NACA TN 3897

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 3897

INCOMPLETE TIME RESPONSE TO A UNIT IMPULSE AND ITS APPLICATION TO LIGHTLY DAMPED LINEAR SYSTEMS

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Washington
December 1956

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SUMMARY

A study is made of the use of the incomplete time response to a unit impulse as a device to analyze lightly damped linear systems. It was found that the incomplete time response to a unit impulse computed from an incomplete time response and complete input adequately describes a lightly damped linear system without requiring an estimated end correction. The incomplete response to a unit impulse may be used with Duhamel's integral to predict the response of the system to an arbitrary input up to at least 90 percent of the cutoff time of the incomplete response.

INTRODUCTION

One result frequently desired from the analysis of flight time-history data is the frequency response of the aircraft. Some of the methods available for accurately obtaining this result are given in references 1 and 2. As shown in references 2 and 3, the information contained in the frequency response can be converted to the time plane in the form of the complete time response to a unit impulse without requiring previous knowledge of the transfer function relating the input and output. This information is valuable in either form (time or frequency) inasmuch as it completely defines the system and thereby permits predicting the system response to any arbitrary input.

In some cases, however, an accurate frequency response is difficult to obtain. Such a case is often experienced when aircraft motions are so lightly damped that it is impractical to continue the recording process until the motion has subsided. The analysis of such systems by Fourier methods requires an estimated end correction to the incomplete-time-response data in order to obtain the system frequency response. Inaccuracies in the estimation of this end correction may introduce significant errors in the computed frequency response and, hence, in the computed response to an arbitrary input.

For a lightly damped higher order linear system where it is impossible to obtain an accurate frequency response because of limited time data, methods of analysis are needed to define the system by using input and output data. It would be desired that such a method use the data as recorded and bypass the need for estimating end corrections.

The primary purpose of this report is to describe one such analysis technique which is called the incomplete time response to a unit impulse, "incomplete" signifying that the response to the unit impulse is obtained from an incomplete time response.

SYMBOLS

$H(i\omega)$	Fourier transform of $h(t)$
$h(t)$	time response to unit impulse
$h(t)_\lambda$	incomplete time response to unit impulse cut off at λ seconds
$h_\beta(t)$	time response to unit impulse in sideslip
R	real component of Fourier transform
t	time, sec
x	output quantity
$x(i\omega)$	Fourier transform, $\int_0^\infty x(t)e^{-i\omega t}dt$
$x(t)$	response of system to arbitrary input
β	sideslip angle, deg
$\beta(t)$	sideslip angle as function of time
δ	input quantity
$\delta(i\omega)$	Fourier transform of input

$\delta(t)$	time history of input to linear system
δ_r	rudder angle, deg
$\delta_r(t)$	time history of rudder input
λ	cutoff time of time history, sec
ϕ	phase angle, deg
$\phi_{x\delta}$	phase angle between output x and input δ of frequency response; negative phase angles indicate lag
$\phi_{\beta\delta_r}$	phase angle between output β and input δ_r
ω	circular frequency, radians/sec
$\left\lvert\frac{x}{\delta}\right\rvert$, $\left\lvert\frac{\beta}{\delta_r}\right\rvert$	amplitude ratios

A dot over a symbol denotes the derivative with respect to time. The absolute value of a term is denoted by $\lvert\ \rvert$.

METHOD AND RESULTS

In order to demonstrate the range of usefulness of the incomplete time response to a unit impulse and to indicate the importance of end corrections, Fourier methods are first applied to a lightly damped mathematical system to show the effect on the results of cutting off the time histories at various points. The computed ratios of the Fourier transforms of output and input are compared as well as the time responses to a unit impulse. The same methods are then applied to a lightly damped response in sideslip as measured in flight, and the incomplete time response to a unit impulse is obtained. This incomplete time response is then used with the actual input to compute a time response which is compared with the measured aircraft time response.

The Fourier transform, as used in this paper, is given as follows:

$$x(i\omega) = \int_0^{\lambda} x(t)e^{-i\omega t}dt + \int_{\lambda}^{\infty} x(t)e^{-i\omega t}dt$$

where the integral with the limits 0 to λ is the incomplete Fourier transform and the integral with the limits λ to ∞ is the Fourier transform of the end correction.

Lightly Damped Mathematical System

Figure 1 shows the exact calculated response of a lightly damped second-order system to a triangular input. This mathematical system is defined by the following equation:

$$\ddot{x}(t) + 0.2\dot{x}(t) + 1.86x(t) = 0.75\delta(t)$$

with the initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$. The exact frequency response obtained analytically for this system is given in figure 2 by the solid line which corresponds to a cutoff time $\lambda_1 = \infty$. This response hereinafter is called case 1. Two other computations representing the Fourier transforms $\frac{x(i\omega)}{\delta(i\omega)}$ are carried out by using the method of reference 2 which is applied to a complete time history of the input and an incomplete time history of the output of figure 1. In one case, the output was cut off at $\lambda_2 = 13.9$ seconds and in the other, at $\lambda_3 = 4.72$ seconds; these cases henceforth are referred to as case 2 and case 3, respectively. In figure 2(a) the amplitude ratio resulting from the ratio of the transforms $\frac{x(i\omega)}{\delta(i\omega)}$ is shown plotted against frequency, and in figure 2(b) the phase angle resulting from the ratio of the transforms is shown. The difference between cases 2 and 3 (the response cut off at $\lambda_2 = 13.9$ seconds and $\lambda_3 = 4.72$ seconds, respectively) and case 1 (the complete system $\lambda_1 = \infty$) indicates the contribution of the Fourier transform of the end corrections of cases 2 and 3 to the complete-system frequency response.

In order to compute the time response to a unit impulse by the method of reference 2, the real part of the Fourier transform $R\left[\frac{x}{\delta}(i\omega)\right]$ must be computed by multiplying the amplitude ratio shown in figure 2(a) by the cosine of the phase angle given in figure 2(b). The result of this computation is shown in figure 3 for all three cases. A staircase function using a frequency interval $\Delta\omega = 0.10$ radian/sec was fitted to $R\left[\frac{x}{\delta}(i\omega)\right]$ for cases 2 and 3, and table I of reference 2 was used to compute the response to a unit impulse.

