A_1 \end{aligned}\right\} \tag{21}$$

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The dependence on η is suppressed because the value of φ_n for a particular η is not related to its value for any other η. These equations, being a set of linear differential equations with constant coefficients, may be solved by the usual methods. As remarked previously, the feature of interest here is the way in which the element imposes angular uniformity upon the incident radiation when Z is large compared with 1 (that is, with λ). This effect is demonstrated in the following approximation to the solution in which all but leading terms are omitted. With such an approximation, the result may be written as

$$\left.\begin{aligned} \varphi_0^-\Big|_{z=0} &= A_0 - (6-4\sqrt{3})\,A_1 \\ \varphi_1^-\Big|_{z=0} &= (7-4\sqrt{3})\,A_1 \\ \varphi_0^+\Big|_{z=Z} &= 0 \\ \varphi_1^+\Big|_{z=Z} &= 0 \end{aligned}\right\} \tag{22}$$

The omitted terms are of the order of A_0 (or A_1) multiplied by $1/Z$. Because Φ must be positive at $z=0$, it follows from equation (14) that A_1 must be less than one-third A_0. If it is noted that $7-4\sqrt{3}$ is about 0.1 and $-(6-4\sqrt{3})$ is about 0.9, it follows that $\varphi_1^-\Big|_{z=0}$ is considerably smaller than $\varphi_0^-\Big|_{z=0}$. Because φ_0^- represents a uniform emergent flux and φ_1^- is associated with radiation displaying a certain degree of asymmetry, this confirms the assertion that the shield acts to promote the uniformity of the radiation. If the next stage of the calculation performed is employed by assuming that $\varphi_n = 0$ for $n > 3$ and substituting the results of the present approximation into the equations thus obtained for φ_2 and φ_3, it follows that, in accord with the original approximation, these last are also small.

The requirement that there be no absorption is reflected in the value of F. In particular, if the absorption were not zero, F would

be less than 1/2 and φ_0^- would be decreased in magnitude relative to the asymmetric terms whose importance would therefore increase. In addition, for an element the constituents of which are of lower atomic weight, the coefficients $f_{nn'}$ do not vanish for n' greater than zero and they also vary with η. The effect of this dependence is to increase the number of scatterings required to produce the same amount of uniformity.

Under the heading of approach 1, a procedure has been developed for dealing with an element in which scattering is the dominant effect. In particular, it has been assumed that λ and h are constant, or nearly so, and that the scattering process is of the S-wave type. Under these circumstances it follows that an appropriate representation is in terms of the intensity numbers $\varphi_n^{\pm}(\eta)$. The expansion thus obtained takes account of the sharp distinction between the forward and backward directions, gives explicit recognition to the expected effects of scattering in producing angular uniformity by segregating terms implying different degrees of angular asymmetry, and takes advantage of the simplification resulting from the use of the Laplace transformation. 2386

Approach 2: Iteration. - Although the expansion discussed in approach 1 may be used in any problem, its practical application is primarily found in those cases where there is elastic scattering and where the element imposes a certain degree of angular uniformity upon the emergent radiation. In the contrary case where the emergent radiation retains essentially the nonuniformity of the incident flux or where such nonuniformity is imposed, it is convenient to use a formalism in which the intensity is expressed directly as a function of the angle and the energy.

This will be the case if multiple scattering with change in direction is not an important process. The procedure used here is an iterative one which again shows the importance of including angular dependence and which permits direct solution of the transport equation for a single element under the circumstances indicated.

The relevant transport equation with no scattering leading to directional changes is (see appendix B)

$$\mu \frac{\partial\psi(E)}{\partial z} + Q(E)\ \psi(E) = \int q_1(E,E')\ \psi(E')\ dE' \tag{23}$$

Here Q is $1/\lambda$ and $q_1(E,E')\ dE$ is the proportion of radiation of energy E' degraded into the range between E and $E+dE$. The function q_1 vanishes for $E > E'$ (no gain of energy). It may be noted that, as this equation stands, μ enters only as a parameter. If ψ is written as

$$\psi(E,\mu,z) = \int_0^\infty T(E,z,E')\ \psi(E',\mu,0)\ dE' \tag{24}$$

equation (23) is satisfied if T fulfills the boundary condition $T(E,Q,E') = \delta(E-E')$ and the equation

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$$\mu\ \frac{\partial T(E,z,E')}{\partial z} + Q(E)\ T(E,z,E') = \int q_1(E,E'')\ T(E'',\ z,E')\ dE'' \tag{25}$$

E' and μ are parameters in this equation.

