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TECHNICAL NOTE 2820

AN ANALYSIS OF THE ERRORS IN CURVE-FITTING PROBLEMS WITH AN APPLICATION TO THE CALCULATION OF STABILITY PARAMETERS FROM FLIGHT DATA

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SUMMARY

The problem of assessing the errors in the parameters obtained from a curve-fitting process is considered, and a scheme which may be applied toward the solution of such problems is obtained. This method is then specialized to the problem of finding the errors in the calculated stability parameters of an airplane, and an example is given.

INTRODUCTION

Curve-fitting procedures have found places in nearly all branches of engineering; in particular, the aeronautical engineer may apply these methods to the calculation, from flight data, of the stability parameters of an airplane (references 1 and 2). Whether least squares or any of the profusion of graphical methods which exists is used for this curve-fitting process, questions of the errors in the calculated parameters are bound to arise.

Although there is a considerable amount of literature on the subject of least squares and curve fitting, comparatively little is to be found on the related subject of errors. What literature does exist (e.g., reference 3) attacks the problem from the point of view of statistics, arriving, finally, at a quantity called the variance. This quantity, while giving a satisfactory reply to the error question when applied to fitting a set of data to a straight line when only one measurement is subject to error, is far from adequate for other curve-fitting problems. One does not have to look far to find the reason for this; it is that no method has as yet been devised for calculating the variance when either the fitted curve is not linear or when more than one measured quantity is subject to error. This latter objection is not pertinent, perhaps, when the problem of calculation of stability parameters is considered, for, although both the input (control-surface deflection) and the response

may be subject to error, it is frequently assumed that only the output is fallible. This approximation is particularly good when only free-oscillation data are analyzed. The first objection is, however, more serious, for, assuming even that all quantities remain within the so-called linear range, so that the response satisfies a linear differential equation, the response will certainly not be a linear function of the parameters.

It is shown in the body of this report that the solution to the error problem depends upon a more or less arbitrary definition, the only criterion as to the choice of the definition being the usefulness of the solution to which it leads. In this respect, it may be said of the method derived herein that it is not difficult to apply and appears to lead to reasonable values of the errors. A relationship between the error formula given herein and the classical formula for the variance (in the linear case when such a formula exists) is established.

ANALYSIS

Relation of the Problem to Aerodynamics

Suppose one has a set of data which represents a time history of, for example, pitching velocity of an airplane in response to an elevator deflection $\delta(t)$. If $q(t)$ represents the pitching velocity, then, under certain simplifying assumptions, it is shown in reference 1 that $q(t)$ satisfies the differential equation

$$\frac{d^2q}{dt^2}+b\frac{dq}{dt}+kq = C_1\frac{d\delta}{dt}+C_0\delta \tag{1}$$

where the constants b, k, C_1 and C_0 are functions of the stability derivatives of the airplane.

It may be verified by differentiation that the following function represents the general solution of equation (1):

$$q(t)=\left[A_1+\frac{C_1\lambda_1+C_0}{\lambda_1-\lambda_2}\int_0^t e^{-\lambda_1\tau}\delta(\tau)d\tau\right]e^{\lambda_1 t}+\left[A_2+\frac{C_1\lambda_2+C_0}{\lambda_2-\lambda_1}\int_0^t e^{-\lambda_2\tau}\delta(\tau)d\tau\right]e^{\lambda_2 t} \tag{2}$$

where A_1 and A_2 are constants depending on the initial conditions $q(o)$, $(dq/dt)_{t=o}$, and $\delta(o)$, while λ_1 and λ_2 are the roots of the characteristic equation

$$\lambda^2 + b\lambda + k = 0$$

It is often desirable to be able to ascertain the "best" values of the constants b and k (or λ_1 and λ_2), C_o and C_1 corresponding to the given time histories of $q(t)$ and $\delta(t)$. This problem is the subject of references 1 and 2.

Suppose then, that by some means these constants have been evaluated. There will be a certain "error" in these values, however, due to the experimental error in the data, the simplifying assumptions mentioned above, and other causes. The question of the magnitudes of these errors, given a certain error in the data, is considered in the present report.

General Discussion of the Problem

In order to state adequately the problem of errors, it is first necessary to give a precise statement of the curve-fitting problem. To this end, consider a physical quantity $q_\epsilon(t)$ which is measured at $t = t_o$, $t_1, \ldots, t_N$, where $t_o < t_1 < \ldots < t_N$. It is assumed that various theoretical considerations would indicate that $q_\epsilon(t)$ should be one of the functions of the set $q(t_1 x_1, x_2, \ldots, x_m)$; that is, there should exist values of the parameters $x_1, x_2, \ldots, x_m$ such that

$$q_\epsilon(t_i) = q(t_i, x_1, x_2, \ldots, x_m) \tag{3}$$

for all $i = 0, 1, \ldots, N$. However, because of certain unknown errors in q_ϵ, equation (3) is not exactly satisfied for all i. It is desirable then to find those values of the parameters x_k which "most nearly" cause equation (3) to be satisfied. One means of doing this is to define the "best" values of $x_1, \ldots, x_m$ as those values of the parameters which make

$$M = \sum_{i=o}^{N} [q(t_i, x_1, x_2, \ldots, x_m) - q_\epsilon(t_i)]^2 \tag{4}$$

a minimum. The process of minimizing M is called curve fitting by least squares, and this general problem is considered in reference 2.

As for the error problem, a careful study of the extant literature on statistics will lead one to the conclusion that the "true" value of a quantity, upon which the intuitive definition of error rests, has never been meaningfully defined. Clearly, however, the values obtained for the errors in a particular problem will depend upon this definition. When the usual probabilistic statement of the error problem is chosen, the quantity called the variance arises, along with the objections raised in the introduction which are concomitant with it.

Various nonprobabilistic statements of the problem are conceivable, and the statement given below has been chosen from among them as a useful one. These nonprobabilistic statements are based on the concept of the "sensitivity" of the function $q(t,x_1,\ldots,x_m)$ with respect to the parameters. That is, the question is asked: If one of the parameters is changed slightly, does q also change only slightly, or does q change by a large amount? If a small change in a parameter ends in a large change in q, q is said to be sensitive to changes in that parameter, and it appears clear that any meaningful theory of errors should lead to relatively small errors in such a parameter. On the other hand, if q changes by only a small amount when a parameter is changed, q is insensitive with respect to that parameter, and the theory should result in a large error. A quantitative discussion follows.

Statement and Solution of the Problem

In place of the elusive term "experimental error," we shall introduce the concept of "residual," defined by the equation

$$\epsilon_i = q(t_i,x_1,x_2,\ldots,x_m) - q_\epsilon(t_i) \tag{5}$$

so that equation (4) may be written in the equivalent form

$$M = \sum_{i=o}^{N} \epsilon_i^2 \tag{4}$$

Suppose the curve-fitting problem has been solved; that is, suppose values $x_1^{(o)}, x_2^{(o)}, \ldots, x_m^{(o)}$ of the parameters $x_1, x_2, \ldots, x_m$, respectively, have been found which minimize M. Let

$$\epsilon_i^{(o)} = q\left[t_i,x_1^{(o)},x_2^{(o)},\ldots,x_m^{(o)}\right] - q_\epsilon(t_i)$$

so that the minimum value of M is

$$M^{(o)} = \sum_{i=o}^{N} \left[\epsilon_i^{(o)}\right]^2$$

One possible statement of the error problem, which, with certain modifications, will be used in this report, is the following. Choose any set of numbers $\Delta x_1, \Delta x_2, \ldots, \Delta x_m$, and let

$$x_k^{(1)} = x_k^{(o)} + \Delta x_k \qquad (k=1,2,\ldots,m)$$

The question then is: How large may the quantities $|\Delta x_k|$ be so that

$$\left|q\left[t_i,x_1^{(1)},x_2^{(1)},\ldots,x_m^{(1)}\right] - q\left[t_i,x_1^{(o)},x_2^{(o)},\ldots,x_m^{(o)}\right]\right| \leq \left|\epsilon_i^{(o)}\right| \quad (6)$$

for all $i = 0, 1, \ldots, N$? This maximum value of $|\Delta x_k|$ may then be defined as the allowable error in x_k. This problem can be solved, but various objections can be raised as to its significance as a statement of the error problem. In particular, it is easy to see that this estimate of the errors may be far too optimistic, for if only m out of the total of $(N+1)$ values of $\epsilon_i^{(o)}$ are zero, then the only way in which inequality (6) may be satisfied for all i is for $x_k^{(1)} = x_k^{(o)}$ for all k. That is, since the number m of parameters is usually far less than the number, $(N+1)$, of data points, we can say that even though most of the residuals be arbitrarily large, the result is that the error in the parameters is zero!

