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DISPLACEMENT EFFECT OF A THREE-DIMENSIONAL BOUNDARY LAYER

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DISPLACEMENT EFFECT OF A THREE-DIMENSIONAL BOUNDARY LAYER

By Franklin K. Moore

SUMMARY

A method is described for determining the "displacement surface" of a known three-dimensional compressible boundary-layer flow in terms of the mass-flow defects associated with the profiles of the two velocity components parallel to the surface.

The height of the displacement surface above the body surface for flow about a yawed infinite cylinder is shown to be equal to the height characterizing the mass-flow defect of the chordwise velocity profile. The displacement surface height is shown to differ, in general, from that associated with the resultant mass-flow defect, even at stagnation points of the secondary flow. Numerical values are found for the known three-dimensional boundary-layer flow about a cone at a small angle of attack to a supersonic stream.

INTRODUCTION

The boundary layer established in the flow of a slightly viscous fluid about a body is normally considered an isolated region wherein the effects of viscosity predominate, and outside of which the motion of the fluid is governed by the laws of nonviscous motion. For large Reynolds number, the boundary layer is assumed to be so thin that the nonviscous portion of the flow occurs as though there were no boundary layer. This assumption is strictly correct in the limit of infinite Reynolds number. For large but finite Reynolds numbers, the growth of the boundary layer causes the stream to be deflected away from the body surface.

This displacement effect of the boundary layer on the nonviscous flow may properly be determined from the behavior of the boundary layer itself, as established either by experiment or by solution of the Prandtl boundary-layer equations for laminar flow.

It does not follow, however, that this revised outer flow may be used to improve the solution for the boundary layer, still using the Prandtl equations. A new set of equations must be used for this purpose taking into account the variation of pressure across the boundary layer, which is neglected in the Prandtl equations. (See Alden's iterative solution for incompressible flat-plate flow, reference 1.)

The customary definition of displacement thickness (reference 2) is applicable to two-dimensional flow and is expressed in the following equation:

$$\int_0^h \rho u \, dy = \rho_1 u_1 (h - \delta^*) \tag{1}$$

where h is some location well outside the boundary layer (fig. 1), at which $\rho u = \rho_1 u_1$, and beyond which the flow may be considered nonviscous. (A list of symbols is provided in the appendix.) Under the assumption of an extremely thin boundary layer, h is so small that $\rho_1 u_1$ may be taken as the evaluation at the body surface $(y=0)$ of the nonviscous flow obtained by neglecting the presence of the boundary layer. Equation (1) equates the actual mass flow near the surface with the mass flow which would be associated with a nonviscous flow that terminates at δ^* rather than extending to the wall. Thus, the nonviscous portion of the flow behaves as if it occurred in the presence of a solid boundary given by the displacement thickness $\delta^*(x)$. Equation (1) may be solved for δ^*:

$$\delta^* = \int_0^h \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy$$

Ordinarily, theoretical boundary-layer solutions for $\rho u/\rho_1 u_1$ asymptotically approach 1 for large values of Reynolds number based on y. Therefore, displacement thickness is often defined as follows:

$$\delta^* = \int_0^\infty \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy$$

For three-dimensional boundary-layer flows, two lengths characterizing mass-flow defect may be defined in terms of the profiles of the two velocity components tangential to the surface (fig. 2),

$$\delta_x \equiv \int_0^h \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy \qquad \delta_z \equiv \int_0^h \left(1 - \frac{\rho w}{\rho_1 w_1}\right) dy \tag{2}$$

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and it is not clear which, if either, defines a displacement surface that properly describes the extent to which the nonviscous flow is deflected by the boundary layer.

Of course, it is expected that there does exist a displacement surface for such flows. The analysis that follows shows that such a surface may be described, using a defining equation more fundamental than equation (1).

The velocity and density profiles are assumed to be known for the three-dimensional flows under consideration. Cases for which this is true include: the laminar boundary layer on yawed infinite cylinders, treated by Prandtl (reference 3), R. T. Jones (reference 4), and by Sears (reference 5); the laminar boundary layer on a cone at a small angle of attack to a supersonic stream (reference 6). The displacement effect of these flows will be treated specifically.

This report may be considered as an addendum to references 6 and 7. This investigation was conducted at the NACA Lewis laboratory.

