REPORT 1257

ON THE KERNEL FUNCTION OF THE INTEGRAL EQUATION RELATING LIFT AND DOWNWASH DISTRIBUTIONS OF OSCILLATING WINGS IN SUPERSONIC FLOW [1]

By Charles E. Watkins and Julian H. Berman

SUMMARY

This report treats the kernel function of the integral equation that relates a known or prescribed downwash distribution to an unknown lift distribution for harmonically oscillating wings in supersonic flow. The treatment is essentially an extension to supersonic flow of the treatment given in NACA Report 1234 for subsonic flow. For the supersonic case the kernel function is derived by use of a suitable form of acoustic doublet potential which employs a cutoff or Heaviside unit function. The kernel functions are reduced to forms that can be accurately evaluated by considering the functions in two parts: a part in which the singularities are isolated and analytically expressed, and a nonsingular part which can be tabulated.

The kernel is treated for the two-dimensional case, and it is shown that the two-dimensional kernel leads to known lift distributions for both steady and oscillating two-dimensional wings. The kernel function for three-dimensional supersonic flow is reduced to the sonic case and is shown to agree with results obtained for the sonic case in NACA Report 1234, and the downwash functions associated with "horseshoe" vortices in supersonic flow are discussed and expressions are derived.

INTRODUCTION

In reference 1 the kernel function of an integral equation relating a known or prescribed downwash distribution to an unknown lift distribution for a harmonically oscillating finite wing of arbitrary plan form was treated for compressible subsonic flow. The purpose of the present report is to extend this treatment of the kernel function to supersonic flow.

The kernel functions under consideration arise when linearized-boundary-value problems for obtaining aerodynamic forces on oscillating wings are reduced to integral equations involving the distribution of pressure or wing loading as the unknown. In such integral equations the kernel functions play the important role of aerodynamic influence functions in that they give the normal induced velocity or downwash at any one point in the plane of the wing due to a unit pressure loading at any other point in the plane of the wing.

As the kernel functions arise in the analysis, they are mathematically defined by rather intricate improper integrals and possess singularities as high as second order. It is therefore desirable to isolate the singularities and determine their explicit nature in order to make the integral equation more amenable to solution, in particular amenable to solution by approximate or numerical procedures.

Approximate lifting-surface theories for finite wings, such as the methods developed by Falkner and Multhopp (refs. 2 and 3) and others, have afforded considerable success in the calculation of aerodynamic coefficients for steady subsonic aerodynamics. Similar approximate methods have been successfully employed to obtain coefficients for two-dimensional oscillating wings in subsonic (compressible) flow (for example, refs. 4 and 5) and are now being extended to the finite oscillating wing in subsonic flow by Harry L. Runyan and Donald S. Woolston of the Langley Aeronautical Laboratory and by W. P. Jones (ref. 6). It is reasonable to expect that these methods can be further extended to apply to finite wings in supersonic flow.

In supersonic flow, solutions of the boundary-value problem for some particular plan forms and downwash conditions can be obtained in the form of infinite series in terms of a parameter involving the frequency of oscillation (see, for example, refs. 7 to 10) or in the form of rather complicated definite integrals (refs. 11 and 12). The infinite-series method furnishes a relatively simple means of obtaining the loading on oscillating wings for low values of the frequency parameter, but for large values of this parameter the series expansions converge so slowly that recourse must be had to other procedures for obtaining the wing loading. One feasible method is to study and develop approximate procedures for solving the integral equations that involve the unknown loading and its associated kernel function. The first step toward such a development is to isolate and determine the explicit nature of the singularities of the kernel function; this step is accomplished in the present report.

The report contains the derivation of the kernel function in the form of an improper integral and a reduction of this integral to proper form. The singularities of the kernel function are isolated and expressed analytically, and the nonsingular parts are reduced to a form readily amenable to numerical evaluation, as was done in reference 1 for subsonic flow. Some expanded forms of the kernel function are derived, and one of these is used to obtain a reduction to two-dimensional flow. In appendix A, the limiting case for sonic flow is derived and shown to agree with the results in reference 1. Appendix B is devoted to certain integrals of

[1] Supersedes NACA Technical Note 3438 by Charles E. Watkins and Julian H. Berman, 1955.

the kernel function. These integrals relate to "horseshoe" vortices in supersonic flow, as treated, for example, in the steady case by Schlichting in reference 13, and may be of interest in certain modes of application.

SYMBOLS

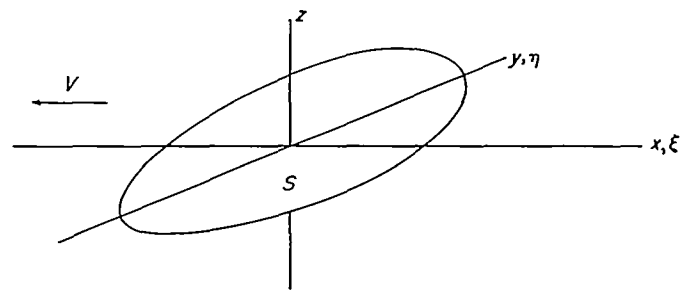
c	velocity of sound
I_0, $I_1(x)$	modified Bessel functions of first kind
$J_n(x)$	Bessel function of first kind
K_0, K_1	modified Bessel functions of second kind
$K(x_0, y_0)$	kernel function for three-dimensional flow
k	reduced-frequency parameter, $l\omega/V$
$K(x_0)$	kernel function for two-dimensional flow
$L(\xi, \eta)$, $L(\xi)$	lift distributions
L_0, L_1	modified Struve function of first order
l	unit length
M	Mach number, V/c
p	perturbation pressure
$r=\sqrt{y_0^2+z^2}$	
S	region of xy-plane occupied by wing
t	time
$U(x)$	unit function
V	forward velocity of wing
$w(x,y,t)$	downwash velocity, $e^{i\omega t}\overline{w}(x,y)$
$\overline{w}(x,y)$	complex amplitude function of prescribed vertical velocity
x,y,z	Cartesian coordinates attached to wing moving in negative x-direction
$x_0=x-\xi$	
$y_0=y-\eta$	
$\beta=\sqrt{M^2-1}$	
$\delta(x)$	Dirac delta function
ξ,η	Cartesian coordinates used to represent space location of doublets
ρ	fluid density
$\phi(x,y,z,t)$	velocity potential, $e^{i\omega t}\overline{\phi}(x,y,z)$
$\overline{\phi}(x,y,z)$	complex amplitude function of velocity potential
$\psi(x,y,z,t)$	acceleration potential, $e^{i\omega t}\overline{\psi}(x,y,z)$
$\overline{\psi}(x,y,z)$	complex amplitude function of acceleration potential
ω	circular frequency of oscillation
$\overline{\omega}=\omega/V\beta^2$	

ANALYSIS

INTEGRAL EQUATION RELATING DOWNWASH AND LIFT DISTRIBUTION

The linearized-boundary-value problem for the determination of the aerodynamic forces on a wing can be immediately reduced to a problem of solving an integral equation that relates downwash and lift distribution. The purpose of this section is to introduce and briefly discuss this equation.

Since the integral equation has the same formal appearance for subsonic and supersonic flow and is derived in various publications (for example, refs. 1 and 14), the equation will not be rederived here but will be formally stated so as to serve as a starting point in the analysis. In keeping with linear theory, the wing is considered as a plane, impenetrable surface S which lies nearly in the xy-plane as indicated in sketch (a).



Sketch (a)

The x,y,z coordinate system and the surface S are assumed to move in the negative x-direction at a uniform velocity V.

In terms of these coordinates, the integral equation may be formally written as

$$\overline{w}(x,y)=\frac{1}{4\pi\rho}\iint_S L(\xi,\eta)K(x-\xi,y-\eta)d\xi\, d\eta \tag{1}$$

where $\overline{w}(x,y)$ is the complex amplitude function of prescribed vertical velocity or downwash at points in S and is defined as follows:

$$w(x,y,t)=e^{i\omega t}\overline{w}(x,y)$$

where ω is the frequency of pulsation or oscillation. The kernel function $K(x_0,y_0)$ physically represents the contribution to the downwash at a point (x,y) in S due to the presence of a pulsating pressure doublet of unit strength located at some other point (ξ,η) in S. It is a function not only of x, y, ξ, and η but also of Mach number and frequency. The function $L(\xi,\eta)$ in equation (1) is the unknown lift distribution or local doublet strength. (Although it is usually convenient to factor out the density term $1/\rho$ as indicated in eq. (1), this was not done in ref. 1.)

Equation (1) pertains formally to either subsonic or supersonic flow; however, separate treatments of the two cases are required because of wide differences associated with flow characteristics. So far as the integral equation is concerned, the differences in the two cases lie mainly in the kernel functions. These differences are associated with the differences in character of doublets for the two cases. Although the main purpose of this analysis is to derive and treat the kernel function for the supersonic case, a necessary first step is to formulate a doublet suitable for such a treatment. In the following section, a desired form, which was arrived at by a convenient use of a cutoff, or Heaviside unit function, is presented in equation (5).

PULSATING DOUBLET MOVING AT SUPERSONIC SPEED

The governing differential equation for linearized unsteady flow at either subsonic or supersonic speeds, which the

doublet potentials must satisfy, is the well-known wave equation referred to a moving coordinate system:

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}-\frac{1}{c^2}\left(V\frac{\partial}{\partial x}+\frac{\partial}{\partial t}\right)^2\psi=0$$

Under the assumption that disturbances vary harmonically with respect to time, this equation becomes

$$\frac{\partial^2\bar{\psi}}{\partial x^2}+\frac{\partial^2\bar{\psi}}{\partial y^2}+\frac{\partial^2\bar{\psi}}{\partial z^2}-\frac{1}{c^2}\left(V\frac{\partial}{\partial x}+i\omega\right)^2\bar{\psi}=0 \tag{2}$$

where $\bar{\psi}$ is a complex amplitude function defined by

$$\psi(x,y,z,t)=e^{i\omega t}\bar{\psi}(x,y,z) \tag{3}$$

It may appear that, since the same differential equation (eq. (2)) is involved, a logical way of obtaining the potential for a pulsating doublet moving at supersonic speed is by simple analogy or continuation from the potential for the doublet moving at subsonic speed. This procedure is applicable only in a broad sense because, as discussed in reference 15 with regard to sources in supersonic flow, the potential of a doublet moving at supersonic speed consists of the sum of two effects corresponding to a retarded-type potential and an advanced-type potential which relate to the two wave fronts encountered by a point at any time; whereas for subsonic speed only the retarded type of potential is admissible. (The advanced-type potential for subsonic or sonic speed does not satisfy the Sommerfeld radiation condition, which requires that disturbances be propagated away from their point of origin.) In the second place, the potential that may be obtained by analogy with the potential for subsonic speed must, as subsequently discussed, be rather severely restricted before it mathematically describes the physical realities of a disturbance moving at supersonic speed. In the following development, a desired form of the doublet potential is arrived at by consideration of these restrictions applied to both a retarded and an advanced type of potential that may be obtained by analogy with results for subsonic flow.

By analogy with results for subsonic speed (for example, eq. (A9) of ref. 1) or, more directly, from the discussion of source potentials in supersonic flow (ref. 15), the sum ψ_D of the retarded and advanced types of potentials required to form the doublet potential for supersonic speeds may be written with the doublet situated at the origin as

$$\psi_D=\frac{\partial}{\partial z}\left[\frac{e^{i\omega\left(t-\frac{Mx}{c\beta^2}-\frac{\sqrt{x^2-\beta^2y^2-\beta^2z^2}}{c\beta^2}\right)}}{\sqrt{x^2-\beta^2y^2-\beta^2z^2}}+\frac{e^{i\omega\left(t-\frac{Mx}{c\beta^2}+\frac{\sqrt{x^2-\beta^2y^2-\beta^2z^2}}{c\beta^2}\right)}}{\sqrt{x^2-\beta^2y^2-\beta^2z^2}}\right]$$

$$=2\frac{\partial}{\partial z}\frac{e^{i(\omega t-M^2\bar{\omega}x)}\cos\left(M\bar{\omega}\sqrt{x^2-\beta^2y^2-\beta^2z^2}\right)}{\sqrt{x^2-\beta^2y^2-\beta^2z^2}} \tag{4}$$

where $M=V/c$, $\beta=\sqrt{M^2-1}$, and $\bar{\omega}=\omega/V\beta^2$. The restrictions that must be placed on this expression are: (a) only real values of the radical term $\sqrt{x^2-\beta^2y^2-\beta^2z^2}$ are to be considered and (b) the values of the expression and its derivatives are to be considered zero when x is negative. These restrictions follow from the physical consideration that small disturbances propagate at sonic speed and in a supersonic stream do not progress forward of their point of origin.

