REPORT 1245

ANALYSIS AND CALCULATION BY INTEGRAL METHODS OF LAMINAR COMPRESSIBLE BOUNDARY LAYER WITH HEAT TRANSFER AND WITH AND WITHOUT PRESSURE GRADIENT

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SUMMARY

A survey of integral methods in laminar-boundary-layer analysis is first given. A simple and sufficiently accurate method for practical purposes of calculating the properties (including stability) of the laminar compressible boundary layer in an axial pressure gradient with heat transfer at the wall is then presented. For flow over a flat plate, the method is applicable for an arbitrarily prescribed distribution of temperature along the surface and for any given constant Prandtl number close to unity. For flow in a pressure gradient, the method is based on a Prandtl number of unity and a uniform wall temperature. A simple and accurate method of determining the separation point in a compressible flow with an adverse pressure gradient over a surface at a given uniform wall temperature is developed. The analysis is based on an extension of the Kármán-Pohlhausen method to the momentum and thermal energy equations in conjunction with fourth- and especially higher degree velocity and stagnation-enthalpy profiles. From the equations derived here, conclusions regarding the effect of pressure gradient, Mach number, and wall temperature on the boundary-layer characteristics are derived and illustrated. In particular the effects on skin-friction, heat-transfer coefficient, separation point in an adverse pressure gradient, and stability of the laminar boundary layer are analyzed.

INTRODUCTION

The purpose of the present report is to present a comprehensive summary of theoretical investigations of compressible laminar boundary layers which have been carried out since 1949 at the Polytechnic Institute of Brooklyn under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. The results of these investigations are contained primarily in references 1 to 7.[1]

Briefly, reference 1 is an investigation of the relative merits of various types of integral methods for the analysis of laminar boundary layers. It is concluded that the one-parameter method based on the Kármán momentum integral equation in conjunction with sixth-degree velocity profiles appears, on the whole, to be the most promising method for analyzing laminar boundary layers in general. On the basis of this conclusion, a simple method of calculating compressible-boundary-layer characteristics, including separation point and stability characteristics, in flow with a pressure gradient without heat transfer at the wall is developed in reference 2. It is further shown in reference 2 that fourth-degree profiles are preferable for analyzing stagnation flows, while the separation point in an adverse pressure gradient can be still more accurately predicted with seventh-degree velocity profiles. In reference 3, it is further verified, by considering the flow over a flat plate with heat transfer, that sixth-degree profiles yield results of sufficient accuracy for stability calculations. In the course of such calculations, certain modifications in the approximate stability criteria of reference 8 were made, and these are shown briefly in references 3 to 5, which are essentially a summary of an unpublished report by Professor M. Bloom of the Polytechnic Institute of Brooklyn entitled "Calculation of Stability of Constant-Pressure Boundary Layers on Isothermal Surfaces With an Integral-Method Mean-Flow Solution." This report is available for loan or reference in the Division of Research Information, National Advisory Committee for Aeronautics, Washington, D. C. In reference 6, a method of calculating the compressible laminar boundary layer in a pressure gradient with heat transfer is developed. This reference includes the calculation of the boundary layer over a flat plate (zero pressure gradient) with a nonuniform wall temperature. Reference 7, finally, applies the equations developed in reference 6 to a general study of the effect of pressure gradient, wall temperature, and Mach number on the skin-friction, heat-transfer, separation, and stability characteristics of laminar boundary layers. A method of calculating the separation point in an adverse pressure gradient with heat transfer is included there. Numerical examples are also included in reference 7 to illustrate in detail the conclusions reached there. The methods of references 6 and 7 are extensions of the corresponding methods of reference 2 to cases of heat transfer at the wall.

The emphasis in the present report will be on the development of methods of calculation of laminar-compressible-boundary-layer characteristics. In particular, a method of calculating the boundary layer over a flat plate with a nonisothermal surface, that is, with a given distribution of temperature along the wall, will be presented. The Prandtl number, although constant, is left arbitrary but must be of the order of magnitude of unity. For flow in a pressure gradient, a method is given of calculating the boundary layer for a given distribution of velocity outside the boundary layer, a given reference Mach number, and a given uniform wall temperature. The Prandtl number is now assumed to be unity. A

[1] Since each of these references contains appreciable material of interest not contained in the present report, the latter does not actually supersede any of these references.

method of calculating the separation point in compressible flow with an adverse pressure gradient over a surface at a given uniform wall temperature is also presented. The mathematical analysis on which these methods are based will be given in sufficient detail to show clearly the logical development of the methods and the main approximating assumptions which are made as well as the range of applicability of the methods.

In addition to the methods of calculation, the implications of the equations developed here regarding the various characteristics of laminar boundary layers as they are affected by parameters such as the pressure gradient, wall temperature, and Mach number will be discussed. The discussion will include stability characteristics. A variety of numerical examples will also be discussed, and details of a few of these will be presented here. Further details of these examples can be found in references 1 to 7.

It may be remarked that, as is well known, the literature on laminar boundary layers and related problems is extensive and rich. Indeed, wherever pertinent, reference will be included in this report to recent work which has appeared either more or less simultaneously with, or since, references 1 to 7. The advantage of the methods of calculation developed in this report is that they combine the merits of adequate accuracy and relative ease of calculation. (An ordinary desk calculator will be found to be more than adequate for all of the calculations. In fact, a large number of the calculations may even be performed by a standard slide rule.) The mathematical analysis will likewise entail practically the minimum number of approximating assumptions required to retain both simplicity and adequate accuracy.

With respect to the pertinent literature, it will suffice, at this point, to mention briefly theoretical investigations on the general case of the compressible laminar boundary layer with pressure gradient and heat transfer. Only a very limited number of exact solutions, that is, solutions (which may be numerical) based on solving directly the original partial differential equations of the boundary layer essentially without any mathematical approximations, appear to have been obtained thus far. Such exact solutions are restricted to particular types of flows. For low-speed (zero Mach number), but nevertheless compressible, flows with heat transfer, numerical solutions for "wedge flows," or flows in which the velocity outside the boundary layer is proportional to a power of the axial distance, have been developed in references 9 to 11. These solutions include a small normal mass flow at the wall. Such solutions have been recently extended for nonnegligible Mach numbers to cases in which the local Mach number outside of the boundary layer is proportional to a power of the axial distance (ref. 12). A class of similar solutions (refs. 13 to 15) for high-speed (i. e., nonzero Mach number) flows has been recently derived, under certain conditions, and calculated with the aid of electronic computors.

In addition to references 6 and 7, several approximate analyses of laminar compressible boundary layers with pressure gradient and heat transfer have been made. An analysis, for example, based on a type of approximation used by Lighthill has been recently made in reference 16, with emphasis on separation. A method based on "internal" and "external" solutions of the compressible-boundary-layer equations, previously introduced by Kármán and Millikan in an analysis for incompressible flow, is developed in reference 17. Reference 18 presents a method based on solutions which have been obtained for wedge flows. References 19 to 21 develop methods based on an extension of the Kármán-Pohlhausen method (with fourth-degree velocity and stagnation-enthalpy profiles) to the thermal-energy, as well as the momentum, partial differential equation. (Refs. 18, 20, and 21 include the case of a small normal mass flow at the wall.) Further analyses based on integral methods are given in references 22 to 24, the latter being a study of the heat-insulating properties of the laminar boundary layer. A small-perturbation type of analysis is developed in reference 25. It may be noted that all of the foregoing approximate analyses, with the exception of references 16 to 18 and 25, are based on integral methods.

The analysis in the present report is, for simplicity, based on the usual assumption of constant specific heats. A means of taking into account variable specific heat with temperature, at least for flow over a flat plate, is discussed, for example, in references 26 and 27. As has already been stated, it is further assumed in this report that for flow with a pressure gradient the Prandtl number is 1. An approximate means, for the case of zero heat transfer, of taking into account a Prandtl number different from unity is discussed in reference 28 and applied in reference 29. In the case of heat transfer at the wall, an approximate means of taking into account a Prandtl number other than 1 would be to multiply the Nusselt number (i. e., heat-transfer coefficient) obtained in accordance with the method given here by the cube root of the Prandtl number (cf., e. g., refs. 30 and 31). Such a correction, however, may be considerably inaccurate at very high Mach numbers (ref. 12). A further assumption in the present analysis of flow *with a pressure gradient* is that the wall temperature is uniform.[2] A summary of investigations on flow over a nonisothermal surface in a pressure gradient (as well as over a flat plate) is given in reference 32. Further information can also be obtained in reference 33. Finally, it must be noted that the present investigation is based on the assumption that the coefficient of viscosity is proportional to the absolute temperature, with the proportionality factor determined so that Sutherland's relation is exactly satisfied at the wall. This is an assumption commonly made (cf. ref. 34) to simplify the analysis and yet retain the main actual influence of the dependence of the viscosity coefficient on temperature, at least for Mach numbers below 5.

The present report is divided into five main sections. The first section discusses concisely the various main types of integral methods in laminar-boundary-layer analysis and their relative merits. The second section develops the basic equations to be used in the present analysis. These equations are valid for an arbitrary constant Prandtl number (close to unity) and a nonuniform wall temperature. In the

[2] The general equations, however, developed in the section "Basic Equations," which pertain to flows with or without a pressure gradient, are valid also for nonuniform wall temperature.

third section, these equations are applied to present a method for the calculation of the boundary layer over a flat plate with a prescribed distribution of wall temperature while the Prandtl number is kept arbitrary. In the fourth section, the basic equations are used to yield a method of calculating the boundary layer in a given pressure gradient over a surface at a prescribed uniform wall temperature. Here the Prandtl number is assumed as unity. The calculation of the separation point in an adverse pressure gradient is included in this section. The fifth section, finally, discusses the various general conclusions on the boundary-layer characteristics which are of physical interest and follow from the analysis presented herein.

SYMBOLS

a_n	coefficient of τ^n in velocity profile (eq. (15))
a_2	given by equation (48)
$\bar{a}_2$	constant average value of a_2
b	positive constant used in reference 2
b_n	coefficient of τ^n in stagnation-enthalpy profile (eq. (16))
b_1	coefficient in thermal profile not determined in advance by boundary conditions
$\bar{b}_1$	constant average value of b_1
C	proportionality factor in temperature-viscosity relation (eqs. (6) and (7))
$\bar{C}$	constant average value of C
$C_1=\int_0^\xi C\,d\xi$	
$\bar{C}_1=\int_0^1 C\,d\xi$	
C_f	average skin-friction coefficient for length L (eq. (35))
C_{fl}	local skin-friction coefficient (eq. (56))
c_p,c_v	specific heats at constant pressure and constant volume, respectively
F_1,F_2,F_3	integrals defined by equations (11)
$\bar{F}_1$	constant average value of F_1
$\bar{F}_{1s}$	constant replacing $\bar{F}_1$ for determination of separation point
$G_2,G_3(\xi),G_4(\xi),G_5(\xi)$	parameters defined by equations (31), (34b), and (40b)
H	stagnation enthalpy, $(u^2/2)+c_pT$
h	ratio of stagnation enthalpy at wall to stagnation enthalpy at outer edge of boundary layer, $H_o/H_1(\xi)$; for $Pr=1$, $h=T_o/T_e$ (cf. also eq. (25))
h_e	value of h for zero heat transfer at wall
k	coefficient of heat conductivity
L	characteristic length
M	Mach number
m,j,l	constants defined by equations (55b)
Nu	Nusselt number
Pr	Prandtl number, $\mu c_p/k$
q	local rate of heat transfer at wall
R_L	Reynolds number based on L, $\rho_\infty u_\infty L/\mu_\infty$
$R_{b,cr}$	minimum critical Reynolds number based on conditions at point b immediately behind shock wave at leading edge of airfoil, $\rho_b u_b L/\mu_b$
$R_{\infty,cr}$	minimum critical Reynolds number based on remote free-stream conditions in supersonic flow over thin airfoil
r	ratio of local skin friction to Nusselt number defined in equation (68)
S	Sutherland constant; $S=216°$ R for air (cf. eq. (7))
T	absolute temperature
T_e	equilibrium wall temperature for zero heat transfer
t	transformation variable, defined by equation (8)
u,v	velocity components in x- and y-directions, respectively
x,y	coordinates parallel and normal to surface, respectively
β	constant defined by equation (34b)
γ	ratio of specific heats, c_p/c_v; $\gamma=1.4$ for air
δ,δ_t	boundary-layer thicknesses in xy and xt planes, respectively
η	recovery factor (eq. (41))
$\lambda=R_L(\delta_t/L)^2$	
$\lambda_s(\xi)$	solution for $\lambda(\xi)$ to be used in determining separation point (eq. (64))
μ	coefficient of viscosity
ξ	dimensionless distance along wall, x/L
ρ	mass density
τ	dimensionless variable, t/δ_t
φ_1	constant defined by equation (53)
φ_{1s}	constant replacing φ_1 in determining separation point

Subscripts:

a	region at which adverse pressure gradient starts
b	value at point outside of boundary layer immediately behind shock wave at leading edge of supersonic airfoil
o	value at wall
s	value used for determining separation point
sep	value at separation point
1	local value at outer edge of boundary layer
∞	value at suitable reference point outside boundary layer; in numerical examples, denotes value in undisturbed (remote) free stream

A prime denotes differentiation with respect to ξ.

