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TECHNICAL MEMORANDUM 1396

FROM LINEAR MECHANICS TO NONLINEAR MECHANICS

By Julien Loeb

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FROM LINEAR MECHANICS TO NONLINEAR MECHANICS*[1]

By Julien Loeb[2]

SUMMARY

It is first recalled how, in the technique of telecommunication, a nonlinear system (the modulator) gives a linear transposition of a signal and then it is shown that a similar method permits linearization of electromechanical devices or nonlinear mechanical devices (relays, directional loops of radiogoniometry, etc.). The function of sweep plays the same role as the carrier wave in radioelectricity.

1. INTRODUCTION

For about 10 years, the introduction of the methods perfected by telecommunication engineers into the calculation of mechanical or electromechanical systems has constituted certain progress.

This is known to be due to the fact that the electrical systems, since they are in first approximation of a linear nature, could be thoroughly studied by means of mathematical tools already developed during the last century, such as matrix calculus, symbolical analysis, etc.

To the extent as they may be considered linear, the mechanical or electromechanical systems can be treated by the same methods.

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*"De la mécanique linéaire a la mécanique non linéaire." Annales des Télécommunications, vol. 5, no. 2, Feb. 1950, pp. 65-71.

[1]The author discussed the subject of this article orally on June 28, 1949 and October 18, 1949, during the Conferences of the Study Center of Flight Mechanics, organized by the Special-Engines Section of the Aeronautical Technical Service. Moreover, the author had partially treated this subject in his Preliminary Note No. 144 of the National Laboratory of Radioelectricity.

[2]Chief Engineer at the C.N.E.T., Head of the Department Remote Control and Counter Measures.

Unfortunately, their characteristics are rarely linear. In particular, the mechanical systems can generally be considered linear only when they are subject to small oscillations. This permits treatment of stability problems, provided, however, that the magnitudes are not too large.

The methods taken over from the technique of electrical networks do not permit, for instance, treating by means of relays, electromechanical systems functioning in a discontinuous fashion.

This is very regrettable since relays constitute powerful and small amplifiers which have only the one fault of not being linear.

The object of the present study is to show that in the arsenal of telecommunications, one may yet find an entirely general method of treating nonlinear problems or, more accurately, of transforming them into linear problems.

The definition of this method is contained in five words: "Utilization of a carrier frequency."

2. MECHANICAL EQUIVALENTS OF THE PHYSICAL QUANTITIES FROM THE TECHNIQUE OF TELECOMMUNICATIONS

2.1. Linear Systems

Electromechanical systems are essentially conceived for transmitting messages (telegraph, telephone, facsimile, television, etc.).

A message is characterized by a band width, that is, by the spectrum of the frequencies it contains which must be passed.

Moreover, the "quantity of information" a channel can transmit is limited by a "magnification" of the amplitude of the signal. Two signals of the same frequency and different amplitudes are actually distinguishable only if the difference of their amplitudes exceeds a predetermined magnitude. This magnitude stems principally from the noise always present in a radio or wire connection. In mechanics, a system of remote control also must convey a message. Here the band width is much smaller than that of the telephone or even of the telegraph, because of the large inertia of the mechanical components in motion. According to the utilization, the band width varies between 1 cycle/second and 10 cycles/second, or even, possibly, up to about 100 cycles/second.

In the case of a regulator, the concept of band width appears less clearly but still applies. Evidently the physical quantity which must

be maintained by the regulator and which is imposed from the outside cannot vary. Nevertheless, the regulator must necessarily possess a response sufficient to re-establish the desired condition when a perturbation has momentarily disturbed it.

An automatic pilot for an airplane is at the same time something of a remote controller and something of a regulator. It must follow a course with sufficient rapidity, even if this course is modified by the pilot, and in addition, it must return rapidly to its position after gusts which play here the role of parasitic impulses.