The resulting incomplete time responses to a unit impulse are shown in figure 4 along with the exact time response for the complete system. Within the cutoff interval the computations could have been made more exact by taking a smaller interval $\Delta\omega$ in fitting the staircase function to $R\left[\frac{x}{\delta}(i\omega)\right]$. The largest error in $h(t)$ occurs at $t = 0$ because $R\left[\frac{x}{\delta}(i\omega)\right]$ does not become zero in the number of intervals used to fit the staircase function to it. Inasmuch as $h(0)$ is equal to the area under $R\left[\frac{x}{\delta}(i\omega)\right]$, the value of $h(0)$ can be made as small as desired by integrating farther out along the scale. The computations near $t = 0$ usually require the most time and effort.

Lightly Damped Airplane Response in Sideslip Angle

In order to demonstrate the use of the incomplete time response to a unit impulse on flight-test data, a typical pulse type of input and response in sideslip of a large modern flexible swept-wing bomber is analyzed. Time histories of the sideslip-angle response $\beta(t)$ and rudder input $\delta_r(t)$ are shown in figure 5. The incomplete response $\beta(t)$ (flight record cutoff at $\lambda = 5.4$ seconds) and the complete time history of the input $\delta_r(t)$ are used. In figure 6, the Fourier transform ratio of the incomplete response $\beta(t)$ and input $\delta_r(t)$ of the maneuver are shown.

In order to compute the incomplete time response to a unit impulse, it was first necessary to compute $R\left[\frac{\beta}{\delta_r}(i\omega)\right]$ (shown in fig. 7) from the data of figure 6. A staircase function was fitted to the curve of $R\left[\frac{\beta}{\delta_r}(i\omega)\right]$ of figure 7 by using a frequency interval $\Delta\omega = 0.25$ radian/sec. This staircase function was then used with table I of reference 2 to compute an incomplete time response to a unit impulse shown in figure 8.

In order to check the accuracy of this incomplete time response to the unit impulse, the data of figure 8 were used with the original rudder pulse input $\delta_r(t)$, shown in figure 5 as the forcing function, to compute a time history of the sideslip angle $\beta(t)$. The Duhamel method outlined in appendix A of reference 3 with a time interval $\Delta t = 0.04$ second was used for these calculations. A comparison of this computed $\beta(t)$ response with the measured $\beta(t)$ response from the flight test is shown in the lower part of figure 5 by the circular symbols. This comparison indicates the magnitude of error involved in the total computation procedure

(transferring the data in time-history form to the response to a unit impulse and finally to the time response to an arbitrary input by means of Duhamel's integral.)

The Duhamel process was also carried out in the frequency plane. This computation was made by using tables I and III of reference 2. The process consisted of forming the Fourier transform of the arbitrary input and multiplying it by the ratio of the Fourier transform of the incomplete response $\beta(t)$ to the complete input $\delta_r(t)$, shown in figure 6, to give the Fourier transform of the desired response. The inverse of this Fourier transform then yields the desired time history of the response up to 90 percent of the cutoff time. The time response thus obtained is shown in figure 5 by the square symbols.

DISCUSSION

In order to discuss the results presented, the manner of estimating end corrections is first considered. The usual procedure is to know the mathematical relation describing the free oscillation of the system and to determine the parameters which fit into this relation from the time history of the response. For example, in figure 1 the relation describing the end correction is of the form $Ce^{-at}\sin(\omega t + \phi)$, where a is the damping parameter and C is a constant. These parameters can readily be determined by means of the logarithmic decrement, period, and phase of the time history of the response. Thus, for this simple case, an accurate frequency response can be obtained. It is important to note, however, in figure 2 that the contribution of the end correction to the frequency response can be considerable. Another interesting property shown in figure 2 is that the cutoff time λ appears in the frequency plane as the period $\frac{2\pi}{\lambda}$ of an oscillation in both the amplitude-ratio and the phase-angle plots when an end correction is not applied. For example, in case 2 the period would be $\frac{2\pi}{13.9} = 0.452$ radian/sec. If the application of an erroneous end correction permits such oscillations to persist, they could be misinterpreted as structural modes in the frequency-response analysis of data obtained from a flexible airplane.

If no end correction is added and the incomplete time response to the unit impulse is computed, the incomplete time response is the same as the complete-system time response to the unit impulse up to 90 percent of the cutoff time λ. (See fig. 4.) This condition means that the incomplete time response to the unit impulse is capable of completely defining the system from zero time up to λ without the need of an end correction.

For many actual cases, such as the one shown in figure 5, the estimation of a valid end correction is difficult, even though its mathematical form may be derived from the equations of motion. The difficulty arises from attempting to determine the parameters which fit into this relation (for example, determining the damping from $\beta(t)$ by using the logarithmic-decrement technique for both the high- and the low-frequency modes). For such a system, the incomplete time response to the unit impulse shown in figure 8 has several advantages. It utilizes the data available without need of an end correction and accurately defines the system in the time plane from zero time up to within 90 percent of the cutoff time. The incomplete time response to the unit impulse can be used in the Duhamel integral to compute the response to any arbitrary input up to the cutoff point.

It has been assumed that the response to the unit impulse exists for the system in question; this assumption is usually the case. A necessary though not sufficient condition for the existence of the inverse Fourier transforms is that

$$\lim_{\omega \to \infty} R\left[H(i\omega)\right] = 0$$

In order to perform some of the computations shown in this paper it was necessary to extend the tables of reference 2.

CONCLUDING REMARKS

In the present investigation it was found that the incomplete time response of a lightly damped dynamic system to a unit impulse could be accurately determined up to at least 90 percent of the cutoff time of the response from a knowledge of the time history of the complete input together with the incomplete time history of the response. In addition, this incomplete time response to a unit impulse may be used to predict, by means of Duhamel's integral, the incomplete time response of the system to an arbitrary input up to at least 90 percent of the cutoff time. Alternatively, the Duhamel process may also be carried out in the frequency plane by using the arbitrary input and incomplete frequency response.