COMPARISON OF INTEGRAL METHODS FOR LAMINAR-BOUNDARY-LAYER ANALYSIS

Since the development in 1921 of the boundary-layer momentum integral equation by Von Kármán (ref. 35) and its first application by Pohlhausen (ref. 36), the Kármán-Pohlhausen method has probably been the most widely

applied and fruitful of the approximate methods used for theoretical analyses of boundary layers.

The Kármán integral equation can be regarded physically as a momentum balance over a fluid element extending across the entire boundary-layer thickness. Mathematically, the equation can be regarded as an integration of the original momentum partial differential equation over the boundary-layer thickness. The advantage of this integral equation for theoretical calculations is that if certain definite forms are assumed for the velocity profiles as functions of the normal distance from the surface then an ordinary differential equation is obtained with axial distance along the surface as independent variable and essentially the boundary-layer thickness as the unknown.

The Kármán-Pohlhausen method in its original form is based on the use of fourth-degree velocity profiles satisfying certain conditions at the wall and at the outer edge of the boundary layer. By means of this particular method a considerable variety of useful results for laminar boundary layers has been obtained, even for cases of a normal mass flow (fluid suction or injection) at the wall with or without heat transfer and pressure gradient (cf., e.g., refs. 30, 37 to 39, and 19 to 22). It has been found, however, that this method has at least two distinct disadvantages in practical cases. It fails to predict accurately the separation point in an adverse pressure gradient, and it often does not yield sufficiently accurate results for derivatives of the profiles for use in laminar-boundary-layer-stability calculations. In view of such limitations, various refinements in the Kármán-Pohlhausen method have been made, and a number of what appeared to be the most important types of refinements were studied and compared in reference 1.

REFINEMENTS OF KÁRMÁN-POHLHAUSEN METHOD

In discussing refinements of the Kármán-Pohlhausen method, it should be first observed that the Kármán momentum integral equation is not actually equivalent to the original partial differential equation. It is, in fact, essentially only an average of this equation over the boundary-layer thickness. Thus, any solution of the partial differential equation will necessarily satisfy the momentum integral equation but not vice-versa. This basic limitation of the integral equation is, however, to some extent overcome in the Kármán-Pohlhausen method by the fact that the velocity profiles which are assumed in this equation are not chosen quite arbitrarily but are chosen as well-behaved functions (namlye, fourth-degree polynomials) satisfying the boundary conditions and certain additional conditions which an exact solution of the governing partial differential equations would necessarily satisfy.

There are two main types of methods of refining the Kármán-Pohlhausen method. One method consists in using integral equations in addition to the Kármán momentum integral obtained by multiplying the original momentum partial differential equation by the axial velocity u or powers of u (e. g., refs. 40, 41, and 29) or by the normal distance y or powers of y (ref. 42) and then by integrating the resulting equations over the boundary-layer thickness. In this type of method, additional unknown parameters as functions of the axial distance x are introduced into the assumed velocity profiles, and these are determined by the additional resulting ordinary differential equations. In most actual applications, only one integral equation in addition to the Kármán momentum integral equation is introduced, and, hence, only two ordinary differential equations for two parameters result. Such methods, in fact, are therefore sometimes called "two-parameter" methods. A detailed discussion of such methods is given in reference 1.

The second main type of refinement of the Kármán-Pohlhausen method is the use of only the Kármán integral equation, but in conjunction with profiles of higher degree than the fourth, satisfying additional conditions at the wall and at the boundary-layer edge which an exact solution of the partial differential equations would necessarily satisfy. In most applications of this type, velocity profiles of the sixth degree (refs. 43, 44, 1 to 3, 6, 7, 23, and 24) are used. However, velocity profiles of higher degree than the sixth have also been used (refs. 41 and 23). Seventh-degree velocity profiles have been found particularly suitable for calculation of the separation point in an adverse pressure gradient (refs. 45, 2, and 7). One-parameter methods with velocity profiles of higher than fourth degree are discussed in some detail in reference 1.

COMPARISON OF METHODS

In view of the variety of specific means of refining the Kármán-Pohlhausen method, a theoretical investigation of the relative merits of these methods was made in reference 1. The methods were compared on the basis of both accuracy and ease of computation. The method of comparison was a posteriori. A relatively simple flow, namely, the incompressible and compressible flow for a Prandtl number of unity in a zero pressure gradient over a surface at a uniform temperature, was calculated on the basis of a number of the foregoing methods, and the results were compared with the accurate method of analysis of reference 34 for flow over a flat plate. The two-parameter methods considered were based on (in addition to the Kármán momentum integral) the integral of the momentum partial differential equation multiplied by u in conjunction with fourth- and fifth-degree velocity profiles. The one-parameter methods were, of course, based on the Kármán momentum integral equation and were applied in conjunction with fourth- (Kármán-Pohlhausen method), fifth-, and sixth-degree velocity profiles. The comparisons were made, in particular, on the basis of calculated skin-friction and heat-transfer coefficients, first and second derivatives of the profiles throughout the boundary-layer thickness, and minimum critical Reynolds numbers for laminar-boundary-layer instability according to the criteria of Lin and Lees (refs. 46 and 8). It is well known that the latter criteria are sensitive to first and second derivatives of the profiles.

The boundary conditions satisfied by the various profiles as well as the detailed comparison of the results of the various methods can be found in reference 1. The results, in brief, indicated that skin-friction and heat-transfer coefficients were predicted with substantially satisfactory accuracy by all of the methods. Moreover, the overall profile shapes ob-

tained by all of the methods were qualitatively correct. However, quantitative differences in the first and especially second derivatives of the profiles were obtained, with corresponding differences in the calculated values of the minimum critical Reynolds numbers. It was concluded that, on the whole, the one-parameter method with sixth-degree profiles gave the most accurate results for the profile derivatives, as well as for the minimum critical Reynolds numbers. In reference 1 the stability calculations were carried out for the case of zero heat transfer at the wall. Subsequent calculations (ref. 3) indicated that reliable results for stability calculations by the one-parameter sixth-degree-profile method are obtainable also for the case of heat transfer at the surface of the flat plate.

In addition to being capable of yielding results of adequate accuracy, it is usually also quite desirable that a method of calculation be simple. In this connection, it must be observed that the one-parameter methods, in general, involve considerably simpler calculations than the two-parameter methods. This advantage of the one-parameter methods may not be very pronounced in the case of flow over a flat plate; however, it becomes quite pronounced for the general case of flow in a pressure gradient with heat transfer. In this case, the thermal-energy partial differential equation must be integrated to yield an integral equation in addition to the momentum integral equation. Consequently, there will be at least two parameters to determine. If, however, both the momentum and the thermal-energy partial differential equations are multiplied, for example, by u and integrated over the boundary-layer thickness, then a total of four ordinary differential equations in four unknown parameters will be obtained. Thus, the so-called two-parameter method would in this case really become a four-parameter method. (The one-parameter method in this general case similarly becomes a two-parameter method.) It is noteworthy, in fact, that in any of the foregoing applications of the two-parameter method only the less general cases of zero pressure gradient, or pressure gradient with zero heat transfer at the wall, have been treated. If it is desired to develop a unified method to be applicable in the more general as well as in the simpler cases, then this would have to be considered a disadvantage of the two-parameter methods.

In view of the foregoing results and considerations, it was concluded in reference 1 that the most promising integral method for laminar-boundary-layer study appeared to be that based on the Kármán integral equation, in conjunction with sixth-degree velocity profiles. This is essentially the method of analysis to be applied in the present report. It should be observed, however, that cases exist in which profiles of other degrees are preferable. In particular, stagnation flows are more satisfactorily treated by fourth-degree profiles (ref. 2), while the separation point in an adverse pressure gradient appears to be determined more accurately by seventh-degree profiles (ref. 2). The latter case will be treated in some detail in the present report.

It may be asked why, in the one-parameter method, profiles of higher degree than the sixth were not considered in the comparison study of reference 1. The reason is that the sixth-degree profiles, as distinguished from fourth-degree profiles, are chosen to satisfy an additional condition at the wall (as well as the outer boundary-layer edge). This condition is obtained by differentiating the partial differential momentum equation with respect to τ. If a velocity profile of higher degree than the sixth is assumed, then the only means of obtaining a further condition at the wall which would be satisfied by an exact solution of the partial differential equations is to differentiate the momentum partial differential equation twice with respect to τ (or y for incompressible flows), and then take values at the wall. This, however, will be found to yield a condition involving partial derivatives with respect to x such as $[\partial^2 u(x,y)/\partial x\,\partial y]_o$, and this condition then becomes essentially an additional ordinary differential equation. Since the sixth-degree profiles have apparently led to satisfactory results, it has not seemed worthwhile to introduce such complications into the analysis by using higher degree velocity profiles.[3] It is noteworthy, in this regard, that although polynomials of as high a degree as the eleventh were applied in reference 41, they satisfied only the same conditions at the wall as the sixth-degree profiles to be used in the present report.

BASIC EQUATIONS

The following equations describe the steady, two-dimensional, laminar-boundary-layer flow of a compressible gas along a slightly curved wall:

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y}=\rho_1 u_1\frac{du_1}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{1}$$

$$\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \tag{2}$$

$$\rho/\rho_1=T_1/T \tag{3}$$

$$\rho u c_p\frac{\partial T}{\partial x}+\rho v c_p\frac{\partial T}{\partial y}=-u\rho_1 u_1\frac{du_1}{dx}+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\mu\left(\frac{\partial u}{\partial y}\right)^2 \tag{4}$$

Equations (1), (2), and (4) are the momentum, continuity, and energy equations, respectively. Equation (3) follows from the ideal-gas law and the assumption that the pressure is constant across the boundary-layer thickness. It will be assumed here that the specific heats c_p and c_v as well as the Prandtl number Pr are constants. By multiplying equation (1) by u and adding the resulting equation to equation (4), the following form of the energy equation is obtained for a constant Prandtl number:

$$Pr\left(\rho u\frac{\partial H}{\partial x}+\rho v\frac{\partial H}{\partial y}\right)=\frac{\partial}{\partial y}\left\{\mu\frac{\partial}{\partial y}\left[H-(1-Pr)(u^2/2)\right]\right\} \tag{5}$$

It will be assumed in the present analysis that the viscosity-temperature relation can be approximated in the form (cf. refs. 34 and 6)

$$\frac{\mu}{\mu_\infty}=C\frac{T}{T_\infty} \tag{6}$$

[3] When seventh-degree velocity profiles are used here for calculation of the separation point in an adverse pressure gradient, the additional condition satisfied at the wall is chosen to be exactly valid only at the separation point.

where

$$C=(T_o/T_\infty)^{1/2}\frac{T_\infty+S}{T_o+S} \tag{7}$$

It is convenient, in this compressible-flow analysis, to replace the normal distance coordinate y by the Dorodnitzyn variable t defined by

$$y=\int_o^t (T/T_1)\,dt \tag{8}$$

By integrating equations (1) and (5) with respect to t over the boundary-layer thickness $t=0$ to $t=\delta_t$ and using the boundary conditions $u=v=0$ at $t=0$ together with smooth transition of the velocity and temperature profiles to their local main-stream values, the following integrodifferential equations are obtained:

$$(F_1/2)\lambda'+\lambda\left\{F_1'+F_1\frac{\rho_1'}{\rho_1}+\frac{u_1'}{u_1}\left[F_1+\left(1+\frac{\gamma-1}{2}M_1^2\right)F_2\right]\right\}$$
$$=C(\rho_\infty/\rho_1)(u_\infty/u_1)(T_1/T_\infty)\left[\frac{\partial}{\partial\tau}(u/u_1)\right]_o \tag{9}$$

$$(F_3/2)\lambda'+\lambda\left[F_3'+F_3\left(\frac{\rho_1'}{\rho_1}+\frac{u_1'}{u_1}\right)\right]$$
$$=C(\rho_\infty/\rho_1)(u_\infty/u_1)(T_1/T_\infty)(1/Pr)\left[\frac{\partial}{\partial\tau}(H/H_1)\right]_o \tag{10}$$

where

$$\left.\begin{aligned}F_1&=\int_0^1(u/u_1)\,[1-(u/u_1)]\,d\tau\\F_2&=\int_0^1[(H/H_1)-(u/u_1)^2]\,d\tau\\F_3&=\int_0^1(u/u_1)\,[1-(H/H_1)]\,d\tau\end{aligned}\right\} \tag{11}$$

and

$$\lambda=(\delta_t/L)^2(\rho_\infty u_\infty L/\mu_\infty) \tag{12}$$

Here $\lambda(\xi)$ is essentially the nondimensional squared boundary-layer-thickness parameter in the xt plane.