2.2. Carrier Frequencies

Most frequently, the signal is not sent just as it is, but is used for modulating an auxiliary current the frequency F of which is called the carrier frequency. A modulator is essentially a nonlinear device in which the carrier frequency and the signal are added. Let

$$P = P_0 \sin 2\pi Ft \qquad \text{be the carrier frequency}$$

$$S = S_0 \sin 2\pi ft \qquad \text{the signal}$$

Assume $\varphi(\sigma)$ to be the function representing the action of the modulator. If one adds in the latter P and S, there results $\varphi(P + S)$. Expanding in Taylor series, one obtains

$$\varphi(P + S) = a_0 + a_1(P + S) + a_2(P + S)^2 + a_3(P + S)^3 + \ldots$$

If one lets the modulator be followed by a filter which transmits only a band centered on F, one obtains the term

$$2a_2PS = 2a_2(S_0 \sin 2\pi ft)P_0 \sin 2\pi Ft$$

Thus one has a carrier frequency $P_0 \sin 2\pi Ft$ the amplitude $S_0 \sin 2\pi ft$ of which represents the signal to be transmitted.

This makes sense only if f is much smaller than F. In practice, f is at most of the order of one-third of F.

Thus it is seen that if one encounters in a chain of transmission a nonlinear element (modulator), one finds a linear function of the signal by operating as follows:

Add a carrier frequency.

Filter a band centered on this carrier frequency.

3. A FEW LINEAR AND NONLINEAR MECHANISMS

We shall examine a certain number of mechanisms regarded as transforming an input quantity into an output quantity. In the most general case, this latter will be a continuous or discontinuous function or a functional.

3.1. Continuous Function

3.11. Odd function.- Here is, for small amplitudes, a linear mechanism: the output quantity is, in magnitude and sign, simply proportional to an input quantity. For instance, the servomechanisms in which the geometrical position of an indicator is governed by the resistance of a potentiometer belong to this category. (See fig. 1.)

The control in this case is effected by means of the slider C_1 of the potentiometer P_1. The fed-back quantity is the position of the slider C_2 of the potentiometer P_2. The input quantity ϵ is the geometrical displacement between the two sliders.

The signal furnished by the error-sensing device is the difference in potential E between C_1 and C_2. E is proportional to ϵ.

Another example is given by figure 2 which represents the classical error-sensing device of an angular Selsyn control.

The control is effected by modification of the angle θ which the single-phase rotor (fed by 50 c/s) forms with a fixed reference. The fed-back quantity is the angle θ'. The error (input quantity) is $\epsilon = \theta - \theta'$.

The output quantity E is an alternating current having a frequency of 50 c/s, the amplitude of which is in magnitude and sign proportional to $\sin \epsilon$.

One considers this system as linear for small values of ϵ.

3.12. The output quantity is an even function of the input quantity.- Let us examine, for instance, the case of radiogoniometry. It is well known that there exist devices which permit automatically guiding an indicator connected with the directional loop toward the source of electromagnetic radiation. They are the radio compasses.

Besides, even if the radiogoniometry is not automatic, the ensemble formed by the directional loop and the operator constitutes a servomechanism (and not one of the better ones, either).

If one plots as abscissa the angular error and as ordinate the detected value E of the current which leaves the directional loop, one obtains the curve of figure 3. Here one has the indication of the absolute value of the angular deviation but one has no longer the sign.

A condition of the same kind was encountered when one attempted to design a local oscillator producing a voltage the frequency of which is to be made equal to a given frequency. Here again the frequency of the beat current indicates the absolute value of the difference, but the sign of the latter does not affect the output current.

3.2. Discontinuous Functions

In mechanisms with "on or off" operation, the output signal increases abruptly from 0 to a given constant value when the input quantity is positive, and from 0 to an opposite value, likewise constant, when that quantity is negative. The diagram of figure 4 which describes the same device as that of figure 1, with one addition, a polarized relay, gives an example for this. In this case, the function is discontinuous and odd (in general, such systems present a sensitivity threshold which the error must exceed for the signal to exist).

Figure 5 gives the curve of E as a function of ϵ.

3.3. Functionals

It happens quite frequently that the output signal is not a function of the error (case where the output voltage depends only on the actual value of the input quantity) but a functional (case where the output voltage depends not only on the actual value of the variable but also on its prior values).

3.31. Continuous functional.- Figure 6 shows the diagram of a system derived from that in figure 1 in which the input signal is amplified by means of an electromechanical amplifier, an "amplidyne" for instance.

Figure 7 gives the curve of E as a function of ϵ.

3.32. Discontinuous functional.- The example for this is furnished by a phase-sensing device used at the Centre National d'Études des Télécommunications; its principle is as follows:

An output signal is to be obtained which is a function of the phase displacement ϵ of two currents of the impulses D and G (fig. 8), taken positively if G lags behind D, and negatively in the opposite case (only small values of ϵ are dealt with here).