The advantage of the incomplete time response to a unit impulse is that this method utilizes the data available and does not require an estimated end correction in order to describe adequately high-order lightly damped linear systems. Erroneous estimates of the end correction

made in obtaining the complete frequency response of a lightly damped linear system may cause a false impression of structural modes to appear in the frequency response of the system.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., September 17, 1956.

REFERENCES

1. Eggleston, John M., and Mathews, Charles W.: Application of Several Methods for Determining Transfer Functions and Frequency Response of Aircraft From Flight Data. NACA Rep. 1204, 1954. (Supersedes NACA TN 2997.)

2. Huss, Carl R., and Donegan, James J.: Method and Tables for Determining the Time Response to a Unit Impulse From Frequency-Response Data and for Determining the Fourier Transform of a Function of Time. NACA TN 3598, 1956.

3. Donegan, James J., and Huss, Carl R.: Comparison of Several Methods for Obtaining the Time Response of Linear Systems to Either a Unit Impulse or Arbitrary Input From Frequency-Response Data. NACA TN 3701, 1956.

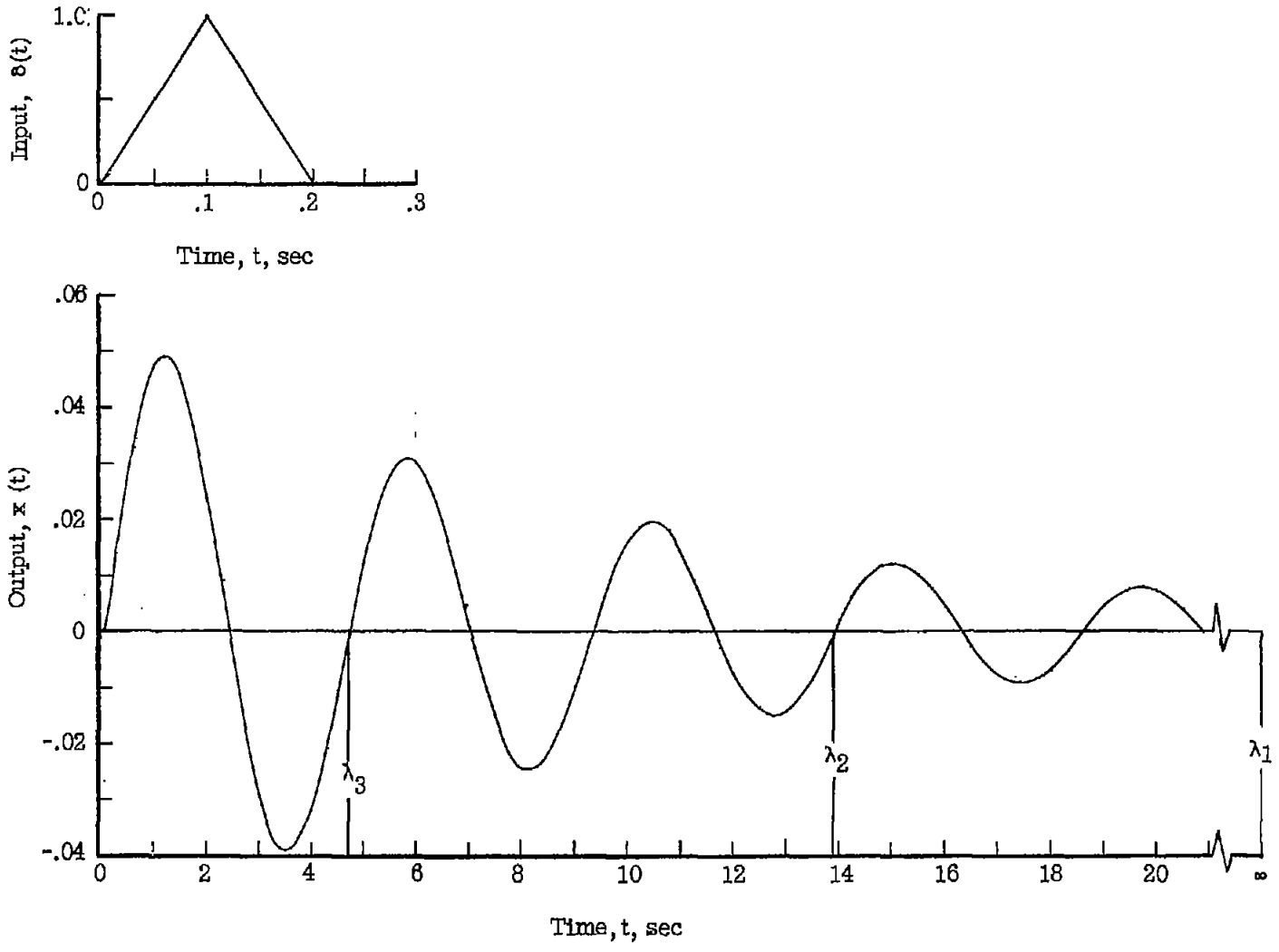
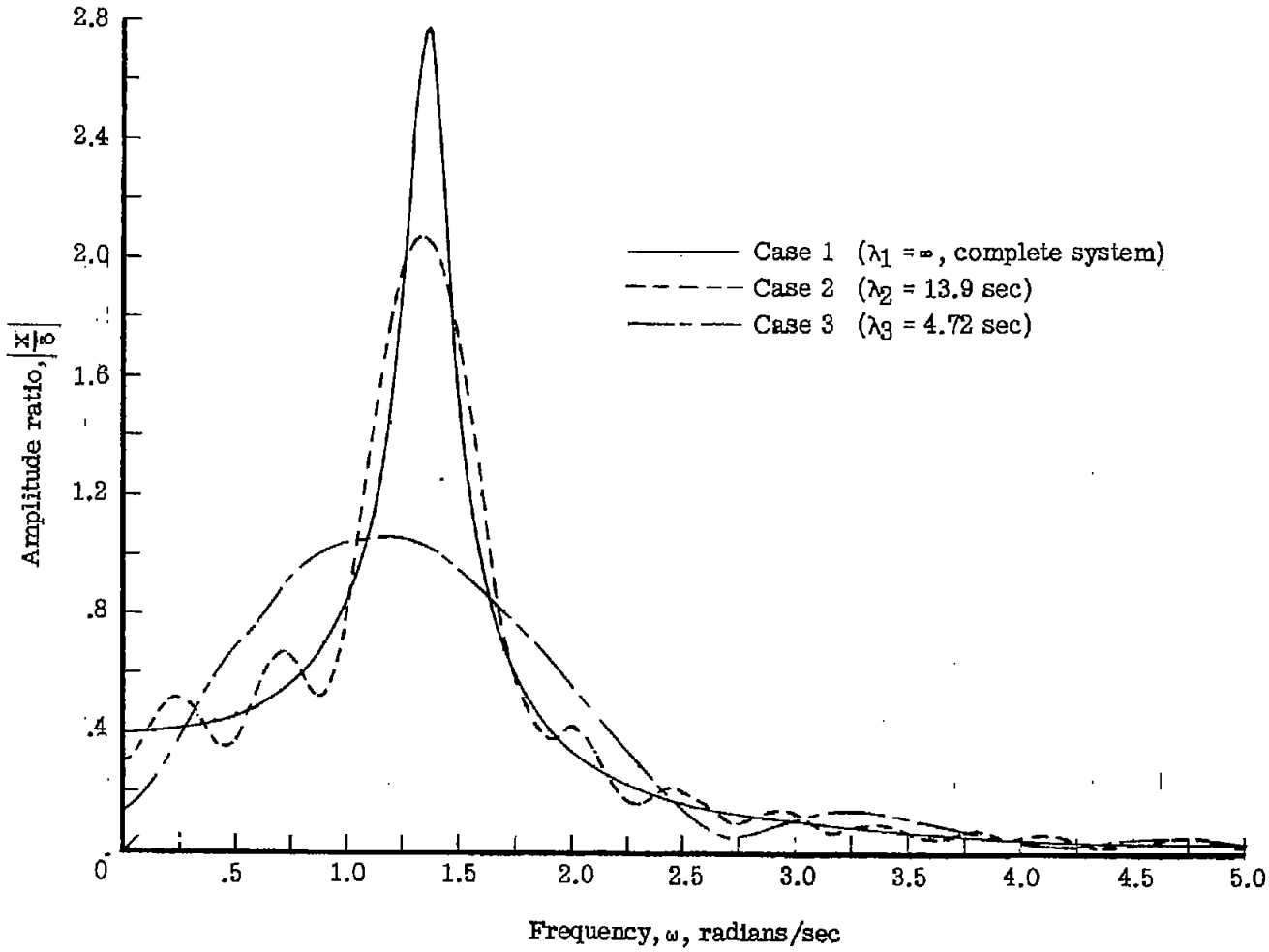
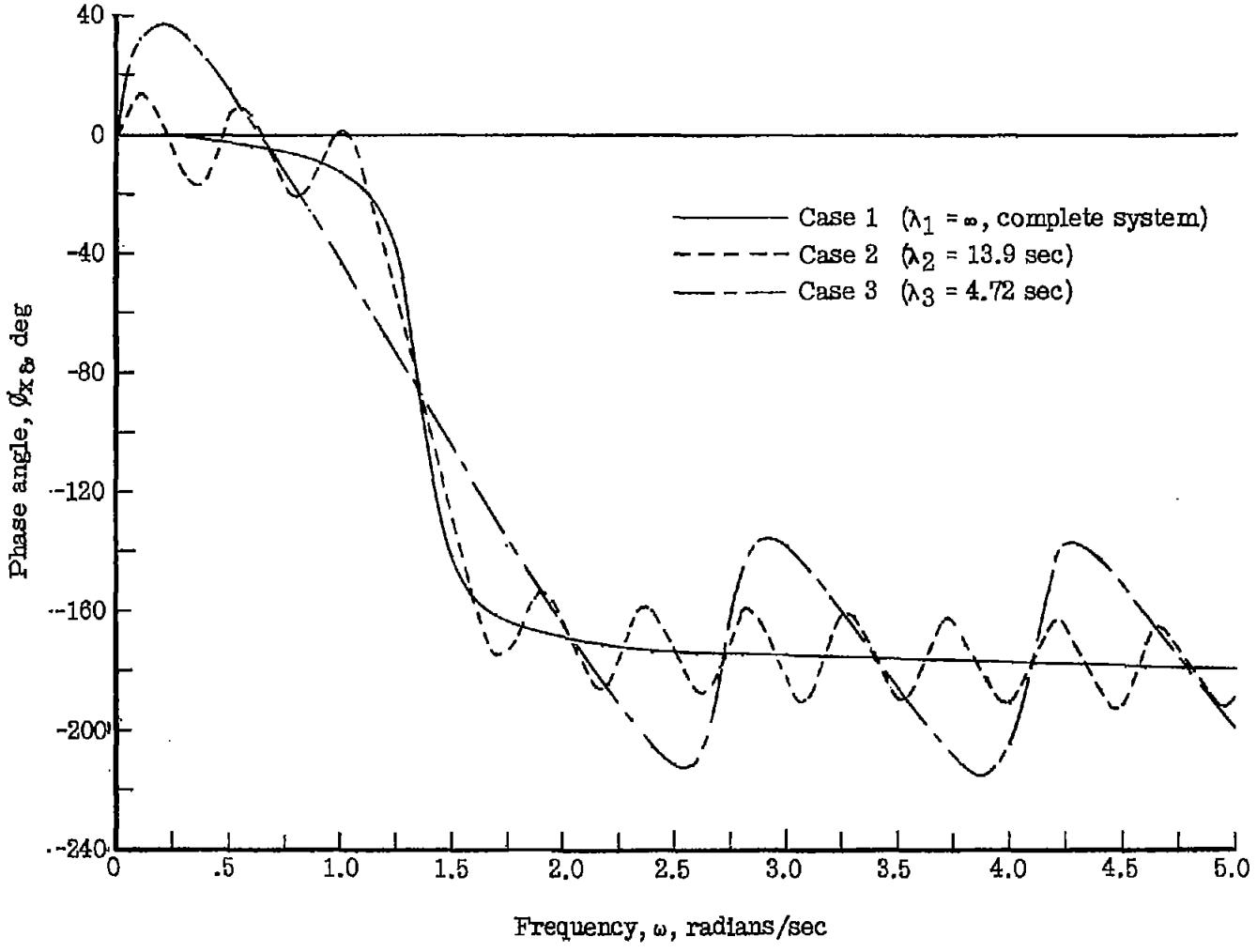


Figure 1.- Time response of a lightly damped simple second-order system to a triangular input.