This objection may be overcome by relaxing the requirement (6), substituting in its place a mean-square inequality obtained by squaring both sides of (6) and summing over i. The resulting inequality will then be required to hold instead of the inequality (6). Symbolically, it will be required that

$$\sum_{i=o}^{N}\left\{q\left[t_i,x_1^{(1)},\ldots,x_m^{(1)}\right] - q\left[t_i,x_1^{(o)},\ldots,x_m^{(o)}\right]\right\}^2 \leq \sum_{i=o}^{N}\left[\epsilon_i^{(o)}\right]^2 = M^{(o)} \quad (7)$$

Thus, condition (6) is made to hold only in a mean-square sense over the whole range of t, while it may not be true for some particular values of t_i. An allowable error in x_k is then defined as any value of Δx_k for which inequality (7) holds.

This last could reasonably be used as the definition of the error in x_k. In the interest of ease in calculation, however, a Taylor's series expansion, with only the first-order terms being retained, will be used to linearize the problem. This may be done since neither the residuals $\epsilon_i^{(o)}$ nor the errors Δx_k may be too large; if they were, it may be said that there is something wrong with the theory which predicted that equation (3) will be approximately true, or with the experiment leading to the data $q_\epsilon(t_i)$, or that the experiment has not been properly designed. In order to shorten the formulas, the following notation will be used:[1]

[1]For the sake of clarity, it should be mentioned that two different types of subscript are used in this report, attached to the same quantity q. Of the experimental data q_ϵ, we have already spoken; in addition, the quantities q_k are now defined by equations (8). There need be no confusion, however, for the subscript ϵ will always be used to denote the data; other subscripts will be used as defined.

$$\left.\begin{aligned} q(t_i) &= q\left[t_i, x_1^{(o)}, \ldots, x_m^{(o)}\right] \\ q_k(t_i) &= \frac{\partial q\left[t_i, x_1^{(o)}, \ldots, x_m^{(o)}\right]}{\partial x_k}, \quad (k=1,2,\ldots,m) \end{aligned}\right\} \tag{8}$$

We shall also write $\Delta q(t_i)$ in place of the difference

$$q\left[t_i, x_1^{(1)}, \ldots, x_m^{(1)}\right] - q\left[t_i, x_1^{(o)}, \ldots, x_m^{(o)}\right]$$

so that the inequality (7) defining the errors in the x_k becomes

$$\sum_{i=o}^{N} \left[\Delta q(t_i)\right]^2 \leq M^{(o)} \tag{7}$$

If the errors in the parameters are not too large, it is clear that the following equation is approximately true:

$$\Delta q(t_i) = \sum_{k=1}^{m} q_k(t_i)\Delta x_k$$

Utilizing this linearization process, inequality (7) becomes

$$\sum_{i=o}^{N} \left[\sum_{k=1}^{m} q_k(t_i)\Delta x_k\right]^2 \leq M^{(o)} \tag{9}$$

The modifications of the definition of errors have finally been completed. We now define the error in x_k as the largest value of $|\Delta x_k|$ for which inequality (9) holds, regardless of the values of the other Δx's.

In order to be able to draw some pictures, it will now be assumed that $m = 2$. The generalization to larger m will be presented afterwards. Thus, it is assumed that the function q is dependent on two parameters only:

$$q = q(t_i, x_1, x_2)$$

For brevity in the formulas, we shall write ξ_k for Δx_k and shall define

$$\left.\begin{aligned} q_{11} &= \sum_{i=0}^{N} [q_1(t_i)]^2 \\ q_{12} &= \sum_{i=0}^{N} q_1(t_i)q_2(t_i) \\ q_{22} &= \sum_{i=0}^{N} [q_2(t_i)]^2 \end{aligned}\right\} \tag{10}$$

The inequality (9) defines a certain domain in the (ξ_1,ξ_2) plane whose boundary is given by the following equation:

$$q_{11}\xi_1^2 + 2q_{12}\xi_1\xi_2 + q_{22}\xi_2^2 = M^{(0)} \tag{11}$$

The graph of equation (11) is either an ellipse, a parabola, or an hyperbola. It is clear that if the definition of error which has been chosen is significant, equation (11) must represent an ellipse, for if it did not, infinite errors would be obtained. It is just as well that the proof that equation (11) does indeed represent an ellipse be given here, for it is very simple. Consider the discriminant

$$q_{11}q_{22} - q_{12}^2$$

of equation (11), which becomes, using equations (10),

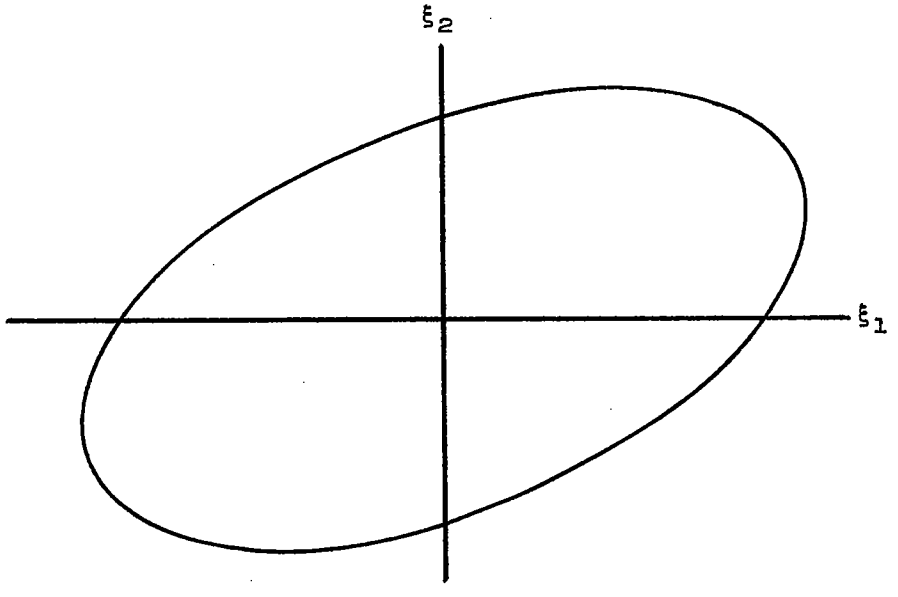
$$q_{11}q_{22} - q_{12}^2 = \sum_{i=0}^{N} [q_1(t_i)]^2 \sum_{i=0}^{N} [q_2(t_i)]^2 - \left[\sum_{i=0}^{N} q_1(t_i)q_2(t_i)\right]^2$$

The curve in question is an ellipse if and only if the discriminant is positive. However, the inequality

$$\sum_{i=0}^{N} [q_1(t_i)]^2 \sum_{i=0}^{N} [q_2(t_i)]^2 - \left[\sum_{i=0}^{N} q_1(t_i)q_2(t_i)\right]^2 \geq 0$$

is precisely the well-known Schwarz inequality. As is also well known, the equality cannot occur unless $q_1(t_i)$ is proportional to $q_2(t_i)$ for all i, and as this implies that the parameters x_1 and x_2 are not independent, it can be assumed, without loss of generality, that $q_1(t_i)$ and $q_2(t_i)$ are not proportional. This completes the proof.