A "seesaw" or "flip-flop" is actuated by these two currents: D acts on the tube at right, G on the tube at left (fig. 9).

The signal E will be the mean value of the voltage between the plates. If ϵ is positive, the flip-flop is in its position G during almost the entire period. If ϵ is negative, it remains in D during the same time.

There occurs therefore an abrupt jump when ϵ passes through zero. In fact, ϵ must slightly exceed the value of zero in order to make the flip-flop operate. One then obtains the diagram represented in figure 10.

4. LINEARIZATION OF NONLINEAR SYSTEMS

One now has to find out whether the systems described above can be transformed into linear systems. Section 2.2 furnishes us the method.

It has been seen that the modulation of a carrier wave by the signal requires the use of a nonlinear network. One then obtains a modulated carrier the amplitude of which is a linear function of the input signal. Here we shall accept our nonlinear system such as it is; we shall superimpose on its circuits the input quantity to be transmitted and a periodic function, of time frequency F (sinusoidal, for instance). A periodic function of the frequency F modulated by the input signal results. Furthermore, after detection it will be possible to obtain a linear function of the input signal. We shall see how this general idea is applied to the concrete cases.

4.1. Method Followed by an Operator

As happens very frequently when one attempts to invent a new automatic device, one must begin by analyzing what motions an operator performs who actuates a known manual device. We shall consider for this analysis the operation of a directional loop in radiogoniometry. The operator ignores the actual direction of the wave which reaches him. He orients his directional loop at random, and generally perceives a signal. This fact informs him that his directionl loop does not occupy the desired position but it cannot tell him in what direction to turn the directional loop in order to nullify the signal. Reasoning is substituted for lacking indications. The operator displaces the directional loop in a certain manner; if this displacement produces a reinforcement of the signal, the operator has made the displacement in the wrong direction; therefore, he will displace the directional loop in the opposite direction which will bring it closer to the correct position.

It will frequently happen that the motion imparted to the directional loop will overshoot the mark, and the error will have changed in sign. If the operator does not have good reflexes, it might even happen that every motion intended to bring the directional loop into the desired position overshoots its mark, and the combination of directional loop plus operator will get into a self-sustained oscillation.

There will be the more chances for the appearance of this oscillation, for the same operator, the greater the required accuracy.

In the language of servomechanisms, one may say that the more the error is amplified, the more the system tends to oscillate by itself.

4.2. General Procedure of Linearization

The above example shows that it might be of advantage to set up mechanisms which reproduce the alternating motion effected by the radiogoniometry operator in search of his mark. That is precisely where the carrier frequency mentioned before comes in. One adds to the input quantity an arbitrary sinusoidal function of time, of the form $\epsilon_0 \sin 2\pi Ft$. Other periodic functions may be used, notably "saw-tooth" curves.

We shall call this function the "sweep function." As in the domain of telecommunications, the frequency F must be placed far beyond the band width of the system.

We shall demonstrate the two following results, valid for small values of the input quantity ϵ:

(1) If the system is odd (representative curve symmetrical with respect to the origin), the mean value of the signal given at its output is proportional in magnitude and in sign to ϵ.

(2) If the system is even, the component of the output current which is at the sweep frequency has an amplitude which is proportional in magnitude and in sign to ϵ.

5. ODD ERROR-SENSING DEVICES

The above theorem is evident when the output voltage of the error-sensing device is an odd and continuous function of the error.

5.1. Odd and Discontinuous Function

When the output voltage of the error-sensing device is an odd and discontinuous function of the error, this result continues to exist as is demonstrated in the report by Mac Coll.[3]

Figure 11 brings, in substance, this demonstration which applies to an "on or off" system.

This system gives a signal:

$$+E \quad \text{as long as} \quad \epsilon > 0$$

$$-E \quad \text{as long as} \quad \epsilon < 0$$

The difference between the durations of the applications of the signals $+E$ and $-E$ is equal to 4τ, with

$$\epsilon_0 \sin \frac{2\pi\tau}{T} = \epsilon$$

Since

$$\epsilon \ll \epsilon_0$$

$$\tau = \frac{T}{2\pi} \frac{\epsilon}{\epsilon_0}$$

and the mean value E of the signal is

$$\overline{E} = E \times \frac{\Delta\tau}{T} = \frac{2\epsilon}{\pi\epsilon_0} E$$

This proportionality continues to exist if the system gives a curve analogous to that of figure 5. We shall not give here the demonstration of this result because this demonstration is entirely similar to the one that will be given further on, with regard to the discontinuous functional.