(a) Amplitude ratio.

Figure 2.- Fourier transform ratio $\frac{x}{\delta}$ of the system for three cutoff points.



(b) Phase angle.

Figure 2.- Concluded.

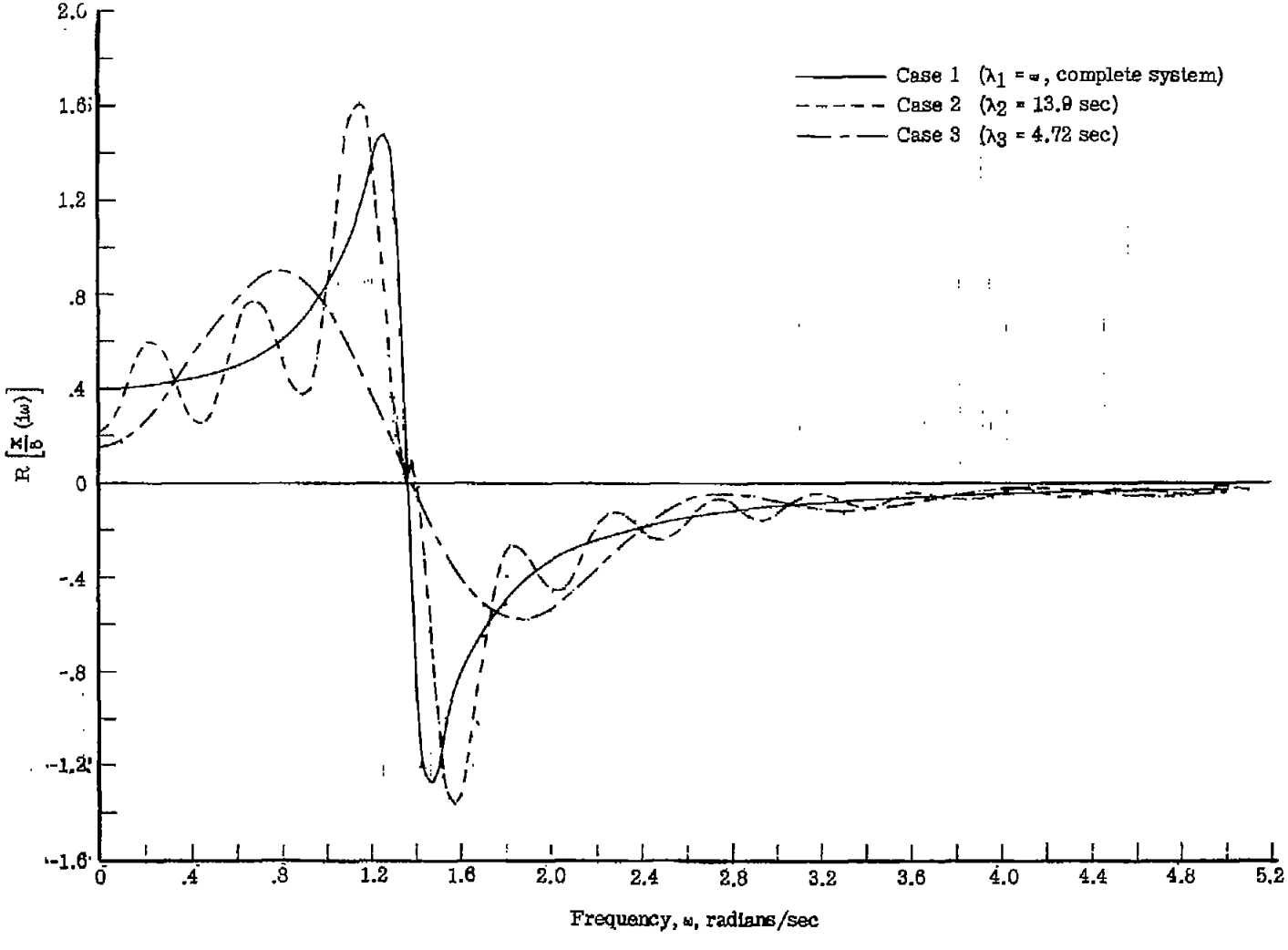


Figure 3.- Real part of the Fourier transform ratio $\frac{X}{\delta}$ of the system for three cutoff points.