The quantities ρ_1/ρ_∞ and M_1 in equations (9) and (10) are related to u_1/u_∞, which is a function of ξ prescribed by the potential flow about the body in question. Thus, in accordance with the usual isentropic-flow relations,

$$\rho_1\rho_\infty=(T_1/T_\infty)^{\frac{1}{\gamma-1}}=\{1+(\gamma-1)(M_\infty^2/2)[1-(u_1/u_\infty)^2]\}^{\frac{1}{\gamma-1}} \tag{13}$$

$$M_1=(u_1/u_\infty)M_\infty(T_1/T_\infty)^{-1/2} \tag{14}$$

In deriving equations (9) and (10) a single boundary thickness has been assumed. This is an alternative to the introduction of two boundary-layer thicknesses, namely, a velocity, or dynamical, and a stagnation-enthalpy, or thermal, boundary-layer thickness (cf. refs. 19 to 21 and 23). The assumption of a single boundary-layer thickness appears feasible for fluids with Prandtl numbers close to unity, since in that case analyses involving both a dynamical and a thermal boundary-layer thickness usually imply that both thicknesses are approximately equal (see, e. g., refs. 30, 20, and 21).[4] Moreover, as explained in reference 6, the use of a single boundary-layer thickness does not necessarily impose any undue restrictions on the thermal profiles, since the latter have here been permitted to contain an additional coefficient not determined in advance by the boundary conditions. This coefficient, to be taken here as b_1, replaces the thermal boundary-layer thickness as the second unknown to be determined by equations (9) and (10). A single boundary-layer thickness has also been used in reference 22.

Equations (9) and (10) can be converted into ordinary differential equations by assuming the velocity and stagnation enthalpy as definite functions of the normal distance variable τ. For this purpose, as explained in the section "Comparison of Integral Methods for Laminar-Boundary-Layer Analysis," the velocity profiles will be chosen as sixth-degree polynomials. The stagnation-enthalpy profiles will similarly be chosen as polynomials but of one degree higher, namely, seventh degree.

Thus, it will be assumed that

$$u/u_1=\sum_{n=0}^{6}a_n\tau^n \tag{15}$$

$$H/H_1=\sum_{n=0}^{7}b_n\tau^n \tag{16}$$

The following boundary conditions must be satisfied:

At $\tau=0$,

$$\left.\begin{aligned}u&=v=0\\H/H_1&=h(\xi)\end{aligned}\right\} \tag{17}$$

where $h(\xi)$ is considered as a prescribed function.

At $\tau=1$,

$$\left.\begin{aligned}u/u_1&=H/H_1=1\\\partial u/\partial\tau&=\partial H/\partial\tau=0\end{aligned}\right\} \tag{18}$$

In addition to these conditions, the following conditions will also be satisfied (cf. ref. 6):

At $\tau=0$,

$$C(T_1/T_\infty)\frac{\partial^2}{\partial\tau^2}(u/u_1)=-\lambda(\rho_1/\rho_\infty)h[1+(\gamma-1)M_1^2/2](u_1/u_\infty)' \tag{19}$$

$$h\frac{\partial^3}{\partial\tau^3}(u/u_1)=\frac{\partial^2(u/u_1)}{\partial\tau^2}\frac{\partial(H/H_1)}{\partial\tau} \tag{20}$$

$$[1+(\gamma-1)M_1^2/2]\frac{\partial^2}{\partial\tau^2}(H/H_1)=(1-Pr)(\gamma-1)M_1^2\left[\frac{\partial(u/u_1)}{\partial\tau}\right]^2 \tag{21}$$

$$(u_1/u_\infty)\lambda Pr(\rho_1/\rho_\infty)\frac{\partial(u/u_1)}{\partial\tau}h'=C(T_1/T_\infty)\left\{\frac{\partial^3}{\partial\tau^3}(H/H_1)-\right.$$
$$\left.3(1-Pr)(\gamma-1)M_1^2\frac{\partial(u/u_1)}{\partial\tau}\frac{\partial^2(u/u_1)}{\partial\tau^2}\Big/[1+(\gamma-1)M_1^2/2]\right\} \tag{22}$$

At $\tau=1$,

$$\frac{\partial^2}{\partial\tau^2}(u/u_1)=\frac{\partial^3}{\partial\tau^3}(u/u_1)=\frac{\partial^2}{\partial\tau^2}(H/H_1)=\frac{\partial^3}{\partial\tau^3}(H/H_1)=0 \tag{23}$$

[4] In the case of flow near a stagnation point, however, it is interesting to note that even for a Prandtl number of 1 the thermal boundary-layer thickness in the xt plane may be around 25 percent greater than the dynamical boundary-layer thickness (ref. 20).

Conditions (19) to (22) follow from equations (1) to (4) and differentiation of each of equations (1) and (4) once with respect to τ, taking values at the wall and taking conditions (17) into account. Conditions (23) follow from differentiation of equations (1) and (4) once and twice with respect to τ, taking values at the local outer boundary-layer edge and taking conditions (18) into account.

After the coefficients a_n and b_n have been determined from conditions (17) to (23) in terms of δ_t (or λ) and b_1, equations (9) and (10) become ordinary differential equations to determine $\lambda(\xi)$ and $b_1(\xi)$. For any given case, the flow outside the boundary layer, as defined by $u_1/u_\infty(\xi)$ and M_∞, is considered as prescribed. Moreover, the temperature distribution along the surface, as defined by $h(\xi)$, is also considered as prescribed here.

The temperature profiles are related, in general, to the stagnation enthalpy and the velocity profiles in accordance with the relation

$$\frac{T}{T_1}=\frac{H}{H_1}\left(1+\frac{\gamma-1}{2}M_1^2\right)-\frac{(\gamma-1)}{2}M_1^2\left(\frac{u}{u_1}\right)^2 \tag{24}$$

Equation (24) follows from the definition of the stagnation enthalpy H. From equation (24) it follows that the wall temperature distribution is related to the stagnation-enthalpy distribution $h(\xi)$ at the wall, in accordance with the relation

$$\frac{T_o}{T_1}(\xi)=h(\xi)\left(1+\frac{\gamma-1}{2}M_1^2\right) \tag{25}$$

Profiles in the xt plane can, if desired, be transformed into profiles in the physical xy plane by determining y as a function of τ and ξ in accordance with equation (8)[5]. Equation (8) can be expressed, in general, in the following nondimensional form:

$$\frac{y}{\delta}=\frac{\int_0^\tau (T/T_1)d\tau}{\int_0^1 (T/T_1)d\tau} \tag{26a}$$

where $\delta(\xi)$ is the physical boundary-layer thickness determined by

$$\frac{\delta}{L}\sqrt{R_L}=\sqrt{\lambda}\int_0^1 (T/T_1)d\tau \tag{26b}$$

In the succeeding two sections, it will be shown how the equations thus far developed can be used to lead to a simple and usually sufficiently accurate method of calculating the laminar-boundary-layer characteristics for certain general types of flows.

FLOW WITHOUT AN AXIAL PRESSURE GRADIENT WITH ARBITRARY (CONSTANT) PRANDTL NUMBER AND VARIABLE WALL TEMPERATURE

In this section, based on the equations thus far derived, a simple and accurate method for calculating the laminar-boundary-layer characteristics of the flow in a zero pressure gradient, such as the flow over a flat plate at zero angle of attack, is developed. The Prandtl number is considered arbitrary but constant and of the order of magnitude of unity, while the wall temperature may vary along the flow; that is, $T_o=T_o(x)$ or $h=h(\xi)$.

GENERAL SOLUTION

For flow without a pressure gradient, that is, $u_1'=0_2$ while $M_1=M_\infty$, $T_1=T_\infty$, and so forth, the integrodifferential equations (9) and (10) become

$$(F_1/2)\lambda'+F_1'\lambda=C\left[\frac{\partial(u/u_1)}{\partial\tau}\right]_o \tag{27}$$

$$(F_3/2)\lambda'+F_3'\lambda=\frac{C}{Pr}\left[\frac{\partial(H/H_1)}{\partial\tau}\right]_o \tag{28}$$

The sixth-degree velocity profile satisfying boundary conditions (17) to (23) in this case is

$$\frac{u}{u_1}=2\tau-5\tau^4+6\tau^5-2\tau^6 \tag{29}$$

The seventh-degree stagnation-enthalpy profile satisfying these conditions in this case is given by equation (16), where [6]

$$\left.\begin{aligned}
b_0&=H_0/H_\infty\equiv h(\xi)\\
b_2&=2G_2\\
b_3&=2G_3\lambda\\
b_4&=35(1-h)-20G_2-8G_3\lambda-20b_1\\
b_5&=-84(1-h)+40G_2+12G_3\lambda+45b_1\\
b_6&=70(1-h)-30G_2-8G_3\lambda-36b_1\\
b_7&=-20(1-h)+8G_2+2G_3\lambda+10b_1
\end{aligned}\right\} \tag{30}$$

where

$$\left.\begin{aligned}
G_2&=\frac{(1-Pr)(\gamma-1)M_\infty^2}{1+\left(\frac{\gamma-1}{2}\right)M_\infty^2}\\
G_3&=\frac{1}{6}Pr\frac{h'}{C}
\end{aligned}\right\} \tag{31}$$

and G_2 is thus a prescribed constant, while G_3 is, in general, a given function of ξ. From equations (11) in conjunction with equations (29) and (30) it is found that

$$\left.\begin{aligned}
F_1&=985/9{,}009\\
F_3&=\frac{31}{126}(1-h)-\frac{953}{180{,}180}G_3\lambda-\frac{302}{9{,}009}G_2-\frac{821}{12{,}012}b_1
\end{aligned}\right\} \tag{32}$$

With F_1 as given by equation (32), the solution of equation (27) for $\lambda(\xi)$ with the condition $\lambda(0)=0$ is

$$\lambda=4C_1/F_1 \tag{33}$$

[5] This is not necessary if it is desired to determine only properties such as skin friction, heat transfer, and separation point which depend only on values at the wall.

[6] The symbol G_1 of references 6 and 7 is replaced here by the somewhat more appropriate symbol h (used also in refs. 20 and 21).

where

$$C_1 \equiv \int_0^{\xi} C\, d\xi$$

Moreover, with F_1, F_3, and λ as given by equations (32) and (33), the solution of equation (28), with the condition of finite b_1 at $\xi=0$, is

$$b_1 = C_1^{-\beta} \int_0^{\xi} C_1^{\beta} G_4(\xi)\, d\xi \tag{34a}$$

where

$$\left.\begin{aligned} &\beta=(1/2)+(985/2{,}463)(1/Pr) \\ &G_4(\xi)=(12{,}012/821)((31/252)(1-h)(C/C_1)-h'\{(31/126)+ \\ &\qquad (953Pr/29{,}550)[(3/2)-(C_1C'/C^2)]\}- \\ &\qquad (953Pr/29{,}550)(C_1/C)h''-(151/9{,}009)(C/C_1)G_2) \end{aligned}\right\} \tag{34b}$$

From equations (34a) and (34b) with any prescribed temperature distribution $h(\xi)$ at the wall, $b_1=b_1(\xi)$ can be readily determined by a single quadrature (for which, e. g., numerical integration may be used). From equations (30) the remaining b_n coefficients can then be found as functions of ξ, and the velocity and stagnation-enthalpy profiles in the $\xi\tau$ plane are then determined by equations (29) and (16). The temperature profiles follow from equation (24). The profiles can be transformed into the physical xy plane by means of equation (8) or (26).