5.2. Odd and Continuous Functional

We may extend this result to include the case where the output current is an odd functional.

In fact, let $\underline{F}(U)$ be this functional. In the cases we are studying (curve of hysteresis), it will lead to a function of (U) which may have two values.

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[3]Mac Coll (L. A.), Eléments de la théorie des servomécanismes (Fundamental theory of servomechanisms). New York, D. Van Nostrand Co., Inc., 1945.

For instance, in the case of figure 7 one has $F(U) = F_1(U)$ when (U) varies from $-U_0$ to $+U_0$, and one has $F(U) = F_2(U)$ when U comes back from $+U_0$ to $-U_0$.

When one has $U = \epsilon + \epsilon_0 \sin \omega t$, the functional $\underline{F}(U)$ becomes a function of t and of ϵ.

In the most general case, the instants when the function changes its value will be continuous functions of ϵ. Consequently, the integral $\int_0^{\frac{2\pi}{\omega}} \underline{F}(\epsilon \pm \epsilon_0 \sin \omega t)dt$ likewise will be a continuous function of ϵ.

If the origin of the coordinates is chosen in such a manner that, for $\epsilon = 0$, this integral becomes zero (symmetry of the hysteresis curve with respect to the origin) and if the first derivative of this integral with respect to ϵ does not become zero, one thus has a signal which is proportional in magnitude and in sign to ϵ.

5.3. Odd and Discontinuous Functional

We shall obtain the same result with the system of figure 10. (See fig. 12.)

The curve I represents the variation of $\epsilon_0 \sin \omega t$ as a function of time. This function reaches the magnitude $+\delta$ at the instant t; then, when it attains in descending the magnitude $+\delta$ by hypotheses, nothing passes because of the hysteresis. At this moment the signal has the value $+C_0$. Only when the function $\epsilon_0 \sin \omega t$ will have attained the value $-\delta$ (instant t_1), the signal will be reversed so as to attain $-C_1$. Likewise, it will become positive again only at the instant t_2.

Since the time intervals t_0, t_1, t_2 are equal, the mean value of the signal is zero.

This is no longer the case when one has, instead of the signal $\epsilon_0 \sin \omega t$

$$\epsilon + \epsilon_0 \sin \omega t \quad \text{(curve II)}$$

At this moment, the instant t_0' where the function attains $+\delta$ comes before t_0, the instant t_1' where the function attains $-\delta$ comes

after t_1, and the instant t_2' where the function exceeds $+\delta$ (exactly one period after t_0') is likewise ahead of t_2.

As a result, the signal is equal to $+C_0$ during a longer time than it is equal to $-C_0$.

The mean value of the signal is therefore no longer zero, and it can easily be seen that, for $\epsilon \ll \delta$, with $\epsilon_0 > \delta$, it is given by the following formula:

$$C = \frac{2\epsilon}{\pi\epsilon_0} C_0 \frac{1}{\sqrt{1 - \frac{\delta^2}{\epsilon_0^2}}}$$

Thus one sees that one does not only obtain an output signal which is a linear function of ϵ but that one has found a method which permits modifying the coefficient of proportionality at will.

For instance, the following experimental curves (fig. 13) were drawn up for various values of ϵ in the phase-sensing device described in section 3.32.

The Sperry compass, with its oscillating "hunting" device, likewise illustrates this method of linearization in the case of an odd functional. In this case, the sweep is "saw-tooth" type.

6. EVEN SYSTEMS

So far we did not encounter error-sensing devices which were at the same time even and discontinuous; their treatment would be the same. The curves given by those one may encounter have, in general, the appearance of figure 14.

Since here the first term of the expansion of F in terms of the increasing powers of its argument is of the form $K(\epsilon + \epsilon_0 \sin \omega t)^2$, one obtains, as frequency term ω, an expression of the form $K\epsilon\epsilon_0 \sin \omega t$.

Therefore, the signal has, at the frequency ω, an amplitude proportional to the error ϵ in magnitude and sign, that is to say, its phase changes by 180° when the error changes in sign. It is analogous to the signal one receives in position-controlling servomechanisms operating on alternating current (section 3.11, fig. 2). This result continues to be valid in the case of an angular curve represented in figure 15.