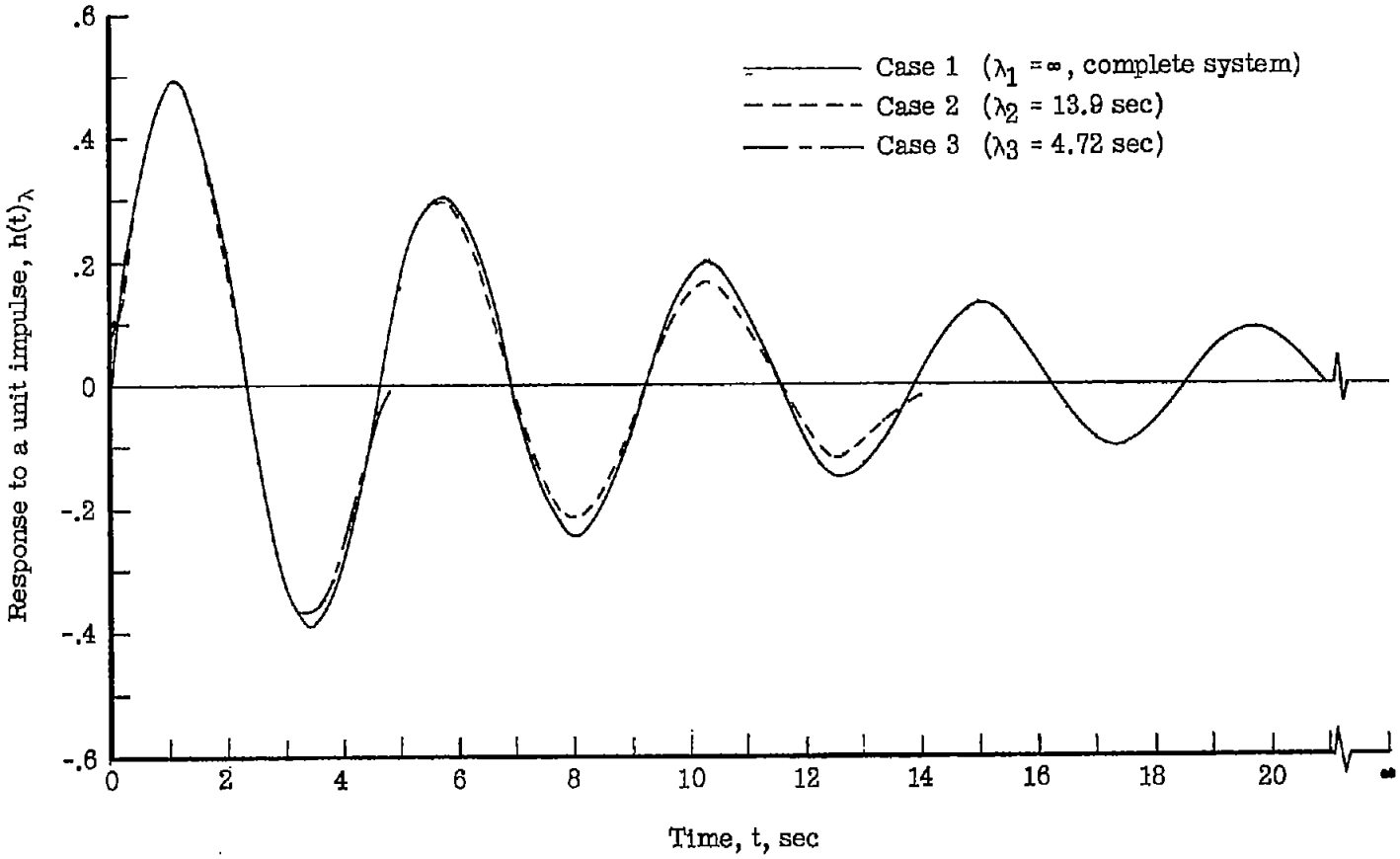


Figure 4.- Time response to a unit impulse of the system for three cutoff points.

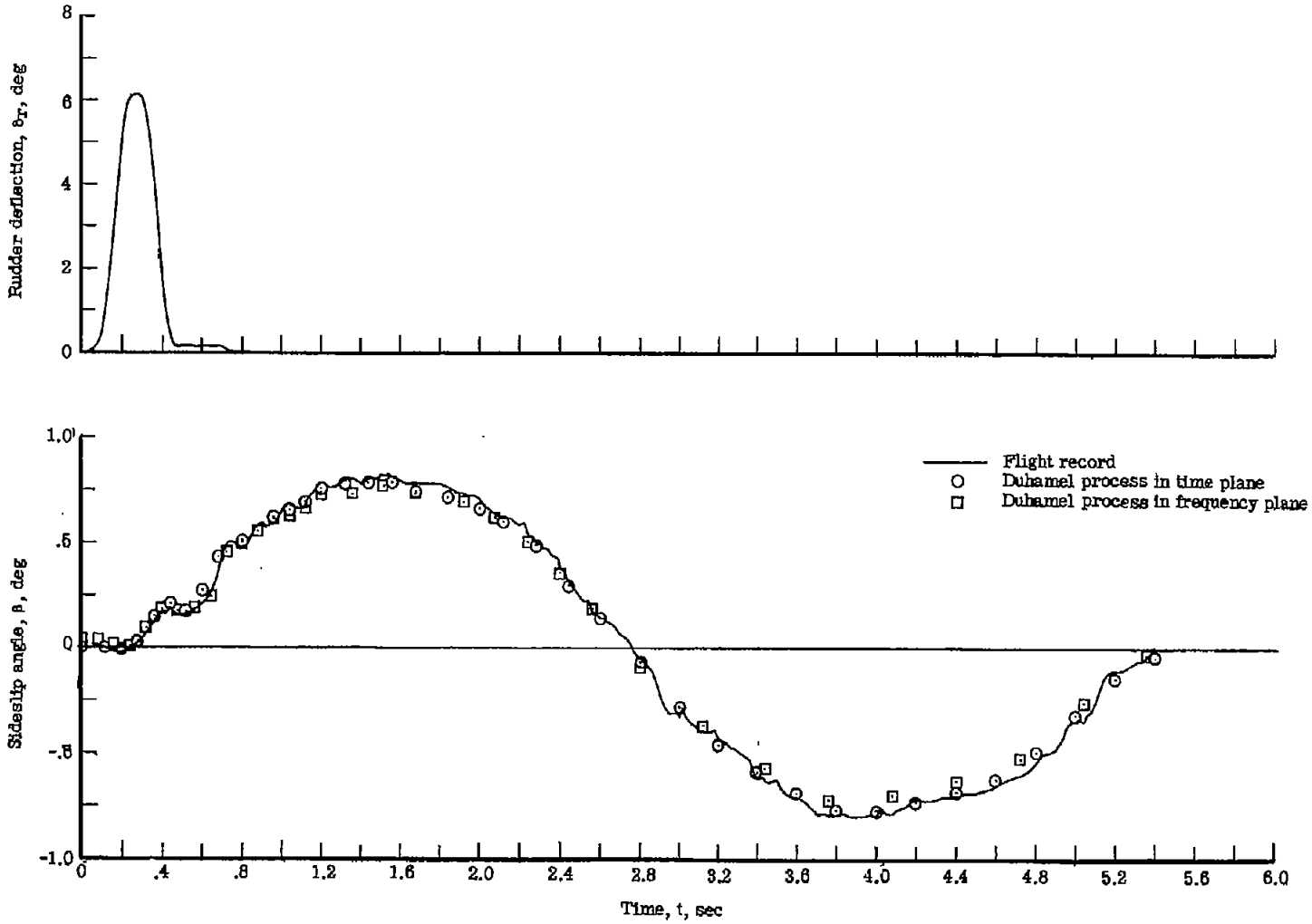


Figure 5.- Typical incomplete time history of a response in sideslip angle due to rudder deflection for a modern flexible swept-wing airplane.

