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RESEARCH MEMORANDUM

PROPAGATION OF SOUND INTO A WIND-CREATED SHADOW ZONE

By David C. Pridmore-Brown

Massachusetts Institute of Technology

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NATIONAL ADVISORY COMMITTEE
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PROPAGATION OF SOUND INTO A WIND-CREATED SHADOW ZONE

By David C. Pridmore-Brown

SUMMARY

The general wave equation is derived governing the propagation of sound in a stratified moving medium, the velocity of which varies only along one coordinate. Under the assumption that the flow velocity is small and slowly varying, a simplified equation is adopted which is satisfactory for the present application. A solution is found to this equation corresponding to the pressure field around a spherical source located above a plane ground in a horizontal wind whose velocity increases with altitude. It is shown that within the acoustic shadow that forms on the upwind side of such a source the pressure field is similar to that which is obtained in the corresponding problem of a sound source in a temperature gradient. An expression is derived for the rate of attenuation within a shadow which is brought about by the presence of both a wind and a temperature gradient.

INTRODUCTION

An important phenomenon associated with the propagation of sound in the atmosphere is that of the refraction of the acoustic rays which can be brought about not only by the presence of a temperature gradient but also by the presence of a wind with a gradient in speed. A uniform wind will have little effect on the sound propagation since its speed will, in general, be much smaller than that of sound. However, the presence of a wind gradient leads to an effective variation in the speed of sound at different points, and this causes a refraction of the rays, just as in the case of a temperature gradient. The important difference between the two phenomena lies in the fact that in a temperature gradient the local speed of sound is determined by the temperature at that point, whereas in the wind case the effective sound speed depends on the direction as well as on the magnitude of the wind velocity. This means that, assuming the gradient in each case to lie always in the same direction, call it the vertical, the sound field around a spherical source in a temperature gradient will be symmetrical about that vertical which passes through the source; in other words, it will depend on only two coordinates. In the

wind case, however, this symmetry clearly no longer exists, and the field will necessarily depend on three coordinates.

This phenomenon leads to an interesting type of shadow formation in the wind case. If a spherical source is situated above ground in a horizontal wind whose velocity increases uniformly with height, then a shadow region into which no acoustic rays penetrate will form on the upwind side of the source but not on the downwind side. In fact, the distance to the shadow will be a minimum directly into the wind and will increase to infinity at right angles to it. This situation is illustrated in figure 1. In the presence of a negative temperature gradient alone, the shadow distance is, of course, the same in all directions.

A consideration of these facts suggested the tentative assumption which was made in a previous report (ref. 1) on the sound field in a wind-created shadow. There it was assumed that an expression for the diffracted field within the wind-created shadow could be obtained from the corresponding expression for the temperature-created shadow by replacing the sonic velocity gradient by the wind velocity gradient times the cosine of the angle between the sound and wind directions. The results of the present study furnish the justification for this assumption.

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SYMBOLS

A_n	constant in $H_{-2/3}^{(1)}\left(A_n e^{-i\pi}\right) = 0$
a	defined in equation (11)
C_1, C_2	contours
c	sonic speed
D	solution to equations (5) or (18) which represents downgoing wave
$H_n^{(\mu)}(x)$	Hankel function of second kind of order n
h	source height
I_m	imaginary

$J_o(x)$	zero-order Bessel function
k	propagation constant, ω/c
M	Mach number of wind, V/c
p	sound pressure
Re	real
r	radial coordinate parallel to ground plane, distance from source
r_o	distance from source at height h to shadow boundary at height z
$s = dM/dz$	
t	time
t_o	travel time from source at height h to shadow boundary at height z
U	solution to equations (5) or (18) which represents upgoing wave
u	particle velocity; function of z, κ, and γ defined in equation (22)
V	wind velocity
W	defined in equation (13)
$\left.\begin{matrix} x,y,z \\ x_1,x_2,x_3 \end{matrix}\right\}$	rectangular coordinates
Z	acoustic impedance of ground
α	wave-number component in x-direction
β	wave-number component in y-direction
$\gamma = \theta - \phi$	
$\epsilon = \pm 1$	
ζ	impedance ratio, $Z/\rho_o c$