This form of signal is advantageous when the desired power exceeds the power the system can furnish directly (error-sensing device in a servomechanism, for instance) because an alternating voltage can be amplified much more easily than a continuous one.

This procedure appears, therefore, to be better than the "false-zero" methods usually employed for avoiding the difficulty presented by even systems, with the possibility of instability of the zero point which characterizes these methods.

An example of these systems (in the present case, an error-sensing device in a servomechanism) has been given above; it is the directional loop in radiogoniometry, with all its possible variants one of the best known of which is the automatic tracking device of radar; here again the angle-sensing device, that is to say, the antenna placed in the focus of a mirror, has a symmetrical diagram; one adds to the angular deviation to be sensed an angular deviation which is a sinusoidal function of time, by making the antenna rotate eccentrically.

A method of the same character is described by Pierre Debraine and Cestmir Simane[4] for the regulation of the magnetic field of a cyclotron.

There one has to control the magnetic field with respect to the ionic output current in such a manner as to maintain it at a value precisely equal to that required by the other characteristics of the apparatus (notably the frequency of the voltage applied to the "Dees"). The curve giving the ionic current as a function of the error ϵ committed in the magnetic field is even, at least in first approximation. The authors applied the principle of linearization by sweep, adding to the magnetic field a component alternating at 3 cycles/second. The ionic current involves an error signal at 3 cycles/second the amplitude of which is proportional to the error, and the phase of which is 0° or 180° according to whether the error is positive or negative. This signal at 3 cycles/second is filtered (in particular, it is separated from the parasitic signal of the frequency 6 cycles/second) and after amplification and detection it is applied to the excitation of the generator which feeds the electromagnet.

7. ARBITRARY SYSTEMS

An arbitrary function or functional may always be considered as the sum of two functions (or functionals) of which one is even and the other odd:

$$2\underline{F}(U) = (\underline{F}(U)) + \underline{F}(-U) + (\underline{F}(U)) - \underline{F}(-U)$$

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[4]Dispositif de synchronisation automatique du cyclotron. C.R. Acad. Sci. Fr., (23 février 1948) 226, no. 8, pp. 648-650.

The procedure of linearization will generally give, if ϵ is a small constant:

(a) A continuous signal proportional to ϵ.

(b) An alternating signal of the form $K\epsilon \sin \omega t$.

Both may be used.

CONCLUSION

We have thus shown that the technique of telecommunications can furnish solutions in the field of mechanics, even in the case of non-linear systems.

It is evident that mechanical and electrical engineers, struggling with their problems, have thought of solutions of the type which were indicated above without waiting for the telecommunication engineers. However, the above outline constitutes an attempt at generalization and classification of these methods.

The analogy thus established between the problems of mechanics and the problems of telecommunications permits, besides, recognizing a limitation imposed on this type of solution by the existence of the allowable frequency bands.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