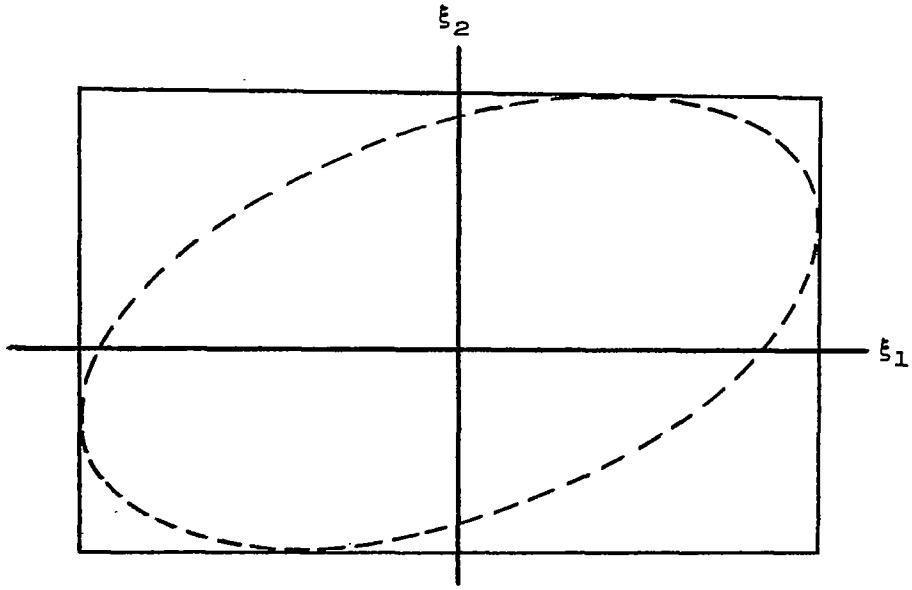
If the ellipse which represents equation (11) should be drawn, a graph similar to the one shown in sketch (a) would be obtained.



Sketch (a)

Since there are no first-order terms in equation (11), the ellipse must be symmetric with respect to the origin. Interpreting this curve, it may be said that any point which lies inside the ellipse has coordinates which define an allowable error in x_1 and x_2. However, in the definition of the error in the parameters, it was said that the error in x_k is the largest value of $|\Delta x_k|$ for which inequality (9) holds, regardless of the values of the other Δx's. For this reason, in order

to define the error in the parameters by means of the diagram, we must enclose the ellipse of sketch (a) in a rectangle, as in sketch (b).



Sketch (b)

Any point lying within this rectangle is considered as defining an allowable error. If Ξ_1 and Ξ_2 denote the errors (i.e., the maximum allowable errors) in x_1 and x_2, respectively, then Ξ_1 and Ξ_2 are obtained as the maximum values of ξ_1 and ξ_2 lying in the rectangle.

If $m = 2$ as in the above example, the errors can always be found graphically. However, if $m > 2$, an analytical method must be used. Such a method will now be found for the case where $m = 2$, and the generalization will then be shown.

It is clear that the sides of the rectangle drawn as solid lines in sketch (b) are the tangents to the ellipse at the points where

$$\left.\begin{aligned} \frac{d\xi_1}{d\xi_2} &= 0 \\ \frac{d\xi_2}{d\xi_1} &= 0 \end{aligned}\right\}$$

Calculating these derivatives from equation (11), we obtain

$$\left.\begin{aligned}\frac{d\xi_1}{d\xi_2} &= -\frac{q_{12}\,\xi_1 + q_{22}\,\xi_2}{q_{11}\,\xi_1 + q_{12}\,\xi_2}\\ \frac{d\xi_2}{d\xi_1} &= -\frac{q_{11}\,\xi_1 + q_{12}\,\xi_2}{q_{12}\,\xi_1 + q_{22}\,\xi_2}\end{aligned}\right\}$$

Setting these derivatives equal to zero results in the equations

$$\left.\begin{aligned}q_{12}\,\xi_1 + q_{22}\,\xi_2 &= 0\\ q_{11}\,\xi_1 + q_{12}\,\xi_2 &= 0\end{aligned}\right\} \tag{12}$$

These equations are not to be solved simultaneously, for they define errors in different parameters. Instead, the first of equations (12) is to be solved simultaneously with equation (11) to find Ξ_1, the maximum value of ξ_1. Similarly, the second of equations (12) is to be solved along with equation (11) to yield the error, Ξ_2, in x_2. Performing these operations, one obtains the following expressions for Ξ_1 and Ξ_2:

$$\left.\begin{aligned}\Xi_1 &= \sqrt{\frac{M^{(o)}q_{22}}{q_{11}\,q_{22} - q_{12}^{\,2}}}\\ \Xi_2 &= \sqrt{\frac{M^{(o)}q_{11}}{q_{11}\,q_{22} - q_{12}^{\,2}}}\end{aligned}\right\} \tag{13}$$

In the general case when $m \geq 2$, the following equation occurs in place of equation (11):

$$\sum_{j,k=1}^{m} q_{jk}\,\xi_j\,\xi_k = M^{(o)} \tag{14}$$

where

$$q_{jk} = \sum_{i=o}^{N} q_j(t_i)\,q_k(t_i) \tag{15}$$

and the quantities $q_j(t_i)$, $q_k(t_i)$ are defined in equation (8). If Ξ_h denotes the maximum value of ξ_h satisfying equation (14), so that Ξ_h is the error in x_h, the problem is to find Ξ_h. It may be solved by setting the (m-1) derivatives of the form

$$\frac{\partial \xi_h}{\partial \xi_j} \qquad (j=1,2,\ldots,h-1,h+1,\ldots,m)$$

equal to zero and solving the resulting equations along with equation (14) for ξ_h. One need not go through this process each time the errors in a problem are to be calculated, for it can be done once and for all as follows: Differentiating equation (14), it may be seen that

$$\frac{\partial \xi_h}{\partial \xi_j} = -\frac{\sum_{k=1}^{m} q_{jk}\xi_k}{\sum_{k=1}^{m} q_{hk}\xi_k}$$

Setting these derivatives equal to zero, we obtain the following (m-1) equations which are to be solved simultaneously along with equation (14) for ξ_h:

$$\sum_{k=1}^{m} q_{jk}\xi_k = 0, \quad j=1,2,\ldots,h-1,h+1,\ldots,m \tag{16}$$

Equation (14) can be written in the form

$$\sum_{j=1}^{m}\left(\sum_{k=1}^{m} q_{jk}\xi_k\right)\xi_j = M^{(o)} \tag{14}$$

Taking equation (16) into account, it may be seen that the parenthesized quantity is zero except when $j=h$, and so we obtain the equation

$$\sum_{k=1}^{m} q_{hk}\xi_k = \frac{M^{(o)}}{\xi_h} \tag{17}$$

This equation and equation (16) are m linear equations to be solved for ξ_h. This may be done, in general, by means of Cramer's rule (the method of determinants).

Let

$$D = \begin{vmatrix} q_{11} & q_{12} & \cdots & q_{1m} \\ q_{21} & q_{22} & \cdots & q_{2m} \\ \cdot & \cdot & \cdot & \cdot \\ q_{m1} & q_{m2} & \cdots & q_{mm} \end{vmatrix} \tag{18}$$

and let D_h be the minor of q_{hh} in D, so that

$$D_h = \begin{vmatrix} q_{11} & q_{12} & \cdots & q_{1(h-1)} & q_{1(h+1)} & \cdots & q_{1m} \\ q_{21} & q_{22} & \cdots & q_{2(h-1)} & q_{2(h+1)} & \cdots & q_{2m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{(h-1)1} & q_{(h-1)2} & \cdots & q_{(h-1)(h-1)} & q_{(h-1)(h+1)} & \cdots & q_{(h-1)m} \\ q_{(h+1)1} & q_{(h+1)2} & \cdots & q_{(h+1)(h-1)} & q_{(h+1)(h+1)} & \cdots & q_{(h+1)m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ q_{m1} & q_{m2} & \cdots & q_{m(h-1)} & q_{m(h+1)} & \cdots & q_{mm} \end{vmatrix} \tag{19}$$

Then, by virtue of equations (16) and (17),

$$\xi_h = \frac{M^{(o)}}{\xi_h} \cdot \frac{D_h}{D}$$

That is,

$$\Xi_h = \sqrt{\frac{M^{(o)} D_h}{D}} \tag{20}$$

Equation (20) is the desired formula for the error in x_k.

Connection of formula (20) with the variance concept.- Although the formula (20) for the errors was obtained without the aid of statistical notions, it may now be shown that a relationship exists between the parametric errors as defined herein and the concept of variance.