The average skin-friction coefficient for the length L, according to equations (29) and (33), will be

$$C_f \equiv \frac{1}{L} \frac{\int_0^L (\mu\, \partial u/\partial y)_o dx}{(1/2)\rho_\infty u_\infty^2} = \frac{1.322}{\sqrt{R_L}} \sqrt{\overline{C}_1} \tag{35}$$

where

$$\overline{C}_1 \equiv \int_0^1 C\, d\xi \tag{36}$$

The Nusselt number, which is a nondimensional measure of the rate of heat transfer at the wall, can in this case be defined as

$$Nu \equiv qL/k_\infty(T_e - T_o) \tag{37}$$

where

$$q=(k\, \partial T/\partial y)_o \tag{38}$$

and T_e is the equilibrium wall temperature for zero heat transfer, that is, for $q=0$. It is appropriate in the determination of heat-transfer rates to replace the temperature parameter h by the parameter T_o/T_e, which is the physically significant temperature parameter in high-speed flows with heat transfer. An expression for the Nusselt number in terms of T_o/T_e can be obtained by first finding the value of h (to be denoted as h_e) for zero heat transfer at the wall. By substituting $h=h_e$ into equation (34b), assuming h_e constant, and setting $G_4=0$ (for zero heat transfer, or $b_1=0$) the following value of h_e in terms of G_2 is obtained:

$$h_e = 1 - \frac{252}{31} \times \frac{151}{9{,}009} G_2 \tag{39}$$

Substituting now $h=h_e(T_o/T_e)$ (cf. eq. (25)) into equation (34b) and also substituting expression (39) for h_e in terms of G_2, it is found, with C replaced by an average constant value $\overline{C}$, for simplicity, that equations (34a), (33), and (37) yield the following expression for the Nusselt number:

$$Nu = 0.297\sqrt{\overline{C}} \left(1 - \frac{T_o}{T_e}\right)^{-1} \xi^{-\beta-1/2} \sqrt{R_L} \int_0^{\xi} G_5\, d\xi \tag{40a}$$

where

$$\begin{aligned} G_5 = \xi^{\beta-1}\{&[1-(T_o/T_e)]-[2+(60{,}039/152{,}675)Pr](T_o/T_e)'\xi- \\ &(40{,}026/152{,}685)Pr(T_o/T_e)''\xi^2\} \end{aligned} \tag{40b}$$

For any prescribed distribution of the wall temperature as given by $T_o/T_e(\xi)$, the local Nusselt number can be readily obtained from equations (40a) and (40b). The actual heat-transfer rate at the wall (in units such as Btu per second per square foot) can then be obtained immediately by solving for q in terms of Nu in accordance with equation (37).

The equilibrium adiabatic wall temperature T_e according to equations (39), (31), and (25) is

$$T_e = T_\infty \left[1 + \left(\frac{\gamma-1}{2}\right) \eta M_\infty^2\right] \tag{41}$$

where η, known as the temperature recovery factor, is found to be

$$\eta = 1 - 0.272(1-Pr) \tag{42}$$

An exact analysis (e. g., ref. 30) shows that for flow without a pressure gradient over an impermeable surface a very good approximation for η (to be denoted here as η_{ex}) is

$$\eta_{ex} = \sqrt{Pr} \tag{43}$$

For $Pr=0.72$, which is essentially the value for air, equation (42) yields $\eta=0.924$ instead of the accurate value $\eta_{ex}=0.845$ (ref. 34). This inaccuracy in the value of η implied by the equations used here, however, will not necessarily affect the accuracy of equation (40a) for the Nusselt number, since the derivation of this equation was actually made independently of the particular value of G_2 (i. e., independently of eq. (31)) and, hence, of η.[7] This is further verified by the agreement obtained with certain exact solutions, to be discussed subsequently. Thus, in applying equations (40a) and (37) for the calculation of heat-transfer rates, the actual value of T_e, as determined either by experiment or by equations (41) and (43), should be used.

It should be observed that the use of the equations developed here is not restricted to any particular type of temperature distribution $h(\xi)$ along the wall. Thus, it is not necessary, in applying the method of calculation described here, that the temperature distribution be expressed as a polynomial in ξ (unlike ref. 34) or as a power of ξ (unlike ref. 47). In the special case, however, in which T_o is expressed as a polynomial in ξ, the calculations indicated by the present method, including the transformation from the xt plane to

[7] If nevertheless desired, a practical and very simple means of modifying the present equations in order virtually to eliminate the implicit discrepancy between η and η_{ex} (for Pr near unity) would be merely to multiply the value of G_2 as given by equation (31) by 2. Equations (40a) and (40b) would remain unchanged by such a correction.

the physical xy plane, can be carried out directly without any quadratures by using the results given in the appendix and figures 4 and 5 of reference 6.

COMPARISON WITH EXACT SOLUTIONS

As a check on the accuracy of the results obtained here, comparison has been made with certain known exact solutions.

For the special case of a uniform wall temperature and a Prandtl number of unity, it is well known that the energy partial differential equation (4) or (5) reduces exactly to a single quadratic relation between the temperature and the velocity throughout the boundary layer. This relation can be expressed in the form

$$\frac{H}{H_1}=h+(1-h)\left(\frac{u}{u_1}\right) \tag{44}$$

By putting h=Constant and $Pr=1$ into equations (31) and (34b) the solution for b_1 as given by equation (34a) is found to be

$$b_1=2(1-h) \tag{45}$$

Substitution into equations (30) for values of b_n and comparison of the resulting stagnation-enthalpy profiles with the velocity profiles (29) then show that relation (44) is exactly satisfied. Thus, the equations used here reduce to the exact integral of the energy partial differential equation in this special case. As already indicated in the section "Comparison of Integral Methods for Laminar-Boundary-Layer Analysis," it has been found (ref. 1), moreover, that the skin-friction and heat-transfer coefficients obtained by the present method in this case agree almost exactly with those obtained by the exact method of reference 34. The present method has also been found to yield results of satisfactory accuracy for stability calculations in this case (refs. 1 and 3).

To check the results of the present method for the more general case of $Pr\neq1$ and variable wall temperature, calculations were carried out for the case

$$T_o/T_e=1.25-0.83\xi+0.33\xi^2$$

This is the case calculated in reference 34 by the exact method of analysis there. The local Nusselt number for this case was calculated by means of equations (40a) and (40b). In addition, temperature and velocity profiles were calculated by means of the present equations. The agreement between the results thus obtained and those in reference 34 was found to be quite close (see ref. 6 for details of the calculations and results).

It is interesting to note that by setting $G_5=0$ and solving the resulting differential equation for $T_o/T_e(\xi)$ it is found that zero heat transfer along the wall can be obtained for a nonuniform (as well as a uniform) wall temperature distribution. This result, in fact, generalizes a result of reference 47 (cf. ref. 6 for details).

From a practical point of view, it should be kept in mind that the solutions developed here are based on the viscosity-temperature relations (6) and (7), which are an approximation to the actual relation for air. Because of relations (6) and (7), the results obtained here, namely, equations (35) and (40a), indicate that, for a fixed wall temperature, the skin-friction coefficient and the Nusselt number will be independent of Mach number. For the Sutherland viscosity-temperature relation, however, this will not be quite valid (cf. ref. 48).

FLOW WITH PRESSURE GRADIENT, PRANDTL NUMBER $Pr=1$, AND UNIFORM WALL TEMPERATURE (h=CONSTANT)

From the equations derived in the section "Basic Equations," a relatively simple and sufficiently accurate method for most practical purposes of calculating laminar boundary-layer characteristics in a pressure gradient with heat transfer will be developed. For this purpose it will be assumed that the Prandtl number of the fluid is unity and that the wall temperature is uniform. These restrictions considerably simplify the mathematical analysis (cf., e. g., eqs. (21) and (22)).

A further advantage of assuming $Pr=1$ here is that in this case it follows from the energy partial differential equation (5) that for zero heat transfer (i. e., for $(\partial T/\partial y)_o=0$ and hence, $(\partial H/\partial y)_o=0$) H=Constant regardless of the pressure gradient. This yields the following well-known value of the equilibrium adiabatic wall temperature T_e for a Prandtl number of 1:

$$Te=T_1\left(1+\frac{\gamma-1}{2}M_1^2\right)=T_\infty\left(1+\frac{\gamma-1}{2}M_\infty^2\right) \tag{46}$$

Consequently,

$$h\equiv\frac{H_0}{H_1}\equiv\frac{c_pT_o}{c_pT_1+\frac{u_1^2}{2}}=\frac{T_o}{T_e} \tag{47}$$

Thus, the parameter h is in this case the physically significant ratio of the actual wall temperature to the equilibrium adiabatic wall temperature. It follows from this that for zero heat transfer $h=1$. It will be seen (cf. eqs. (45), (55a), and (55b)) that this condition is exactly satisfied by the approximate equations and solutions used here.

A brief discussion of methods for cases of $Pr\neq1$ and/or nonuniform wall temperature has been given in the introduction. The development given here will be essentially the same as that in references 6 and 7.

GENERAL APPROXIMATE SOLUTION

With $Pr=1$ and h constant, while $u_1/u_\infty(\xi)$ is arbitrary, the coefficients a_n and b_n in equations (15) and (16), by virtue of boundary conditions (19) to (23), can all be expressed in terms of a_2 and b_1, where b_1 remains arbitrary, while a_2 is given by:

$$a_2=-\frac{1}{2C}\left(\frac{T_1}{T_\infty}\right)^{\frac{2-\gamma}{\gamma-1}}\left(\frac{u'_1}{u_\infty}\right)h\left(1+\frac{\gamma-1}{2}M_1^2\right)\lambda \tag{48}$$

The profiles in terms of a_2 and b_1 are then:

$$\frac{u}{u_1}=(2\tau-5\tau^4+6\tau^5-2\tau^6)+(a_2/5)[-2\tau+5\tau^2-10\tau^4+10\tau^5-3\tau^6+(b_1/6h)(-\tau+10\tau^3-20\tau^4+15\tau^5-4\tau^6)] \tag{49}$$

$$\frac{H}{H_1}=h+(1-h)(35\tau^4-84\tau^5+70\tau^6-20\tau^7)+b_1(\tau-20\tau^4+45\tau^5-36\tau^6+10\tau^7) \tag{50}$$

With profiles (49) and (50) the following explicit expressions for F_1, F_2, and F_3, are obtained:

$$\left.\begin{aligned}
F_1&=0.1093+0.00211a_2-0.000622a_2^2+\\
&\quad 0.000412(b_1a_2/h)-0.0000095(b_1a_2/h)^2-\\
&\quad 0.000153(b_1a_2^2/h)\\
F_2&=0.395-0.500(1-h)+0.107b_1+\\
&\quad 0.0212a_2-0.00062a_2^2+0.0028(b_1a_2/h)-\\
&\quad 0.00015(b_1a_2^2/h)-0.0000095(b_1a_2/h)^2\\
F_3&=(1-h)[0.246-0.015a_2-0.00181(b_1a_2/h)]-\\
&\quad b_1[0.0683-0.00324a_2-0.00041(b_1a_2/h)]
\end{aligned}\right\}\tag{51}$$

With expressions (49), (50), and (51) inserted in equations (9) and (10), two ordinary differential equations for $\lambda(\xi)$ and $b_1(\xi)$ are obtained. Although these can be solved numerically for a given distribution of $u_1/u_\infty(\xi)$, the process may be tedious. A relatively simple general approximate solution of these equations will, therefore, be derived.

Equation (9) can be solved approximately for λ by assuming that F_1 and F_2 can be replaced there by constant "average" values $\overline{F}_1$ and $\overline{F}_2$ over the distance ξ. This is justified by the fact that the variable terms there, which are proportional to a_2 and b_1, are relatively small (cf. eqs. (51)). This is equivalent to replacing a_2 and b_1 by constant average values $\overline{a}_2$ and $\overline{b}_1$ for this purpose. With equation (49) for the velocity profile and equation (48) for a_2, equation (9) then becomes the following linear ordinary differential equation in λ:

$$(\overline{F}_1/2)\lambda'+\lambda\left\{\overline{F}_1(\rho_1'/\rho_1)+(u_1'/u_1)\left[\varphi_1+\frac{\gamma-1}{2}M_1^2(\varphi_1-\overline{F}_1)\right]\right\}=2C(\rho_\infty/\rho_1)(T_1/T_\infty)(u_\infty/u_1)\tag{52}$$

where φ_1 is a constant given by:

$$\varphi_1=0.3h+0.0905\overline{b}_1+0.00438+0.0232\overline{a}_2-0.00124\overline{a}_2^2+(0.0838-0.00458\overline{a}_2)(\overline{a}_2\overline{b}_1/30h)\tag{53}$$

With relations (13) and (14), the solution of equation (52) satisfying the condition $\lambda=0$ or a finite value (if $u_1=0$ at $\xi=0$) at the leading edge $\xi=0$ is found to be:

$$\lambda=(4/\overline{F}_1)C\frac{\displaystyle\int_0^\xi (u_1/u_\infty)^{\frac{2}{\overline{F}_1}\varphi_1-1}(T_1/T_\infty)^{\frac{2\gamma-1}{\gamma-1}\frac{\varphi_1}{\overline{F}_1}}d\xi}{(u_1/u_\infty)^{\frac{2}{\overline{F}_1}\varphi_1}(T_1/T_\infty)^{\frac{\gamma+1}{\gamma-1}\frac{\varphi_1}{\overline{F}_1}}}\tag{54}$$

Equation (54) is similar in form to equations obtained for zero heat transfer in references 2 and 49 and to those obtained for heat transfer, but with fourth-degree profiles and two boundary-layer thicknesses, in references 19 to 21. It is interesting to observe that for zero heat transfer it is possible to derive forms like equation (54) (cf. refs. 29 and 49) by applying the Stewartson-Illingworth transformation (refs. 50 and 51). However, by the present (approximate) method of analysis, it is seen that with the use of only the Dorodnitsyn transformation (8) such a form can be straightforwardly derived even for the case of heat transfer, but uniform temperature, along the wall (cf. also ref. 20).