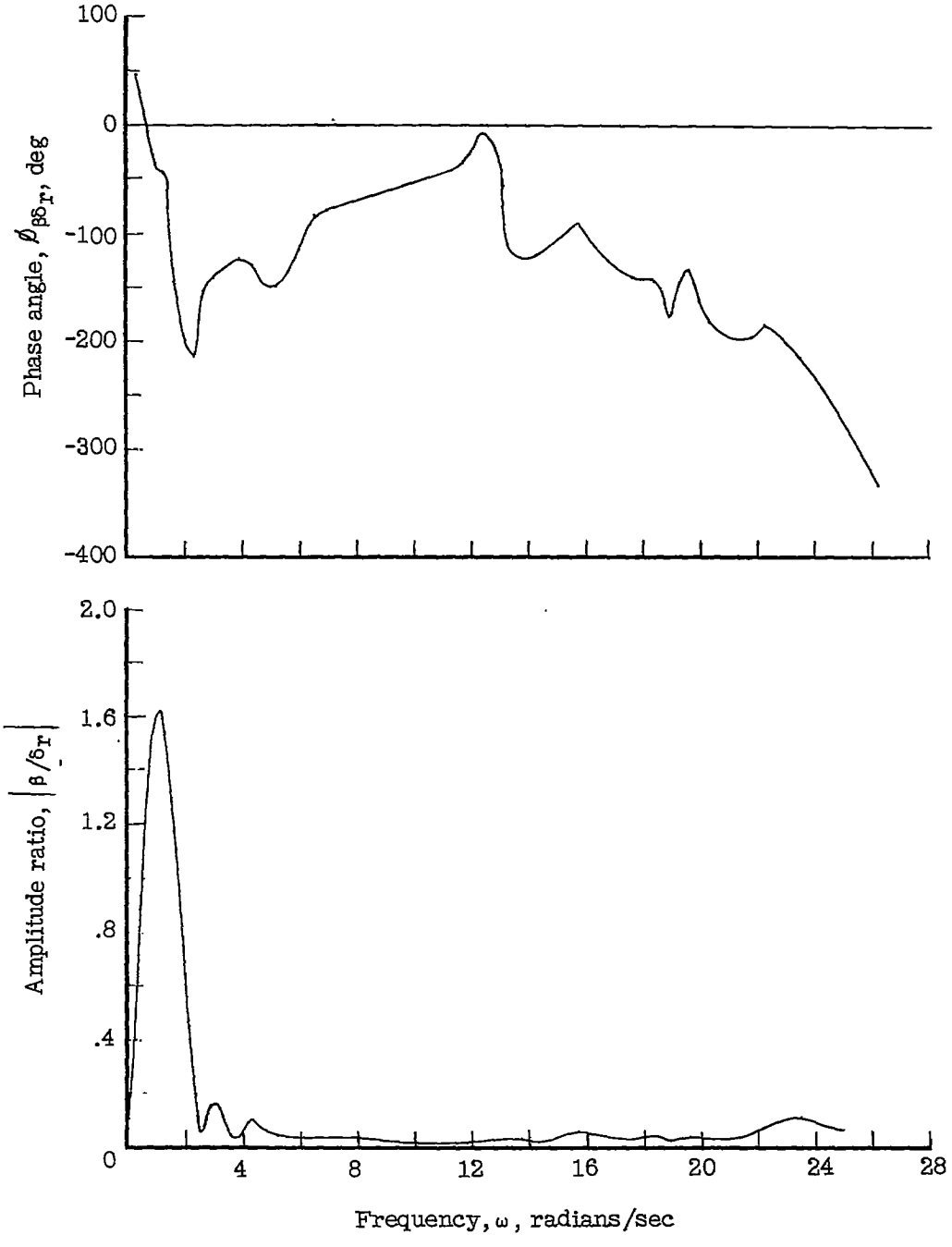


Figure 6.- Fourier transform ratio of incomplete response $\beta(t)$ (cut off at $\lambda = 5.4$ sec) to the complete input $\delta_r(t)$.