A convenient way of writing the expression for ψ_D with these restrictions accounted for, as previously mentioned, is to employ a cutoff or unit function as a factor. Thus, if ψ_0 represents the restricted value of ψ_D, the amplitude function of ψ_0 may be written as

$$\bar{\psi}_0=2\frac{\partial}{\partial z}e^{-iM^2\bar{\omega}x}U(x-\beta\sqrt{y^2+z^2})\frac{\cos\left(M\bar{\omega}\sqrt{x^2-\beta^2y^2-\beta^2z^2}\right)}{\sqrt{x^2-\beta^2y^2-\beta^2z^2}} \tag{5}$$

where

$$\left.\begin{aligned}U(x-\beta\sqrt{y^2+z^2})&=1 \qquad (x>\beta\sqrt{y^2+z^2})\\ U(x-\beta\sqrt{y^2+z^2})&=0 \qquad (x\leqq\beta\sqrt{y^2+z^2})\end{aligned}\right\} \tag{6}$$

and only positive values of the radical $\sqrt{y^2+z^2}$ are considered. A method whereby this form of potential can be determined in a more direct manner is discussed in detail in reference 16. The discussion in this reference is in connection with the Green functions associated with the dispersion of sound waves in an n-dimensional medium in which a pulsating source exists. When appropriate changes are made in notation, the expression for $\bar{\psi}_0$ given in equation (5) agrees essentially with results for the dispersion of waves in a three-dimensional space given in equation (55), chapter XVI of reference 16.

With regard to the unit function $U(x)$, in many applications where this function is employed it need not be defined as having any particular value when its argument is zero. In other applications, especially where the unit function is involved in a Fourier analysis, it must be defined as having a value of ½ when its argument is zero. In the present case, it is conveniently defined, as may be noted in equation (6), as having zero value when its argument is zero.

Derivatives of the unit function give rise to an impulse function called the Dirac delta function. For example,

$$\frac{\partial}{\partial x}U(x)=\delta(x)=0 \qquad (x\neq 0)$$

$$\frac{\partial}{\partial x}U(x)=\delta(x)=\infty \qquad (x=0)$$

A useful integral property of this delta function is

$$\int_0^b f(x)\,\delta(x)\,dx=f(0)$$

The next step in the analysis is to make use of the doublet potential (eq. (5)) to derive the kernel function for supersonic speed.

DERIVATION AND REDUCTION OF KERNEL FUNCTION

In this section the kernel function is derived and presented.

The function is given in terms of an improper integral by equation (13) and in a reduced form with no improper integrals by equation (15). As it is frequently desirable to present results in terms of nondimensional length variables, the results given in equation (15) are presented in this manner in equation (16).

In order to derive the kernel function, the function $\bar{\psi}$ of equation (3) is considered as the complex amplitude of the acceleration potential. As such, ψ is directly proportional to a perturbation pressure field $p=e^{i\omega t}\bar{p}$, through the simple relation

$$\bar{p}=-\rho\bar{\psi} \tag{7}$$

and to a velocity potential

$$\phi=e^{i\omega t}\bar{\phi}$$

through the equation

$$V\frac{\partial\bar{\phi}}{\partial x}+i\omega\bar{\phi}=\bar{\psi} \tag{8}$$

By differentiation of equation (8) with respect to z and integration of the result with respect to x, the vertical velocity associated with the acceleration potential ψ is obtained. Thus, when ψ is considered as the potential of a pressure doublet, equation (8) affords a straightforward means for obtaining an equation for $K(x_0,y_0)$, namely:

$$K(x_0,y_0)=\frac{\partial}{\partial z}\bar{\phi}(x_0,y_0,z)_{z=0} \tag{9}$$

Details of the procedure are as follows:

The result of the differentiation of equation (8) with respect to z may be written as

$$V\frac{\partial}{\partial x}\frac{\partial\bar{\phi}}{\partial z}+i\omega\frac{\partial\bar{\phi}}{\partial z}=\frac{\partial\bar{\psi}}{\partial z} \tag{10}$$

When this equation is considered as an ordinary differential equation with dependent variable $\frac{\partial\bar{\phi}}{\partial z}$ and independent variable x, a complete solution is

$$\frac{\partial\bar{\phi}}{\partial z}=\frac{1}{V}e^{-\frac{i\omega x}{V}}\int_{-\infty}^{x}\frac{\partial}{\partial z}\bar{\psi}(\lambda,y,z)\,e^{\frac{i\omega\lambda}{V}}\,d\lambda \tag{11}$$

where the lower limit of integration is employed in place of a constant of integration and is chosen so as to satisfy the condition that $\bar{\phi}$ vanish far ahead of the origin $\lambda=0$. Thus, from equation (9) there is obtained

$$K(x_0,y_0)=\lim_{z\to 0}\frac{1}{V}e^{-\frac{i\omega x_0}{V}}\int_{-\infty}^{x_0}\frac{\partial}{\partial z}\bar{\psi}_0(\lambda,y_0,z)\,e^{\frac{i\omega\lambda}{V}}\,d\lambda \tag{12}$$

or, after substitution of the expression for $\bar{\psi}_0$ (eq. (5)) into equation (12), the results may be written as

$$K(x_0,y_0)=\frac{2}{V}\lim_{z\to 0}e^{-\frac{i\omega x_0}{V}}\int_{\beta r}^{x_0}e^{-i\bar{\omega}\lambda}\frac{\partial^2}{\partial z^2}\left[\frac{U(\lambda-\beta r)\cos\left(M\bar{\omega}\sqrt{\lambda^2-\beta^2r^2}\right)}{\sqrt{\lambda^2-\beta^2r^2}}\right]d\lambda \tag{13}$$

where $r=\sqrt{y_0^2+z^2}$ and, since the integrand is zero for $\lambda<\beta r$, the lower limit of integration has been changed from $-\infty$ to βr.

It is apparent upon examination of equation (13) that, if the indicated differentiation under the integral sign is carried out, the integrand has singular and perhaps troublesome terms. The indicated differentiation with respect to z, however, can be replaced by equivalent operations and followed by integrations by parts that lead to a reduced form of the kernel function containing no improper integrals. These steps follow.

Reduced form of kernel function.—As may be directly verified, the indicated differentiation with respect to z in equation (13) is, in the limit $z\to 0$, identical with

$$\begin{aligned}
&e^{-i\bar{\omega}\lambda}\lim_{z\to 0}\frac{\partial^2}{\partial z^2}\left[\frac{U(\lambda-\beta r)\cos\left(M\bar{\omega}\sqrt{\lambda^2-\beta^2r^2}\right)}{\sqrt{\lambda^2-\beta^2r^2}}\right]\\
&=-\frac{1}{y_0^2}\frac{\partial}{\partial\lambda}\left\{U(\lambda-\beta|y_0|)e^{-i\bar{\omega}\lambda}\left[\frac{\lambda\cos\left(M\bar{\omega}\sqrt{\lambda^2-\beta^2y_0^2}\right)}{\sqrt{\lambda^2-\beta^2y_0^2}}+\right.\right.\\
&\qquad\left.\left.\frac{i}{M}\sin\left(M\bar{\omega}\sqrt{\lambda^2-\beta^2y_0^2}\right)\right]\right\}-\frac{e^{-i\bar{\omega}\lambda}}{y_0^2}\left[\frac{\beta^2\bar{\omega}}{M}U(\lambda-\beta|y_0|)\right.\\
&\qquad\sin\left(M\bar{\omega}\sqrt{\lambda^2-\beta^2y_0^2}\right)-\frac{i}{M}\delta(\lambda-\beta|y_0|)\sin\left(M\bar{\omega}\sqrt{\lambda^2-\beta^2y_0^2}\right)-\\
&\qquad\left.\frac{\lambda-\beta|y_0|}{\sqrt{\lambda^2-\beta^2y_0^2}}\delta(\lambda-\beta|y_0|)\cos\left(M\bar{\omega}\sqrt{\lambda^2-\beta^2y_0^2}\right)\right]
\end{aligned} \tag{14}$$

Since the coefficients of $\delta(\lambda-\beta|y_0|)$ in equation (14) vanish at $\lambda=\beta|y_0|$, it follows from the integral properties of $\delta(\lambda-\beta|y_0|)$ that, when equation (14) is substituted into equation (13), the integrals involving the delta function vanish and equation (13) becomes:

$$\begin{aligned}
K(x_0,y_0)=-\frac{2}{Vy_0^2}e^{-\frac{i\omega x_0}{V}}\Bigg[&\frac{x_0e^{-i\bar{\omega}x_0}U(x_0-\beta|y_0|)}{\sqrt{x_0^2-\beta^2y_0^2}}\cos\left(M\bar{\omega}\sqrt{x_0^2-\beta^2y_0^2}\right)+\\
&\frac{i}{M}e^{-i\bar{\omega}x_0}U(x_0-\beta|y_0|)\sin\left(M\bar{\omega}\sqrt{x_0^2-\beta^2y_0^2}\right)+\\
&\frac{\beta^2\bar{\omega}}{M}\int_{\beta|y_0|}^{x_0}e^{-i\bar{\omega}\lambda}U(\lambda-\beta|y_0|)\sin\left(M\bar{\omega}\sqrt{x_0^2-\beta^2y_0^2}\right)d\lambda\Bigg]
\end{aligned} \tag{15}$$

Equation (15) provides an expression for the kernel function that involves no improper integrals. Except for the integral, the terms of the expression can be quite easily evaluated with the aid of trigonometric tables except at $y_0=0$, where the function is singular, and at $x_0=\beta|y_0|$, where the function is indeterminate. The integral is well behaved and can be accurately evaluated by numerical or approximate procedures. The singularities and indeterminate values are isolated and discussed in a later section, but it is desirable first to express the function $K(x_0,y_0)$ in terms of nondimensional length variables. As a check on the correctness of equation (15), the expression for $K(x_0,y_0)$ is reduced to the limiting value for $M=1$ and compared in appendix A with the corresponding limiting value for the subsonic case.

The kernel function in terms of nondimensional length variables.—Although the preceding results contain dimensional length variables, it is usually desirable to have such results in terms of nondimensional length variables. By

employing the variables x_0 and y_0 in a new sense to mean that they have been referred to some chosen length l and by introducing the reduced-frequency parameter $k=l\omega/V$, the length variables may be made nondimensional. (In flutter theory the reference length normally is selected as a semichord b.) The variables are used in this sense throughout the rest of this report. The kernel function (eq. (15)) can be written in terms of these nondimensional variables as

$$K(x_0,y_0)=-\frac{2}{Vl^2y_0^2}e^{-ikx_0}\left[\frac{x_0e^{-\frac{ikx_0}{\beta^2}}U(x_0-\beta|y_0|)}{\sqrt{x_0^2-\beta^2y_0^2}}\cos\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)+\frac{i}{M}e^{-\frac{ikx_0}{\beta^2}}U(x_0-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)+\right.$$
$$\left.\frac{k}{M}\int_{\beta|y_0|}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda\right] \tag{16a}$$

An equivalent expression for $K(x_0,y_0)$ which will be useful in subsequent considerations is

$$K(x_0,y_0)=-\frac{2}{Vl^2y_0^2}\left\{e^{-\frac{iM^2kx_0}{\beta^2}}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}\right)\left[U(x_0-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)\right]+\right.$$
$$\left.\frac{k}{M}e^{-ikx_0}\int_{\beta|y_0|}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda\right\} \tag{16b}$$

Another equivalent expression for $K(x_0,y_0)$ that is more concise and, for many purposes, more attractive than equation (16a) is

$$K(x_0,y_0)=-\frac{2}{Vl^2y_0^2}e^{-ikx_0}U(x_0-\beta|y_0|)\left[\frac{x_0e^{-\frac{ik}{\beta^2}x_0}}{\sqrt{x_0^2-\beta^2y_0^2}}\cos\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)+\frac{ik|y_0|}{2}\int_{\frac{1}{\beta^2|y_0|}(x_0-M\sqrt{x_0^2-\beta^2y_0^2})}^{\frac{1}{\beta^2|y_0|}(x_0+M\sqrt{x_0^2-\beta^2y_0^2})}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}d\tau\right] \tag{16c}$$

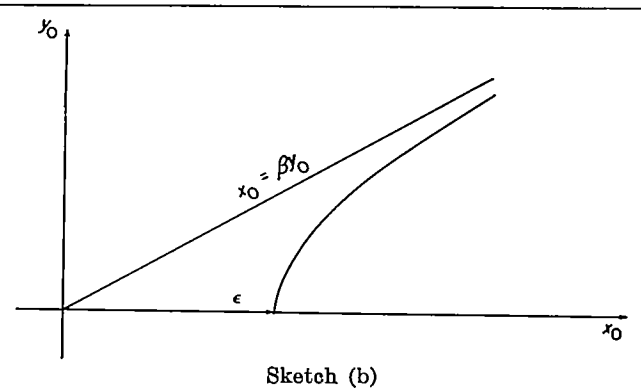
The reduction of equation (16a) to (16c) is given in appendix A. One noteworthy feature of equation (16c) is that the integral which remains to be evaluated has the same form as the integral occurring in the expression for the kernel function for subsonic speeds as presented in appendix D of reference 1.