THEORY

Defining Relation for Displacement Surface

The boundary-layer solution (assumed known) yields a certain distribution of velocity $v_{bl}(x,z)$ normal to the body surface at the outer edge $h(x,z)$ of the boundary layer, where ρ, u, and w may be taken essentially equal to ρ_1, u_1, and w_1 (see fig. 2). Under the assumption that the nonviscous flow is altered only slightly by displacement, the most direct way to compute this effect would be to suppose that the nonviscous equations hold for $y \geq h(x,z)$ and to impose the following boundary condition on the normal velocity v_{of} in the outer flow:

$$v_{of} = v_{bl}(x,z) \quad \text{at} \quad y = h(x,z) \tag{3}$$

However, since the boundary conditions usually encountered in nonviscous flow specify an impermeable surface, it is convenient to recast the boundary condition given in equation (3) in answer to the question: What impermeable surface $y = \Delta(x,z)$ would deflect a nonviscous fluid in such a way as to produce a normal velocity satisfying condition (3)? This fictitious surface may be called the displacement surface.

Because $\Delta(x,z)$ is imagined to be a stream surface, it is necessary to specify that at $y=\Delta$ the resultant velocity vector (u,v,w) be tangent to the surface $y=\Delta$. Thus, at $y=\Delta$ the ratio of v to the magnitude of $\underline{q}$, which is defined as the vector (u,w), must be set equal to the slope of the surface $y=\Delta$, measured in the direction of the vector $\underline{q}$, or, equivalently, equal to the component in the direction of $\underline{q}$ of the vector grad Δ. In vector notation, therefore, the normal velocity v which would be produced in a nonviscous fluid at an impermeable surface $y=\Delta$ is $\underline{q} \cdot \text{grad } \Delta$. The vector $\underline{q}$ may be obtained by evaluating the velocity vector of the unrevised nonviscous flow at the body surface $y=0$, under the related assumptions that the velocity vector varies only slightly over distances of the order of the actual boundary-layer thickness, and that the revision required to take account of displacement is slight. Thus, at $y=\Delta$ (see fig. 3),

$$v_{of} = \underline{q}_1 \cdot \text{grad } \Delta$$

The increment in v_{of} between Δ and h is approximately $(h-\Delta)\partial v_{of}/\partial y$, again assuming a thin boundary layer and using only the first term in a Taylor series.

To the order of approximation contemplated in this analysis, $\partial v_{of}/\partial y$ may be obtained from the unrevised nonviscous flow evaluated at $y=0$. Thus, the fictitious impermeable surface Δ would produce, at $y=h$, a normal velocity (see fig. 3)

$$v_{of} = \underline{q}_1 \cdot \text{grad } \Delta + (h-\Delta)\left(\frac{\partial v_{of}}{\partial y}\right)_{y=0}$$

Introducing this result into the boundary condition given in equation (3) yields the defining relation for $\Delta(x,z)$:

$$(v_{bl})_{y=h} = (h-\Delta)\left(\frac{\partial v_{of}}{\partial y}\right)_{y=0} + \underline{q}_1 \cdot \text{grad } \Delta \tag{4}$$

The boundary-layer solution yields v_{bl}.

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Expression of Δ in Terms of Mass-Flow Defects

The displacement surface Δ may be related to the mass-flow defects (equations (2)) which characterize the boundary layer as follows: in a Cartesian coordinate system (fig. 2), the equation of continuity for both the boundary-layer and nonviscous flow is

$$\frac{\partial \rho v}{\partial y} = -\frac{\partial \rho u}{\partial x} - \frac{\partial \rho w}{\partial z} \tag{5}$$

Under the Prandtl boundary-layer assumptions, the Cartesian equations of motion, and hence equation (5), may be applied in an orthogonal curvilinear coordinate system in which the surface of the body is given by $y=0$, provided the radius of curvature of the body is everywhere large as compared with the boundary-layer thickness. Integrating equation (5) across the boundary layer yields

$$\begin{aligned}(\rho v_{bl})_{y=h} &= -\int_0^h \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho w}{\partial z}\right) dy \\ &= \int_0^h \left[\frac{\partial}{\partial x}(\rho_1 u_1 - \rho u) + \frac{\partial}{\partial z}(\rho_1 w_1 - \rho w)\right] dy - \\ &\quad \int_0^h \left(\frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 w_1}{\partial z}\right) dy\end{aligned}$$