A general approximate solution for $b_1(\xi)$ can be obtained in a comparatively simple manner by eliminating λ' from differential equations (9) and (10) in conjunction with the same type of simplifying approximations concerning the a_2 terms as made in deriving equation (54). (See ref. 6 for details.) A quadratic equation in b_1 is thereby obtained, with the solution

$$b_1=-\frac{j}{2m}\pm\left[\left(\frac{j}{2m}\right)^2+\frac{l}{m}\right]^{1/2}\tag{55a}$$

where

$$\left.\begin{aligned}
m&=\frac{2a_2}{h}\times10^{-4}\left[63.90-4.496a_2+\frac{a_2}{h}(2.705-0.2096a_2)\right]\\
j&=0.24602+a_2(0.07917-0.005868a_2)+\frac{2a_2}{h}\times10^{-4}\times\\
&\quad\left[-284.5+(42.26-2.067a_2)a_2-(7.906-0.9086a_2)\frac{a_2}{h}\right]\\
l&=2(1-h)(0.24602-0.01496a_2)\left\{1+\frac{a_2}{h}\left[-0.10495+\right.\right.\\
&\quad\left.\left.0.3h+a_2(0.02116-0.0006216a_2)\right]\right\}
\end{aligned}\right\}\tag{55b}$$

The physically appropriate root in equation (55a) will, in general, be that which is closer to the value $2(1-h)$.

After $\lambda(\xi)$ has been obtained by means of equation (54), the coefficient $a_2(\xi)$ follows from equation (48), and $b_1(\xi)$ can then be directly calculated by means of equations (55). For objects with sharp leading edges, for which $\lambda=0$ at $\xi=0$, it will ordinarily be found that an approximate value of b_1 according to equations (55) is that given by equation (45), which is valid exactly for the case $a_2\equiv0$. This is illustrated in detail in reference 7 by numerical example for the supersonic flow over a thin biconvex airfoil.

The general approximate solutions given by equations (54) and (55) are quite convenient for actual calculations and involve, at most, numerical integration. These solutions will be approximately valid as long as the a_2 terms in expressions (51) are indeed relatively small either individually or collectively. Such is expected to be ordinarily the case in practice. In cases for which the a_2 terms become relatively large, however, the ordinary differential equations (9) and (10) may have to be solved numerically.

In evaluating $\overline{F}_1$ and φ_1 a reasonable average value $\overline{a}_2$ for a_2 for any given $u_1/u_\infty(\xi)$, h, and M_∞ can usually be obtained by considering equation (48) for $(a_2C/h\lambda)$ and equation (54) for λ/C. A satisfactory average value $\overline{b}_1$ for b_1 in evaluating $\overline{F}_1$ and φ_1 will ordinarily be that given by equation (45).

In reference 2, numerical examples based on the case $u_1/u_\infty=1-b\xi$ (where b is a positive constant) for $M_\infty=0$, 1, and 3 and zero heat transfer at the wall ($h=1$) were carried out to determine the accuracy of approximate solution (54) of ordinary differential equation (52). Comparison of the solutions obtained by means of equation (54) was made with numerical solutions of differential equation (52) without the use of any of the approximating assumptions made in deriving equation (54). The comparison indicated, on the whole, satisfactory agreement for practical purposes (including stability calculations) between the results of

equation (54) and the numerical solution of equation (52). Details are given in reference 2. Similar comparisons have also been carried out in reference 21 for the cases $u_1/u_\infty = 1 \pm b\xi$ with heat transfer at the wall. The agreement between the type of approximate solution given by equation (54) and the numerical solution of the original ordinary differential equation was, again, found to be on the whole satisfactory.[8]

SKIN FRICTION, HEAT TRANSFER, VELOCITY, AND TEMPERATURE PROFILES

With $\lambda(\xi)$ and $b_1(\xi)$ determined, the boundary-layer characteristics can all be straightforwardly calculated. The local skin-friction coefficient will be

$$C_{fl} \equiv \frac{(\mu\, \partial u/\partial y)_o}{\frac{1}{2}\rho_\infty u_\infty^2} = 4[1-(a_2/5)-(b_1 a_2/60h)]\,(C/\sqrt{\lambda})(T_1/T_\infty)(u_1/u_\infty)R_L^{-1/2} \quad (56)$$

The Nusselt number giving local heat-transfer properties at the wall will be

$$Nu \equiv \frac{(k\, \partial T/\partial y)_o L}{k_\infty (T_e - T_o)} = (C/\sqrt{\lambda})[b_1/(1-h)](T_1/T_\infty)\sqrt{R_L} \quad (57)$$

The velocity and temperature profiles follow from equations (49), (50), and (24) in conjunction with equation (8) for transforming to the physical plane. For zero heat transfer at the wall, an explicit expression for y as a function of τ in terms of a_2 is given in appendix A of reference 2. This expression can be conveniently written in the form

$$\frac{y(\xi,\tau)}{\delta_t} = \tau + \left(\frac{\gamma-1}{2}\right) M_1^2\,[g_1(\tau)+a_2 g_2(\tau)+a_2^2 g_3(\tau)]$$

where g_1, g_2, and g_3 are definite functions (polynomials) of τ only which remain the same for all cases. These functions can, if desired, be evaluated and plotted once for all. A similar expression can be obtained for the case of heat transfer at the wall, except that additional terms, such as those proportional to $a_2 b_1/h$, will be included. The universal functions of τ thus obtained can, if desired, also be evaluated once for all. For a given value of ξ, y or y/δ (cf. eqs. (26a) and (26b)) can then, in any given case, be found quite straightforwardly for values of τ from $\tau=0$ to $\tau=1$.

A numerical example to check the accuracy of the results obtained by the equations developed in this section was carried out in reference 2. This example, as previously indicated, was the case of flow with a linearly decreasing velocity outside of the boundary layer. Velocity profiles, local skin-friction coefficient, and minimum critical Reynolds numbers for laminar instability were calculated by this means, and the results for incompressible flow ($h=1$ and $M_\infty=0$) thus obtained were compared with those based on the series solution in reference 52 of the original partial differential equation (1). The agreement was in all cases found to be satisfactory for practical purposes. (Details are given in ref. 2.)

The solutions presented here require some modification in two important special cases: (a) Flow near a forward stagnation point and (b) calculation of the separation point in an adverse pressure gradient.

STAGNATION FLOWS

The case

$$u_1/u_\infty = b\xi \quad (58)$$

where b is a positive constant represents physically the flow in the vicinity of a forward stagnation point, such as the subsonic flow over the leading edge of a blunt object. For zero Mach number, an exact solution of the ordinary differential equations (9) and (10) (with uniform wall temperature) can be obtained in the form λ=Constant and b_1=Constant. Equations (9) and (10) then become algebraic equations for λ and b_1. For the special case of zero heat transfer ($h=1$ and $b_1=0$), however, it has already been found (refs. 43 and 2) that these equations will not yield any physically significant real roots. In reference 2 it was shown that an approximate solution can still be obtained in this case by writing the algebraic equation as $f(\lambda)=0$ and taking the value of λ for which $f(\lambda)$ has a local maximum value relatively close to the λ-axis. The root $\lambda=9.481$ was thus obtained. This solution, however, is unsatisfactory in principle. Consequently, the use of fourth-degree, instead of sixth-degree, velocity profiles for this case was investigated in reference 2. The profiles were chosen to satisfy the usual Kármán-Pohlhausen conditions. A physically significant real root, namely, $\lambda=7.052$, was now obtained, and the accuracy of the resulting solution was compared with the results of an exact solution (ref. 53). In particular, skin friction, velocity profiles, and minimum critical Reynolds number were compared. The comparison indicated that the results obtained by the use of the fourth-degree profiles led on the whole to results of satisfactory accuracy. It was therefore concluded that the boundary-layer characteristics in flow near a forward stagnation point can be determined with satisfactory accuracy by the Karman-Pohlhausen method with fourth-degree profiles.

To calculate the boundary layer near a forward stagnation point for the more general case of heat transfer at the wall, in particular for a prescribed uniform wall-temperature ratio h or T_o/T_∞, the method of reference 2 can be generalized by introducing fourth-degree stagnation-enthalpy, as well as velocity, profiles. This has been carried out in reference 20 with the introduction of a thermal, in addition to a dynamical, boundary-layer thickness. Two algebraic equations in essentially the two (constant) boundary-layer thicknesses are obtained. These equations can, in general, be solved either numerically for a given h or by using the values in figures 1 and 2 of reference 20. Although reference 20 is based on flow over a sweat-cooled surface and, hence, includes a normal mass flow at the wall ($v=v_o$ at $\tau=0$), the results there can also be used for an impermeable wall by simply putting $C=0$ and letting h be arbitrary. (This C is not to be confused with the temperature-viscosity factor used in the present paper.) An example of low-speed ($M_\infty=0$) flow in a favorable pressure gradient with a stagnation point at $\xi=0$, representing subsonic flow over a turbine blade, was carried out in reference 20 on the basis of the method presented there.

[8] In reference 21, the analysis was based on fourth-degree profiles in conjunction with both a thermal and a dynamical boundary-layer thickness. An equation quite analogous to equation (54) for $\lambda(\xi)$, however, was obtained by analogous approximating assumptions.

An alternative method of calculating flows near a forward stagnation point, based on the use of a single boundary-layer thickness, is given in reference 22.

CALCULATION OF SEPARATION POINT

The equations thus far developed in this section can be used to calculate the laminar separation point in an adverse pressure gradient. The results thus obtained will generally be more accurate than those obtained by the use of fourth-degree profiles. By an analysis for incompressible flow for the case of a linearly diminishing velocity outside the boundary layer, it was found (ref. 45), however, that still greater accuracy for the location of the separation point is obtainable by the special use, for this purpose, of seventh-degree velocity profiles satisfying an additional condition involving the fourth derivative of the velocity at the wall at the separation point. This condition would necessarily be satisfied by an exact solution of the original partial differential equations. This method of calculating the separation point was subsequently extended to compressible flow with zero heat transfer in reference 2 and to compressible flow with heat transfer in reference 7. The method of analysis to be presented here is essentially that of reference 7.

It may be mentioned that a considerable number of methods of calculating the laminar separation point have been developed. No attempt will be made here to summarize or evaluate all of these methods. For incompressible flow, a method which has been found to yield results of satisfactory accuracy in addition to that of reference 45 is that of reference 54. For compressible flow with zero heat transfer (which, of course, includes incompressible flow) recent methods, in addition to that of reference 2, are those of references 29, 50, 55, and 56. For compressible flow with heat transfer, the only studies of laminar separation which appear to have been made, in addition to that of reference 7, are those of references 13 to 16. The advantage of the method to be presented here is, once again, not only that it appears to yield results of adequate accuracy but that the analysis is kept relatively simple, although it is based on a minimum of what might be termed mathematically "arbitrary" assumptions. The method of analysis developed here is indeed sufficiently simple and flexible to be applicable to a wide variety of conditions. (The method has, in fact, been quite recently extended to the case of compressible flow over a transpiration-cooled surface (ref. 57).) The calculations to be performed according to the method presented here will be relatively simple and will involve, at most, numerical integration.