An approximate solution of equation (25) may now be obtained by the use of iteration. This approach is especially appropriate because it includes as a special case the method of breaking the energy range into intervals and replacing the integral on the right side by a sum essentially employed in reference 1. In order to carry out the iteration, T is expressed in the form

$$T = \sum_{j=0}^{\infty} T^{(j)} \tag{26}$$

where $T^{(j)}$ satisfies the boundary condition

$$T^{(j)}\ (E,0,E') = \delta_{j0}\ \delta(E-E') \tag{27}$$

and represents the contribution to T of particles that have started out with energy E' and have been scattered j times. It follows that $T^{(j)}$ is of the order of $q_1{}^j$ because the latter is essentially the probability of j scatterings. Consequently, if equation (26) is substituted into equation (27) and terms of the same order in q_1 are equated, there results

$$\left.\begin{aligned} &\mu\ \frac{\partial T^{(0)}(E,z,E')}{\partial z} + Q(E)\ T^{(0)}(E,z,E') = 0 \\ &\mu\ \frac{\partial T^{(j)}(E,z,E')}{\partial z} + Q(E)\ T^{(j)}(E,z,E') = \int_0^{E_{max}} q_1(E,E'')\ T^{(j-1)}(E'',z,E')\ dE'' \\ &j \neq 0 \end{aligned}\right\} \tag{28}$$

the solution of which, subject to the boundary condition (27), is (appendix C)

$$\left.\begin{aligned}
T^{(0)}(E,z,E') &= \delta(E-E')\, e^{-Q(e)\frac{z}{\mu}} \\
T^{(1)}(E,z,E') &= \frac{q_1(E,E')}{Q(E)-Q(E')}\left[e^{-Q(E')\frac{z}{\mu}} - e^{-Q(E)\frac{z}{\mu}}\right] \\
T^{(j)}(E^{(0)},z,E^{(j)}) &= \underbrace{\int\cdots\int}_{j-1} \prod_{s=0}^{j-1} q_1(E^{(s+1)},E^{(s)})\,\mathrm{X} \\
&\quad \left\{\sum_{e=0}^{j} \prod_{\substack{r=0\\ r\neq e}}^{j} \frac{e^{-Q(E^{(e)})z/\mu}}{Q(E^{(r)})-Q(E^{(e)})} \prod_{k=1}^{j-1} dE^{(k)}\right\} \quad j > 1
\end{aligned}\right\} \tag{29}$$

In case two of the Q's are equal, the expression is to be evaluated by initially treating them as distinct, grouping all terms in which their difference is contained in the denominator, and then taking the limit. In case the set of energies which the particle can lose upon scattering is discrete or in case the integral is approximated by a sum, the function q_1 can be represented as a sum of δ functions. If this sum contains t terms, then the series for T breaks off after $T^{(t)}$. (This depends upon the fact that $q_1(E,E')$ is zero for E larger than E'.)

A variation of the preceding treatment can be applied to the case where there exists a scattering process which produces a change in both direction and energy as well as the original process which affects only the energy. In this case the differential equation may be written (appendix B)

$$\mu \frac{\partial\psi(E,\mu,z)}{\partial z} + Q(E)\,\psi(E,\mu,z) = \int_0^{E_{max}} q_1(E,E')\,\psi(E',\mu,z)\,dE' + \int_{-1}^{1}\int_0^{E_{max}} q_2(E,\mu;E',\mu')\,\psi(E',\mu',z)\,dE'\,d\mu' \tag{30}$$

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where $q_2(E,\mu;E',\mu')\,dE\,d\mu$ is the proportion of particles with energy E' and direction μ' scattered as a result of the second process so as to have energies in the range between E and $E+dE$ and directions in the range between μ and $\mu+d\mu$. Just as was done in equation (24), ψ may be expressed in the form

$$\psi(E,\mu,z) = \iint \overline{T}(E,\mu,z;E',\mu')\psi(E',\mu',0)\,dE'\,d\mu' \tag{31}$$

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where $\overline{T}$ satisfies the boundary conditions

$$\begin{aligned} \overline{T}(E,\mu,0;E',\mu') &= \delta(\mu-\mu')\,\delta(E-E') \qquad \mu > 0 \\ \overline{T}(E,\mu,Z;E',\mu') &= 0 \qquad \mu < 0 \end{aligned} \tag{32}$$

and the equation

$$\mu\,\frac{\partial\overline{T}(E,\mu,z;E',\mu')}{\partial z} + Q(E)\,\overline{T}(E,\mu,z;E',\mu') =$$

$$\int q_1(E,E'')\,\overline{T}(E'',\mu,z;E',\mu')\,dE'' + \iint q_2(E,\mu,E'',\mu'')\,\overline{T}(E'',\mu'',z;E',\mu')\,dE''\,d\mu'' \tag{33}$$

If the second integral on the right side of equation (33) is small in comparison with either the second term on the left side of equation (33) or the first integral on the right side of equation (33), then the approximate solution (equation (29)) obtained for equation (25) may be adapted for use as the first approximation for $\overline{T}$ by applying the boundary conditions (32). The resulting function, which may be denoted as $\overline{T}^{(0)}(E,\mu,z;E',\mu')$, satisfies

$$\mu\,\frac{\partial\overline{T}^{(0)}(E,\mu,z;E',\mu')}{\partial z} + Q(E)\,\overline{T}^{(0)}(E,\mu,z;E',\mu') =$$

$$\int q_1(E,E'')\,\overline{T}^{(0)}(E'',\mu,z;E',\mu')\,dE'' \tag{34}$$

and the boundary conditions

$$\begin{aligned} \overline{T}^{(0)}(E'',\mu,0;E',\mu') &= \delta(\mu-\mu')\,\delta(E-E') \qquad \mu > 0 \\ \overline{T}^{(0)}(E'',\mu,Z;E',\mu) &= 0 \qquad \mu < 0 \end{aligned}$$