or, inasmuch as h is outside the boundary layer where $\rho u = \rho_1 u_1$ and $\rho w = \rho_1 w_1$,

$$\rho_1 (v_{bl})_{y=h} = -h\left(\frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 u_1}{\partial z}\right) + \frac{\partial}{\partial x}(\rho_1 u_1 \delta_x) + \frac{\partial}{\partial z}(\rho_1 w_1 \delta_z) \tag{6}$$

where δ_x and δ_z are the mass-flow defects defined in equations (2). For the nonviscous flow, $v_{of} = 0$ at the body surface $(y=0)$, and equation (5) becomes

$$\rho_1 \left(\frac{\partial v_{of}}{\partial y}\right)_{y=0} = -\left(\frac{\partial \rho_1 u_1}{\partial x} + \frac{\partial \rho_1 w_1}{\partial z}\right) \tag{7}$$

Introducing equations (6) and (7) into equation (4) yields

$$\frac{\partial}{\partial x}\left[\rho_1 u_1(\Delta-\delta_x)\right] + \frac{\partial}{\partial z}\left[\rho_1 w_1(\Delta-\delta_z)\right] = 0 \tag{8a}$$

In a Cartesian coordinate system, the displacement surface is related to the mass-flow defects by equation (8a). When cases arise for which other coordinate systems must be used, the following generalization in vector form of equation (8a) may be used: 2536

$$\operatorname{div}\left[\rho_1 \underline{q}_1 \Delta - \int_0^h (\rho_1 \underline{q}_1 - \rho \underline{q})\, dy\right] = 0 \tag{8b}$$

where y is the distance normal to the body surface and where the divergence operator involves differentiation only with respect to the two coordinates parallel to the body surface.

Examples

Plane flow. - Equation (8a) may be integrated to yield, for plane flow $(\partial/\partial z = 0)$:

$$\Delta = \delta_x + \frac{K}{\rho_1 u_1} \tag{9}$$

where K is a constant of integration.

The appearance of this constant means, in general, that the revised boundary condition on the nonviscous flow near the wall may be applied at any surface in the boundary-layer region; for example, along the wall itself. Of course, if there is a stagnation point on the body where u_1 vanishes, then K must be taken equal to zero and the revised boundary condition must be imposed at the location $\Delta = \delta_x$, at least near the stagnation point.

Stagnation point of secondary flow. - In the vertical plane of symmetry of the flow about a body of revolution at an angle of attack, the circumferential velocity component vanishes in the boundary layer as well as in the nonviscous outer region. Such stagnation of the secondary motion would occur in a variety of cases, in particular, wherever an essentially three-dimensional flow has a plane of symmetry. When w_1 is designated as the component of secondary flow (circumferential velocity for a body of revolution) at the outer edge of the boundary layer, and equation (8a) is written in the form

$$\frac{\partial}{\partial x}\left[\rho_1 u_1(\Delta-\delta_x)\right] + w_1 \frac{\partial}{\partial z}\left[\rho_1(\Delta-\delta_z)\right] + \rho_1(\Delta-\delta_z)\frac{\partial w_1}{\partial z} = 0$$

it is clear that Δ cannot equal δ_x, in general, even if both w_1 and the boundary-layer profile of w vanish, unless $\partial w_1/\partial z$ also vanishes.

Flow about yawed infinite cylinder. - In this case, there is a spanwise boundary-layer velocity profile and an associated spanwise mass-flow defect. (See references 3 to 5.) If x is taken in the chordwise direction (fig. 4), the entire flow depends only on x. Thus, derivatives with respect to the spanwise coordinate z vanish; and, hence, from equation (8a), the plane-flow result (equation (9)) applies. Accordingly, the spanwise mass-flow defect represented by δ_z does not enter into the determination of the displacement surface.