ISOLATION AND DISCUSSION OF SINGULARITIES OF KERNEL FUNCTION

As previously mentioned and as may be noted in equations (15) and (16a), the kernel function becomes singular at $y_0=0$ and is of an indeterminate nature when $x_0=\beta|y_0|$. It is therefore desirable to make special treatment of the function in the neighborhood of these values of x_0 and y_0 in order to be able to express the function in a form which is more amenable for calculations. The indeterminate condition arises from the first term of equation (16a) because of the manner in which the unit function has been defined for this analysis. (The denominator of the first term vanishes at $x_0=\beta y_0$. The presence of the unit function in the numerator, however, renders this singularity indeterminate.)

In the next few sections the forms of the singularities are extracted (see eq. (24)) and the aforementioned indeterminate forms of the kernel function are explicitly determined (see eq. (20)). A form of the kernel function more suitable for calculation purposes, since the troublesome points are isolated, is presented in equation (26). A manner of integrating the kernel function across its singularities is given in equation (27). The singularities of the supersonic and subsonic cases are then compared.

Indeterminate form.—Consideration is first given to the indeterminate form, and it is convenient for this purpose to consider the value of $K(x_0,y_0)$ at points on the positive branch of a hyperbola. (See sketch (b)).



Sketch (b)

The equation of the hyperbola may be written as

$$\left.\begin{aligned}x_0&=\epsilon\cosh\theta\\ \beta|y_0|&=\epsilon\sinh\theta\end{aligned}\right\} \tag{17}$$

In these equations $\epsilon=0$ corresponds to $x_0=\beta y_0$, since elimination of θ gives $x_0^2-\beta^2y_0^2=\epsilon^2$.

After substitution of these expressions for x_0 and $\beta|y_0|$ into equation (16a), the results may be written as

$$K(\epsilon,\theta)=-\frac{2\beta^2e^{-ik\epsilon\cosh\theta}}{Vl^2\epsilon^2\sinh^2\theta}\left[U(\epsilon e^{-\theta})e^{-\frac{ik\epsilon\cosh\theta}{\beta^2}}\cosh\theta\cos\frac{Mk\epsilon}{\beta^2}+\right.$$
$$\frac{i}{M}U(\epsilon e^{-\theta})e^{-\frac{ik\epsilon\cosh\theta}{\beta^2}}\sin\frac{Mk\epsilon}{\beta^2}+$$
$$\left.\frac{k}{M}\int_{\epsilon\sinh\theta}^{\epsilon\cosh\theta}e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\epsilon\sinh\theta)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\epsilon^2\sinh^2\theta}\right)d\lambda\right] \tag{18}$$

To obtain a limiting value of this equation for small values of ϵ, the trigonometric and exponential terms can be replaced by terms up to the second power of ϵ in a series expansion.

If the result of performing this expansion is denoted by K', the equation

$$K'(\epsilon,\theta)=-\frac{2\beta^2 e^{-ik\epsilon\cosh\theta}}{Vl^2}U(\epsilon e^{-\theta})\left(\frac{\cosh\theta}{\epsilon^2\sinh^2\theta}-\frac{ik}{\beta^2\epsilon}-\frac{k^2\cosh\theta}{2\beta^4}-\frac{k^2}{2\beta^2}\log\frac{\cosh\theta+1}{\sinh\theta}\right) \tag{19}$$

is obtained, which, in terms of the original coordinates x_0 and y_0, is

$$K'(x_0,y_0)=-\frac{e^{-ikx_0}}{Vl^2}U(x_0-\beta|y_0|)\left[\frac{x_0}{\sqrt{x_0^2-\beta^2y_0^2}}\left(\frac{2}{y_0^2}-\frac{2ik}{x_0}-\frac{k^2}{\beta^2}\right)-k^2\log\frac{x_0+\sqrt{x_0^2-\beta^2y_0^2}}{\beta|y_0|}\right] \tag{20}$$

Although these equations were obtained in order to reveal the form of the indeterminate value of the kernel function, they are found to contain singularities at $y_0=0$. Prior to any further discussion of this result, it is desirable to consider the limiting form of $K(x_0,y_0)$ as y_0 approaches zero, to determine all the singularities at $y_0=0$.

Singularities at $y_0=0$.—For the purpose of obtaining a limiting value of the kernel function for vanishingly small values of y_0, the integral appearing in equation (16a) may be written as the sum of two integrals, namely:

$$\int_{\beta|y_0|}^{x_0} e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda$$
$$=\int_{\beta|y_0|}^{\infty} e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda-\int_{x}^{\infty} e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda \tag{21}$$

The first of these integrals may be evaluated from the table of Laplace transforms of reference 17 and has the following value:

$$\int_{\beta|y_0|}^{\infty} e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda=M|y_0|K_1(k|y_0|) \tag{22}$$

where K_1 is the modified Bessel function of the second kind, of first order. In the second integral, the integrand may be replaced by terms up to the second power of y_0 in a series expansion. Thus,

$$\int_{x_0}^{\infty} e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda$$
$$\approx\int_{x_0}^{\infty} e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\left(\sin\frac{Mk\lambda}{\beta^2}-\frac{Mky_0^2}{2\lambda}\cos\frac{Mk\lambda}{\beta^2}\right)d\lambda$$
$$=U(x_0-\beta|y_0|)\left\{\frac{1}{k}e^{-\frac{ikx_0}{\beta^2}}\left(M\cos\frac{Mkx_0}{\beta^2}+i\sin\frac{Mkx_0}{\beta^2}\right)+\frac{Mky_0^2}{4}\left[\mathrm{Ci}\left(\frac{kx_0}{M+1}\right)+\mathrm{Ci}\left(\frac{kx_0}{M-1}\right)+i\,\mathrm{Si}\left(\frac{kx_0}{M+1}\right)-i\,\mathrm{Si}\left(\frac{kx_0}{M-1}\right)\right]\right\} \tag{23}$$

where Ci and Si denote the cosine integral function and sine integral function, respectively, which are defined as follows:

$$\mathrm{Ci}\,(x)=-\int_x^{\infty}\frac{\cos t}{t}\,dt \qquad \mathrm{Si}\,(x)=\frac{\pi}{2}-\int_x^{\infty}\frac{\sin t}{t}\,dt$$

Substituting equations (22) and (23) into equation (16a) gives, as a limiting value of $K\,(x_0,y_0)$,

$$\lim_{y_0\to 0}K(x_0,y_0)=\lim_{y_0\to 0}-\frac{2e^{-ikx_0}}{Vl^2y_0^2}\left\{k|y_0|K_1(k|y_0|)-\frac{k^2y_0^2}{4}\left[\mathrm{Ci}\left(\frac{kx_0}{M+1}\right)+\mathrm{Ci}\left(\frac{kx_0}{M-1}\right)+i\,\mathrm{Si}\left(\frac{kx_0}{M+1}\right)-i\,\mathrm{Si}\left(\frac{kx_0}{M-1}\right)\right]\right\}U(x_0-\beta|y_0|)$$
$$=\lim_{y_0\to 0}-\frac{e^{-ikx_0}}{Vl^2}U(x_0-\beta|y_0|)\left\{\frac{2}{y_0^2}+k^2\left(\gamma-\frac{1}{2}\right)+k^2\log\frac{k|y_0|}{2}-\frac{k^2}{2}\left[\mathrm{Ci}\left(\frac{kx_0}{M+1}\right)+\mathrm{Ci}\left(\frac{kx_0}{M-1}\right)+i\,\mathrm{Si}\left(\frac{kx_0}{M+1}\right)-i\,\mathrm{Si}\left(\frac{kx_0}{M-1}\right)\right]\right\} \tag{24}$$

where the following series expression for $K_1(z)$ (see ref. 18) is employed:

$$K_1(z)=\left(\gamma+\log\frac{z}{2}\right)\left(\frac{z}{2}+\frac{z^3}{16}+\frac{z^5}{384}+\cdots\right)+\frac{1}{z}-\left(\frac{z}{4}+\frac{5z^3}{64}+\frac{5z^5}{1152}+\cdots\right) \tag{25}$$

where γ is Euler's constant ($\gamma=0.5772157$). Examination of equation (24) shows that the only singular terms are the same as those which appear in equation (20), namely $-\frac{2e^{-ikx_0}}{y_0^2}$ and $-k^2e^{-ikx_0}\log|y_0|$. Thus, for the purpose of isolating the singularities of the kernel function, only $K'(x_0,y_0)$, as defined in equation (20), need be considered. Nevertheless, the results given in equation (24) may be useful in some applications since they provide a ready means for evaluating the nonsingular part of $\lim_{y_0\to 0}K\,(x_0,y_0)$.

Form of kernel function suitable for calculations.—As in the subsonic case, with knowledge of the critical values of the kernel function, an expression can be written in which the kernel function is separated into two parts, one of which contains no singularities or indeterminate values and the other of which contains all the singularities and critical values of the kernel function. This expression is

$$K(x_0,y_0)\equiv[K(x_0,y_0)-K'(x_0,y_0)]+K'(x_0,y_0) \tag{26}$$

where $K(x_0,y_0)$ is defined in equations (15) or (16), and $K'(x_0,y_0)$ is defined in equation (20). The term $K(x_0,y_0)-K'(x_0,y_0)$ in equation (26) has no singular or indeterminate values. The term $K'(x_0,y_0)$ is singular at $y_0=0$ and indeterminate when $x_0=\beta y_0$.

Integration of singularities of kernel function.—Since integration of the kernel function is often necessary, a few remarks on how to circumvent its inherent singularities are in order. Each term of $K'(x_0,y_0)$ in equation (20) possesses a simple indefinite integral with respect to the variable $\eta=y-y_0$. In performing integrations with respect to η that involve a passage across the line $\eta=y$, a principal value is to be taken. For example,

$$\fint_{-1}^{1}\frac{U(x_0-\beta|y_0|)x_0\,d\eta}{(y-\eta)^2\sqrt{x_0^2-\beta^2(y-\eta)^2}}=\frac{1}{x_0}\left[\frac{U(x_0-\beta|y-1|)\sqrt{x_0^2-\beta^2(y-1)^2}}{y-1}-\frac{U(x_0-\beta|y+1|)\sqrt{x_0^2-\beta^2(y+1)^2}}{y+1}\right] \tag{27}$$

where the symbol $\fint$ indicates that the singular integral is to be considered simply as a function of its limits. A justification for this consideration is that it leads to results that could, with considerable labor, be rigorously established by maintaining the variable z in the analysis until all operations are performed.

Comparison with singularities of subsonic case.—It may be of interest to compare the above results with corresponding results for the subsonic case, that is, for $x_0>0$ and $y_0\approx 0$. Results for the subsonic case may be obtained from equation (31) of reference 1 as follows:

$$\begin{aligned}\lim_{y_0\to 0} K'(x_0,y_0)_{M<1}&=\lim_{y_0\to 0}\frac{1}{Vl^2}e^{-ikx_0}\left\{-\frac{\sqrt{x_0^2+(1-M^2)y_0^2}+x_0}{y_0^2\sqrt{x_0^2+(1-M^2)y_0^2}}+\right.\\&\frac{ik}{\sqrt{x_0^2+(1-M^2)y_0^2}}-\frac{k^2}{2(1-M^2)}\frac{x_0-M\sqrt{x_0^2+(1-M^2)y_0^2}}{\sqrt{x_0^2+(1-M^2)y_0^2}}-\\&\left.\frac{k^2}{2}\log\frac{k[\sqrt{x_0^2+(1-M^2)y_0^2}-x_0]}{2(1-M)}\right\}\\&=\frac{e^{-ikx_0}}{Vl^2}\left[-\frac{2}{y_0^2}+\frac{ik}{x_0}-\frac{k^2}{2(1+M)}-\frac{k^2}{2}\log\frac{k(1+M)y_0^2}{4x_0}\right]\end{aligned} \tag{28}$$

The singular terms of this expression for $x_0>0$ and $M<1$ are

$$-\frac{2}{Vl^2}e^{-ikx_0}\left(\frac{1}{y_0^2}+\frac{k^2}{z}\log|y_0|\right)$$

Comparison of this result with equation (20) shows that the singularities for subsonic and supersonic flow are of identical form.