A better approximation is obtained by putting

$$\overline{T} = \overline{T}^{(0)} + \overline{T}^{(1)} \tag{35}$$

where $\overline{T}^{(0)}$ is as before and $\overline{T}^{(1)}$ satisfies the conditions

$$\overline{T}^{(1)}(E,\mu,0;E';\mu') = 0 \qquad \mu > 0$$

$$\overline{T}^{(1)}(E,\mu,Z;E',\mu') = 0 \qquad \mu < 0$$

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If equation (35) is substituted into equation (33) and if the product $q_2\overline{T}^{(1)}$ is neglected in comparison with $q_2\overline{T}^{(0)}$, the following equation is obtained for $\overline{T}^{(1)}$:

$$\mu \frac{\partial\overline{T}^{(1)}(E,\mu,z;E',\mu')}{\partial z} + Q(E)\ \overline{T}^{(1)}(E,\mu,z;E',\mu') =$$

$$\int q_1(E,E'')\ \overline{T}^{(1)}(E'',\mu,z;E',\mu')\ dE'' +$$

$$\iint q_2(E,\mu;E'',\mu'')\ \overline{T}^{(0)}(E'',\mu'',z;E',\mu')\ dE''\ d\mu'' \tag{36}$$

This equation in turn may be treated by iteration. If the symbol

$$S(E,\mu,z;E',\mu') = \iint q_2(E,\mu\ E'',\mu'')\ \overline{T}^{(0)}(E'',\mu'',z;E'\mu')\ dE''\ d\mu''$$

is introduced, a first approximation to $\overline{T}^{(1)}$, obtained by neglecting the first integral on the right side, is given by

$$\left.\begin{aligned} &\overline{T}^{(1,1)}(E,\mu,z;E',\mu') = \\ &\frac{1}{\mu}\int_0^z \exp\left[\frac{Q(E)(\zeta-z)}{\mu}\right] S(E,\mu,\zeta;E',\mu')\ d\zeta \qquad \text{for} \quad \mu > 0 \\ &\text{and} \\ &\frac{1}{\mu}\int_Z^z \exp\left[\frac{Q(E)(\zeta-z)}{\mu}\right] S(E,\mu,\zeta;E',\mu')\ d\zeta \qquad \text{for} \quad \mu < 0 \end{aligned}\right\} \tag{37}$$

This may then be substituted into the first integral on the right side of equation (36) and a better approximation for $\overline{T}^{(1)}$ obtained. The iteration may, of course, be continued along these general lines, but its usefulness is, in general, greatly diminished as the length of the process is increased. In any case, the early approximation will be good if the contribution from the integral involving q_2 is not too large; that is, if multiple scattering with angular deviation does not occur to any great extent.

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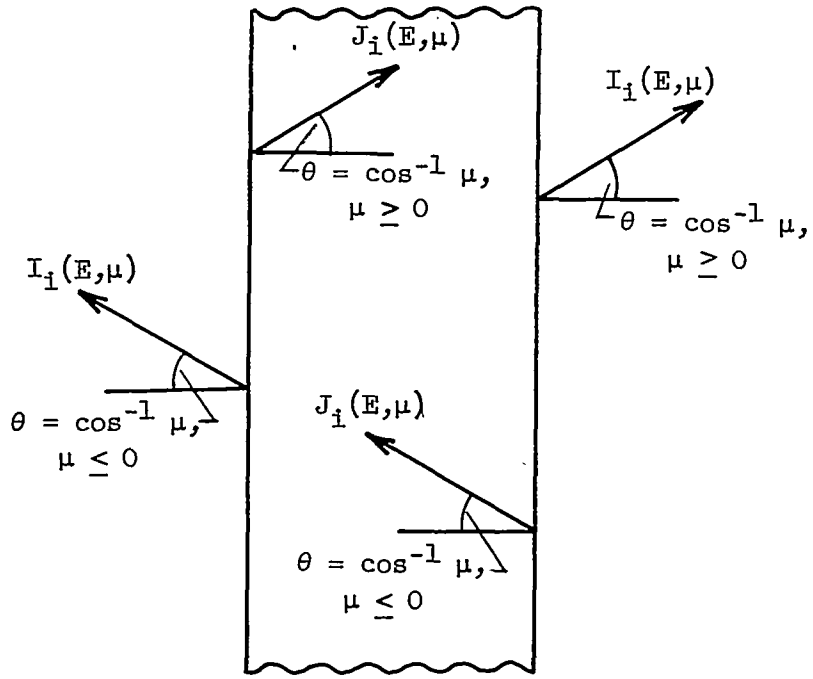
It is apparent that in the two cases considered (q_2 small or essentially zero) the emergent flux will depend strongly on angle for an element of reasonable thickness; indeed, for a sufficiently thick absorber, it follows from the form of $T^{(0)}$ that this flux is mainly concentrated within a cone of half-angle of order $\sqrt{2/QZ}$ where the value of Q to be used is a typical one. Consequently, an analysis in which the angular dependence is given by specifying the amount of radiation within each of a set of ranges into which μ is divided is appropriate.