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Supersonic flow about cone at small angle of attack. - When the coordinate system shown in figure 5 is used, equation (8b) becomes

$$\sin\Theta \frac{\partial}{\partial x}\left[\rho_1 u_1 x(\Delta-\delta_x)\right] + \frac{\partial}{\partial \varphi}\left[\rho_1 w_1(\Delta-\delta_\varphi)\right] = 0 \tag{11}$$

where

$$\delta_x = \int_0^h \left(1 - \frac{\rho u}{\rho_1 u_1}\right) dy \qquad \delta_\varphi = \int_0^h \left(1 - \frac{\rho w}{\rho_1 w_1}\right) dy \tag{12}$$

Because the outer flow is conical, ρ_1, u_1, and w_1 are functions only of φ; and equation (11) may be written

$$\rho_1 u_1 \sin\Theta \frac{\partial}{\partial x}\left[x(\Delta-\delta_x)\right] + \frac{\partial}{\partial \varphi}\left[\rho_1 w_1(\Delta-\delta_\varphi)\right] = 0 \tag{13}$$

In the case of conical outer flow, the associated boundary-layer profiles show similarity of the Blasius type in meridional planes (see reference 7 or 8). Thus, in a meridional plane, Δ, δ_x, and δ are proportional to $\sqrt{x}$. Incorporation of this information into equation (13) gives

$$\frac{3}{2}\rho_1 u_1 \sin\Theta\,(\Delta-\delta_x) + \frac{\partial}{\partial \varphi}\left[\rho_1 w_1(\Delta-\delta_\varphi)\right] = 0 \tag{14}$$

For a cone at small angle of attack α, u_1 is nearly equal to $\bar{u}$, the velocity on the cone surface at zero angle of attack. The quantities ρ_1, Δ, δ_x, and δ_φ vary only slightly with angle of attack, whereas

$$w_1 = \alpha\bar{u} A_2 \sin \varphi \tag{15}$$

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where A_2 depends only on the cone vertex angle and the flow Mach number and is defined in reference 6 as follows:

$$A_2 = -\frac{z}{\bar{u}} - \frac{2x}{\bar{u} \sin \Theta}$$

The quantities z, x, and $\bar{u}$ are in the notation of reference 9, wherein they are tabulated.

To the first order in α, the substitution of equation (15) into equation (14) yields the result

$$\Delta = \delta_x + \frac{2}{3} \frac{\alpha}{\sin \Theta} A_2(\delta_\varphi - \delta_x) \cos \varphi \tag{16}$$

The analysis of reference 6 yields the values of δ_x and δ_φ.

Clearly, Δ differs from δ_x in the plane of symmetry $\varphi = 0, \pi$, where the circumferential velocity w vanishes. It might, however, be noted that in the plane $\varphi = \frac{\pi}{2}, \frac{3\pi}{2}$, $\Delta = \delta_x$. Figure 6 shows the proportional rate of increase of displacement thickness Δ with angle of attack in the plane of symmetry $\varphi = \pi$ at zero angle of attack. The corresponding rate of increase of the mass-flow defect δ_x is shown for comparison. These curves are obtained from equation (16) and the results of reference 6. The sine of the semivertex angle is introduced as a factor primarily to permit presentation of the case $\Theta = 0$ as the limit of an indeterminate form. For a stream Mach number of 2, the change in displacement thickness appears to be of the order of 50 percent larger than the change in the meridional mass-flow defect.

CONCLUDING REMARKS

The foregoing analysis deals only with the displacement effect of a known boundary layer on the nonviscous outer flow, and hence can be applied only if the boundary-layer behavior has been determined either theoretically or experimentally. The latter approach might possibly

find application in the correction of nozzle contours for boundary-layer development. In this connection, perhaps, it should be noted that the analysis is not restricted to laminar flows.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, March 6, 1952

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APPENDIX - SYMBOLS

The following symbols are used in this report:

A_2 function of cone angle and Mach number (equation (15))

h height above wall at which ρ, u, $w = \rho_1$, u_1, w_1, and beyond which nonviscous equations apply (equation (3))

M_0 stream Mach number

$\underline{q}$ velocity vector composed of components parallel to body surface u,w

u velocity component in x-direction

$\bar{u}$ meridional velocity component at surface of cone at zero angle of attack

v velocity component in y-direction

v_{bl} boundary-layer solution for velocity normal to surface, evaluated at outer edge of boundary layer

v_{of} nonviscous solution for velocity normal to surface, evaluated near the surface

w velocity component in z-direction

x, z coordinates in body surface

y coordinate normal to surface

α angle of attack

Δ height above body of displacement surface

δ^* displacement thickness in plane flow

δ_x length characterizing mass-flow defect of u-profile (equations (2), (12))

δ_z length characterizing mass-flow defect of w-profile (equation (2))

δ_φ length characterizing mass-flow defect of w-profile (equation (12))

Θ semivertex angle of cone

ρ density

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φ angular coordinate (fig. 5)

Subscript 1 denotes evaluation of nonviscous flow at body surface, taken equivalent to conditions at outer edge of a boundary layer of infinitesimal thickness.