By differentiating the momentum partial differential equation (1) it can be shown (ref. 7), under the present assumption of a Prandl number of 1 and a linear viscosity-temperature relation, that, at the separation point, with or without heat transfer at the wall,

$$(\partial^4 u/\partial t^4)_o=0 \tag{59}$$

The seventh-degree velocity profile satisfying condition (59) in addition to conditions (17) to (23) is [9]

$$\frac{u}{u_1}=\left(\frac{7}{4}\tau-\frac{21}{4}\tau^5+7\tau^6-\frac{5}{2}\tau^7\right)+a_2\left(-\frac{1}{2}\tau+\tau^2-\frac{5}{2}\tau^5+3\tau^6-\tau^7\right)+(a_2b_1/3h)\left(-\frac{1}{5}\tau+\tau^3-3\tau^5+\frac{16}{5}\tau^6-\tau^7\right) \tag{60}$$

where a_2 is given by equation (48). Separation occurs where $(\partial u/\partial y)_o=0$ and, hence, where $(\partial u/\partial \tau)_o=0$. Therefore, according to equation (60), separation will occur where $a_2(\xi)$ has the value (denoted by a_{2s})

$$a_{2s}=\frac{3.5h}{h+\frac{2}{15}b_1} \tag{61}$$

From equations (48) and (61) it follows that the value λ_{sep} of λ at the separation point will, in general, be

$$\lambda_{sep}=-7C\frac{(T_1/T_\infty)^{-\left(\frac{2-\gamma}{\gamma-1}\right)}}{(u_1'/u_\infty)\left(1+\frac{\gamma-1}{2}M_1^2\right)}\frac{1}{h+\frac{2}{15}b_1} \tag{62}$$

A satisfactory approximation for b_1 in equation (62) will, in general, be that given by equation (45) (cf. also ref. 7). With this expression for b_1, equation (62) becomes

$$\lambda_{sep}=-105C\frac{(T_1/T_\infty)^{-\left(\frac{2-\gamma}{\gamma-1}\right)}}{(u_1'/u_\infty)\left(1+\frac{\gamma-1}{2}M_1^2\right)}\frac{1}{11h+4} \tag{63}$$

By inserting profile (60) into differential equation (9) and assuming, as in the foregoing analysis, that the a_2 and b_1 terms in F_1 and F_2 may be replaced by constant values, an ordinary differential equation of the same form as equation (52) is obtained, except that the explicit expressions for $\overline{F}_1$ and φ_1 (to be denoted now as $\overline{F}_{1s}$ and φ_{1s}) are modified, while the factor 2 on the right side of equation (52) is replaced by 7/4. Comparison, accordingly, with the solution (eq. (54)) of equation (52) yields the following solution for $\lambda(\xi)$ (denoted now as λ_s):

$$\lambda_s(\xi)=\frac{7}{2\overline{F}_{1s}}C\frac{\int_0^\xi (u_1/u_\infty)^{\left(\frac{2}{\overline{F}_{1s}}\varphi_{1s}-1\right)}(T_1/T_\infty)^{\frac{2\gamma-1}{\gamma-1}-\frac{\varphi_{1s}}{\overline{F}_{1s}}}d\xi}{(u_1/u_\infty)^{\frac{2}{\overline{F}_{1s}}\varphi_{1s}}(T_1/T_\infty)^{\frac{\gamma+1}{\gamma-1}-\frac{\varphi_{1s}}{\overline{F}_{1s}}}} \tag{64}$$

Taking the constant value of a_2 as that at the separation point (as in refs. 2 and 7), the espressions for $\overline{F}_{1s}$ and φ_{1s} are found to be

$$\left.\begin{aligned}\overline{F}_{1s}&=0.1159+0.002525a_{2s}-0.001454a_{2s}^2-0.0000572(b_1a_{2s}/h)^2-0.000574(b_1a_{2s}^2/h)+0.000887(b_1a_{2s}/h)\\ \varphi_{1s}&=0.25h+0.0437+0.0738b_1+0.0348a_{2s}-0.00291a_{2s}^2+0.00773(b_1a_{2s}/h)-0.001147(b_1a_{2s}^2/h)-0.0001145(b_1a_{2s}/h)^2\end{aligned}\right\} \tag{65}$$

where a_{2s} and b_1 are given by equations (61) and (45), respectively. The quantities $\overline{F}_{1s}$ and φ_{1s} are functions of h only and are shown in figure 1.

[9] This seventh-degree profile is to be used only for calculation of the separation point. For other purposes, the sixth-degree profile (eq. (49)) should be used, even in an adverse pressure gradient (cf. ref. 2).

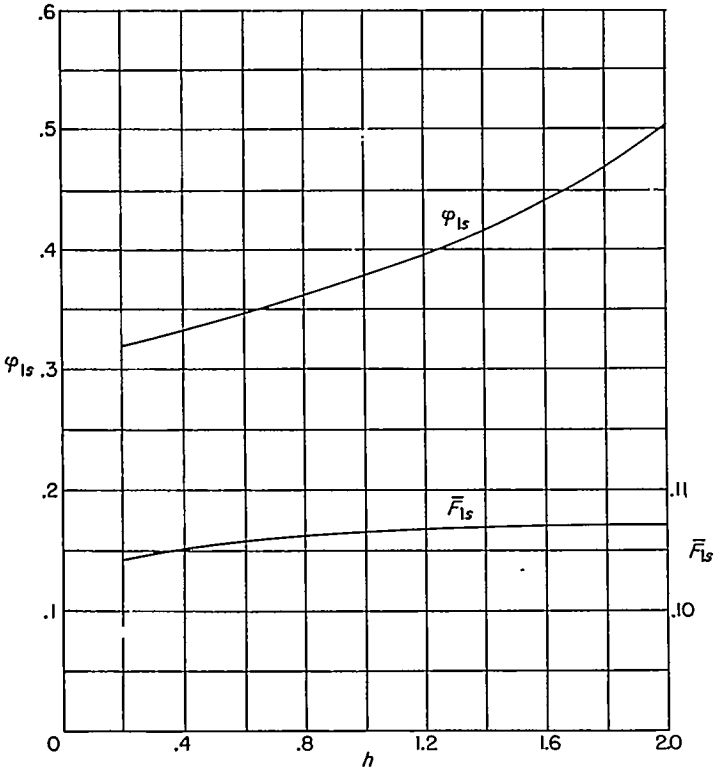

FIGURE 1.—$\bar{F}_{1s}$ and φ_{1s} as functions of h.

For any given reference Mach number M_∞ and uniform wall temperature ratio h, the separation point in a region of given adverse pressure gradient, as specified by $u_1/u_\infty(\xi)$, will be the station ξ at which the right sides of equations (63) and (64) are equal. Thus, it is necessary, in general, only to plot λ versus ξ, in the anticipated vicinity of separation, in accordance with both equations (63) and (64) and to determine the point of intersection of these two curves. The separation point will evidently be independent of C, so that for the purpose of determining the separation point one may set $C=1$.

In case the region of adverse pressure gradient starts at some point $\xi=\xi_a$ downstream of the leading edge, equation (64) can still be applied directly in calculating the separation point. Greater accuracy, however, might be obtained in such a case by applying equation (64) only for the region of adverse pressure gradient. For this purpose, equation (64) must be modified to satisfy the boundary condition $\lambda=\lambda_a$ at $\xi=\xi_a$. Thus,

$$\left.\begin{aligned} I(\xi)\lambda_s(\xi)&=I(\xi_a)\lambda_a+\frac{7}{2}\int_{\xi_a}^{\xi}\left(\frac{u_1}{u_\infty}\right)^{\frac{2}{\bar{F}_{1s}}\varphi_{1s}-1}\left(\frac{T_1}{T_\infty}\right)^{\frac{2\gamma-1}{\gamma-1}\frac{\varphi_{1s}}{\bar{F}_{1s}}}d\xi \\ &\text{where} \\ I(\xi)&=(u_1/u_\infty)^{\frac{2}{\bar{F}_{1s}}\varphi_{1s}}(T_1/T_\infty)^{\frac{\gamma+1}{\gamma-1}\frac{\varphi_{1s}}{\bar{F}_{1s}}} \end{aligned}\right\} \quad (66)$$

and where λ_a can be obtained as the value of λ at $\xi=\xi_a$ based on equation (54) for the region $0\leqq\xi\leqq\xi_a$ of favorable pressure gradient.

For purposes of calculating the separation point for various values of the temperature ratio T_o/T_∞ and of the reference Mach number M_∞, equation (63) may be replaced by the following equivalent equation for the value of λ at the separation point:

$$\lambda_{sep}=-105C\frac{(T_1T_\infty)^{-0.5}}{[4+11(T_o/T_\infty)+0.8M_\infty^2](u_1'/u_\infty)} \quad (67)$$

Equation (67) follows from equation (63) by inserting relations (13), (14), (46), and (47) there, with $\gamma=1.4$.

Numerical examples for flow with a linearly decreasing velocity at the outer edge of the boundary layer are illustrated in detail in reference 7, and these will be discussed briefly in the following section. An example based on a stagnation flow followed by an adverse pressure gradient is also discussed in detail in reference 7. For the case of a linearly decreasing velocity outside of the boundary layer with zero heat transfer at the wall the separation point was calculated by the method presented here for Mach numbers M_∞ ranging from 0 to 10. These results are compared in table I with those of reference 50, and the agreement is seen to be extremely close.

It may be recalled that the method of calculating the separation point presented here is based on the assumption of a linear viscosity-temperature relation $\mu\propto T$ and of a Prandtl number Pr of unity. It is noteworthy, in this connection, that it has been concluded in a recent analysis (ref. 56) that for $\mu\propto T^\omega$ and $\omega<1$, the separation point for $Pr>0.7$ occurs at roughly the same position as for $Pr=\omega=1$.

DISCUSSION OF SKIN-FRICTION, HEAT-TRANSFER, SEPARATION, AND STABILITY CHARACTERISTICS

To conclude this report, a summary will be given in this section of the implications of the equations developed here regarding the effect of wall temperature, Mach number, and pressure gradient on the laminar-boundary-layer characteristics. These conclusions have been derived and illustrated in detail especially in reference 7.

SKIN-FRICTION AND HEAT-TRANSFER COEFFICIENTS

The effect of wall temperature on the skin-friction and heat-transfer coefficients will depend on the nature (favorable or adverse) of the pressure gradient. This follows from the fact that in ordinary differential equation (52) and in expression (48) for a_2 the temperature parameter h appears primarily in a form multiplied by the velocity gradient u_1'. The effect of the wall temperature on the skin-friction coefficient arising from the $(u_1'h)$ term in a_2 is particularly important. Thus, equations (56) and (48) show that, without the effect of the temperature-viscosity factor C, lowering the wall temperature tends to diminish the local skin friction in a favorable pressure gradient (negative u_1') and to increase it in an adverse pressure gradient. It can be shown (ref. 7) that a similar, but much smaller, effect on the Nusselt number will also tend to occur.

Since the velocity gradient u_1' in the equations developed here (cf., especially, eqs. (48), (52), and (53)) appears in a form multiplied by the wall-temperature ratio h, it can be inferred that a lowering of the wall temperature has a tendency to diminish the direct effect of a given pressure gradient, that is, the effect of u_1' as such, on the boundary-layer properties. This is explainable physically by the increased

importance of the inertia forces relative to the pressure gradient because of the increase of the fluid density when the wall temperature is decreased.[10] A clear illustration of this will be seen subsequently in the analysis of laminar separation. It must be observed, however, that the effect of a pressure gradient also appears indirectly, namely, in the variation of u_1/u_∞ and T_1/T_∞ with ξ. For Mach numbers above 1, in fact, the T_1/T_∞ terms in λ (eq. (54)) may become particularly important, so that in such a case the net effect of the pressure gradient may actually be increased by a lowering of the wall temperature. This is illustrated in detail in reference 7 by a numerical example for the supersonic flow over a thin airfoil (especially at $M_\infty=3$).

From equations (56) and (57) it follows that the ratio of local skin friction to Nusselt number can be expressed in the form

$$r\equiv\frac{C_{fl}}{Nu}\left(\frac{u_\infty}{u_1}\right)R_L=\frac{4(1-h)}{b_1}\left[1-\frac{a_2}{5}\left(1+\frac{b_1}{12h}\right)\right] \tag{68}$$

For flow along a flat plate ($u_1/u_\infty=1$, $a_2=0$, and $b_1=2(1-h)$), equation (68) implies that $r=2$. For flow in a pressure gradient, however, since ordinarily $b_1\approx2(1-h)$, it follows from equations (68) and (48) that $r>2$ along the flow in a favorable pressure gradient ($u_1'>0$), and $r<2$ in an adverse pressure gradient ($u_1'<0$). Moreover, it also follows from these equations that lowering the wall-temperature parameter h will tend to bring r closer to its value for flow without a pressure gradient. This illustrates the diminution of the direct effect of a pressure gradient by cooling of the wall.

From equations (54), (56), and (57) it follows that both the skin friction and Nusselt number will be proportional to $\sqrt{C}$. Thus, an effect of wall temperature on the skin-friction and heat-transfer coefficients follows from the viscosity-temperature coefficient C arising from the particular viscosity-temperature relation (eqs. (6) and (7)) assumed here. This effect is independent of the pressure gradient. From equation (7) it follows that if, as will ordinarily be the case, $T_o>S$, that is, $T_o>216°$ R, then a lowering of the ratio T_o/T_∞ will increase C_{fl} and Nu. For a fixed ratio h of T_o/T_e, it follows from equation (25) that a Mach number effect will also appear in C. Thus, if $T_o>S$, then for a fixed value of h an increase of Mach number M_∞ will diminish C and hence will tend, as far as C is concerned, to diminish both the skin-friction and heat-transfer coefficients in proportion to $\sqrt{C}$.

From equation (54), as has already been noted, it will be found that in the presence of a pressure gradient λ/C may be appreciably affected by the Mach number because of the values of $T_1/T_\infty(\xi)$. Consequently, it can be inferred that a pressure gradient will, in general, tend to enhance the effect of Mach number on both the skin-friction and heat-transfer coefficients. This effect will depend on the nature of the pressure gradient. For a favorable pressure gradient, for example, for which $u_1/u_\infty>1$ and hence $T_1/T_\infty<1$, an increase of Mach number will tend to increase λ/C and, hence, to decrease both the skin-friction coefficient and the Nusselt number.

Since λ will ordinarily be only little affected by the wall temperature, equation (26b) implies that cooling of the wall will, in general, tend to diminish the physical boundary-layer thickness δ. However, for a given value of $T_o/T_e=h$, the boundary-layer thickness δ will tend to increase with Mach number, especially in a favorable pressure gradient.