The results obtained here serve to elucidate the case in which scattering with angular deviation is not a very significant process so that it may be treated as no more than a perturbation. An exact solution is obtained by the iteration method for the case where the direction of motion of the particles is unchanged; but, unless Q and q_1 have a particularly simple form, its application is obviously inconvenient where there is much multiple scattering for the reason that, under such circumstances, a large number of terms must be retained in the series for T. The process for dealing with the perturbing effects of scattering with angular deviation is described.

Semiempirical Method

The results of the previous section, entitled "Individual Element," find application to the treatment of shields by a modification of the semiempirical method. In this formalism, a shield intercepting a flux of particles is to be represented in the following fashion. Radiation of an intensity independent of position on the face falls upon the first of a set of n plane parallel elements. As a result, there is incident upon both faces of each element a flux of radiation; consequently, there emerges from each face a certain quantity of radiation. Let $J_i(E,\mu)$ represent the number of particles with energy E and direction μ per unit range of E per unit range of μ per unit time incident upon each unit area of the ith element. Similarly, let $I_i(E,\mu)$ represent the flux of radiation emergent from the ith element. The following diagram depicts the ith element of the shield:



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Positive μ indicates radiation incident upon the front face, and negative μ, that incident upon the rear face. For I, then, negative μ is associated with radiation leaving the front face and positive μ, that leaving the rear face. The manner in which the presence of the element modifies the radiation is given by the equation

$$I_i(E,\mu) = \iint K_i(E,\mu;\mu_i,E_i)\ J_i(\mu_i,E_i)\ dE_i\ d\mu_i \tag{38}$$

where $K_i(E,\mu;\mu_i,E_i)\ dE\ d\mu$ represents the proportion of particles with energy E_i and direction μ_i which is modified by the ith element so as to emerge with energy between E and $E+dE$ and direction between μ and $\mu+d\mu$.

In addition, the I's and J's satisfy the following equations:

$$\left.\begin{aligned} I_i(E,\mu) &= J_{i+1}(E,\mu) \qquad \mu > 0 \\ J_i(E,\mu) &= I_{i+1}(E,\mu) \qquad \mu < 0 \\ J_n(E,\mu) &= 0 \qquad \mu < 0 \end{aligned}\right\} \tag{39}$$

and $I_1(E,\mu)$ is known for $\mu > 0$. It follows from symmetry considerations that

$$K_i(E,\mu;E_i,\mu_i) = K_i(E,-\mu;E_i,-\mu_i) \tag{40}$$

The previous set of equations is, as noted in the introduction, best treated by some method of approximation. In reference 1 this was done by replacing the integral by a sum; as shown previously (see remarks following equation (37)), this is appropriate where the dominant processes are absorption and loss of energy without scattering.

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However, in the case when elastic scattering is of primary importance, another procedure suggests itself (see Approach 1). In this case it is natural to expand I and J in terms of the polynomials $P_n^{\pm}(\pm 2\mu-1)$ and to use the Laplace transforms of the coefficients so obtained as the intensity numbers. This may be done by defining the quantities

$$\left.\begin{aligned} \overline{I}_{in}^{+}(\eta) &= \int_{E_{max}}^{0}\int_{0}^{1} P_n^{+}(2\mu-1)\ e^{-\eta u}\ I_i(E,\mu)\ d\mu\ dE \\ \overline{I}_{i}^{-}(\eta) &= \int_{E_{max}}^{0}\int_{-1}^{0} P_n^{-}(-2\mu-1)\ e^{-\eta u}\ I_i(E,\mu)\ d\mu\ dE \end{aligned}\right\} \tag{41}$$

with analogous expressions for $\overline{J}^{+}$ and $\overline{J}^{-}$. In terms of these quantities, equations (38) and (39) become, respectively, when λ is independent of E and there is no absorption,

$$\left.\begin{aligned} \overline{I}_{in}^{+}(\eta) &= \sum_{n'}\left[k_{inn'}^{++}(\eta)\ \overline{J}_{in'}^{+}(\eta) + k_{inn'}^{+-}(\eta)\ \overline{J}_{in'}^{-}(\eta)\right] \\ \overline{I}_{in}^{-}(\eta) &= \sum_{n'}\left[k_{inn'}^{++}(\eta)\ \overline{J}_{in'}^{-}(\eta) + k_{inn'}^{+-}\ \overline{J}_{in'}^{+}(\eta)\right] \end{aligned}\right\} \tag{42}$$

and

$$\left.\begin{aligned} \overline{I}_{in}^{+}(\eta) &= \overline{J}_{i+1,n}^{+}(\eta) \qquad i < n \\ \overline{J}_{in}^{-}(\eta) &= \overline{I}_{i+1,n}^{-}(\eta) \qquad i < n \\ \overline{J}_{n}^{-}(\eta) &= 0 \end{aligned}\right\} \tag{43}$$