Suppose that the given approximating function $q(t)$ were linear in the parameters, so that

$$q(t_i) = q_1(t_i)x_1 + q_2(t_i)x_2 + \cdots + q_m(t_i)x_m$$

where the functions $q_j(t_i)$ are independent of the parameters x_k. Differentiating this last equation, we have that

$$\frac{\partial q(t_i)}{\partial x_k} = q_k(t_i) \qquad (k=1, \ldots, m)$$

and, utilizing equation (15), that

$$q_{jk} = \sum_{i=0}^{N} q_j(t_i)q_k(t_i)$$

In reference 3, the familiar normal equations obtained from the least-squares process are shown to be

$$\sum_{j=1}^{m} q_{jk}x_j = q_{\epsilon,k} \tag{21}$$

where[2]

$$q_{\epsilon,k} = \sum_{i=0}^{N} q_\epsilon(t_i)q_k(t_i)$$

and $q_\epsilon(t_i)$ represents the given data. Let D denote the determinant of the coefficients in equation (21), and let D_h be the minor of q_{hh} in D, so that D and D_h are given by equations (18) and (19), respectively. Then, if σ^2 denotes the mean-square error in the data, while $\sigma_h{}^2$ denotes the mean-square error to be expected in the parameter x_h (i.e., σ^2 and $\sigma_h{}^2$ are the variances of the data and of x_h, respectively), it is shown in reference 3 that if $q(t_i)$ is a linear function of the parameters,

$$\sigma_h = \sigma\sqrt{\frac{D_h}{D}} \tag{22}$$

[2]The comma is placed after the subscript ϵ to indicate that it is a different sort of symbol from the other subscripts. Thus, j and k are to vary, assuming the values 1, ..., m; however, the subscript ϵ is used merely to indicate that the experimental data $q_\epsilon(t_i)$ is used in the definition of $q_{\epsilon,k}$.

Noting now that σ^2 affords an estimate for the error in the data when the statistical conception of the error problem is considered, while $M^{(o)}$ occupies a similar position in the present theory, a relationship between the variance in a parameter and what has herein been called the allowable error in that parameter can be established. Comparing equations (20) and (22), it may be seen that the ratio of the variance in a parameter to the statistical measure of the error in the data is the same as the ratio of the error in a parameter to the nonstatistical measure of the error in the data. In symbols,

$$\frac{\Xi_h^2}{M^{(o)}} = \frac{\sigma_h^2}{\sigma^2} \tag{23}$$

This is a remarkable fact that in formulating a theory of errors which abstains from the use of probability theory, a notion of error has still been defined which bears as intimate a relation to the statistical ideas as that described by equation (23). Of course, this relationship has been proved only in the case of linear curve-fitting problems. From this, however, it might appear reasonable that even for general nonlinear problems equations (20) and (23) (and, therefore, (22)) afford an expression for the variance. This leaves an interesting problem for future research.

There is an important conclusion to be drawn from equation (23). If, as before, $N + 1$ is the number of data points, then σ^2 and $M^{(o)}$ are related (reference 3) by the equation

$$\sigma^2 = \frac{M^{(o)}}{N+1}$$

Thus, from equation (23),

$$\Xi_h^2 = (N+1)\sigma_h^2$$

which implies that Ξ_h is larger than σ_h. This result should not be too surprising, for it will be recalled that σ_h^2 is a measure of the <u>probable</u> error in x_h, while we have tried, in Ξ_h, to define some sort of <u>maximum</u> error.

APPLICATION TO AN EXAMPLE

Referring to equation (2), if there is some value t_α of time such that $\delta(t) = 0$ for all $t \geq t_\alpha$ (i.e., if a pulse elevator input has been

applied to the airplane), it may be seen that for $t \geq t_\alpha$, the pitching velocity is a sum of two exponentials

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t}$$

t_i	$q_\epsilon(t_i)$
0.4	0.224
.5	.120
.6	.020
.7	-.057
.8	-.112
.9	-.148
1.0	-.160
1.1	-.150
1.2	-.127
1.3	-.097
1.4	-.062
1.5	-.032
1.6	-.005
1.7	.017
1.8	.030
1.9	.036
2.0	.035
2.1	.032
2.2	.027
2.3	.020
2.4	.015
2.5	.011
2.6	.008
2.7	.005
2.8	.003
2.9	.001
3.0	0
3.1	0
3.2	0

where B_1, B_2, λ_1 and λ_2 are constants depending on the stability derivatives of the airplane.

In general, such data are oscillatory; that is, $\lambda_1, \lambda_2, B_1$ and B_2 are complex. Let $\lambda_1 = l + l'i$, $B_1 = \frac{1}{2}(\beta + \beta' i)$, $i^2 = -1$. Since q is real, $\lambda_2 = l - l'i$, $B_2 = \frac{1}{2}(\beta - \beta' i)$, and

$$q = e^{lt}(\beta \cos l't - \beta' \sin l't) \tag{24}$$

The data given on this page are actual flight data which represent the pitching velocity of a test airplane in response to an elevator deflection which was zero for all $t \geq 0.4$ ($t = 0$ is taken at the beginning of the pulse). These data are fitted to a sum of exponentials in reference 2, where the results

$$\left.\begin{aligned} l &= -1.366 & \beta &= 0.614 \\ l' &= 3.071 & \beta' &= -0.208 \\ & & M^{(o)} &= 0.000895 \end{aligned}\right\} \tag{25}$$

were obtained. The errors in these parameters will now be calculated, utilizing equation (20) and substituting l, l', β, and β' for x_1, x_2, x_3, and x_4, respectively.

Calculation of the errors in l, l', β, and β'.- Taking derivatives from equation (24), one may see that

$$\left.\begin{aligned} \frac{\partial q}{\partial l} &= te^{lt}(\beta \cos l't - \beta' \sin l't) \\ \frac{\partial q}{\partial l'} &= -te^{lt}(\beta \sin l't + \beta' \cos l't) \\ \frac{\partial q}{\partial \beta} &= e^{lt} \cos l't \\ \frac{\partial q}{\partial \beta'} &= -e^{lt} \sin l't \end{aligned}\right\} \tag{26}$$

Identifying l with the parameter x_1, l' with x_2, β with x_3, and β' with x_4, it may be seen that $q_1(t_i)\left[=\frac{\partial q(t_i)}{\partial l}\right]$ is given in column 11 of table I and $q_2(t_i)$ is given by minus column 15, while $q_3(t_i)$ and $q_4(t_i)$ are given, respectively, by column 6 and by minus column 7. Therefore, referring to equations (15), and letting circled numbers refer to columns in table I, we have that

$$q_{11} = \Sigma\,(11)^2 = 0.168$$

$$q_{12} = q_{21} = -\Sigma\,(11) \times (15) = 0.006$$

$$q_{13} = q_{31} = \Sigma\,(6) \times (11) = 0.228$$

$$q_{14} = q_{41} = -\Sigma\,(7) \times (11) = -0.091$$

$$q_{22} = \Sigma\,(15)^2 = 0.212$$

$$q_{23} = q_{32} = -\Sigma\,(6) \times (15) = 0.117$$

$$q_{24} = q_{42} = \Sigma\,(7) \times (15) = 0.389$$

$$q_{33} = \Sigma\,(6)^2 = 0.415$$

$$q_{34} = q_{43} = -\Sigma\,(6) \times (7) = 0.051$$

$$q_{44} = \Sigma\,(7)^2 = 0.985$$

Inserting these numbers into the expression (18) for D, we obtain

$$D = \begin{vmatrix} 0.168 & 0.006 & 0.228 & -0.091 \\ .006 & .212 & .117 & .389 \\ .228 & .117 & .415 & .051 \\ -.091 & .389 & .051 & .985 \end{vmatrix} = 0.000343$$

Furthermore,

$$D_1 = \begin{vmatrix} 0.212 & 0.117 & 0.389 \\ .117 & .415 & .051 \\ .389 & .051 & .985 \end{vmatrix} = 0.0145$$

$$D_2 = \begin{vmatrix} 0.168 & 0.228 & -0.091 \\ .228 & .415 & .051 \\ -.091 & .051 & .985 \end{vmatrix} = 0.0115$$

$$D_3 = \begin{vmatrix} 0.168 & 0.006 & -0.091 \\ .006 & .212 & .389 \\ -.091 & .389 & .985 \end{vmatrix} = 0.00744$$

$$D_4 = \begin{vmatrix} 0.168 & 0.006 & 0.228 \\ .006 & .212 & .117 \\ .228 & .117 & .415 \end{vmatrix} = 0.00177$$

Substituting these numbers and the value of $M^{(o)}$ given in equations (25) into equation (20), we obtain

$$\left.\begin{aligned} |\Delta l| &= 0.194 \\ |\Delta l'| &= .173 \\ |\Delta \beta| &= .139 \\ |\Delta \beta'| &= .068 \end{aligned}\right\} \tag{27}$$

where Δl denotes the error in l, $\Delta l'$ in l', etc. On a percentage basis, this implies that l, l', β, and β', respectively, are known to within about 14, 6, 23, and 33 percent.