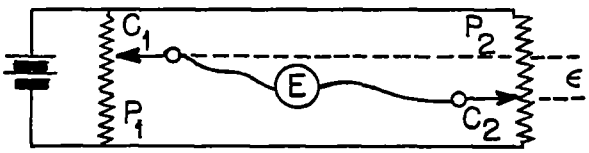

Figure 1.-

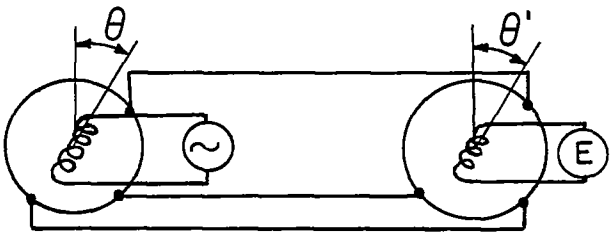

Figure 2.-

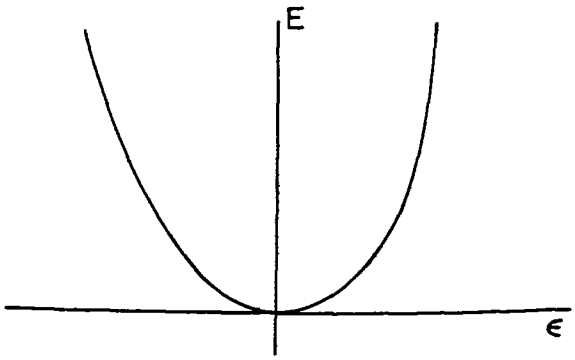

Figure 3.-

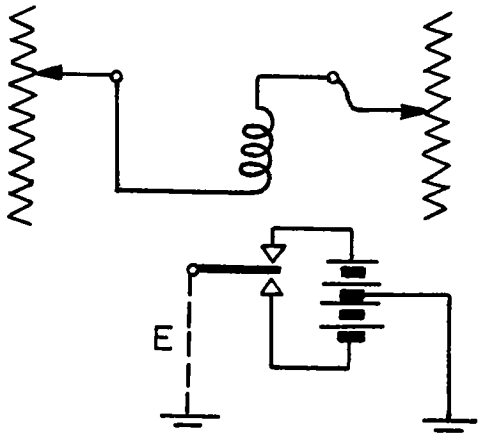

Figure 4.-

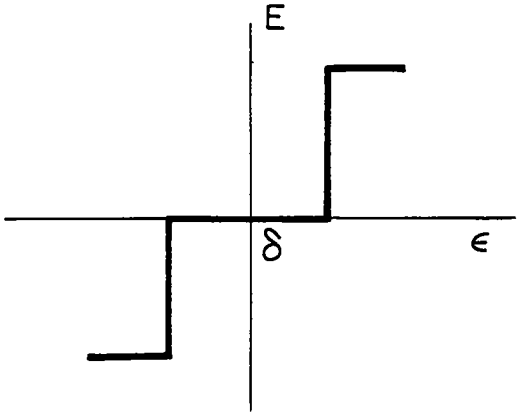

Figure 5.-

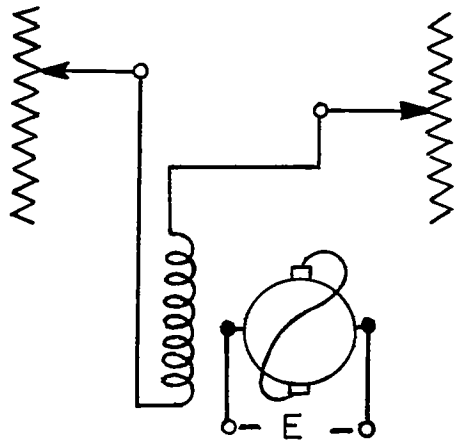

Figure 6.-

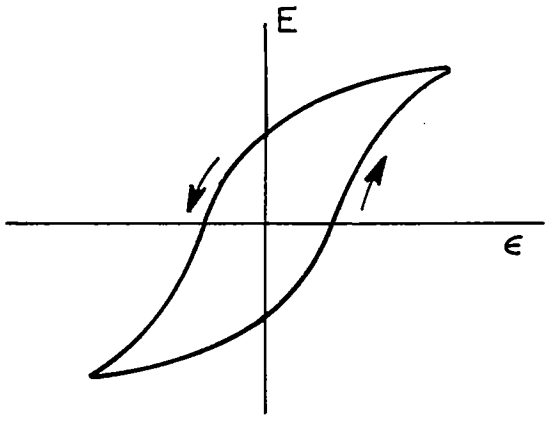

Figure 7.-

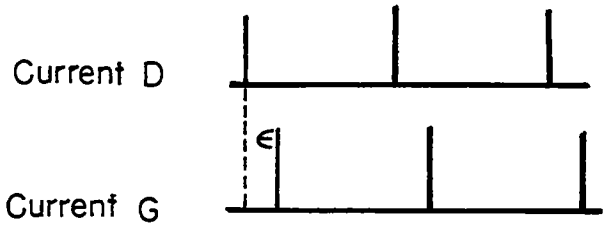