SOME INFINITE-SERIES EXPANSIONS PERTINENT TO THE KERNEL FUNCTION

The kernel function can be expressed as a series by various expansion procedures. Some particular expansions, which should be useful in applications, are discussed in succeeding paragraphs. These are the power-series expansion in terms of the reduced-frequency parameter (see eq. (29)) and an expansion in terms of Bessel functions. The latter expansion is used in a later section to obtain the kernel function for two-dimensional flow from that for three-dimensional flow.

Power-series expansion with respect to k.—As in the case of subsonic flow, the kernel function can be expanded into a power series with respect to k that, in the present case, is useful for small values of k/β^2, a combination of reduced frequency and Mach number that is inherent in such an expansion of the supersonic kernel. The terms of the expansion may be simply obtained by expanding the terms of equation (16a) that are functions of k and collecting the results. The first few terms are

$$\begin{aligned}K(x_0,y_0)=-\frac{2e^{-ikx_0}}{Vl^2y_0^2}U(x_0-\beta|y_0|)\Bigg[&\frac{x_0}{\sqrt{x_0^2-\beta^2y_0^2}}-\frac{ik}{\beta^2}\frac{\beta^2y_0^2}{\sqrt{x_0^2-\beta^2y_0^2}}-\\&\frac{1}{2}\left(\frac{k}{\beta^2}\right)^2\left(\frac{\beta^2y_0^2x_0}{\sqrt{x_0^2-\beta^2y_0^2}}+\beta^4y_0^2\cosh^{-1}\frac{x_0}{\beta|y_0|}\right)+\\&\frac{i}{6}\left(\frac{k}{\beta^2}\right)^3\frac{(3M^2-1)\beta^2y_0^2x_0^2+(2-3M^2)\beta^4y_0^4}{\sqrt{x_0^2-\beta^2y_0^2}}+0\left(\frac{k}{\beta^2}\right)^4\Bigg]\end{aligned} \tag{29}$$

Although this power-series expansion converges to the appropriate value of $K(x_0,y_0)$ for all finite values of k/β^2, a great number of terms are required unless k/β^2 is small. These first few terms of the expansion can be considered to represent the kernel function for values of k in the range of magnitudes generally encountered in dynamic-stability studies and, therefore, they are pertinent for obtaining time-dependent stability derivatives. A noteworthy feature of the expansion is that each term can be integrated, in the sense that it contains a simple indefinite integral, with respect to the variable $\eta=y-y_0$. When such integrations involve a passage across the line $\eta=y$, a principal value is to be taken in the sense described after equation (28).

Expansions in terms of Bessel functions.—The trigonometric terms appearing in the expression for $K(x_0,y_0)$ in equations (15), (16a), and (16b) can be expanded into infinite series involving Bessel functions of the first kind. Such expansions have good convergence properties, even for large values of the parameter k/β^2, and each term possesses a simple indefinite integral with respect to η. Such Bessel function series are therefore useful for deriving an expansion for the indefinite integral of $K(x_0,y_0)$ with respect to η. The indefinite integral of $K(x_0,y_0)$ leads to the downwash associated with pulsating vortex lines ("horseshoe" vortices) and, as previously indicated, to the kernel function for two-dimensional flow. It might be useful to point out that the expansion of the cosine term into a series involving Bessel functions is also useful for studying distributions of pulsating sources.

For the purpose of expanding the trigonometric terms under discussion, consider the expressions

$$U(\lambda-a)\frac{\cos b\sqrt{\lambda^2-a^2}}{\sqrt{\lambda^2-a^2}} \tag{30}$$

and

$$U(\lambda-a)\sin\left(b\sqrt{\lambda^2-a^2}\right)=-\frac{\partial}{\partial b}\left[U(\lambda-a)\frac{\cos\left(b\sqrt{\lambda^2-a^2}\right)}{\sqrt{\lambda^2-a^2}}\right] \tag{31}$$

where a, b, and λ are positive.

By making use of a known Fourier transform relation, expression (30) can be equated to an infinite integral involving a Bessel function of the first kind (see, for example, p. 33 of ref. 17):

$$\left.\begin{aligned}\int_0^\infty J_0\left(\lambda\sqrt{\tau^2+b^2}\right)\cos a\tau\,d\tau&=\frac{\cos\left(b\sqrt{\lambda^2-a^2}\right)}{\sqrt{\lambda^2-a^2}} &&(\lambda>a)\\ \int_0^\infty J_0\left(\lambda\sqrt{\tau^2+b^2}\right)\cos a\tau\,d\tau&=0 &&(\lambda<a)\end{aligned}\right\} \tag{32}$$

By use of the addition formula for Bessel functions (see, for example, p. 358 of ref. 18), the Bessel function appearing in this equation can be written as an infinite sum of products of Bessel functions as follows:

$$J_0\left(\lambda\sqrt{\tau^2+b^2}\right)=J_0(\tau\lambda)J_0(b\lambda)+2\sum_{n=1}^{\infty}(-1)^nJ_{2n}(\tau\lambda)J_{2n}(b\lambda) \tag{33}$$

Thus,

$$\begin{aligned}\int_0^\infty J_0\left(\lambda\sqrt{\lambda^2+b^2}\right)\cos a\tau\,d\tau&=U(\lambda-a)\frac{\cos\left(b\sqrt{\lambda^2-a^2}\right)}{\sqrt{\lambda^2-a^2}}\\ &=\int_0^\infty\Bigg[J_0(\tau\lambda)J_0(b\lambda)+\\ &\qquad 2\sum_{n=1}^{\infty}(-1)^nJ_{2n}(\tau\lambda)J_{2n}(b\lambda)\Bigg]\cos a\tau\,d\tau\end{aligned} \tag{34}$$

In view of the relation (see ref. 17, p. 37)

$$\int_0^\infty J_{2n}(\tau\lambda)\cos a\tau\,d\tau=\frac{U(\lambda-a)\cos\left(2n\sin^{-1}\frac{a}{\lambda}\right)}{\sqrt{\lambda^2-a^2}} \tag{35}$$

the indicated integration on the right-hand side of equation (34) can be carried out term by term so that

$$\begin{aligned}&\frac{U(\lambda-a)\cos\left(b\sqrt{\lambda^2-a^2}\right)}{\sqrt{\lambda^2-a^2}}\\ &=\frac{U(\lambda-a)}{\sqrt{\lambda^2-a^2}}\left[J_0(b\lambda)+2\sum_{n=1}^{\infty}(-1)^nJ_{2n}(b\lambda)\cos\left(2n\sin^{-1}\frac{a}{\lambda}\right)\right]\end{aligned} \tag{36}$$

Substituting the expression on the right-hand side of equation (36) into equation (31) gives

$$\begin{aligned}U(\lambda-a)\sin\left(b\sqrt{\lambda^2-a^2}\right)&=\frac{\lambda U(\lambda-a)}{\sqrt{\lambda^2-a^2}}\left\{J_1(b\lambda)-\sum_{n=1}^{\infty}(-1)^n\left[J_{2n-1}(b\lambda)-J_{2n+1}(b\lambda)\right]\cos\left(2n\sin^{-1}\frac{a}{\lambda}\right)\right\}\\ &=\frac{2\lambda U(\lambda-a)}{\sqrt{\lambda^2-a^2}}\left\{\cos\left(\sin^{-1}\frac{a}{\lambda}\right)\sum_{n=1}^{\infty}(-1)^{n-1}J_{2n-1}(b\lambda)\cos\left[(2n-1)\sin^{-1}\frac{a}{\lambda}\right]\right\}\end{aligned} \tag{37}$$

But, since

$$U(\lambda-a)\frac{\lambda}{\sqrt{\lambda^2-a^2}}\cos\left(\sin^{-1}\frac{a}{\lambda}\right)\equiv U(\lambda-a)\qquad(a\geqq 0)$$

the expression for $U(\lambda-a)\sin\left(b\sqrt{\lambda^2-a^2}\right)$ may be written as

$$U(\lambda-a)\sin\left(b\sqrt{\lambda^2-a^2}\right)=2U(\lambda-a)\sum_{n=1}^{\infty}(-1)^{n-1}J_{2n-1}(b\lambda)\cos\left[(2n-1)\sin^{-1}\frac{a}{\lambda}\right] \tag{38}$$

By direct comparison, equations (36) and (38) can be used to write expanded forms of the trigonometric terms appearing in equations (15), (16a), and (16b). Expansions thus obtained will now be used to derive the kernel function for two-dimensional supersonic flow.

In contrast to three-dimensional flow, a physical interpretation of the kernel function for two-dimensional flow is that it represents the downwash at a given field point due to a pulsating bound vortex line of infinite length. This kernel function may be obtained by integrating the kernel function for three-dimensional flow from $-\infty$ to ∞ or, in view of the role of the unit function, from one Mach line to the other, with respect to the variable $\eta=y-y_0$. Pulsating "horseshoe" vortices may be obtained by integrating $K(x_0,y_0)$ over an arbitrarily finite range with respect to η.

DERIVATION OF KERNEL FUNCTION FOR TWO-DIMENSIONAL FLOW

In this section the kernel function for three-dimensional flow is reduced to the function for two-dimensional flow, and the final results of the reduction are given in equation (49). For the purpose of derivation, $K(x_0,y_0)$ will be considered as given in equation (16b). The two-dimensional kernel function can then be expressed as

$$\int_{-\infty}^{\infty} K(x_0,y_0)d\eta=-\frac{2}{Vl}\left\{e^{\frac{iM^2kx_0}{\beta^2}}\fint_{-\infty}^{\infty}\frac{\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}}{y_0^2}\left[U(x_0-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)\right]d\eta+\right.$$
$$\left.\frac{k}{M}e^{-ikx_0}\fint_{-\infty}^{\infty}\frac{d\eta}{y_0^2}\int_{\beta|y_0|}^{x_0}e^{\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda\right\}$$
$$=K(x_0) \tag{39}$$

or

$$K(x_0)=-\frac{2}{Vl}\left(e^{\frac{iM^2kx_0}{\beta^2}}I_1+\frac{k}{M}e^{-ikx_0}I_2\right) \tag{40}$$

where

$$I_1=\fint_{-\infty}^{\infty}\frac{\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}\right)}{(y-\eta)^2}\left[U(x_0-\beta|y-\eta|)\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2(y-\eta)^2}\right)\right]d\eta \tag{41a}$$

or

$$I_1=\beta\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}\right)\fint_{-x_0}^{x_0}\frac{U(x_0-|\eta|)}{\eta^2}\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\eta^2}\right)d\eta \tag{41b}$$

and

$$I_2=\beta\fint_{-x_0}^{x_0}\frac{1}{\eta^2}\left[\int_{|\eta|}^{x_0}e^{\frac{ik\lambda}{\beta^2}}U(\lambda-|\eta|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\eta^2}\right)d\lambda\right]d\eta \tag{42}$$

In equations (41) for I_1 and equation (42) for I_2, infinite limits with respect to the integration of η are not necessary. In view of the role of the unit function in the integrands, limits of $\pm x_0$ include all values of η for which the integrands are different from zero. The symbol $\fint$ indicates that the integrals are to be considered simply as functions of their limits or that the singularity at $\eta=0$ is to be ignored.