where the quantities $k^{++}_{inn'}(\eta)$ and $k^{+-}_{inn'}(\eta)$ are defined by:

$$k^{++}_{inn'}(\eta) =$$

$$(2n'+1)\int_{E_{max}}^{0}\int_{0}^{1}\int_{0}^{1} P_n^+(2\mu-1)P_{n'}^+(2\mu_i-1)\ e^{-\eta u}\ K_i(E,\mu,E_{max},\mu_i)\ d\mu\ du_i\ dE \tag{44}$$

and

$$k^{+-}_{inn'}(\eta) =$$

$$(2n'+1)\int_{E_{max}}^{0}\int_{0}^{1}\int_{-1}^{0} P_n^+(2\mu-1)P_{n'}^-(-2\mu_i-1)\ e^{-\eta u}\ K_i(E,\mu,E_{max},\mu_i)\ d\mu\ d\mu_i\ dE$$

The equations here contain η only as a parameter; because the element has the effect of producing a distribution of intensity nearly independent of μ, it should be sufficient to consider only a few values of n. In case the mean free path is not constant, the right side will depend on other values of η besides the one appearing on the left side (see equation (7)). In this case, if the average rate of variation over a wide region is not too large, it is possible to apply successive approximation by first omitting these extra terms. If this is impractical, the best procedure is simply to expand I and J in terms of Legendre polynomials as before, but with the coefficients functions of energy rather than of η. The integrals may then be replaced by sums as in reference 1.

Even in the case in which the expression in terms of the Laplace transform expedites the solution, there remains the problem of interpreting the results obtained, which will give quantities like $\overline{I}^+_{in}(\eta)$. The corresponding function of u and μ may be obtained by performing the inverse Laplace transformation with regard to η, multiplying the resulting function of u by $(2n+1)P_n^+$, and summing over n. Each of these procedures, and particularly the first, is likely to be quite tedious. It is therefore desirable to avoid the actual calculation; and this may sometimes be done because, as the following considerations show, the quantities $\overline{I}^+_{in}(\eta)$ themselves contain considerable information of physical significance. If the dependence on n and μ is suppressed, the relation between I and $\overline{I}$ is given by

$$\bar{I}(\eta) = \int_0^\infty e^{-\eta u} I(u)\, du \tag{45}$$

Here $I(u)$ may be taken as representing the rate of flow of particles per unit area per unit time per unit range of u. In view of the connection between u and E, $I(\eta)$ may be rewritten in the form

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$$\bar{I}(\eta) = \int_0^\infty \left(\frac{E}{E_{max}}\right)^\eta I(u)\, du \tag{46}$$

Thus, $\bar{I}(0)$ is the rate of flow of particles per unit area per unit time, $E_{max}\bar{I}(1)$ is the average rate of flow of energy per unit area per unit time, etc. The quantity $I(\eta)$ therefore finds interpretation in terms of particle current density, of energy current density, and of the moments of the energy flow. If $I(u)$ itself must be known, the values of u of interest will usually be large because the important question will be whether the particles have been sufficiently slowed down to reach an absorption band in a succeeding element. In this case, use may be made of known methods for approximating the inverse transformation for large u (see reference 3, p. 204). If the dependence on n and μ is now considered, it follows in a similar fashion that much may be deduced about the directional flow of particles directly from $\bar{I}_n$. Thus the total current in the forward direction is (from equation (46)):

$$\int_{E_{max}}^0 \int_0^1 I(E,\mu)\mu\, d\mu\, dE = \sum_n (2n+1)\, \bar{I}_n^+(0) \int_0^1 P_n(2\mu-1)\mu\, d\mu$$

$$= \frac{1}{2}\left[\bar{I}_0^+(0) + \bar{I}_1^+(0)\right] \tag{47}$$

It is therefore possible to learn a good deal about the particle distribution without transforming from the variables (n,η) to the variables (μ,E).

Up to now, the results of the section "Individual Element" have been considered from the viewpoint of their significance in the solution of a problem involving a number of elements whose separate effects upon the incident radiation have been determined experimentally. However, the results obtained in approach 2 of that section may also be used to determine, to a good approximation, the effect of a thin element of a given material for which the constants associated with each possible

process are known. The semiempirical method may then be used to obtain the required data for thick shields. Because the term "thin", as used herein, means only sufficiently thin that multiple scattering need not be considered, it would appear that this procedure would provide a convenient and reasonably flexible way of obtaining the needed constants for a variety of elements.