Calculation of the error in the stability parameters.- It has been shown how the errors in the parameters l, l', β, and β' may be found. However, the problem of interest to the aerodynamicist is the calculation of the errors in the stability parameters b, k, C_1, and C_0 of equation (1). While $q(t)$ can be written directly as a function of b, k, C_1, and C_0 instead of the parameters l, l', β, and β' and the method described above applied, it is believed that the following general considerations simplify the calculations.

Suppose, as before, that $q(t)$ is a function of the m parameters x_k, $k=1,\ldots,m$, and that the errors Δx_k in the x_k have been found. Suppose further that there are n other parameters $y_1,\ldots,y_n$, each of which is a function of x_1, $x_2,\ldots$, and x_m. Thus,

$$y_j = y_j(x_1,\ldots,x_m), \quad j=1,\ldots,n$$

The error Δy_j in y_j may then be estimated from the formula

$$\left|\Delta y_j\right| = \sum_{k=1}^{m} \left|\frac{\partial y_j}{\partial x_k} \Delta x_k\right| \tag{28}$$

It should be noted that equation (28) gives a quite pessimistic value of the errors.

From the definition of l and l', we may now write

$$\left.\begin{aligned} b &= -2l \\ k &= l^2 + l'^2 \end{aligned}\right\} \tag{29}$$

and so,

$$\frac{\partial b}{\partial l} = -2, \frac{\partial b}{\partial l'} = \frac{\partial b}{\partial \beta} = \frac{\partial b}{\partial \beta'} = 0$$

Also,

$$\frac{\partial k}{\partial l} = 2l \qquad \frac{\partial k}{\partial \beta} = 0$$

$$\frac{\partial k}{\partial l'} = 2l' \qquad \frac{\partial k}{\partial \beta'} = 0$$

Thus, applying equation (28) using the error values (27), we see that

$$|\Delta b| = 2\,|\Delta l| = 0.388$$

$$|\Delta k| = 2\,|l\Delta l| + 2\,|l'\Delta l'| = 1.59$$

and since, from equations (29),

$$\left.\begin{aligned} b &= 2.732 \\ k &= 11.30 \end{aligned}\right\}$$

it follows that b and k are known to within about 14 percent.

This same scheme will now be applied to the computation of the errors in C_1 and C_0. It will be noticed that since C_1 and C_0 are quite complicated functions of l and l', our task will be considerably more difficult than it was when computing $|\Delta b|$ and $|\Delta k|$.

As before, suppose the input $\delta(t)$ is zero for all t greater than t_α. Make the following definitions:

$$\left.\begin{aligned}
S_i &= \int_{t_i}^{t_\alpha} e^{-l\tau}\delta(\tau)\cos l'\tau d\tau \\
S_i' &= \int_{t_i}^{t_\alpha} e^{-l\tau}\delta(\tau)\sin l'\tau d\tau \\
R_i &= \int_{t_i}^{t_\alpha} \tau e^{-l\tau}\delta(\tau)\cos l'\tau d\tau \\
R_i' &= \int_{t_i}^{t_\alpha} \tau e^{-l\tau}\delta(\tau)\sin l'\tau d\tau
\end{aligned}\right\} \tag{30}$$

so that

$$\left.\begin{aligned}
\frac{\partial S_i}{\partial l} &= -\frac{\partial S_i'}{\partial l'} = -R_i \\
\frac{\partial S_i}{\partial l'} &= \frac{\partial S_i'}{\partial l} = -R_i'
\end{aligned}\right\} \tag{31}$$

Also define

$$\left.\begin{aligned}
\alpha_{oi} &= e^{lt_i}(S_i'\cos l't_i - S_i\sin l't_i) \\
\alpha_{1i} &= -e^{lt_i}[(l'S_i - lS_i')\cos l't_i + (lS_i + l'S_i')\sin l't_i] \\
\alpha_i &= l'[q_\epsilon(t_i) - e^{lt_i}(\beta\cos l't_i - \beta'\sin l't_i)]
\end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned}
a_{oo} &= \sum_{i=o}^{N}\alpha_{oi}^2 \qquad a_{o1} = \sum_{i=o}^{N}\alpha_{oi}\alpha_{1i} \qquad a_o = \sum_{i=o}^{N}\alpha_{oi}\alpha_i \\
a_{1o} &= \sum_{i=o}^{N}\alpha_{1i}\alpha_{oi} \qquad a_{11} = \sum_{i=o}^{N}\alpha_{1i}^2 \qquad a_1 = \sum_{i=o}^{N}\alpha_{1i}\alpha_i
\end{aligned}\right\} \tag{33}$$

With these definitions, it is shown in reference 2 that C_o and C_1 satisfy the equations

$$\left.\begin{aligned}
a_{oo}C_o + a_{o1}C_1 &= a_o \\
a_{1o}C_o + a_{11}C_1 &= a_1
\end{aligned}\right\} \tag{34}$$

Therefore, if p denotes any one of the four parameters l, l', β, β', the derivatives $\partial C_o/\partial p$ and $\partial C_1/\partial p$ can be found from the equations

$$\left.\begin{aligned} a_{oo}\frac{\partial C_o}{\partial p} + a_{o1}\frac{\partial C_1}{\partial p} &= \frac{\partial a_o}{\partial p} - C_o\frac{\partial a_{oo}}{\partial p} - C_1\frac{\partial a_{o1}}{\partial p} \\ a_{1o}\frac{\partial C_o}{\partial p} + a_{11}\frac{\partial C_1}{\partial p} &= \frac{\partial a_1}{\partial p} - C_o\frac{\partial a_{1o}}{\partial p} - C_1\frac{\partial a_{11}}{\partial p} \end{aligned}\right\} \tag{35}$$

The problem before us is the calculation of these derivatives.

From the definitions (32) and from equations (31), it follows that

$$\left.\begin{aligned} \frac{\partial \alpha_{oi}}{\partial l} &= e^{lt_i}[(t_iS_i' - R_i')\cos l't_i - (t_iS_i - R_i)\sin l't_i] \\ \frac{\partial \alpha_{1i}}{\partial l} &= t_i\alpha_{1i} + \alpha_{oi} + e^{lt_i}[(l'R_i - lR_i')\cos l't_i + (lR_i + l'R_i')\sin l't_i] \\ \frac{\partial \alpha_i}{\partial l} &= -l't_ie^{lt_i}(\beta\cos l't_i - \beta'\sin l't_i) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{\partial \alpha_{oi}}{\partial l'} &= e^{lt_i}[(R_i - t_iS_i)\cos l't_i + (R_i' - t_iS_i')\sin l't_i] \\ \frac{\partial \alpha_{1i}}{\partial l'} &= -e^{lt_i}\Big\{[S_i + l(t_iS_i - R_i) + l'(t_iS_i' - R_i')]\cos l't_i + \\ &\qquad [S_i' - l'(t_iS_i - R_i) + l(t_iS_i' - R_i')]\sin l't_i \\ \frac{\partial \alpha_i}{\partial l'} &= \frac{\alpha_i}{l'} + l't_ie^{lt_i}(\beta\sin l't_i + \beta'\cos l't_i) \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{\partial \alpha_{oi}}{\partial \beta} &= 0 \\ \frac{\partial \alpha_{1i}}{\partial \beta} &= 0 \\ \frac{\partial \alpha_i}{\partial \beta} &= -l'e^{lt_i}\cos l't_i \end{aligned}\right\}$$

$$\left.\begin{aligned} \frac{\partial \alpha_{oi}}{\partial \beta'} &= 0 \\ \frac{\partial \alpha_{1i}}{\partial \beta'} &= 0 \\ \frac{\partial \alpha_i}{\partial \beta'} &= l'e^{lt_i}\sin l't_i \end{aligned}\right\}$$