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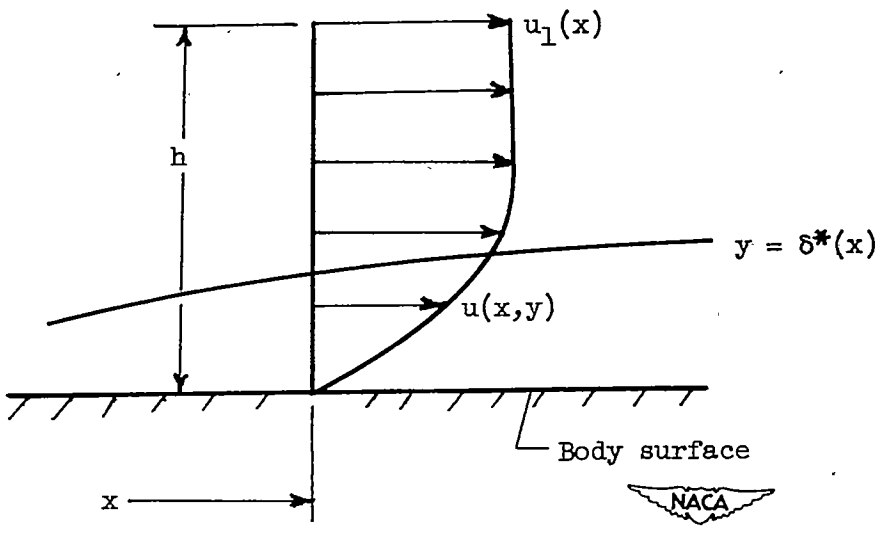

Figure 1. - Plane boundary-layer flow.

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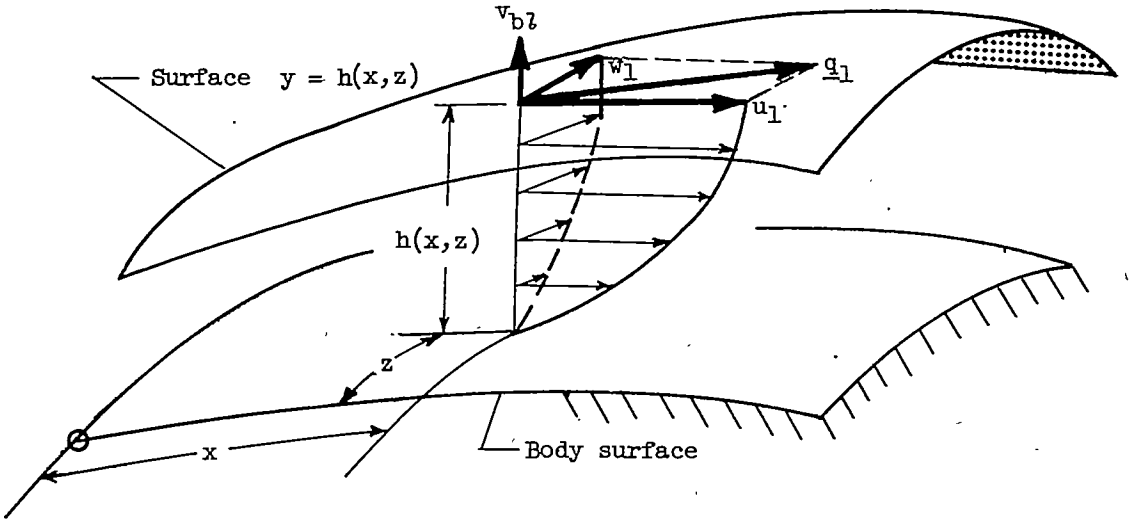

Figure 2. - Three-dimensional boundary-layer flow (Cartesian coordinates).