First consider equations (41) for I_1 and then perform an integration by parts. The expression for I_1 may then be written as

$$I_1=\beta\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}\right)\left[-\frac{1}{\eta}U(x_0-|\eta|)\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\eta^2}\right)\Big|_{-x_0}^{x_0}-\right.$$
$$\left.\frac{Mk}{\beta^2}\int_{-x_0}^{x_0}\frac{U(x_0-|\eta|)}{\sqrt{x_0^2-\eta^2}}\cos\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\eta^2}\right)d\eta\right]$$
$$=-2\beta\left(\frac{\partial}{\partial x_0}+\frac{ik}{\beta^2}\right)\int_0^{x_0}\frac{U(x_0-|\eta|)}{\sqrt{x_0^2-\eta^2}}\cos\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\eta^2}\right)d\eta \tag{43}$$

By direct comparison with equation (36), equation (43) may be written in expanded form as

$$I_1=-2\beta\left(\frac{\partial}{\partial x_0}+\frac{ik}{\beta^2}\right)\int_0^{x_0}\frac{U(x_0-|\eta|)}{\sqrt{x_0^2-\eta^2}}\left[J_0\left(\frac{Mk}{\beta^2}x_0\right)+\right.$$
$$\left.2\sum_{n=1}^{\infty}(-1)^nJ_{2n}\left(\frac{Mk}{\beta^2}x_0\right)\cos\left(2n\sin^{-1}\frac{\eta}{x_0}\right)\right]d\eta \tag{44}$$

In this equation, the terms involving J_{2n} do not contribute to the integral because

$$\int_0^{x_0}\frac{U(x_0-|\eta|)}{\sqrt{x_0^2-\eta^2}}\cos\left(2n\sin^{-1}\frac{\eta}{x_0}\right)d\eta$$
$$=\frac{1}{2n}\left[U(x_0-|\eta|)\sin\left(2n\sin^{-1}\frac{\eta}{x_0}\right)\Big|_0^{x_0}+\right.$$
$$\left.\int_0^{x_0}\frac{\eta}{|\eta|}\delta(x_0-|\eta|)\sin\left(2n\sin^{-1}\frac{\eta}{x_0}\right)d\eta\right]\equiv 0$$

Hence, since

$$\int_0^{x_0} \frac{U(x_0-|\eta|)}{\sqrt{x_0^2-\eta^2}}\,d\eta=U(x_0-|\eta|)\sin^{-1}\frac{\eta}{x_0}\Big|_0^{x_0}+\int_0^{x_0}\frac{\eta}{|\eta|}\,\delta(x_0-|\eta|)\sin^{-1}\frac{\eta}{x_0}\,d\eta=\frac{\pi}{2}U(x_0)$$

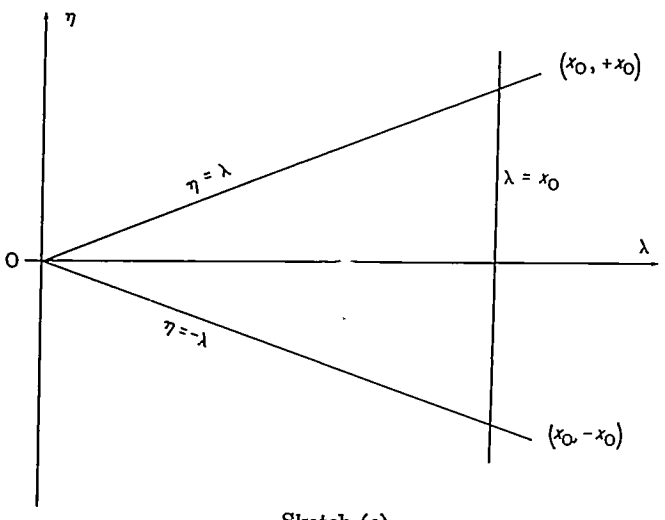
the expression for I_1 can be written as

$$I_1=-\pi\beta\left(\frac{\partial}{\partial x_0}+\frac{ik}{\beta^2}\right)U(x_0)J_0\left(\frac{Mk}{\beta^2}x_0\right)$$
$$=-\pi\beta\left[\delta(x_0)+\frac{ik}{\beta^2}U(x_0)J_0\left(\frac{Mk}{\beta^2}x_0\right)-\frac{Mk}{\beta^2}U(x_0)J_1\left(\frac{Mk}{\beta^2}x_0\right)\right] \tag{45}$$

Now consider equation (42) for I_2, namely

$$I_2=\beta\fint_{-x_0}^{x_0}\frac{1}{\eta^2}\left[\int_{|\eta|}^{x_0}e^{\frac{ik\lambda}{\beta^2}}U(\lambda-|\eta|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\eta^2}\right)d\lambda\right]d\eta$$

The double integral in this expression can be considered as a surface integral over a triangular region of the $\lambda\eta$-plane cut out by the lines $\eta=\lambda$, $\eta=-\lambda$, and $\lambda=x_0$, as shown in sketch (c).



Sketch (c)

By a change in the order of integration, which is admissible since the singularity at $\eta=0$ is to be ignored, the expression for I_2 may be written as

$$I_2=\beta\int_0^{x_0}e^{\frac{ik\lambda}{\beta^2}}d\lambda\fint_{-\lambda}^{\lambda}\frac{U(\lambda-|\eta|)}{\eta^2}\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\eta^2}\right)d\eta \tag{46}$$

The inner integral in this equation is identical in form to the integral in equation (41). Hence, by observation of and comparison with the results obtained for I_1 in equations (43), (44), and (45), it is found that

$$\fint_{-\lambda}^{\lambda}\frac{U(\lambda-|\eta|)}{\eta^2}\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\eta^2}\right)d\eta=-\frac{\pi Mk}{\beta^2}U(\lambda)J_0\left(\frac{Mk}{\beta^2}\lambda\right) \tag{47}$$

The expression for I_2 can therefore be written

$$I_2=-\frac{\pi Mk}{\beta}\int_0^{x_0}e^{\frac{ik\lambda}{\beta^2}}U(\lambda)J_0\left(\frac{Mk}{\beta^2}\lambda\right)d\lambda \tag{48}$$

Substituting this result and the results given in equation (45) for I_1 into equation (40) gives a desired form of the kernel function for two-dimensional supersonic flow:

$$K(x_0)=\frac{2\pi\beta}{Vl}\left\{e^{-\frac{iM^2kx_0}{\beta^2}}\left[\delta(x_0)+\frac{ik}{\beta^2}U(x_0)J_0\left(\frac{Mk}{\beta^2}x_0\right)-\frac{Mk}{\beta^2}U(x_0)J_1\left(\frac{Mk}{\beta^2}x_0\right)\right]+\frac{k^2}{\beta^2}e^{-ikx_0}\int_0^{x_0}e^{-\frac{ik\lambda}{\beta^2}}U(\lambda)J_0\left(\frac{Mk}{\beta^2}\lambda\right)d\lambda\right\} \tag{49}$$

Examination of equation (49) shows that the only singularity involved in the kernel function for two-dimensional supersonic flow is the δ-function. At zero frequency, all the terms of $K(x_0)$ except the δ-function vanish. The kernel function required to treat two-dimensional wings at steady angle-of-attack conditions is therefore proportional to this δ-function, and, as shown in the following section, leads in a very simple manner to the well-known Ackeret results.

The integral that remains to be evaluated in equation (49) is well behaved and similar to integrals, treated by Schwarz (ref. 19) and others, that arise in the velocity-potential approach for treating two-dimensional wings.

APPLICATION OF KERNEL FUNCTION TO LIFT DISTRIBUTIONS FOR TWO-DIMENSIONAL WINGS

The results obtained in the previous section for the two-dimensional kernel function are now employed to obtain the lift distribution on oscillating and steady two-dimensional wings moving at supersonic speed. (See eqs. (56) and (61).) Since the lift distributions so obtained agree with the Ackeret results for a steady wing and also with known results for the oscillating wing (ref. 15), they serve as a check on the correctness of the expressions for both the two-dimensional and three-dimensional kernel functions.

The integral equation that must be solved to obtain the lift distribution for two-dimensional wings in supersonic flow is particularly simple since it involves a single integral of the convolution type:

$$\overline{w}(x)=\frac{l}{4\pi\rho}\int_0^x L(\xi)K(x_0)d\xi=\frac{l}{4\pi\rho}\int_0^x L(\xi)K(x-\xi)d\xi \tag{50}$$

Integral equations of this type can be readily solved by Laplace transform procedures since the Laplace transform of a convolution integral is the product of the transforms of the functions that compose the integrand. In the present case, if s represents the Laplace transform operator defined by

$$L[f(x)]=\int_0^\infty e^{-sx}f(x)dx=f(s) \tag{51}$$

the transform of equation (50) may be written as

$$\overline{w}(s)=\frac{l}{4\pi\rho}L(s)K(s) \tag{52}$$

Solving this equation for $L(s)$ gives the Laplace transform

of the lift distribution:

$$L(s)=\frac{4\pi\rho}{l}\frac{\overline{w}(s)}{K(s)} \tag{53}$$

Inversion of the transform on the right-hand side of this equation gives the lift distribution.

For the case of a steady two-dimensional wing,

$$\overline{w}(x)=V\alpha \qquad \overline{w}(s)=\frac{V\alpha}{s}$$

$$K(x)=\frac{2\pi\beta}{Vl}\delta(x) \qquad K(s)=\frac{2\pi\beta}{Vl}$$

Then

$$L(s)=\frac{2\rho V^2\alpha}{\beta s} \tag{54}$$

The inverse transform of equation (54) gives for the lift distribution:

$$L(x)=\frac{2\rho V^2\alpha}{\beta}U(x) \tag{55}$$

From this result, the total lift per unit of span is

$$l\int_0^{\frac{Chord}{l}} L(x)dx=\frac{2\rho V^2\alpha\times Chord}{\beta} \tag{56}$$

This result agrees with the well-known Ackeret result.

Now consider the unsteady case for oscillatory translation,

$$\overline{w}(x)=i\omega l\overline{h}=iVk\overline{h} \tag{57}$$

where $\overline{h}$ is the amplitude of displacement referred to l, and the Laplace transform of $\overline{w}(x)$ is

$$\overline{w}(s)=\frac{iVk\overline{h}}{s} \tag{58}$$

The Laplace transforms of the different terms of $K(x)$ (eq. (49)) can be simply derived or they may be obtained from Laplace transform tables (for example, ref. 16). After combining the transforms of the different terms, the results can be written as

$$K(s)=\frac{2\pi\beta}{Vl}\frac{\sqrt{\left(s+\frac{iM^2k}{\beta^2}\right)^2+\frac{M^2k^2}{\beta^4}}}{s+ik} \tag{59}$$

Substituting equations (58) and (59) into equation (53) gives for the transform of the lift distribution

$$L(s)=\frac{2i\rho V^2k\overline{h}}{\beta}\frac{s+ik}{s\sqrt{\left(s+\frac{iM^2k}{\beta^2}\right)^2+\frac{M^2k^2}{\beta^4}}} \tag{60}$$

The inverse of this transform gives for the lift distribution

$$L(x)=\frac{2i\rho V^2k\overline{h}}{\beta}\left[U(x)e^{-\frac{iM^2kx}{\beta^2}}J_0\left(\frac{Mkx}{\beta^2}\right)+ik\int_0^x U(\xi)e^{-\frac{iM^2k\xi}{\beta^2}}J_0\left(\frac{Mk\xi}{\beta^2}\right)d\xi\right] \tag{61}$$

This result can easily be shown to check with the results of reference 15. Moreover, if $ik\overline{h}$ is set equal to α, and then k is allowed to approach zero, equation (61) reduces to the result for the steady case.

CONCLUDING REMARKS

The main purpose of this report was to derive and present in a form that could be numerically evaluated the kernel function of the integral equation relating downwash and lift distributions for oscillating wings in supersonic flow. This purpose has been achieved for three-dimensional flow, and the results have been converted to a form more suitable for calculation by isolating the singular or critical points. The kernel function for two-dimensional supersonic flow has been presented and the results show that the only singularity is a Dirac delta function, which appeared in such a manner that further reduction with regard to singularities is not required.

The results presented in this report for supersonic flow together with those previously obtained for subsonic flow provide a kernel function that is capable of being evaluated at any Mach number. As experience develops it is expected that use can be made of the kernel function to develop approximate procedures, that will be more or less uniform throughout the Mach number range, for calculating aerodynamic forces on oscillating (or steady) wings of arbitrary plan form and with arbitrary downwash conditions. The labor involved in such approximate or numerical procedures will indeed be prodigious and will require the use of modern high-speed computing equipment.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., *February 15, 1955.*

APPENDIX A

DERIVATION OF EQUATION (16c) AND REDUCTION OF THE KERNEL FUNCTION TO THE SONIC CASE

The purpose of this appendix is first to reduce equation (16a) to equation (16c) and then to reduce equation (16c) to the sonic case. The reduction to the sonic case, by comparison with results obtained from consideration of subsonic speeds for the sonic case in reference 1, provides a partial check on the correctness of results obtained for the supersonic case.

DERIVATION OF EQUATION (16c)

In order to reduce equation (16a) to equation (16c) consider the integral

$$\frac{k}{M}\int_{\beta|y_0|}^{x_0} e^{-\frac{ik}{\beta^2}\lambda}\, U(\lambda-\beta|y_0|)\, \sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right) d\lambda$$

$$=\frac{k}{2iM}\int_{\beta|y_0|}^{x_0} e^{-\frac{ik}{\beta^2}\lambda}\, U(\lambda-\beta|y_0|)\left(e^{\frac{iMk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}}-e^{-\frac{iMk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}}\right)d\lambda$$

$$=\frac{k}{2iM}(I_1-I_2) \tag{A1}$$

where

$$I_1=\int_{\beta|y_0|}^{x_0} U(\lambda-\beta|y_0|)e^{-\frac{ik}{\beta^2}\left(\lambda-M\sqrt{\lambda^2-\beta^2y_0^2}\right)}\, d\lambda \tag{A2}$$

and

$$I_2=\int_{\beta|y_0|}^{x_0} U(\lambda-\beta|y_0|)e^{-\frac{ik}{\beta^2}\left(\lambda+M\sqrt{\lambda^2-\beta^2y_0^2}\right)}\, d\lambda \tag{A3}$$

In these expressions λ takes on only positive values, that is, $\beta\,|y_0|\leqq\lambda\leqq x_0$. Hence, consider for each integral the single-valued substitution

$$\lambda=|y_0|(+M\sqrt{1+\tau^2}-\tau) \tag{A4}$$

which, for $M\geqq 1$, leads only to positive values of λ.