CONCLUDING REMARKS

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The problem of a layered shield consisting of a number of plane parallel elements of infinite extent but arbitrary thickness has been treated. By use of the transport equation, an explicit approximate solution has been obtained for a single element for the case where multiple scattering with angular deviation is not dominant and the form of the solution investigated in the case where multiple scattering with angular deviation is the principal effect. In each case, an expansion is given for the angular dependence which is well suited to the boundary conditions of the problem and is rapidly convergent. It is pointed out that the use of an expansion of this sort and of the Laplace transform with regard to energy (where scattering with angular deviation is dominant) should considerably reduce the labor in a semiempirical treatment of shielding. On the other hand, the semiempirical approach may be used to extend the single scattering result to a thicker element where multiple scattering is of importance.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, August 8, 1950

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APPENDIX A

SYMBOLS

The following symbols are used in this report:

$A_n(\eta)$ expansion coefficient for $A(\eta,\mu)$ in series

$$A(\eta,\mu) = \sum_{n=0}^{\infty} (2n+1)A_n(\eta)P_n(2\mu-1)$$

$A(\eta,\mu)$ value of Φ at $z=0$ for $\mu > 0$

E energy of a particle

E_{max} maximum energy of any particle entering shield

$F(\mu,\mu',\eta)$ Laplace transform with respect to u of $f_1(\mu,\mu',u)$

$f_1(\mu,\mu',u-u')$ $= \int_0^{2\pi} f(\mu_0,u-u')\, d\varphi'$

$f(\mu_0,u-u')$ proportion of particles with direction (θ',φ') and energy E' scattered through the angle Θ into the direction (θ,φ) with energy E per unit solid angle per unit range of u

$f_{nn'}^{\pm\pm}(\eta)$ expansion coefficient for expression of F in terms of $P_n^+(2\mu-1)$, $P_{n'}^+(2\mu-1)$, $P_n^-(-2\mu-1)$, and $P_{n'}^-(-2\mu-1)$ divided by $(2n+1)$

H_j coefficient of e^{-ju} in expansion $h(u) = \sum_j H_j\, e^{-ju}$

$h(u)$ $\dfrac{\lambda(u)}{\lambda_s(u)}$

h_j coefficient of E^j in expansion $h(u) = \sum_j h_j E^j$

$I_i(E,\mu)$ number of particles per unit time per unit area leaving i^{th} element of shield with energy E and angle $\cos^{-1}\mu$ per unit range of energy and unit range of μ

$$\bar{I}_{in}^{+}(\eta) = \int_{E_{max}}^{0}\int_{0}^{1} P_n(2\mu-1)e^{-\eta\mu}\, I_i(E,\mu)\, d\mu\, dE$$

$$\bar{I}_{in}^{-}(\eta) = \int_{E_{max}}^{0}\int_{-1}^{0} P_n(-2\mu-1)e^{-\eta\mu}\, I_i(E,\mu)\, d\mu\, dE$$

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$J_i(E,\mu)$ number of particles per unit time per unit area incident upon i^{th} element of shield with energy E and angle $\cos^{-1}\mu$ per unit range of energy and unit range of μ

$$\bar{J}_{in}^{+}(\eta) = \int_{E_{max}}^{0}\int_{0}^{1} P_n(2\mu-1)e^{-\eta\mu}\, J_i(E,\mu)\, d\mu\, dE$$

$$\bar{J}_{in}^{-}(\eta) = \int_{E_{max}}^{0}\int_{-1}^{0} P_n(-2\mu-1)e^{-\eta\mu}\, J_i(E,\mu)\, d\mu\, dE$$

$K_i(E,\mu';\mu_i,E_i)$ proportion of particles incident upon i^{th} element whose original energy is E_i and direction, μ_i scattered into energy E per unit range of E and direction μ per unit range of μ

$$k_{inn'}^{++}(\eta) = (2n'+1)\int_{E_{max}}^{0}\int_{0}^{1}\int_{0}^{1} P_n(2\mu-1)P_{n'}(2\mu-1)e^{-\eta\mu}\, K_i(E,\mu;E_{max},\mu_i)\, d\mu_i\, d\mu\, dE$$

$$k_{inn'}^{+-}(\eta) = (2n'+1)\int_{E_{max}}^{0}\int_{0}^{1}\int_{-1}^{0} P_n(2\mu-1)P_{n'}(-2\mu_i-1)e^{-\eta\mu}\, K_i(E,\mu;E_{max},\mu_i)\, d\mu_i\, d\mu\, dE$$

$P_n(x)$ Legendre polynomial of degree n

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$P_n^+(2\mu-1)$ $= \begin{cases} P_n(2\mu-1) & \mu \geq 0 \\ 0 & \mu < 0 \end{cases}$

$P_n^-(-2\mu-1)$ $= \begin{cases} P_n(-2\mu-1) & \mu \leq 0 \\ 0 & \mu > 0 \end{cases}$

$Q(E)$ $= \dfrac{1}{\lambda(u)}$

$q_1(E,E')$ proportion of particles with energy E' scattered into state with energy E per unit range of E

$q_2(E,\mu,E',\mu')$ proportion of particles with angle $\cos^{-1}\mu'$ and energy E' scattered into state with angle $\cos^{-1}\mu$ and energy E per unit range of E per unit range of μ

$S(E,\mu,z;E',\mu')$ $= \displaystyle\int_{-1}^{1}\int_{0}^{E_{max}} q_2(E,\mu;E'',\mu'')\overline{T}^0(E'',\mu'',z;E',\mu')\, dE''\, d\mu''$