The quantities $e^{-lt}\delta(t) \cos l't$, $e^{-lt}\delta(t) \sin l't$, $te^{-lt}\delta(t) \cos l't$, $te^{-lt}\delta(t) \sin l't$ must be integrated to find S_i, S_i', R_i, and R_i'. These quantities are calculated in table II and are plotted versus time in figure 1. They were integrated by means of a planimeter to obtain the values displayed in table II. Referring to table II and equations (33), it may be seen that

$$a_{oo} = \Sigma\, 21^2 = 0.000399$$

$$a_{o1} = a_{1o} = \Sigma\, 21 \times 30 = -0.00292$$

$$a_{11} = \Sigma\, 30^2 = 0.0241$$

$$a_o = \Sigma\, 21 \times 36 = -0.0615$$

$$a_1 = \Sigma\, 30 \times 36 = 0.494$$

and that

$$\frac{\partial a_{oo}}{\partial l} = 2\,\Sigma\, 21 \times 43 = -0.000152$$

$$\frac{\partial a_{o1}}{\partial l} = \frac{\partial a_{1o}}{\partial l} = \Sigma\, 21 \times 55 + \Sigma\, 30 \times 43 = 0.001388$$

$$\frac{\partial a_{11}}{\partial l} = 2\,\Sigma\, 30 \times 55 = -0.01264$$

$$\frac{\partial a_o}{\partial l} = \Sigma\, 21 \times 57 + \Sigma\, 36 \times 43 = 0.00772$$

$$\frac{\partial a_1}{\partial l} = \Sigma\, 30 \times 57 + \Sigma\, 36 \times 55 = -0.0851$$

$$\frac{\partial a_{oo}}{\partial l'} = 2\Sigma\, 21 \times 60 = 0.000232$$

$$\frac{\partial a_{o1}}{\partial l'} = \frac{\partial a_{1o}}{\partial l'} = \Sigma\, 21 \times 69 + \Sigma\, 30 \times 60 = -0.001571$$

$$\frac{\partial a_{11}}{\partial l'} = 2\,\Sigma\, 30 \times 69 = 0.01226$$

$$\frac{\partial a_o}{\partial l'} = \Sigma\, 21 \times 74 + \Sigma\, 36 \times 60 = -0.0381$$

$$\frac{\partial a_1}{\partial l'} = \Sigma\, 30 \times 74 + \Sigma\, 36 \times 69 = 0.280$$

$$\frac{\partial a_{00}}{\partial \beta} = 0 \qquad \frac{\partial a_{00}}{\partial \beta'} = 0$$

$$\frac{\partial a_{01}}{\partial \beta} = \frac{\partial a_{10}}{\partial \beta} = 0 \qquad \frac{\partial a_{01}}{\partial \beta'} = \frac{\partial a_{10}}{\partial \beta'} = 0$$

$$\frac{\partial a_{11}}{\partial \beta} = 0 \qquad \frac{\partial a_{11}}{\partial \beta'} = 0$$

$$\frac{\partial a_0}{\partial \beta} = \Sigma\, (21) \times (75) = -0.104 \qquad \frac{\partial a_0}{\partial \beta'} = \Sigma\, (21) \times (76) = 0.0163$$

$$\frac{\partial a_1}{\partial \beta} = \Sigma\, (30) \times (75) = 0.899 \qquad \frac{\partial a_1}{\partial \beta'} = \Sigma\, (30) \times (76) = -0.224$$

Substituting these numbers in the appropriate places in equations (34) and (35), it follows that

$$C_0 = -37.90$$

$$C_1 = 15.88$$

while

$$\frac{\partial C_0}{\partial l} = 6.65 \qquad \frac{\partial C_1}{\partial l} = 7.79$$

$$\frac{\partial C_0}{\partial l'} = -27.38 \qquad \frac{\partial C_1}{\partial l'} = -2.25$$

$$\frac{\partial C_0}{\partial \beta} = 108.93 \qquad \frac{\partial C_1}{\partial \beta} = 50.50$$

$$\frac{\partial C_0}{\partial \beta'} = -239.79 \qquad \frac{\partial C_1}{\partial \beta'} = -38.35$$

Using equations (28) and (27), we obtain, finally,

$$|\Delta C_0| = 37.5$$

$$|\Delta C_1| = 11.5$$

Therefore, C_1 is known to within about 72 percent; C_0 is only known to within 99 percent.

The large errors in some of the parameters are worthy of special comment. It should be noted first of all that the particularly large error in C_0 is completely consistent with what has been found for some time by empirical means. It has been found that upon repeating a set of flight records, making the two sets identical as far as that is possible, two entirely different values of C_0 have often been obtained.

Secondly, it will be recalled that in the derivation of the formula for the errors it was required that the errors be small; this is certainly not the case for some of the errors. For this reason, the error of 37.5 in C_0 particularly may not be considered as definitive. The theory developed herein fails for such a large error, and the actual error in C_0 may be much larger than that calculated. However, this is not serious, for an error as large as 99 percent renders the calculated parameter value meaningless in any case, and it matters not at all whether the error is 99 or 199 percent.

Finally and most important is the following conclusion regarding the entire philosophy of calculation of stability parameters from pitching velocity data alone in response to an elevator pulse: This experiment is ill designed for the calculation[3] of C_0. In general, when a pulse is applied to produce a set of pitching velocity data, no more should really be expected from the analysis than the period and the damping parameters k and b. If, in addition, the analysis results in a value of C_1 whose error is within reasonable limits, this should be regarded as fortuitous. It should be stated that no example has yet been found for which C_0 may be calculated at all accurately.

Such a negative comment as the preceding deserves a remark on the possibility of the most accurate calculation of C_0. If one were to measure a step response rather than a pulse response, the analysis of these data would surely lead to more reliable values of C_0. The reason for this lies in the easily proved proportionality of the steady-state value of $q(t)$ with C_0.

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[3]This conclusion is drawn for the parameter C_0 alone and not also for C_1, as would seem to be indicated by the error of 72 percent in C_1, primarily because the error in C_1 is not always as large as this. With some data which have been analyzed, the error in C_1 has been far less, of the order of 30 percent.

CONCLUDING REMARKS

A formula has been given which may be used to find the errors in the parameters obtained from a curve-fitting process. The method of derivation did not use the concepts of probability theory, since the latter ideas lead to quantities for which there is no known method of solution. However, the result that the formula obtained bears a close relationship to the classical probabilistic formula in case the curve-fitting problem is linear (the only case for which such a formula has been derived and proved valid) is proved.

As may be seen by studying the example given, the ease of application of the method is directly dependent on the simplicity of the form of the function which has been fitted. Thus, if b, k, C_1, and C_0 are the aerodynamic parameters occurring in the differential equation (1), the errors in b and k may be calculated fairly rapidly; the errors in C_1 and C_0 require more time. However, if the method of least squares (reference 2) is used for the curve-fitting problem, a great many of the computations which are needed to find the errors will have already been performed in the process of finding the parameters.

The method actually weights favorably the method of least squares in another manner. The errors, as given by equation (20), are proportional to the square root of the sum of the squares of the residuals, which quantity is minimized by the least-squares process. However, it is not believed that this weighting is a serious limitation, for no matter what means are used for fitting the curve, if this latter method is to have significance and is to lead to a good fit of the data, the sum of the squares of the residuals will also be near its minimum.

Finally, it has been shown that pitching velocity data in response to a pulse are not alone adequate to compute the parameter C_0 (and, to a lesser extent, C_1) occurring in the differential equation of motion. Usually, all that may be obtained from such data are the parameters b and k, which determine the damping and the period of the oscillation. In some cases, C_1 may also be obtained with reasonable accuracy.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., July 30, 1952

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TABLE I.- THE ERRORS IN l, l', β AND β'