Solving this expression for τ gives

$$\tau=\frac{\lambda\pm M\sqrt{\lambda^2-\beta^2y_0^2}}{\beta^2|y_0|} \tag{A5}$$

Thus, if the substitution

$$\left(\lambda-M\sqrt{\lambda^2-\beta^2y_0^2}\right)=\beta^2|y_0|\tau \tag{A6}$$

is made in I_1 and the substitution

$$\left(\lambda+M\sqrt{\lambda^2-\beta^2y_0^2}\right)=\beta^2|y_0|\tau \tag{A7}$$

is made in I_2, there is obtained

$$I_1=|y_0|\int_{\frac{1}{\beta}}^{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)} U\left(\tau-\frac{1}{\beta}\right)\left(\frac{M\tau}{\sqrt{1+\tau^2}}-1\right)e^{-ik|y_0|\tau}d\tau \tag{A8}$$

and

$$I_2=|y_0|\int_{\frac{1}{\beta}}^{\frac{1}{\beta^2|y_0|}\left(x_0+M\sqrt{x_0^2-\beta^2y_0^2}\right)} U\left(\tau-\frac{1}{\beta}\right)\left(\frac{M\tau}{\sqrt{1+\tau^2}}-1\right)e^{-ik|y_0|\tau}d\tau$$

$$=|y_0|\int_{\frac{1}{\beta}}^{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)} U\left(\tau-\frac{1}{\beta}\right)\left(\frac{M\tau}{\sqrt{1+\tau^2}}-1\right)e^{-ik|y_0|\tau}d\tau+$$

$$|y_0|U(x_0-\beta|y_0|)\int_{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)}^{\frac{1}{\beta^2|y_0|}\left(x_0+M\sqrt{x_0^2-\beta^2y_0^2}\right)}\left(\frac{M\tau}{\sqrt{1+\tau^2}}-1\right)e^{-ik|y_0|\tau}d\tau \tag{A9}$$

Combining these results gives for $\frac{k}{2iM}(I_1-I_2)$

$$\frac{k}{2iM}(I_1-I_2)=$$

$$-\frac{k|y_0|U(x_0-\beta|y_0|)}{2iM}\int_{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)}^{\frac{1}{\beta^2|y_0|}\left(x_0+M\sqrt{x_0^2-\beta^2y_0^2}\right)}\left(\frac{M\tau}{\sqrt{1+\tau^2}}-1\right)e^{-ik|y_0|\tau}d\tau \tag{A10}$$

or since

$$\int_{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)}^{\frac{1}{\beta^2|y_0|}\left(x_0+M\sqrt{x_0^2-\beta^2y_0^2}\right)} e^{-ik|y_0|\tau}d\tau=-\frac{2e^{-\frac{ik}{\beta^2}x_0}}{k}\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right) \tag{A11}$$

$$\frac{k}{2iM}(I_1-I_2)=-|y_0|U(x_0-\beta|y_0|)\left[\frac{ie^{-\frac{ik}{\beta^2}x_0}}{M}\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)+\right.$$

$$\left.\frac{k}{2i}\int_{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)}^{\frac{1}{\beta^2|y_0|}\left(x_0+M\sqrt{x_0^2-\beta^2y_0^2}\right)}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}d\tau\right] \tag{A12}$$

Substituting this result into equation (16a) gives equation (16c) or

$$K(x_0,y_0)=-\frac{2}{vl^2y_0^2}e^{-ikx_0}U(x_0-\beta|y_0|)\left[\frac{x_0e^{-\frac{ik}{\beta^2}x_0}}{\sqrt{x_0^2-\beta^2y_0^2}}\cos\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)+\frac{ik|y_0|}{2}\int_{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)}^{\frac{1}{\beta^2|y_0|}\left(x_0+M\sqrt{x_0^2-\beta^2y_0^2}\right)}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}d\tau\right] \tag{16c}$$

REDUCTION OF EQUATION (16c) TO THE SONIC CASE

In order to reduce this result to the sonic case it is first necessary to discard terms arising from the advanced-type potentials employed in deriving the doublet potentials for supersonic flow. This may be accomplished by replacing $\cos\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)$ with $\frac{1}{2}e^{\frac{iMk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}}$. The limiting form of $K(x_0,y_0)$ as $M\to 1$ can then be written as

$$K(x_0,y_0)_{M=1}=\lim_{M\to 1}-\frac{1}{vl^2y_0^2}e^{-ikx_0}U(x_0-\beta|y_0|)\left[\frac{x_0}{\sqrt{x_0^2-\beta^2y_0^2}}e^{\frac{ik}{\beta^2}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)}+ik|y_0|\int_{\frac{1}{\beta^2|y_0|}\left(x_0-M\sqrt{x_0^2-\beta^2y_0^2}\right)}^{\frac{1}{\beta^2|y_0|}\left(x_0+M\sqrt{x_0^2-\beta^2y_0^2}\right)}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau\right] \tag{A13}$$

When the limit $M=1$ is approached from the supersonic side, the term M is conveniently replaced by

$$M=1+\frac{1}{2}\epsilon$$

where ϵ is infinitesimally small, so that

$$\beta^2=(M-1)(M+1)=\frac{1}{2}\epsilon\left(2+\frac{1}{2}\epsilon\right)\approx\epsilon \tag{A14}$$

With this approximation equation (A13) can be written as

$$K(x_0,y_0)_{M=1}=\lim_{\epsilon\to 0}-\frac{e^{-ikx_0}}{vl^2y_0^2}U(x_0-\sqrt{\epsilon}|y_0|)\left\{\frac{x_0}{\sqrt{x_0^2-\epsilon y_0^2}}e^{-\frac{ik}{\epsilon}\left[x_0-\left(1+\frac{\epsilon}{2}\right)\sqrt{x_0^2-\epsilon y_0^2}\right]}+ik|y_0|\int_{\frac{1}{\epsilon|y_0|}\left[x_0-\left(1+\frac{\epsilon}{2}\right)\sqrt{x_0^2-\epsilon y_0^2}\right]}^{\frac{1}{\epsilon|y_0|}\left[x_0+\left(1+\frac{\epsilon}{2}\right)\sqrt{x_0^2-\epsilon y_0^2}\right]}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau\right\}$$

$$=\lim_{\epsilon\to 0}-\frac{e^{-ikx_0}}{vl^2y_0^2}U(x_0)\left\{e^{-\frac{ik}{\epsilon}\left[x_0-\left(1+\frac{\epsilon}{2}\right)\left(x_0-\frac{\epsilon y_0^2}{2x_0}\right)\right]}+ik|y_0|\int_{\frac{1}{\epsilon|y_0|}\left[x_0-\left(1+\frac{\epsilon}{2}\right)\left(x_0-\frac{\epsilon y_0^2}{2x_0}\right)\right]}^{\frac{1}{\epsilon|y_0|}\left[x_0+\left(1+\frac{\epsilon}{2}\right)\left(x_0-\frac{\epsilon y_0^2}{2x_0}\right)\right]}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau\right\} \tag{A15}$$

or

$$K(x_0,y_0)_{M=1}=-\frac{e^{-ikx_0}}{vl^2y_0^2}U(x_0)\left\{e^{\frac{ik}{2}\left(x_0-\frac{y_0^2}{x_0}\right)}+ik|y_0|\int_{\frac{1}{2|y_0|}\left(x_0-\frac{y_0^2}{x_0}\right)}^{\infty}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau\right\} \tag{A16}$$

In order to show that this result is equivalent to that given for the sonic case in reference 1, it is convenient to first express the integral term as the difference of two integrals

$$ik|y_0|\int_{\frac{1}{2|y_0|}\left(x_0-\frac{y_0^2}{x_0}\right)}^{\infty}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau=ik|y_0|\int_0^{\infty}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau-ik|y_0|\int_0^{\frac{1}{2|y_0|}\left(x_0-\frac{y_0^2}{x_0}\right)}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau \tag{A17}$$

The first integral on the right of equation (A17) can be evaluated by comparison with the following results:

$$\begin{aligned}\int_0^{\infty}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ia\tau}d\tau&=\int_0^{\infty}\frac{\tau}{\sqrt{1+\tau^2}}\cos a\tau\,d\tau-i\int_0^{\infty}\frac{\tau}{\sqrt{1+\tau^2}}\sin a\tau\,d\tau\\&=\frac{\partial}{\partial a}\left(\int_0^{\infty}\frac{\sin a\tau}{\sqrt{1+\tau^2}}d\tau+i\int_0^{\infty}\frac{\cos a\tau}{\sqrt{1+\tau^2}}d\tau\right)\\&=\frac{\partial}{\partial a}\left\{\frac{\pi}{2}[I_0(a)-L_0(a)]+iK_0(a)\right\}\\&=\frac{\pi}{2}[I_1(a)-L_1(a)]-1-iK_1(a)\end{aligned} \tag{A18}$$

where I_0, I_1, K_0, K_1 are modified Bessel functions and L_0 and L_1 are modified Struve functions (see ref. 18, p. 172 and p. 332 for reductions of similar integrals). Thus

$$ik|y_0|\int_0^\infty \frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau=k|y_0|K_1(k|y_0|)+\frac{\pi ik|y_0|}{2}\left[I_1(k|y_0|)-L_1(k|y_0|)\right]-ik|y_0| \tag{A19}$$

In the second integral on the right of equation (A17) make the substitution

$$\tau=\frac{1}{2|y_0|}\left(\frac{y_0^2}{\lambda}-\lambda\right)$$

or

$$\lambda=|y_0|\left(\sqrt{1+\tau^2}-\tau\right)$$

This gives for the second integral

$$\begin{aligned}-ik|y_0|\int_0^{\frac{1}{2|y_0|}\left(x_0-\frac{y_0^2}{x_0}\right)}\frac{\tau}{\sqrt{1+\tau^2}}e^{-ik|y_0|\tau}\,d\tau&=ik\int_{|y_0|}^{x_0}\left(\frac{y_0^2+\lambda^2}{2\lambda^2}-1\right)e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2}{\lambda}\right)}d\lambda\\&=e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2}{\lambda}\right)}\Big|_{|y_0|}^{x_0}-ik\int_{|y_0|}^{x_0}e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2}{\lambda}\right)}d\lambda\\&=e^{\frac{ik}{2}\left(x_0-\frac{y_0^2}{x_0}\right)}-1-ik\int_{|y_0|}^{x_0}e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2}{\lambda}\right)}d\lambda\end{aligned} \tag{A20}$$

Substituting this result and that given in equation (A19) for the integral in equation (A16) gives

$$K(x_0,y_0)_{M=1}=-\frac{e^{-ikx_0}}{vl^2y_0^2}U(x_0)\left\{2e^{\frac{ik}{2}\left[x_0-\frac{y_0^2}{x_0}\right]}-1-ik|y_0|+k|y_0|K_1(k|y_0|)+\frac{\pi ik|y_0|}{2}\left[I_1(k|y_0|)-L_1(k|y_0|)\right]-ik\int_{|y_0|}^{x_0}e^{\frac{ik}{2}\left(\lambda-\frac{y_0^2}{\lambda}\right)}d\lambda\right\} \tag{A21}$$

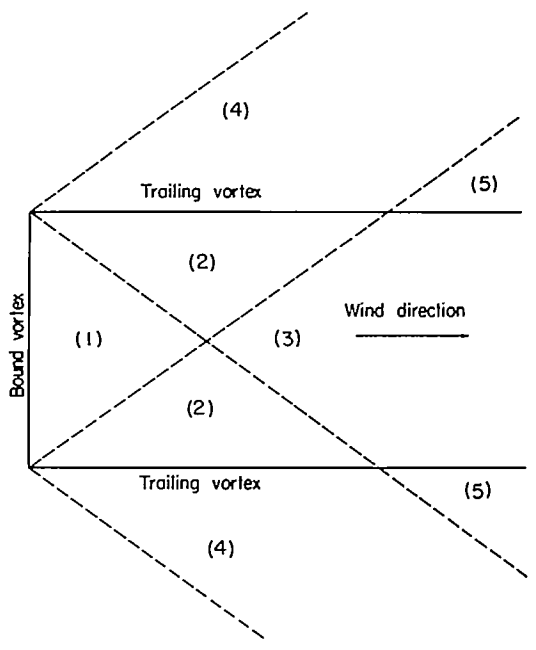
A comparison of this result with the result given in equation (47a) of reference 1 shows that the two equations are equivalent.