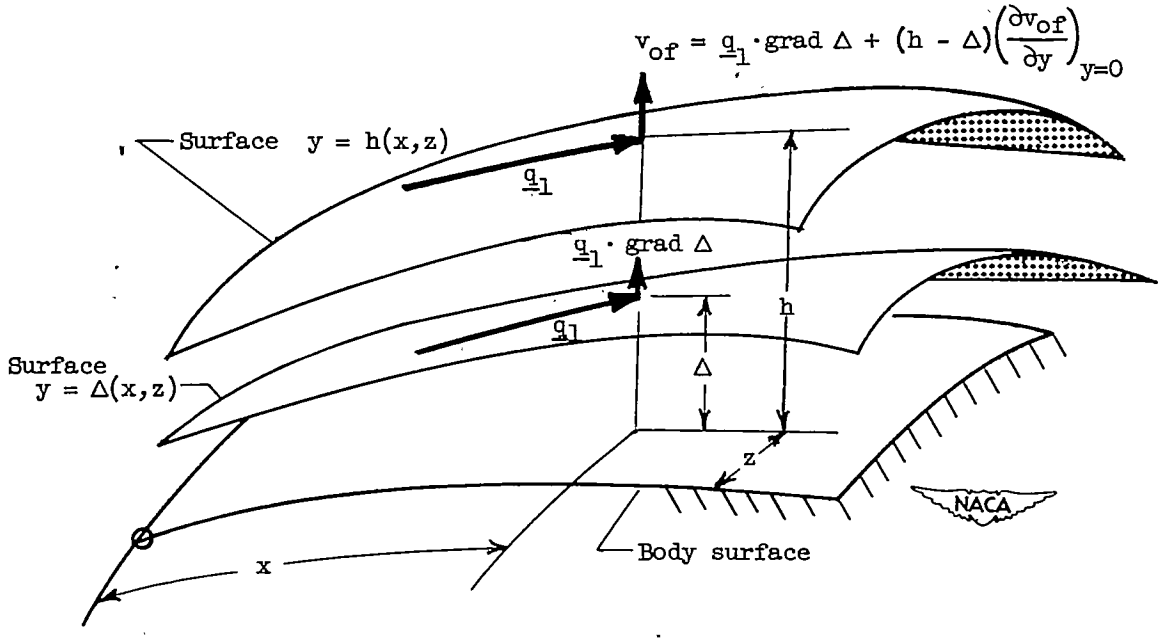

Figure 3. - Displacement surface in three-dimensional flow (Cartesian coordinates).

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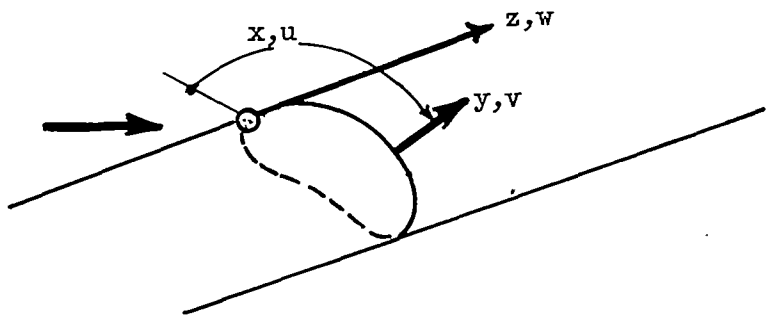

Figure 4. - Coordinate system for yawed infinite cylinder.

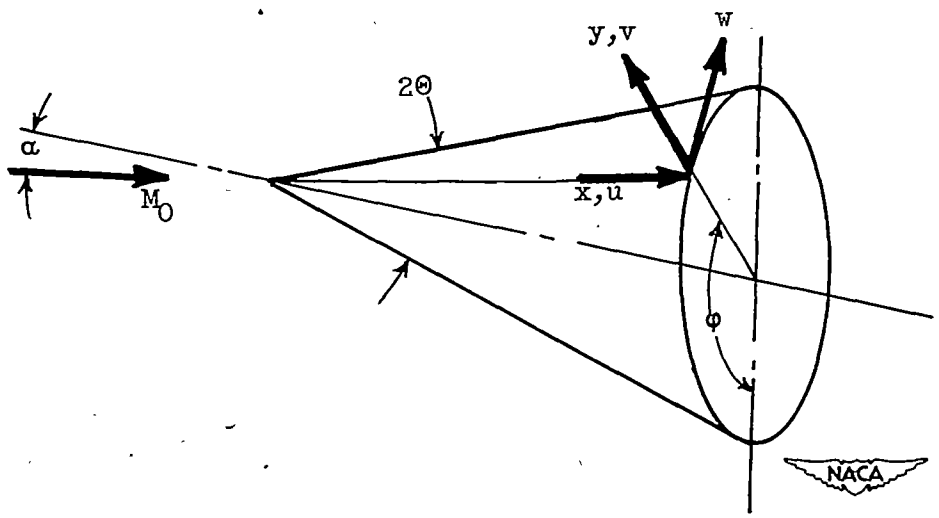

Figure 5. - Coordinate system for cone at angle of attack.