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Row	t_i	e^{lt_i}	$\cos l't_i$	$\sin l't_i$	③×④	③×⑤	⑥× β	⑦× β'	⑧-⑨	②× ⑩	⑦× β	⑥× β'	⑫ + ⑬	②× ⑭
1	0.4	0.579	0.336	0.942	0.194	0.545	0.119	-0.114	0.233	0.093	0.335	-0.041	0.294	0.118
2	.5	.505	.035	.999	.018	.505	.011	-.105	.116	.058	.310	-.004	.306	.153
3	.6	.441	-.269	.963	-.118	.424	-.073	-.088	.016	.009	.261	.025	.285	.171
4	.7	.384	-.547	.837	-.210	.322	-.129	-.067	-.062	-.044	.198	.044	.241	.169
5	.8	.335	-.775	.632	-.260	.212	-.160	-.044	-.115	-.092	.130	.054	.184	.147
6	.9	.292	-.930	.369	-.272	.108	-.167	-.022	-.144	-.130	.066	.057	.123	.111
7	1.0	.255	-.998	.070	-.254	.018	-.156	-.004	-.152	-.152	.011	.053	.064	.064
8	1.1	.223	-.972	-.235	-.216	-.052	-.133	.011	-.144	-.158	-.032	.045	.013	.014
9	1.2	.194	-.856	-.518	-.166	-.100	-.102	.021	-.123	-.148	-.062	.035	-.027	-.033
10	1.3	.169	-.659	-.752	-.112	-.127	-.064	.027	-.095	-.124	-.078	.023	-.055	-.071
11	1.4	.148	-.401	-.916	-.059	-.135	-.036	.028	-.065	-.090	-.083	.012	-.071	-.099
12	1.5	.129	-.105	-.995	-.014	-.128	-.008	.027	-.035	-.053	-.079	.003	-.076	-.114
13	1.6	.112	.200	-.980	.023	-.110	.014	.023	-.009	-.015	-.068	-.005	-.072	-.116
14	1.7	.098	.487	-.873	.048	-.086	.029	.018	.011	.020	-.053	-.010	-.063	-.106
15	1.8	.086	.728	-.685	.062	-.059	.038	.012	.026	.047	-.036	-.013	-.049	-.088
16	1.9	.075	.902	-.433	.067	-.032	.041	.007	.035	.066	-.020	-.014	-.034	-.064
17	2.0	.065	.990	-.140	.064	-.009	.040	.002	.038	.076	-.006	-.013	-.019	-.038
18	2.1	.057	.987	.166	.056	.009	.034	-.002	.036	.076	.006	-.012	-.006	-.012
19	2.2	.050	.890	.456	.044	.023	.027	-.005	.032	.070	.014	-.009	.005	.010
20	2.3	.043	.710	.704	.031	.030	.019	-.006	.025	.058	.019	-.006	.012	.028
21	2.4	.038	.464	.886	.018	.033	.011	-.007	.018	.043	.021	-.004	.017	.040
22	2.5	.033	.175	.985	.006	.032	.004	-.007	.010	.026	.020	-.001	.019	.047
23	2.6	.029	-.131	.991	-.004	.028	-.002	-.006	.004	.009	.017	.001	.018	.047
24	2.7	.025	-.425	.905	-.011	.023	-.007	-.005	-.002	-.005	.014	.002	.016	.044
25	2.8	.022	-.678	.735	-.015	.016	-.009	-.003	-.006	-.016	.010	.003	.013	.036
26	2.9	.019	-.869	.495	-.017	.009	-.010	-.002	-.008	-.024	.006	.003	.009	.027
27	3.0	.017	-.978	.209	-.016	.004	-.010	-.001	-.009	-.027	.002	.003	.006	.017
28	3.1	.015	-.995	-.096	-.014	-.001	-.009	0	-.009	-.028	-.001	.003	.002	.007
29	3.2	.013	-.920	-.393	-.012	-.005	-.007	.001	-.008	-.026	-.003	.002	-.001	-.002

Σ ⑪^2 = 0.168

Σ ⑪ × ⑮ = -0.006

Σ ⑥× ⑪ = 0.228

Σ ⑦× ⑪ = 0.091

Σ ⑮^2 = 0.212

Σ ⑥× ⑮ = -0.117

Σ ⑦× ⑮ = 0.389

Σ ⑥^2 = 0.415

Σ ⑥× ⑦ = -0.051

Σ ⑦^2 = 0.985

TABLE II.- THE ERRORS IN C_1 AND C_0

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Row	t_1	$\delta(t_1)$	e^{-lt_1}	$\cos l't_1$	$\sin l't_1$	④×⑤	④×⑥	③×⑦	③×⑧	S_1	S_1'	②×⑨	②×⑩	R_1	R_1'	⑨/④	⑥/④	⑫×⑰	⑪×⑱	⑲ = ⑳ - ⑳
1	0	0	1.000	1.000	0	1.000	0	0	0	0.02172	0.01498	0	0	0.004186	0.003234	1.000	0	0.01498	0	0.01498
2	.05	.005	1.071	.988	.153	1.058	.164	.005	.001	.02167	.01497	.00025	.00005	.004184	.003234	.923	.143	.01382	.00310	.01072
3	.10	.045	1.146	.953	.302	1.093	.347	.049	.016	.02044	.01468	.00430	.00160	.004074	.003210	.832	.264	.01221	.00540	.00681
4	.15	.105	1.227	.896	.445	1.099	.546	.115	.057	.01638	.01278	.01725	.00855	.003504	.002978	.730	.363	.00933	.00595	.00338
5	.20	.125	1.314	.817	.576	1.074	.757	.134	.095	.00996	.00900	.02680	.01900	.002410	.002098	.622	.438	.00560	.00436	.00124
6	.25	.100	1.407	.719	.695	1.012	.977	.101	.098	.00360	.00410	.02525	.02450	.001050	.001148	.511	.494	.00210	.00178	.00032
7	.30	.030	1.507	.605	.796	.911	1.200	.027	.036	.00040	.00076	.00810	.01080	.000150	.000128	.401	.528	.00031	.00021	.00010
8	.35	0	1.613	.476	.860	.768	1.419	0	0	0	0	0	0	0	0	.295	.546	0	0	0
9	.40	0	1.727	.336	.942	.580	1.627	0	0	0	0	0	0	0	0	.195	.545	0	0	0

	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38
Row	⑪ × l'	⑫ × l	㉒ - ㉓	⑰ × ㉔	⑪ × l	⑫ × l'	㉖ + ㉗	⑬ × ㉘	α_{11} = -[㉕ + ㉙]	⑰ × β	⑱ × β'	㉛ - ㉜	$q_e(t_1)$	㉞ - ㉝	α_1 = ㉟ × l'	② × ⑫	㊱ - ㊲
1	0.06670	-0.02046	0.08716	0.08720	-0.02967	0.04500	0.01633	0	-0.06720	0.614	0	0.614	0	-0.614	-1.886	0	-0.003234
2	.06655	-.02045	.08700	.08021	-.02960	.04597	.01637	.00234	-.08255	.567	-.030	.597	.008	-.589	-1.809	.000749	-.002485
3	.06277	-.02005	.08282	.06897	-.02792	.04508	.01716	.00453	-.07350	.511	-.055	.566	.095	-.471	-1.446	.001468	-.001742
4	.05030	-.01746	.06776	.04549	-.02238	.03925	.01687	.00612	-.05561	.448	-.076	.524	.200	-.324	-.995	.001917	-.001061
5	.03059	-.01289	.04288	.02688	-.01361	.02764	.01403	.00615	-.03283	.382	-.091	.473	.310	-.163	-.501	.001800	-.000438
6	.01106	-.00560	.01666	.00853	-.00492	.01259	.00767	.00379	-.01232	.314	-.103	.417	.360	-.057	-.175	.001025	-.000123
7	.00123	-.00104	.00227	.00088	-.00055	.00233	.00178	.00094	-.00182	.246	-.110	.356	.320	-.036	-.111	.000228	.000100
8	0	0	0	0	0	0	0	0	0	.181	-.114	.295	.272	-.023	-.071	0	0
9	0	0	0	0	0	0	0	0	0	.120	-.113	.233	.224	-.009	-.028	0	0

	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55
Row	⑰ × ㊳	② × ⑬	㊴ - ㊵	⑱ × ㊶	$\frac{\partial\alpha_{01}}{\partial l}$ = ㊷ - ㊷	② × ㊸	㉓ + ㊹	⑬ × l'	⑭ × l	㊻ - ㊼	⑰ × ㊽	⑬ × l	⑭ × l'	㊾ + ㊿	⑱ × 51	52 - 53	$\frac{\partial\alpha_{11}}{\partial l}$ = 54 + 55
1	-0.003234	0	-0.004186	0	-0.003234	0	0.01498	0.01286	-0.00442	0.01728	0.01728	-0.00572	0.00993	0.00421	0	0.01728	0.03226
2	-.002294	.00108	-.003100	-.000443	-.001851	-.00413	.00659	.01285	-.00442	.01727	.01594	-.00572	.00993	.00421	.00060	.01654	.02313
3	-.001449	.00214	-.002030	-.000536	-.000913	-.00735	-.00054	.01251	-.00438	.01689	.01405	-.00557	.00986	.00429	.00113	.01518	.01464
4	-.000775	.00257	-.001047	-.000380	-.000395	-.00834	-.00496	.01076	-.00407	.01483	.01083	-.00479	.00915	.00436	.00158	.01241	.00745
5	-.000313	.00198	-.000418	-.000183	-.000127	-.00697	-.00533	.00740	-.00314	.01054	.00656	-.00329	.00706	.00377	.00165	.00821	.00688
6	-.000063	.00090	-.000150	-.000074	.000011	-.00308	-.00276	.00323	-.00157	.00480	.00245	-.00143	.00353	.00210	.00104	.00349	.00073
7	.000040	.00012	-.000030	-.000016	.000056	-.00055	-.00045	.00046	-.00017	.00063	.00025	-.00020	.00033	.00019	.00010	.00035	-.00010
8	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