Figure 8.-

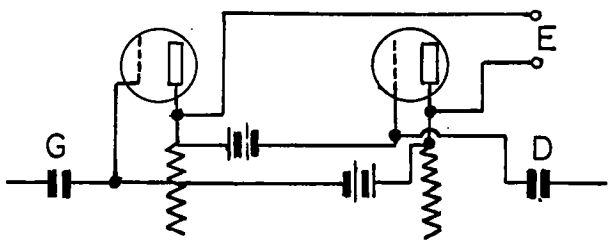

Figure 9.-

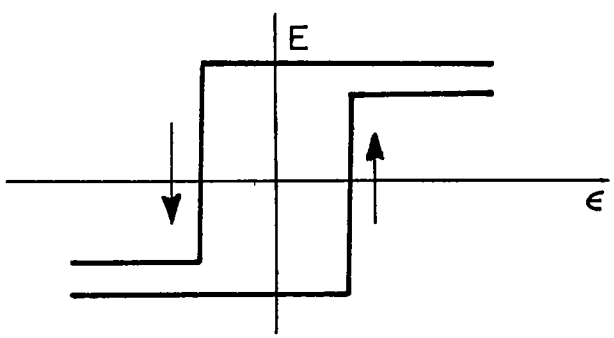

Figure 10.-

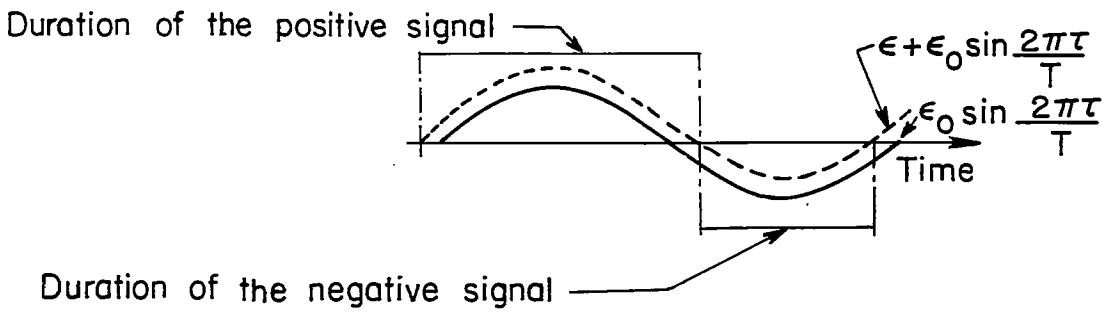

Figure 11.-

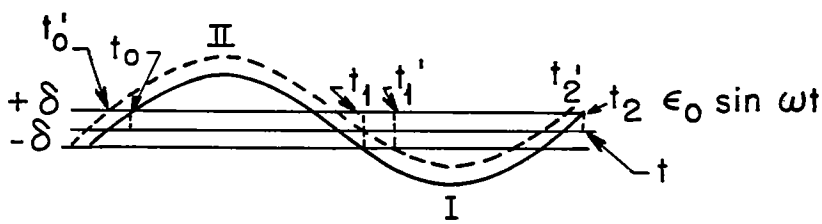

Figure 12.-

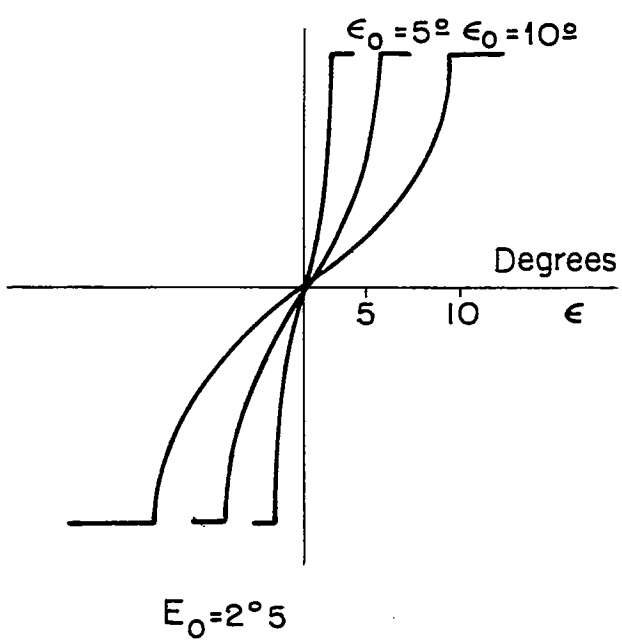

Figure 13.-

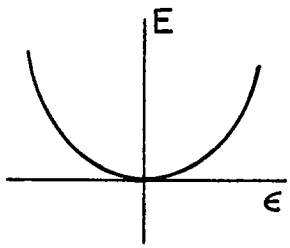

Figure 14.-

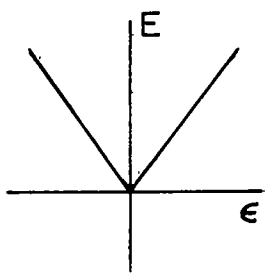

Figure 15.-