SEPARATION

For a fixed velocity distribution $u_1/u_\infty(\xi)$ outside the boundary layer and a fixed Mach number M_∞, diminishing the wall temperature will tend to delay separation by moving the separation point downstream. This can be seen particularly from equation (63), according to which the value of λ required for separation λ_{sep} will increase as h is diminished. This is a further illustration of the diminution of the direct effect of a pressure gradient (in this case, an adverse pressure gradient) by cooling of the wall.

The effect of Mach number on the separation point for a fixed distribution of $u_1/u_\infty(\xi)$ and either a fixed value of $h=T_o/T_e$ or a fixed value of T_o/T_∞ cannot be so readily predicted from the equations developed here, since an increase of Mach number in the adverse pressure gradient will tend to decrease both λ_{sep} (eq. (63)) and $\lambda_s(\xi)$ (eq. (64)). However, numerical examples carried out for the case $u_1/u_\infty=1-\xi$ have indicated that for a fixed ratio h of wall temperature to equilibrium adiabatic wall temperature, including the case of an insulated wall ($h=1$), an increase of Mach number tends to enhance separation by moving the separation point upstream (cf. table I and refs. 2, 21, 16, and 50). Lowering the (fixed) value of h, however, tends to diminish this unfavorable effect of Mach number on separation (cf. ref. 21).

If the ratio T_o/T_∞ of wall temperature to free-stream or reference temperature instead of that of wall temperature to equilibrium adiabatic wall temperature h is kept fixed, the effect of Mach number on the separation point is changed. This is essentially due to the fact that for a fixed value of T_o/T_∞ the temperature ratio h decreases with Mach number (cf. eq. (25)) and hence λ_{sep} will no longer tend to be so greatly decreased by an increase in M_∞ (cf. eq. (67)). Consequently, the effect of an increase in Mach number is, in general, much less unfavorable in this case and may, in certain cases, move the separation point downstream, especially at high fixed values of T_o/T_∞.

Figure 2 shows the separation point for the case $u_1/u_\infty=1-\xi$ as a function of the wall-temperature ratio T_o/T_∞ for $M_\infty=0$. The favorable effect of cooling of the wall is clearly seen here. Figure 3 shows the separation point as a function of Mach number for a fixed ratio of wall to free-stream reference temperature $T_o/T_\infty=2$. An increase of Mach number is seen in this case actually to move the separation point downstream, in contrast with its effect, also shown in figure 3, at zero heat transfer. Figures 2 and 3 are based on the equations developed here, and further details of the calculations can be found in reference 7.

STABILITY CHARACTERISTICS

It has already been pointed out that the methods developed here may be expected to yield sufficiently accurate results

[10] This conclusion and the foregoing conclusions on the effect of wall temperature on skin friction have been similarly derived in reference 21. It is noteworthy, moreover, that such conclusions have also been derived in references 12 to 16 by considerably different methods of analysis. The physical explanation for the lessened effect of a pressure gradient by cooling of the wall has been given independently in references 21 and 16.

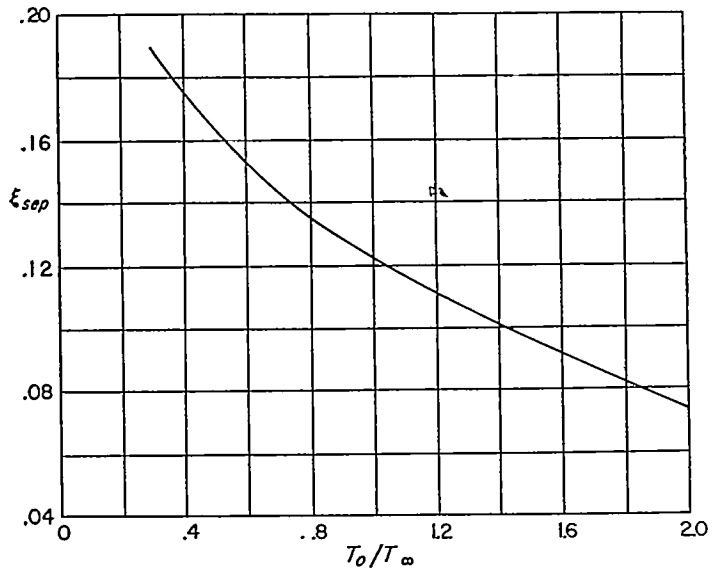

FIGURE 2.—Separation point as a function of wall temperature. Incompressible flow; $u_1/u_\infty=1-\xi$; $M_\infty=0$.

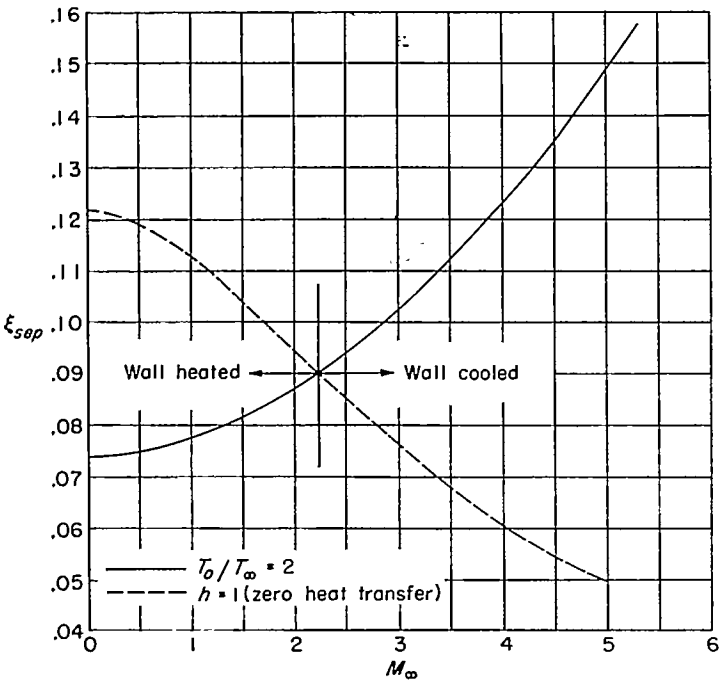

FIGURE 3.—Separation point as a function of Mach number. $u_1/u_\infty=1-\xi$.

for laminar-boundary-layer stability calculations. In fact, although stability calculations and their results are shown to some extent in references 1 and 2, the chief purpose of these calculations was to show that the results obtained by the approximate methods presented here compare sufficiently closely with those obtained by known exact solutions. Such was also, at first, the purpose of reference 3. Thus, it was found that for compressible flow without a pressure gradient, such as flow over a flat plate, the minimum critical Reynolds numbers for various Mach numbers at zero heat transfer, as well as at various uniform wall temperatures, were predicted with satisfactory accuracy by the solutions obtained by the methods presented here. Moreover, it was shown that the maximum wall temperatures (to be called here the "critical temperatures") required to stabilize the flow completely were also calculated as functions of the Mach number with satisfactory accuracy on the basis of these methods (presented in the section "Flow Without an Axial Pressure Gradient With Arbitrary (Constant) Prandtl Number and Variable Wall Temperature"). The minimum critical Reynolds number for incompressible flow in the vicinity of a forward stagnation point as calculated by the Kármán-Pohlhausen method (cf. the subsection "Stagnation Flows" in the section "Flow With Pressure Gradient, Prandtl Number $Pr=1$, and Uniform Wall Temperature (h=Constant)") was found to agree well with that calculated by the exact solution of reference 53. Finally, for incompressible flow with a linearly diminishing velocity outside the boundary layer, the present method of calculation (cf. the section "Flow With Pressure Gradient, Prandtl Number $Pr=1$, and Uniform Wall Temperature (h=Constant)") was found to lead to a minimum critical Reynolds number in satisfactory agreement with that calculated from the solution in reference 52.

Most of the stability calculations which have been carried out in this country have been based on the analysis and criteria developed by Lin (ref. 58) for incompressible flow and subsequently extended by Lin and Lees (refs. 8 and 46) to compressible flow. In reference 8, simplified approximate two-dimensional stability criteria for compressible flow have been developed, whereby, without much difficulty, it is possible to calculate, for a given type of flow, the minimum critical Reynolds numbers as well as the wall temperature required for infinite minimum critical Reynolds number. As will be explained subsequently, these criteria have recently been modified. The minimum critical Reynolds number $R_{b,cr}$ is the minimum Reynolds number necessary for the possibility that very small disturbances in the boundary layer may be amplified with time; that is, $R_{b,cr}$ is the minimum Reynolds number required for instability of the laminar boundary layer with respect to small disturbances of at least certain wavelengths. The wall temperature for infinite values of $R_{b,cr}$ is then usually interpreted as the highest temperature for which the (compressible) laminar boundary layer will be completely stable for all Reynolds numbers. The analyses in references 58, 46, and 8 and subsequent analyses based on them are of practical interest, since under the condition of a sufficiently low free-stream turbulence a necessary (though not sufficient) condition for transition from a laminar to a turbulent boundary layer appears to be an instability of the laminar layer. A survey (as of 1952) of theoretical and experimental investigations on laminar-boundary-layer stability can be found in reference 59.

The purpose of the present subsection is to summarize the theoretical investigations on laminar-boundary-layer stability performed at the Polytechnic Institute of Brooklyn by using the mean-flow (or steady-state) solutions obtained by the methods presented in this report. In an unpublished report entitled "Calculation of Stability of Constant-Pressure Boundary Layers on Isothermal Surfaces With an Integral-Method Mean-Flow Solution" Professor Martin Bloom de-

veloped certain modifications of Lees' approximate stability criteria (ref. 8) and applied these to the calculation of the stability of the laminar boundary layer over a flat plate at uniform wall temperature. This report is available for loan or reference in the Division of Research Information, National Advisory Committee for Aeronautics, Washington, D. C. This work is summarized in references 3 to 5. Minimum critical Reynolds numbers for given wall temperatures and Mach numbers were calculated. Moreover, the wall temperature required to stabilize the flow completely was also calculated as a function of the Mach number. Similar types of calculations with similar results were carried out independently by Van Driest (ref. 60), and these are now well known.[11] Briefly, the results indicate the stabilizing effect of cooling of the wall by increasing the minimum critical Reynolds number for a given Mach number. Moreover, for a Prandtl number Pr of 1, it is found that the boundary layer can be completely stabilized by sufficiently low wall-temperature ratios T_o/T_∞ for Mach numbers M_∞ between 1 and approximately 5. (For $Pr=0.72$, this can be theoretically accomplished for $1<M_\infty<9$). At higher Mach numbers, particularly in the hypersonic range, the validity of the theoretical approach has not been established.

The stability of the laminar compressible boundary layer in a pressure gradient has been analyzed in reference 44 for zero heat transfer at the wall. Calculations there for the supersonic flow over a thin biconvex airfoil indicated the stabilizing influence of the favorable pressure gradient. This stabilizing influence, however, was found to be considerably diminished at higher free-stream Mach numbers $M_\infty=4$. The stabilizing influence of a favorable pressure gradient can also be clearly illustrated by comparing the minimum critical Reynolds number $R_{\infty,cr}$ (namely, $R_{\infty,cr}\xi^2=2.40\times10^8$ (ref. 2)) for the incompressible flow $u_1/u_\infty=\xi$ in the vicinity of a forward stagnation point with the much smaller value $R_{\infty,cr}\xi=7.3\times10^4$ (ref. 1) for incompressible flow over a flat plate. The destabilizing effect, in the case of zero heat transfer, of an adverse pressure gradient is readily illustrated by considering the case $u_1/u_\infty=1-\xi$ (ref. 2). The minimum critical Reynolds numbers for this case for $M_\infty=0$ and 1 are compared, in table II, with the larger values for flow over a flat plate taken from reference 1.

For compressible flows with heat transfer and pressure gradient, the only stability calculations which appear to have been made thus far are those in references 7 and 62. In both of these references only Mach numbers of 3 or lower were considered. (Cf. footnote 11.) In reference 62, the small-perturbation solutions of reference 25 are used, while reference 7 uses solutions based on the methods of analysis presented in the present report. Reference 62 shows that

[11] Bloom's first calculations (refs. 3 and 4) gave results quite similar to the well-known results of Van Driest (ref. 60). Further modifications of the stability criteria, however, led to rather complicated curves with several branches of critical temperature ratio versus Mach number (ref. 5). These were apparently due to large values of the stability parameter λ (not related to the λ of the present report) as defined in reference 8, for large values of M_∞ or small values of T_o/T_∞. Dunn and Lin (ref. 61), however, have quite recently made basic refinements in the analysis of reference 46 and have developed a more accurate set of both two-dimensional and three-dimensional stability criteria. Calculations for flow over a flat plate based on this set yielded results quite similar to those of Van Driest or of Bloom's first calculations. According to the Dunn-Lin criteria, the values of λ (now redefined) remained quite small for flow over a flat plate even at high Mach numbers. The new two-dimensional criteria do not appear to yield results appreciably different from those of Bloom or Lees for Mach numbers below approximately 3.