	56	57	58	59	60	61	62	63	64	65	66	67	68	69	70	71	72
Row	② × l'	$\frac{\partial\alpha_1}{\partial l'}$ = - 55 × 56	⑰ × 57	⑱ × 58	$\frac{\partial\alpha_{01}}{\partial l'}$ = -[58 + 59]	43 × l	58 × l'	⑪ + 61 + 62	⑰ × 63	61 × l'	58 × l	⑫ - 62 + 64	⑬ × 67	$\frac{\partial\alpha_{11}}{\partial l'}$ = -[64 + 65]	⑱ × β	⑰ × 69	70 + 71
1	0	0	-0.004186	0	0.004186	0.005718	-0.00993	0.01751	0.01751	-0.01286	0.00442	0.03226	0	-0.01751	0	-0.208	-0.208
2	.154	-.092	-.002861	-.000355	.002149	.004235	-.00763	.01828	.01687	-.00952	.00339	.02788	.00399	-.02086	.088	-.192	-.104
3	.307	-.174	-.001689	-.000460	.002149	.002773	-.00535	.01786	.01486	-.00623	.00238	.02389	.00615	-.02101	.162	-.173	-.011
4	.461	-.242	-.000764	-.000385	.001149	.001430	-.00326	.01455	.01062	-.00322	.00145	.01745	.00633	-.01695	.223	-.152	.071
5	.614	-.290	-.000260	-.000218	.000478	.000571	-.00153	.00900	.00560	-.00128	.00068	.01096	.00480	-.01040	.269	-.129	.140
6	.768	-.320	-.000077	-.000061	.000138	.000205	-.00038	.00343	.00175	-.00046	.00017	.00473	.00234	-.00409	.303	-.106	.197
7	.921	-.328	-.000012	.000053	-.000041	.000041	.00031	.00075	.00030	-.00009	-.00014	.00071	.00037	-.00067	.324	-.083	.241
8	1.075	-.317	0	0	0	0	0	0	0	0	0	0	0	0	.335	-.061	.274
9	1.228	-.286	0	0	0	0	0	0	0	0	0	0	0	0	.335	-.041	.294

	73	74	75	76
Row	56 × 72	$\frac{\partial\alpha_1}{\partial l'}$ = 73 + 73	$\frac{\partial\alpha_1}{\partial\beta}$ = -17 × l'	$\frac{\partial\alpha_1}{\partial\beta'}$ = 18 × l'
1	0	-0.614	-3.071	0
2	-.016	-.605	-2.835	.439
3	-.003	-.474	-2.559	.811
4	.033	-.291	-2.242	1.115
5	.086	-.077	-1.910	1.345
6	.151	.094	-1.569	1.517
7	.222	.186	-1.231	1.621
8	.295	.272	-.906	1.677
9	.361	.352	-.599	1.674

Σ ㉑^2 = 0.000399 Σ ㉑ × ㊸ = -0.0000760 Σ ㊱ × ㊸ = 0.0112 Σ ㊱ × ㊽ = -0.000870 Σ ㊱ × ㊿ = 0.124

Σ ㉑ × ㉚ = -0.00292 Σ ㉑ × 52 = 0.000860 Σ ㊱ × 57 = 0.0479 Σ ㊱ × 62 = 0.00613 Σ ㉑ × 72 = -0.104

Σ ㉑ × ㉜ = -0.0615 Σ ㊱ × 43 = 0.000528 Σ ㊱ × 52 = -0.133 Σ ㉑ × 74 = -0.0199 Σ ㊱ × 72 = 0.899

Σ ㉚^2 = 0.0241 Σ ㊱ × 52 = -0.00632 Σ ㉑ × 64 = 0.000116 Σ ㊱ × 60 = -0.0182 Σ ㉑ × 76 = 0.0163

Σ ㊱ × ㊿ = 0.494 Σ ㉑ × 57 = -0.00348 Σ ㉑ × 62 = -0.000701 Σ ㊱ × 74 = 0.156 Σ ㊱ × 76 = -0.224

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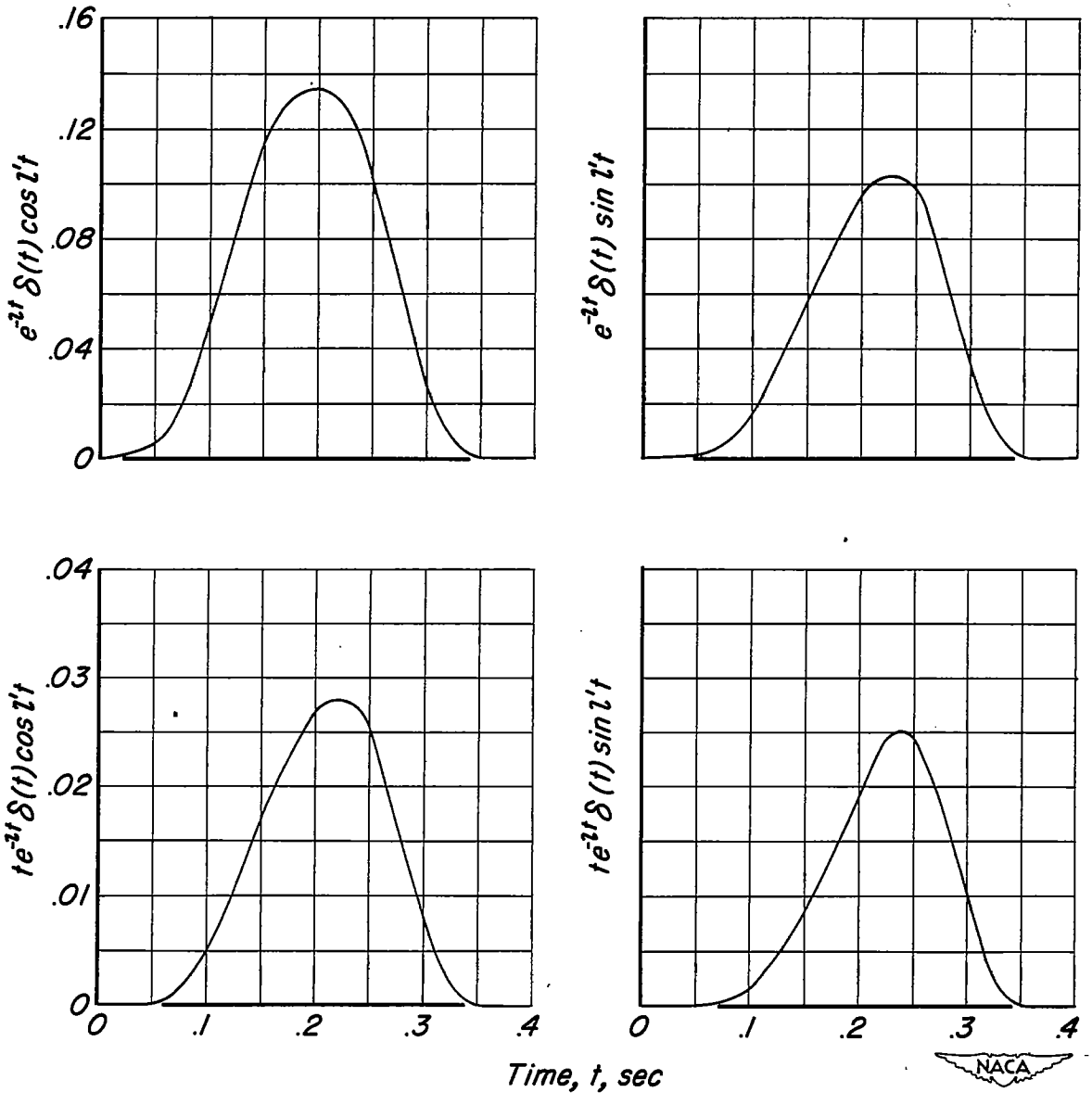

Figure 1.- The variation with time of four quantities required for the calculation of C_0 and C_1.