$T(E,z,E')$ proportion of particles with energy E' incident upon face of element reaching energy E per unit range E upon penetrating to depth z for $q_2 = 0$

$T(E,\mu,z;E',\mu')$ proportion of particles with energy E' and angle $\cos^{-1}\mu'$ incident upon surface of layer reaching energy E per unit range of E and angle $\cos^{-1}\mu$ per unit range of μ upon penetrating to depth z

$T^{(j)}$ j^{th} approximation to T

$\overline{T}^{(j)}$ j^{th} approximation to $\overline{T}$

u $\ln \dfrac{E_{max}}{E}$

Z thickness of layer

z distance of point inside layer from the front face of the layer

$\delta(x)$ Direc delta function

δ_{ij}	Kronecker delta
η	variable of Laplace transform
Θ	angle between direction of scattered particle and direction of particle before scattering
θ	colatitude measured with respect to normal to face of shield upon which radiation is incident
Λ_j	coefficient of e^{-ju} in expansion $\lambda(u) = \sum_j \Lambda_j e^{-ju}$
$\lambda(u)$	total mean free path
λ_j	coefficient of E^j in expansion $\lambda(u) = \sum_j \lambda_j E^j$
$\lambda_s(u)$	scattering mean free path
μ	cosine θ
μ_0	cosine Θ
$\Phi(z,\eta,\mu)$	Laplace transform of ψ with respect to u
φ	longitude about normal to surface
$\varphi_n^{\pm}$	expansion coefficient in expression for Φ in terms of $(2n+1)\ P_n^{\pm}(\pm 2\mu-1)$
$\psi(E,\mu,z)$	number of particles per unit time per unit area per unit range of E per unit solid angle of energy E and direction $\cos^{-1}\mu$ and location z divided by $\lambda(u)$
$\psi(u,\mu,z)$	number of particles per unit time per unit area per unit range of u per unit solid angle of energy E, direction $\cos^{-1}\mu$, and location z divided by $\lambda(u)$
Ω	solid angle

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All integrals for which no limits are specified are to be extended over the entire range of the variable concerned.

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APPENDIX B

DERIVATION AND SIGNIFICANCE OF TRANSPORT EQUATION

The transport equation simply expresses the fact that any particle entering a given region must, in the steady state, be absorbed, be scattered, or else leave the region unchanged, and that any particle leaving the region must, in the steady state with no internal sources, either be replaced by a like particle entering the region or else be produced when a particle with a different momentum is scattered. In order to state this in mathematical terms, it is necessary only to consider the balance within a region R bounded by a surface S. The following contributions and losses occur:

The number of particles with energies between E and $E+dE$ (where $E = E_{max}e^{-u}$ and $dE = - E\,du$) moving in directions lying within a cone of solid angle $d\Omega$ around the direction specified by the unit vector $\vec{n}$ which enter R through the surface S per unit time is

$$- \oint_S \lambda \psi \vec{n} \cdot d\vec{S}\, du\, d\Omega$$

where $d\vec{S}$ is a vector element of area. This holds because $\lambda\psi\vec{n}$ is the vector current density of particles per unit time per unit area per unit solid angle per unit range of u.

The rate at which particles of this energy and direction disappear within R is

$$\int_R \psi\, dv\, du\, d\Omega$$

where dv is an element of volume. This follows from the fact that the linear rate of decrease of the density of particles is simply the density divided by the mean free path and that the time rate of decrease of density is equal to the linear rate multiplied by the speed; but this last product is just ψ.

The rate at which particles with other momenta are scattered within R so as to achieve an energy between E and $E+dE$ and a direction within the solid angle $d\Omega$ centered around $\vec{n}$ is

$$\int_0^u du' \int d\Omega' \int_R dv\ \psi(u',\mu',z) f(\mu_0,u-u') h(u')\ du\ d\Omega$$

This may be seen as follows: $h(u')\ f(\mu_0,u-u')\ du\ d\Omega$ is the proportion of particles either absorbed or scattered which originally have an energy E' and a direction $\vec{n}'$ and which are scattered so as to achieve an energy lying in the range between E and $E+dE$ and a direction lying within the solid angle $d\Omega$ centered around $\vec{n}$. Thus when this product is multiplied by ψ and integrated over all directions $d\Omega'$, over u' between 0 and u ($f=0$ for $u < u'$ because of the assumed degradation of energy), and over R, the desired rate is obtained. A few other remarks about f may be in order: (1) Because every scattered particle must attain some direction and energy, f satisfies the equation 2386

$$\int_{u'}^{\infty} du \int d\Omega\ f(\mu_0,u-u') = 1 \tag{B1}$$

(2) The dependence of f upon $u-u'$ results from the assumption of classical elastic or quantum mechanical S-wave scattering (that is, scattering symmetrical with regard to a coordinate system located at the center of gravity of the system participating in the scattering collision). (See reference 3; pp. 187, 188.)