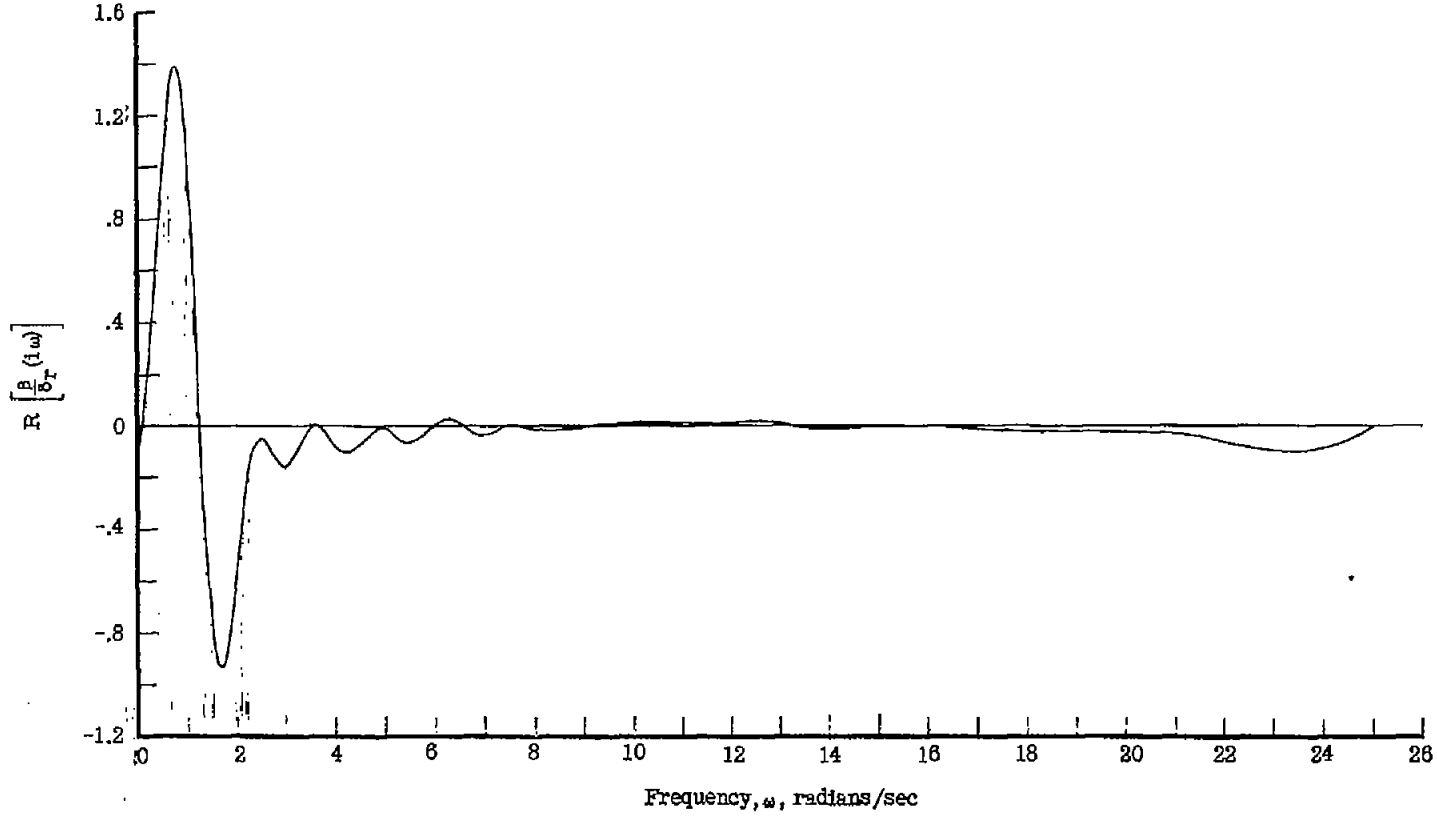


Figure 7.- Real part of the Fourier transform ratio β/δ_r for $\beta(t)$ cut off at $\lambda = 5.4$ seconds.

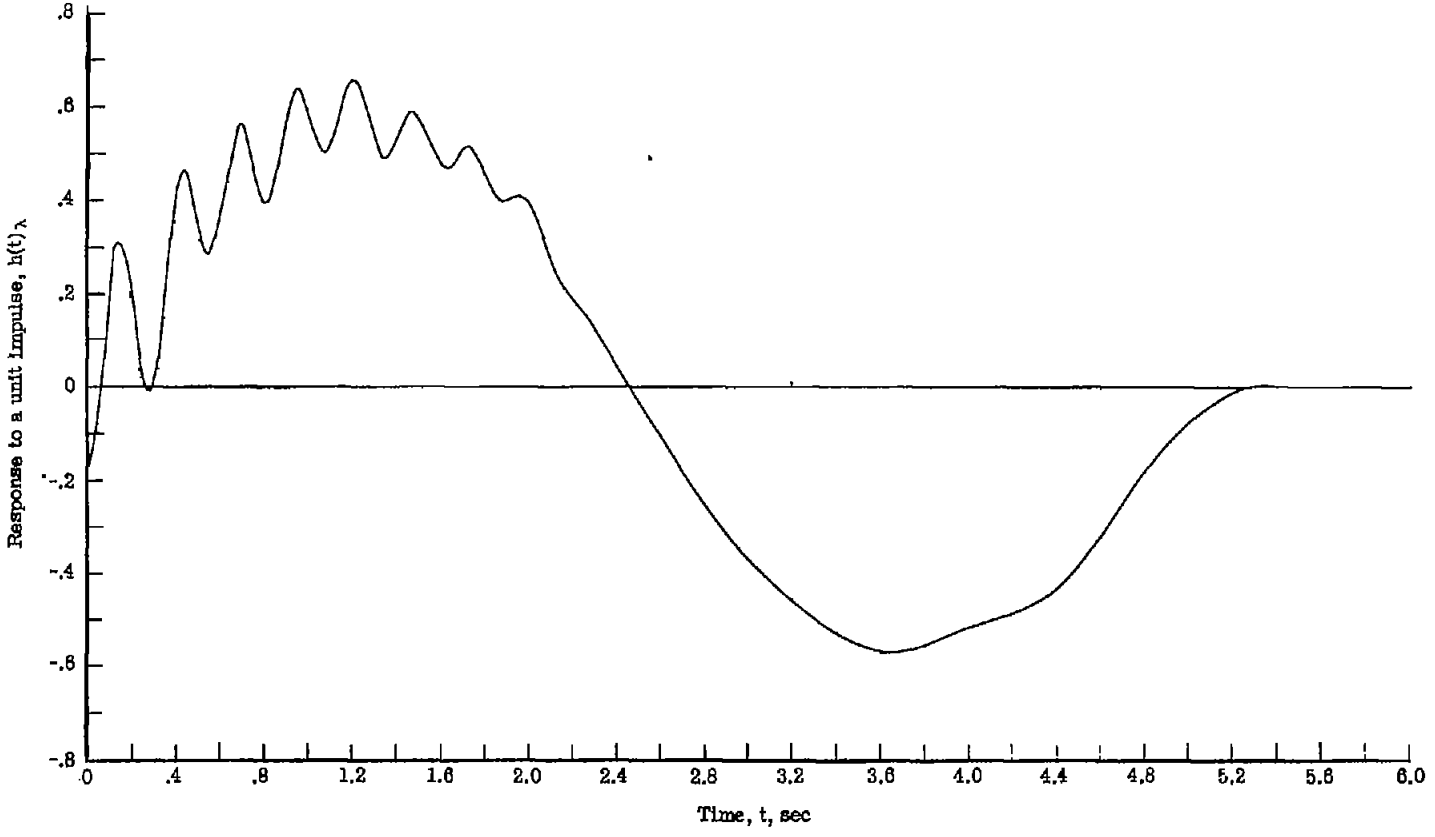


Figure 8.- Incomplete time response to a unit impulse as obtained from the data of figure 5 (λ = 5.4 sec).

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