APPENDIX B

DERIVATION OF DOWNWASH FUNCTIONS ASSOCIATED WITH "HORSESHOE" VORTICES IN SUPERSONIC FLOW

The downwash associated with a vortex line can be obtained by an integration, between appropriate limits, of the kernel function $K(x_0,y_0)$ with respect to $\eta=y-y_0$. In order to perform such an integration analytically, recourse must be had to term-by-term integrations of an expanded form of $K(x_0,y_0)$. In this regard, use can be made of the expansions given in equations (36) and (38) of the analysis to obtain expanded forms of the downwash functions for vortex lines that have very good convergence properties, especially for the range of values of the parameter Mk/β^2 that would usually be of interest in applications. Expressions so obtained will be cumbersome and will require high-speed computing equipment to make them very useful.

In regard to "horseshoe" vortices in supersonic flow, there are five different significant regions in which a field point may be considered to be located (see sketch (d)).



Sketch (d)

Region (1) is between the Mach cones emanating from the end points of the bound-vortex line. The downwash at a point in this region is not affected by the trailing vortices but is created by the bound vortex alone. Therefore, the downwash is the same as would be produced by a bound vortex of infinite length and corresponds to the kernel function for two-dimensional flow discussed in the analysis. Region (2) is between the trailing-vortex lines and is within the Mach cone emanating from one end of the bound vortex but outside the Mach cone emanating from the other end. The downwash at a point in this region is created by the bound vortex and one of the trailing-vortex lines. The other trailing vortex has no effect on the downwash. Region (3) is between the trailing-vortex lines and is within the Mach cones emanating from both ends of the bound vortex. Downwash in this region is created by the bound vortex and both trailing-vortex lines. Region (4) is outside the trailing-vortex lines and is within the Mach cone emanating from one end of

the bound vortex. The downwash is created by the bound vortex and only one of the trailing-vortex lines. Region (5) is outside the trailing-vortex line but within the Mach cones emanating from both ends of the bound vortex. The downwash is created by the bound vortex and both trailing-vortex lines.

For any of the five regions discussed in the preceding paragraph, the integral corresponding to the downwash function may be formally written, with use of equation(16b), as

$$\int_{\eta_1}^{\eta_2} K(x_0,y_0)d\eta=-\frac{2}{Vl}\left[e^{-\frac{M^2kx_0}{\beta^2}}\fint_{\eta_1}^{\eta_2}\frac{\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}}{y_0^2}U(x_0-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)d\eta+\right.$$
$$\left.\frac{k}{M}e^{-ikx_0}\fint_{\eta_1}^{\eta_2}\frac{d\eta}{y_0^2}\int_{\beta|y_0|}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\beta^2y_0^2}\right)d\lambda\right]$$
$$=\frac{2\beta}{Vl}\left(e^{-\frac{iM^2kx_0}{\beta^2}}I_3+\frac{k}{M}e^{-ikx_0}I_4\right) \tag{B1}$$

where use of the substitution $\beta y_0=\zeta$ gives

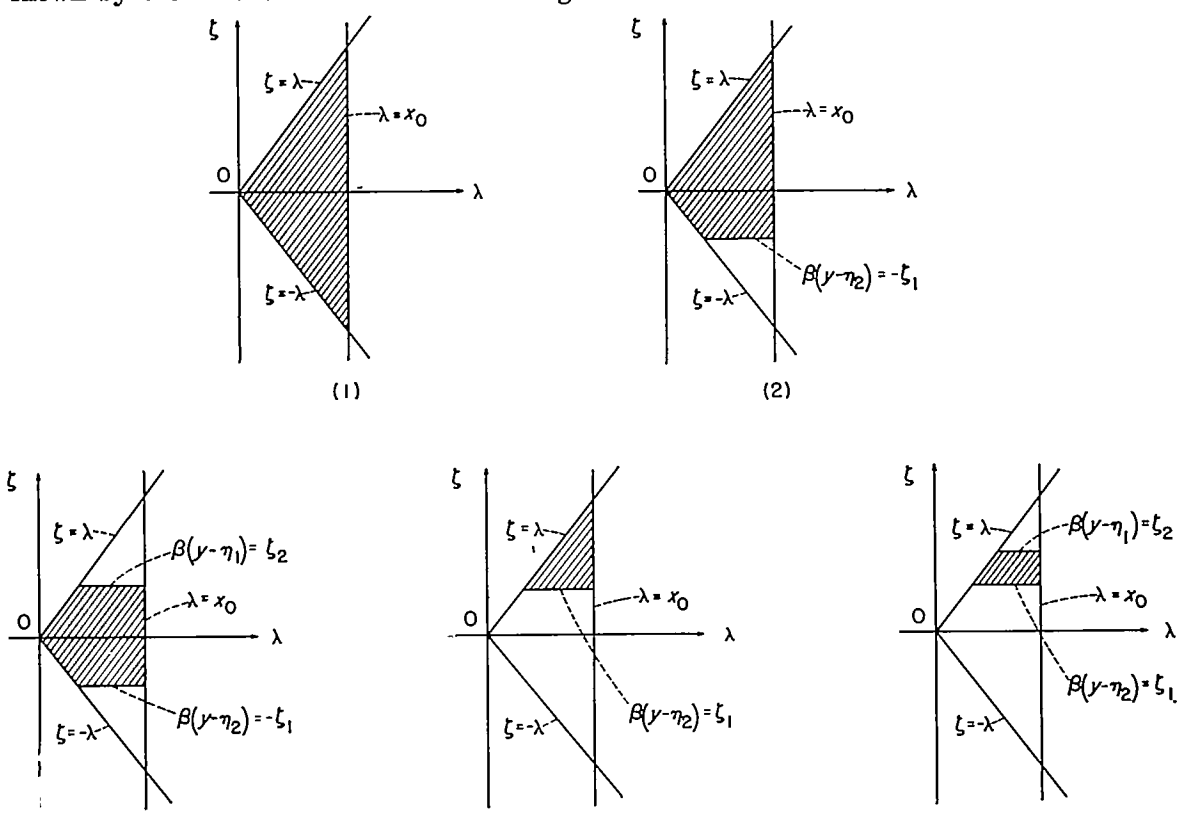
$$I_3=\frac{1}{\beta}\fint_{\eta_1}^{\eta_2}\frac{\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}}{y_0^2}U(x_0-\beta|y_0|)\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\beta^2y_0^2}\right)d\eta$$
$$=\int_{\beta(y-\eta_1)}^{\beta(y-\eta_2)}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial x_0}+\frac{i}{M}\right)\frac{U(x_0-\beta|y_0|)}{\zeta^2}\sin\left(\frac{Mk}{\beta^2}\sqrt{x_0^2-\zeta^2}\right)d\zeta \tag{B2}$$

and

$$I_4=\fint_{\beta(y-\eta_2)}^{\beta(y-\eta_1)}\frac{1}{\zeta^2}\left[\int_{|\zeta|}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}U(\lambda-|\zeta|)\sin\left(\frac{Mk}{\beta^2}\sqrt{\lambda^2-\zeta^2}\right)d\lambda\right]d\zeta \tag{B3}$$

In equations (B2) and (B3), a principal part—as described after equation (27) in the analysis—is to be taken when the integrations are carried across the line $\zeta=0$. The purpose now is to reduce these expressions to forms amenable to numerical evaluation. The first step in this procedure is a reduction of the expression for I_4 (eq. (B3)). The double integral in this expression can be considered as a surface integral in the $\lambda\zeta$-plane where the order of integration is first with respect to λ and then with respect to ζ. The steps in the reduction are first to delineate the area of integration for each of the five different cases under consideration, and then to change the order of integration in the surface-integral representation of I_4.

From the description of the different cases to be considered and by examination of the limits of integration in equations (B1), (B2), and (B3), the area of integration for the case of a field point in each of the aforementioned regions may be considered as shown by the hatched areas in the following sketches:



Sketch (e)

Expressions for I_4 for the five different regions or cases may then be expressed as simple integrals as follows:

Case (1):

$$I_4=\int_0^{x_0} e^{-\frac{ik\lambda}{\beta^2}} F(\lambda,-\lambda)d\lambda \tag{B4}$$

Case (2):

$$I_4=\int_0^{\zeta_1} e^{-\frac{ik\lambda}{\beta^2}} F(\lambda,-\lambda)d\lambda+\int_{\zeta_1}^{x_0} e^{-\frac{ik\lambda}{\beta^2}} F(\lambda,-\zeta_1)d\lambda \tag{B5}$$

Case (3):

$$I_4=\int_0^{\zeta_1} e^{-\frac{ik\lambda}{\beta^2}} F(\lambda,-\lambda)d\lambda+\int_{\zeta_1}^{\zeta_2} e^{-\frac{ik\lambda}{\beta^2}} F(\lambda,-\zeta_1)d\lambda+\int_{\zeta_2}^{x_0} e^{-\frac{ik\lambda}{\beta^2}} F(\zeta_2,-\zeta_1)d\lambda \tag{B6}$$

Case (4):

$$I_4=\int_{\zeta_1}^{x_0} e^{-\frac{ik\lambda}{\beta^2}} F(\lambda,\zeta_1)d\lambda \tag{B7}$$

Case (5):

$$I_4=\int_{\zeta_1}^{\zeta_2} e^{-\frac{ik\lambda}{\beta^2}} F(\lambda,\zeta_1)d\lambda+\int_{\zeta_2}^{x_0} e^{-\frac{ik\lambda}{\beta^2}} F(\zeta_2,\zeta_1)d\lambda \tag{B8}$$

The expression

$$F(\lambda,-\lambda)=\fint_{-\lambda}^{\lambda} \frac{U(\lambda-|\zeta|)}{\zeta^2} \sin\frac{Mk}{\beta^2}\sqrt{\lambda^2-\zeta^2}\,d\zeta \tag{B9}$$

is evaluated in the text in connection with the derivative of the two-dimensional kernel function and is found to reduce (see eq. (47)) to

$$F(\lambda,-\lambda)=\frac{\pi Mk}{\beta^2}U(\lambda)J_0\left(\frac{Mk}{\beta^2}\lambda\right) \tag{B10}$$

The F-function for other arguments can be obtained by substituting appropriate limits in an integration by parts of $F(\lambda,-\zeta_1)$, namely (see the development following eq. (41) in the analysis):

$$\begin{aligned}F(\lambda,-\zeta_1)&=\fint_{-\zeta_1}^{\lambda} \frac{U(\lambda-|\zeta|)}{\zeta^2} \sin\frac{Mk}{\beta^2}\sqrt{\lambda^2-\zeta^2}\,d\zeta\\
&=-\frac{1}{\zeta}U(\lambda-|\zeta|)\sin\frac{Mk}{\beta^2}\sqrt{\lambda^2-\zeta^2}\Big|_{-\zeta_1}^{\lambda}-\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)U(\lambda-|\zeta|)\sin^{-1}\frac{\zeta}{\lambda}\Big|_{-\zeta_1}^{\lambda}-\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)\int_{-\zeta_1}^{\lambda}\frac{\zeta}{|\zeta|}\delta(\lambda-|\zeta|)\sin^{-1}\frac{\zeta}{\lambda}\,d\zeta-\\
&\quad\frac{Mk}{\beta^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}J_{2n}\left(\frac{Mk\lambda}{\beta^2}\right)\left[U(\lambda-|\zeta|)\sin\left(2n\sin^{-1}\frac{\zeta}{\lambda}\right)\Big|_{-\zeta_1}^{\lambda}+\int_{-\zeta_1}^{\lambda}\frac{\zeta}{|\zeta|}\delta(\lambda-|\zeta|)\sin\left(2n\sin^{-1}\frac{\zeta}{\lambda}\right)d\zeta\right]\end{aligned} \tag{B11}$$

After the first term on the right-hand side of equation (B11) has been expanded by comparison with the expansion given in equation (38) of the analysis and the limits of integration have been substituted, this expression may be written as

$$\begin{aligned}F(\lambda,-\zeta_1)=\frac{\pi Mk}{\beta^2}U(\lambda)J_0\left(\frac{Mk\lambda}{\beta^2}\right)-U(\lambda-|\zeta_1|)\Bigg\{&\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)\sin^{-1}\frac{\zeta_1}{\lambda}+\sum_{n=1}^{\infty}\Bigg[\frac{2(-1)^{n-1}}{\zeta_1}J_{2n-1}\left(\frac{Mk\lambda}{\beta^2}\right)\cos\left[(2n-1)\sin^{-1}\frac{\zeta_1}{\lambda}\right]+\\
&\frac{Mk}{\beta^2}\frac{(-1)^n}{n}J_{2n}\left(\frac{Mk\lambda}{\beta^2}\right)\sin\left(2n\sin^{-1}\frac{\zeta_1}{\lambda}\right)\Bigg]\Bigg\}\end{aligned} \tag{B12}$$