The resulting equation for balance is:

$$-\oint_S \lambda(u)\ \psi(u,\mu,z)\ \vec{n} \cdot d\vec{S} = \int_R \psi\ dv$$

$$-\int_R dv \int_0^u du' \int d\Omega' \psi(u',\mu',z)\ f(\mu_0,u-u')\ h(u') \tag{B2}$$

If the left side of equation (B2) is converted to a volume integral by Gauss's theorem and the integrands in the integrals over R occurring in the equation set equal in view of the fact that R is arbitrary, the following result is obtained:

$$\lambda(u)\ \vec{n} \cdot \nabla\psi(u,\mu,z) + \psi$$

$$= \int_0^u du' \int d\Omega' \psi(u',\mu',z)\ f(\mu_0,u-u')\ h(u') \tag{B3}$$

Because ψ depends on position only through z and because the component of $\vec{n}$ in the z direction is $\cos\theta$, $\vec{n}\cdot\nabla\psi$ reduces to $\cos\theta\,\frac{\partial\psi}{\partial z}$ and equation (B3) becomes identical with equation (1).

Similarly, equation (23) is obtained by writing ψ as a function of E rather than u and by assuming that a particle does not change direction on scattering so that $\int f(\mu_0,u-u')\,h(u')\,du'$ becomes $\int \frac{q_1(E,E')\delta(\mu-\mu')\,dE'}{2\pi}$. Equation (20) is the same as equation (23) except that in equation (20) $\int f(\mu_0,u-u')\,h(u')\,du'$ is replaced by the more general function

$$\int\left[\frac{q_1(E,E')\delta(\mu-\mu')+q_2(E,\mu;E',\mu')}{2\pi}\right]dE'$$

The q_1 term represents all the scattering without angular deviation, as before, and the q_2 term allows for an additional scattering process which produces a change in direction.

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APPENDIX C

DERIVATION OF EQUATIONS (29)

It may be verified by direct substitution that the expressions for $T^{(0)}$ and $T^{(1)}$ given by equations (29) satisfy equation (28) subject to condition (27). For values of j greater than one, the correctness of equations (29) may be checked by induction. Because the general expression given for $T^{(j)}$ is correct for $j=1$, it remains only to show that, if equations (27) and (28) are used to compute $T^{(j+1)}$ on the assumption that $T^{(j)}$ is given by equations (29), then the resulting expression for $T^{(j+1)}$ agrees with that obtained from equations (29). The formal solution of equations (27) and (28) for $T^{(j+1)}$ with $T^{(j)}$ given is 2386

$$T^{(j+1)}(E^0,z,E^{(j+1)}) = \int_0^{E_{max}} q_1(E^{(j+1)},E^{(j)}) \int_0^z \frac{e^{-\left[Q(E^{(j+1)})z - Q(E^{(j+1)})\zeta\right]\frac{1}{\mu}}}{\mu} T^{(j)}(E^{(0)},\zeta,E^{(j)})\, d\zeta\, dE^{(j)} \tag{C1}$$

Here $T^{(j)}$ stands for the function specified in equations (29); when the explicit substitution is performed, the right side of equation (C1) takes the form of a multiple integral with respect to $dE^{(1)} \ldots dE^{(j-1)}$. The integrand of this multiple integral may be written, upon integration with respect to ζ, as

$$\int_0^{E_{max}} dE^{(j)} \prod_{s=0}^{j} q_1\left(E^{(s+1)},E^{(s)}\right) \left\{ \sum_{n=0}^{j} \prod_{\substack{r=0\\ r\neq n}}^{j+1} \frac{e^{-Q(E^{(n)})\frac{z}{\mu}}}{Q(E^{(r)}) - Q(E^{(n)})} + \right.$$

$$\left. + \sum_{n=0}^{j} \prod_{\substack{r=0\\ r\neq n}}^{j} \frac{e^{-Q(E^{(j+1)})\frac{z}{\mu}}}{\left[Q(E^{(r)}) - Q(E^{(n)})\right]\left[Q(E^{(n)}) - Q(E^{(j+1)})\right]} \right\} \tag{C2}$$

In order that this result agree with equation (29) it is necessary only that the following relation hold:

$$\sum_{n=0}^{j} \prod_{\substack{r=0 \\ r\neq n}}^{j} \frac{1}{\left[Q(E^{(r)}) - Q(E^{(n)})\right]\left[Q(E^{(n)}) - Q(E^{(j+1)})\right]}$$

$$= \prod_{r=0}^{j} \frac{1}{Q(E^{(r)}) - Q(E^{(j+1)})} \tag{C3}$$

Equation (C3), in turn, is valid if the following expression vanishes:

$$\sum_{n=0}^{j} \prod_{\substack{r=0 \\ r\neq n}}^{j} \frac{Q(E^{(r)}) - Q(E^{(j+1)})}{Q(E^{(r)}) - Q(E^{(n)})} - 1 \tag{C4}$$

Equation (C4), however, is just a polynomial of degree $(j-1)$ or less in $Q(E^{(j+1)})$ which vanishes whenever $Q(E^{(j+1)})$ takes on one of the values $Q(E^{(0)}), \ldots, Q(E^{(j)})$ and, hence must be identically zero. This completes the verification of equations (29).

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