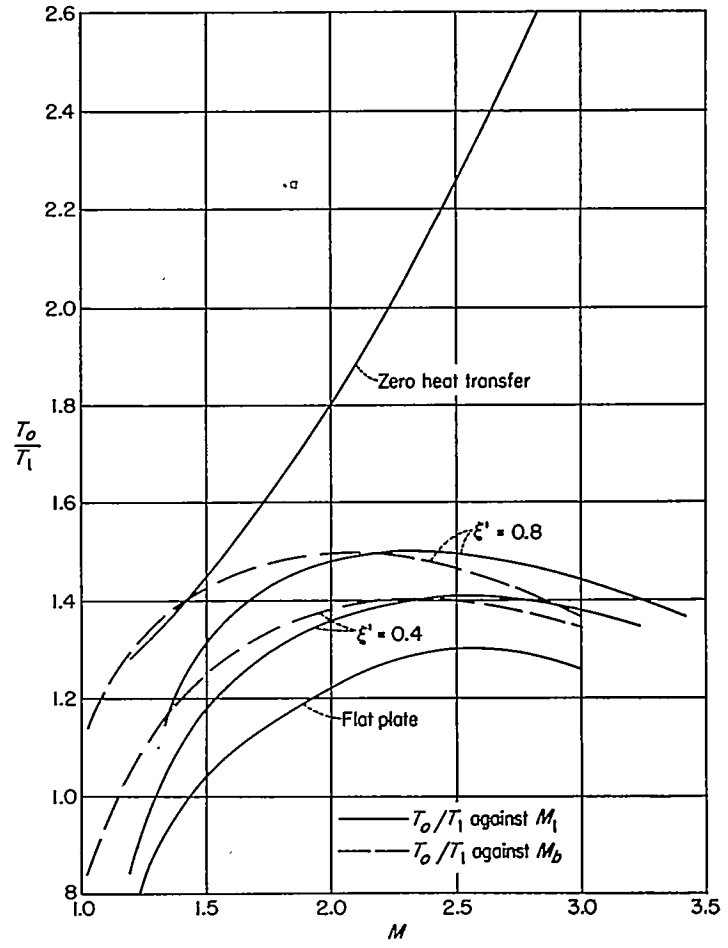

FIGURE 4.—Ratios of wall to local main-stream temperature required for infinite minimum critical Reynolds number over thin biconvex supersonic airfoil.

the critical wall temperatures required to stabilize the laminar boundary layer completely are, for a given Mach number, increased by a favorable pressure gradient and decreased by an adverse pressure gradient. Further calculations also indicate the greater amount of cooling required to stabilize completely the flows with adverse pressure gradients than that required for those with favorable pressure gradients. This illustrates in a further fashion the stabilizing influence of a favorable, and the destabilizing influence of an adverse, pressure gradient. In reference 7, critical wall temperatures have been determined for the supersonic flow over a thin biconvex airfoil at two given stations along the flow, and these have been compared with the corresponding results for flow over a flat plate. The results are shown in figure 4, wherein it is seen that higher critical temperature ratios T_o/T_1 are obtained for the flow with the favorable pressure gradient than for the flow over a flat plate. It may be observed, in this connection, that for a given reference temperature T_b at a point immediately behind the shock wave at the leading edge of the supersonic airfoil, the critical wall temperature may, at the higher Mach numbers, be greater for the favorable-pressure-gradient case than for the flat-plate case. This is due simply to the fact that the local temperature T_1 outside the boundary layer over the airfoil diminishes along the flow (see ref. 7 for details).

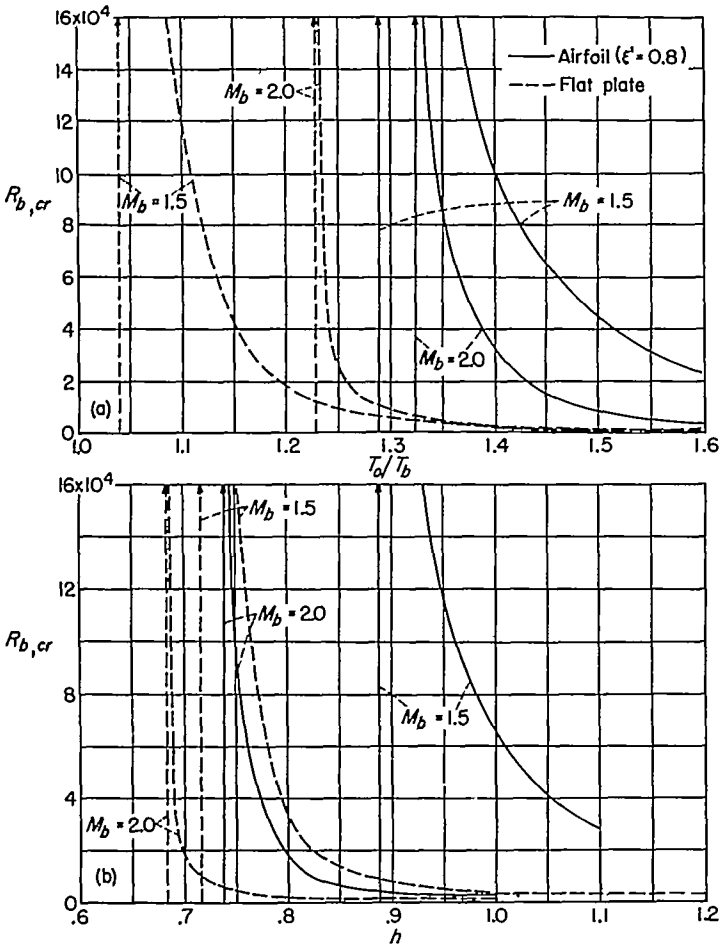

FIGURE 5.—Minimum critical Reynolds numbers $R_{b,cr}$ versus Mach number M_b, wall-reference-temperature ratio T_o/T_b, and wall-equilibrium-temperature ratio h. Vertical lines with arrows indicate corresponding asymptotes; compare with table III.

A second type of stability calculation carried out in reference 7 was the determination of minimum critical Reynolds numbers for the laminar boundary layer at a given station of the supersonic airfoil for various values of the (uniform) wall temperature and free-stream Mach number. The results are shown in table III and figure 5, where comparison is also made with the flow over a flat plate. The stabilizing effect of cooling of the wall and of the favorable (negative) pressure gradient here can be clearly seen.

From table III and figure 5, the effect of Mach number on the stability characteristics is seen to depend on the pressure gradient and whether the ratio $h=T_o/T_e$ of wall temperature to equilibrium adiabatic wall temperature or the ratio T_o/T_b of wall temperature to reference temperature is held fixed. From figure 5 it is seen that for a fixed h an increase of Mach number from 1.5 to 2.0 destabilizes the boundary layer both over a flat plate and over the airfoil. This effect is seen, in fact, to be enhanced by the negative pressure gradient here. For a fixed value of the ratio T_o/T_b, however, an increase of Mach number is now seen, from figure 5, to have a stabilizing influence on the flow without a pressure gradient, especially at the lower wall temperatures. For the flow over the airfoil, however, figure 5 (cf. also table III(a)) now indicates that an increase of Mach number has a stabilizing effect only at wall temperatures close to the critical temperature and that for (fixed) higher wall-temperature ratios of T_o/T_b an increase of Mach number has a clear destabilizing effect similar to the case of fixed h.

CONCLUSIONS

From the analysis of compressible laminar boundary layers with heat transfer and with and without pressure gradient presented herein under the assumption of a linear temperature-viscosity relation, the following conclusions can be drawn:

1. For flow without a pressure gradient, such as flow over a flat plate, the boundary-layer characteristics can be easily determined from the equations developed here for a given constant Prandtl number (of the order of magnitude of unity), a given Mach number, and a given wall-temperature distribution.

2. For flow with a pressure gradient, the boundary-layer characteristics can also be easily determined from the equations developed here, provided the Prandtl number is unity and the wall temperature is uniform. Here, the velocity distribution outside the boundary layer and the free-stream Mach number, as well as the wall temperature, are considered as prescribed. The equations are also valid for zero heat transfer at the wall ($h=1$ where h is the ratio of stagnation enthalpy at the wall to stagnation enthalpy at the outer edge of the boundary layer).

3. A relatively simple method of calculating the separation point in a given subsonic or supersonic adverse pressure gradient over a wall at any specified uniform temperature has been developed here. This method is also applicable for zero heat transfer ($h=1$).

4. A comparison of the results of the methods in conclusions 1, 2, and 3 with known exact solutions for various types of flows indicates that the methods of calculation developed here may be expected, in general, to yield results of sufficient accuracy for practical purposes, including stability calculations.

5. From the equations developed here, it can be shown that cooling of the wall tends to diminish the Nusselt number and especially the skin-friction coefficient in a favorable (negative) pressure gradient and to increase the coefficients in an adverse pressure gradient. Because of the proportionality factor in the viscosity-temperature relation assumed here, it also follows that lowering the ratio of wall to free-stream temperature will, independently of the pressure gradient, ordinarily tend to increase both the Nusselt number and the skin-friction coefficient.

6. The equations developed here further imply that cooling of the wall tends, in general, to diminish the direct effect of a pressure gradient, while heating tends to enhance it. A particularly clear example of this is the delay of separation in an adverse pressure gradient by cooling of the wall.

7. The results of a numerical example for a fixed linearly decreasing velocity outside the boundary layer indicate, in addition to the delaying of separation by cooling of the wall, that for a fixed ratio h of wall temperature to equilibrium adiabatic wall temperature an increase of free-stream Mach number moves the separation point upstream, while for a fixed ratio of wall temperature to free-stream temperature

T_o/T_∞ an increase of Mach number has, in general, a less unfavorable effect and in the case $T_o/T_\infty=2$ actually moves the separation point downstream.

8. While cooling of the wall tends, in general, to stabilize the laminar boundary layer, it is shown theoretically that at moderate supersonic Mach numbers sufficient cooling may completely stabilize the boundary layer. At higher Mach numbers, particularly in the hypersonic range, the validity of the theoretical approach has not been established. A favorable pressure gradient has, in general, a stabilizing effect on the laminar boundary layer, while an adverse pressure gradient has a destabilizing effect. A numerical example for supersonic flow over a thin airfoil illustrates in detail these and other effects of Mach number, wall temperature, and pressure gradient on the stability of the laminar boundary layer.

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TABLE I

SEPARATION POINT CALCULATED AS A FUNCTION OF MACH NUMBER FOR ZERO HEAT TRANSFER ($h=1$) AND $u_1/u_\infty=1-\xi$

Method	ξ_{sep} for M_∞ of—			
	0	1	3	10
Present report	0.122	0.113	0.0768	0.023
Ref. 50	0.120	0.110	0.077	0.024

TABLE II

MINIMUM CRITICAL REYNOLDS NUMBERS $R_{\infty,cr}=(\rho_\infty u_\infty L/\mu_\infty)_{cr}$ FOR ADVERSE PRESSURE GRADIENT COMPARED WITH FLOW IN ZERO PRESSURE GRADIENT WITH ZERO HEAT TRANSFER ($h=1$), $S/T_\infty=0.5$, AND $\xi=0.0496$

Flow	$R_{\infty,cr}$ for M_∞ of—	
	0	1
$u_1/u_\infty=1-\xi$ (adverse pressure gradient; data from ref. 2)	330×10^3	104×10^3
$u_1/u_\infty=1$ (zero pressure gradient; data from ref. 1)	2,390	381

TABLE III

MINIMUM CRITICAL REYNOLDS NUMBERS OF LAMINAR BOUNDARY LAYER OVER THIN SUPERSONIC BICONVEX AIRFOIL AND OVER A FLAT PLATE

(a) Values over airfoil; data taken from reference 7; $\xi'=0.8$

M_b	$h=T_w/T_e$	T_w/T_b	$R_{b,cr}$
1.5	1.1	1.595	2.69×10^4
	1.0	1.450	6.58
	.93	1.348	48.6
	.91	1.320	128.6
	.90	1.305	261.3
	.888	1.283	∞
2.0	1.0	1.8	0.217
	.9	1.62	.389
	.8	1.44	1.777
	.77	1.386	4.448
	.737	1.327	∞

(b) Values over flat plate; $\xi'=\xi=0.8$

M_b	$h=T_w/T_e$	T_w/T_b	$R_{b,cr}$
1.5	1.2	1.740	0.0804×10^4
	1.0	1.450	.282
	.95	1.378	.426
	.90	1.305	.641
	.75	1.088	16.68
	.72	1.044	322.2
	.7166	1.039	∞
2.0	1.00	1.80	0.0506
	.90	1.62	.0903
	.75	1.35	.481
	.70	1.26	1.648
	.683	1.229	∞