Substitution of ζ_2 for λ in the limits of equation (B11) gives

$$\begin{aligned}F(\zeta_2,-\zeta_1)=-U(\lambda-|\zeta_2|)\Bigg\{&\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)\sin^{-1}\frac{\zeta_2}{\lambda}+\sum_{n=1}^{\infty}\Bigg[\frac{2(-1)^{n-1}}{\zeta_2}J_{2n-1}\left(\frac{Mk\lambda}{\beta^2}\right)\cos\left[(2n-1)\sin^{-1}\frac{\zeta_2}{\lambda}\right]+\\
&\frac{Mk}{\beta^2}\frac{(-1)^n}{n}J_{2n}\left(\frac{Mk\lambda}{\beta^2}\right)\sin\left(2n\sin^{-1}\frac{\zeta_2}{\lambda}\right)\Bigg]\Bigg\}-U(\lambda-|\zeta_1|)\Bigg\{\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)\sin^{-1}\frac{\zeta_1}{\lambda}+\\
&\sum_{n=1}^{\infty}\Bigg[\frac{2(-1)^{n-1}}{\zeta_1}J_{2n-1}\left(\frac{Mk\lambda}{\beta^2}\right)\cos\left[(2n-1)\sin^{-1}\frac{\zeta_1}{\lambda}\right]+\frac{Mk}{\beta^2}\frac{(-1)^n}{n}J_{2n}\left(\frac{Mk\lambda}{\beta^2}\right)\sin\left(2n\sin^{-1}\frac{\zeta_1}{\lambda}\right)\Bigg]\Bigg\}\end{aligned} \tag{B13}$$

If $-\zeta_1$ is replaced by ζ_1 in the limits of equation (B11), then

$$F(\lambda,\zeta_1)=\frac{\pi Mk}{\beta^2}U(\lambda)J_0\left(\frac{Mk\lambda}{\beta^2}\right)+$$
$$U(\lambda-|\zeta_1|)\left\{\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)\sin^{-1}\frac{\zeta_1}{\lambda}+\right.$$
$$\sum_{n=1}^{\infty}\left[\frac{2(-1)^{n-1}}{\zeta_1}J_{2n-1}\left(\frac{Mk\lambda}{\beta^2}\right)\cos\left[(2n-1)\sin^{-1}\frac{\zeta_1}{\lambda}\right]+\right.$$
$$\left.\left.\frac{Mk}{\beta^2}\frac{(-1)^n}{n}J_{2n}\left(\frac{Mk\lambda}{\beta^2}\right)\sin\left(2n\sin^{-1}\frac{\zeta_1}{\lambda}\right)\right]\right\} \tag{B14}$$

Substitution of ζ_2 for λ and ζ_1 for $-\zeta_1$ in equation (B11) gives

$$F(\zeta_2,\zeta_1)=-U(\lambda-|\zeta_2|)\left\{\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)\sin^{-1}\frac{\zeta_2}{\lambda}+\right.$$
$$\sum_{n=1}^{\infty}\left[\frac{2(-1)^{n-1}}{\zeta_2}J_{2n-1}\left(\frac{Mk\lambda}{\beta^2}\right)\cos\left[(2n-1)\sin^{-1}\frac{\zeta_2}{\lambda}\right]+\right.$$
$$\left.\left.\frac{Mk(-1)^n}{\beta^2 n}J_{2n}\left(\frac{Mk\lambda}{\beta^2}\right)\sin\left(2n\sin^{-1}\frac{\zeta_2}{\lambda}\right)\right]\right\}+$$
$$U(\lambda-|\zeta_1|)\left\{\frac{Mk}{\beta^2}J_0\left(\frac{Mk\lambda}{\beta^2}\right)\sin^{-1}\frac{\zeta_1}{\lambda}+\right.$$
$$\sum_{n=1}^{\infty}\left[\frac{2(-1)^{n-1}}{\zeta_2}J_{2n-1}\left(\frac{Mk\lambda}{\beta^2}\right)\cos\left[(2n-1)\sin^{-1}\frac{\zeta_2}{\lambda}\right]+\right.$$
$$\left.\left.\frac{Mk(-1)^n}{\beta^2 n}J_{2n}\left(\frac{Mk\lambda}{\beta^2}\right)\sin\left(2n\sin^{-1}\frac{\zeta_1}{\lambda}\right)\right]\right\} \tag{B15}$$

When equation (B10) and equations (B12) to (B15) are substituted into equations (B4) to (B8), respectively, they give the reduced forms of I_4 for the five cases under consideration.

After the reduction of I_4, the corresponding reduction of I_3 is considered. As may be found by examination of the expression for I_3 (eq. (B2)) and the sketches showing the areas of integration for the different cases, reductions of I_3 corresponding to those of I_4 can be obtained from the F-functions (eqs. (B10) to (B15)). Results for the different cases may be expressed as follows:

Case (1):

$$I_3=\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\lambda,-\lambda) \tag{B16}$$

Case (2):

$$I_3=\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\lambda,-\zeta_1) \tag{B17}$$

Case (3):

$$I_3=\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\zeta_2,-\zeta_1) \tag{B18}$$

Case (4):

$$I_3=\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\lambda,\zeta_1) \tag{B19}$$

Case (5):

$$I_3=\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\zeta_2,\zeta_1) \tag{B20}$$

When the expressions for I_3 (eqs. (B16) to (B20)) and I_4 (eqs. (B4) to (B8)) that are associated with each particular case are substituted into equation (B1), expressions for the downwash at each of the five significant field-point locations may be obtained in terms of the F-functions (eqs. (B10) to (B15)) as follows:

Case (1):

$$\int_{\eta_1}^{\eta_2}K(x_0,y_0)d\eta=-\frac{2\beta}{l}\left[e^{-\frac{iM^2kx_0}{\beta^2}}\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\lambda,-\lambda)+\right.$$
$$\left.\frac{k}{M}e^{-ikx_0}\int_0^{x_0}e^{-\frac{ik\lambda}{\beta^2}}F(\lambda,-\lambda)d\lambda\right] \tag{B21}$$

Case (2):

$$\int_{\eta_1}^{\eta_2}K(x_0,y_0)d\eta=-\frac{2\beta}{l}\left\{e^{-\frac{iM^2kx_0}{\beta^2}}\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\lambda,-\zeta_1)+\right.$$
$$\frac{k}{M}e^{-ikx_0}\left[\int_0^{\zeta_1}e^{-\frac{ik\lambda}{\beta^2}}F(\lambda,-\lambda)d\lambda+\right.$$
$$\left.\left.\int_{\zeta_1}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}F(\lambda,-\zeta_1)d\lambda\right]\right\} \tag{B22}$$

Case (3):

$$\int_{\eta_1}^{\eta_2}K(x_0,y_0)d\eta=-\frac{2\beta}{l}\left\{e^{-\frac{iM^2kx_0}{\beta^2}}\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\zeta_2,-\zeta_1)+\right.$$
$$\frac{k}{M}e^{-ikx_0}\left[\int_0^{\zeta_1}e^{-\frac{ik\lambda}{\beta^2}}F(\lambda,-\lambda)d\lambda+\right.$$
$$\int_{\zeta_1}^{\zeta_2}e^{-\frac{ik\lambda}{\beta^2}}F(\lambda,-\zeta_1)d\lambda+$$
$$\left.\left.\int_{\zeta_2}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}F(\zeta_2,-\zeta_1)d\lambda\right]\right\} \tag{B23}$$

Case (4):

$$\int_{\eta_1}^{\eta_2}K(x_0,y_0)d\eta=-\frac{2\beta}{l}\left\{e^{-\frac{iM^2kx_0}{\beta^2}}\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\lambda,\zeta_1)+\right.$$
$$\left.\frac{k}{M}e^{-ikx_0}\int_{\zeta_1}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}F(\lambda,\zeta_1)d\lambda\right\} \tag{B24}$$

Case (5):

$$\int_{\eta_1}^{\eta_2}K(x_0,y_0)d\eta=-\frac{2\beta}{l}\left\{e^{-\frac{iM^2kx_0}{\beta^2}}\lim_{\lambda\to x_0}\left(\frac{\beta^2}{Mk}\frac{\partial}{\partial\lambda}+\frac{i}{M}\right)F(\zeta_2,\zeta_1)+\right.$$
$$\frac{k}{M}e^{-ikx_0}\left[\int_{\zeta_1}^{\zeta_2}e^{-\frac{ik\lambda}{\beta^2}}F(\lambda,\zeta_1)d\lambda+\right.$$
$$\left.\left.\int_{\zeta_2}^{x_0}e^{-\frac{ik\lambda}{\beta^2}}F(\zeta_2,\zeta_1)d\lambda\right]\right\} \tag{B25}$$

The results for case (1) (eq. (B21)) agree with results obtained for the two-dimensional kernel function given in equation (49) of the analysis.

REFERENCES

1. Watkins, Charles E., Runyan, Harry L., and Woolston, Donald S.: On the Kernel Function of the Integral Equation Relating the Lift and Downwash Distributions of Oscillating Finite Wings in Subsonic Flow. NACA Rep. 1234, 1955. (Supersedes NACA TN 3131.)
2. Falkner, V. M.: The Calculation of Aerodynamic Loading on Surfaces of Any Shape. R. & M. No. 1910, British A.R.C., Aug. 1943.
3. Multhopp, H.: Die Berechnung der Auftriebsverteilung von Tragflügeln. Luftfahrtforschung, Bd. 15, Lfg. 4, Apr. 6, 1938, pp. 153–169. (Available as R.T.P. Translation No. 2392, British M.A.P.)
4. Possio, Camillo: Aerodynamic Forces on an Oscillating Profile in a Compressible Fluid at Sub-Sonic Speed. Air Ministry Translation No. 830, British A.R.C., Nov. 24, 1938. (From Volume Commemorative del XXV Annuale del Laboratorio de Aeronautica del R. Politecnico di Torino. Turin 14–17, Oct. 1937, pp. 152–169.)
5. Schade, [Th.]: Numerische Lösung der Possioschen Integralgleichung der schwingenden Tragfläche in ebener Unterschallströmung. I.—Analylischer Teil. UM Nr. 3209, Deutsche Luftfahrtforschung (Berlin-Adlershof), 1944.
6. Jones, W. P.: The Calculation of Aerodynamic Derivative Coefficients for Wings of Any Plan Form in Non-Uniform Motion. R. & M. No. 2470, British A.R.C., 1946.
7. Watkins, Charles E.: Effect of Aspect Ratio on the Air Forces and Moments of Harmonically Oscillating Thin Rectangular Wings in Supersonic Potential Flow. NACA Rep. 1028, 1951.
8. Nelson, Herbert C., Rainey, Ruby A., and Watkins, Charles E.: Lift and Moment Coefficients Expanded to the Seventh Power of Frequency for Oscillating Rectangular Wings in Supersonic Flow and Applied to a Specific Flutter Problem. NACA TN 3076, 1954.
9. Watkins, Charles E., and Berman, Julian H.: Air Forces and Moments on Triangular and Related Wings With Subsonic Leading Edges Oscillating in Supersonic Potential Flow. NACA Rep. 1099, 1952.
10. Nelson, Herbert C.: Lift and Moment on Oscillating Triangular and Related Wings With Supersonic Edges. NACA TN 2494, 1951.
11. Stewartson, K.: On Linearized Potential Theory of Unsteady Supersonic Motion. Quarterly Jour. Mech. and Appl. Math., vol. III, pt. 2, June 1950, pp. 182–199.
12. Miles, John W.: The Oscillating Rectangular Airfoil at Supersonic Speeds. NAVORD Rep. 1170, NOTS 226, U. S. Naval Ordnance Test Station (Inyokern, Calif.), July 21, 1949.
13. Schlichting, H.: Airfoil Theory at Supersonic Speed. NACA TM 897, 1939.
14. Küssner, H. G.: General Airfoil Theory. NACA TM 979, 1941.
15. Garrick, I. E., and Rubinow, S. I.: Theoretical Study of Air Forces on an Oscillating or Steady Thin Wing in a Supersonic Main Stream. NACA Rep. 872, 1947. (Supersedes NACA TN 1383.)
16. Van der Pol, Balth., and Bremmer, H.: Operational Calculus Based on the Two-Sided Laplace Integral. Cambridge Univ. Press, 1950.
17. Magnus, Wilhelm, and Oberhettinger, Fritz: Formulas and Theorems for the Special Functions of Mathematical Physics. Chelsea Pub. Co. (New York), 1949.
18. Watson, G. N.: A Treatise on the Theory of Bessel Functions. Second ed., The Macmillan Co., 1944.
19. Schwarz, L.: Untersuchung einiger mit den Zylinderfunktionen nullter Ordnung verwandter Funktionen. Luftfahrtforschung, Bd. 20, Lfg. 12, Feb. 8, 1944, pp. 341–372.