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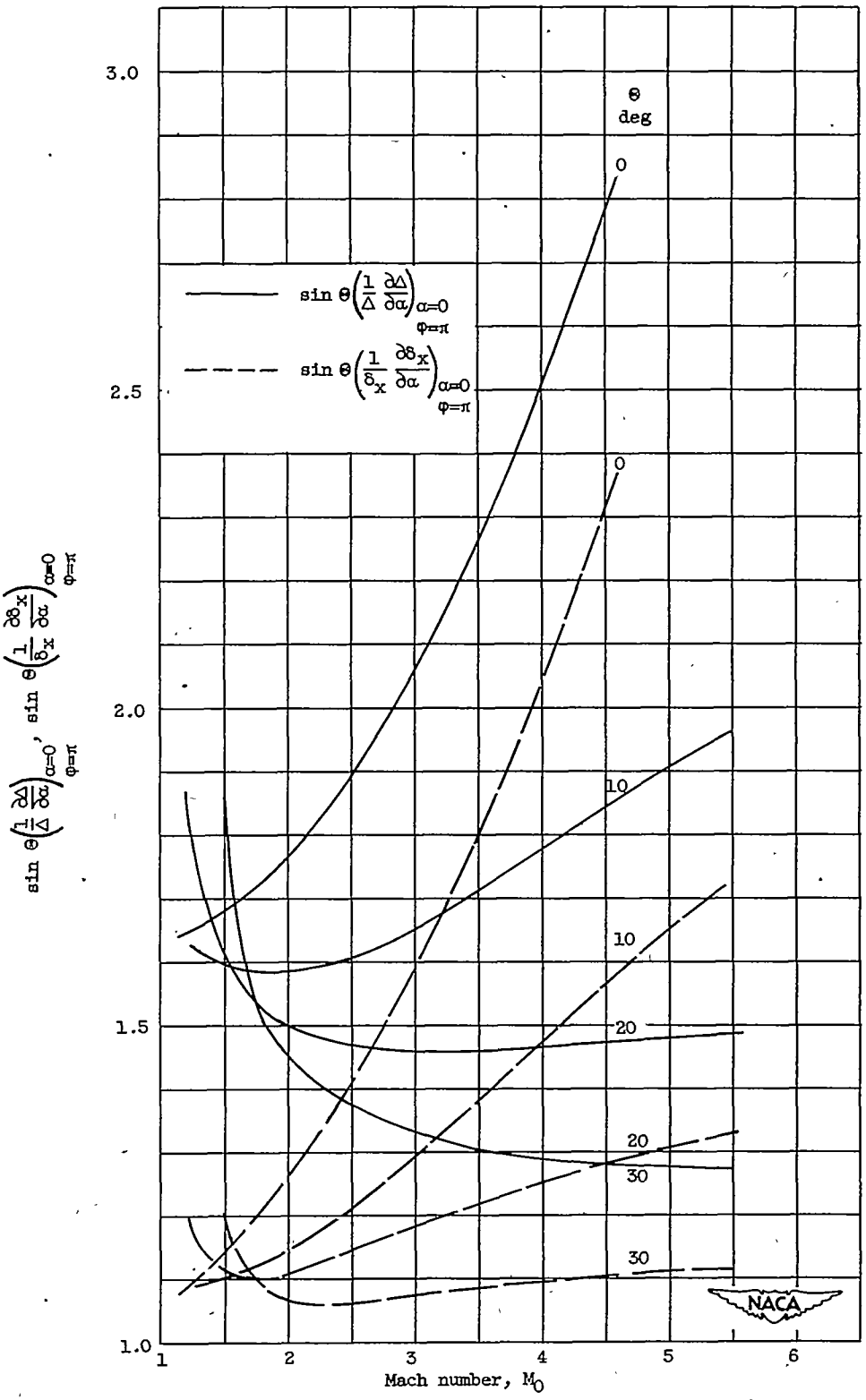

Figure 6. - Proportional rate of increase of displacement thickness with angle of attack on a cone.

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