θ polar angle of wave number, $\tan^{-1}(\beta/\alpha)$

κ wave-number component in horizontal plane, $\sqrt{\alpha^2 + \beta^2}$

$\mu = 1$ or 2

ρ density fluctuations

ρ_o density of medium

ϕ polar angle

ω angular frequency

ANALYSIS

Derivation of Wave Equation

In an inviscid medium moving with the velocity V the linearized Navier-Stokes equations for the conservation of mass and momentum become, in the absence of sources,

$$\frac{\partial \rho}{\partial t} + V \frac{\partial \rho}{\partial x_1} + \rho_o \frac{\partial u_1}{\partial x_i} = 0 \tag{1}$$

$$\frac{\partial u_i}{\partial t} + u_3 \frac{\partial \left(V\delta_{i1}\right)}{\partial x_3} + V \frac{\partial u_i}{\partial x_1} = - \frac{1}{\rho_o} \frac{\partial p}{\partial x_i} \tag{2}$$

provided the wind velocity is assumed to lie in the x_1-direction and to be a function only of x_3. Here ρ_o represents the static density of the medium which is assumed constant; the density fluctuations ρ, the sound pressure p, and the particle velocity u_i are considered to be small quantities of the first order. If the further assumption is made that $p = c^2\rho$, that is, that the pressure and density variations are adiabatically related, then these equations yield

$$\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = \left(1 - M^2\right)\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \frac{2M}{c} \frac{\partial^2 p}{\partial x\, \partial t} + 2\rho_o c \frac{\partial M}{\partial z} \frac{\partial u_3}{\partial x} \tag{3}$$

where $M = V/c$ and the coordinates x_i have been relabeled x, y, and z.

It is next assumed that the sound pressure p and the vertical component of the particle velocity u_3 vary harmonically in time as e^{-ikct} and can be Fourier analysed in the x- and y-directions:

$$\left.\begin{aligned} p &= e^{-ikct} \iint_{-\infty}^{\infty} e^{i(\alpha x+\beta y)} F(\alpha,\beta,z)\,d\alpha\,d\beta \\ u_3 &= e^{-ikct} \iint_{-\infty}^{\infty} e^{i(\alpha x+\beta y)} G(\alpha,\beta,z)\,d\alpha\,d\beta \end{aligned}\right\} \tag{4}$$

The Fourier transforms are related by equation (2) which gives

$$\frac{dF}{dz} = i\rho_o c(k - \alpha M)G$$

Substitution of these relations into original differential equation (3) then shows that F must satisfy

$$\frac{d^2F}{dz^2} + \frac{2\alpha M'}{k - \alpha M}\frac{dF}{dz} + \left[(k - \alpha M)^2 - \alpha^2 - \beta^2\right]F = 0 \tag{5}$$

together with appropriate boundary and source conditions. Here $M' = dM/dz$.

It is convenient to rewrite expressions (4) in polar coordinates by putting

$$x = r \cos \phi$$

$$y = r \cos \phi$$

$$\alpha = \kappa \cos \theta$$

$$\beta = \kappa \sin \theta$$

$$d\alpha\,d\beta = \kappa\,d\kappa\,d\theta$$

so that p becomes, with $\gamma = \theta - \phi$,

$$p = \iint e^{i\kappa r \cos \gamma} F(\kappa,\gamma,z)\kappa \, d\kappa \, d\gamma \tag{6}$$

The boundary condition at the ground surface, $z = 0$, can be specified by requiring that the ratio of the pressure p to the normal component of the particle velocity u in this plane be equal to a complex impedance Z independent of the angle of incidence. A point source of sound is considered to be located a height h above this plane.

The source condition can be derived in the usual way by writing p as the sum of two functions, namely, $p = p_o + p_1$, where p_o represents the solution in an unbounded medium and p_1 is an everywhere-regular solution to wave equation (3) chosen so that $p_o + p_1$ satisfy the boundary conditions. Since one of the requirements on p_o is that it represent outgoing radiation at great heights on the upwind side of the source, it will be convenient to label the two independent solutions of equation (5) as $D(z)$ and $U(z)$, where $D(z)$ represents a downcoming wave and $U(z)$ an upgoing one for large values of z. One can then write p_o in the form given by equation (6), where the corresponding F_o is given below the source by

$$F_o = AD(\kappa,\gamma,z) \tag{7a}$$

and above the source by

$$F_o = BU(\kappa,\gamma,z) \tag{7b}$$

which corresponds to $e^{i\sqrt{k^2-\kappa^2}|z-h|}$ in a stationary medium. The two constants A and B are determined by the conditions at the source, namely, by requiring that the pressure be continuous across the plane of the source, $z = h$, and that the particle velocity (or pressure gradient) suffer a discontinuity across this plane, which is determined by the source strength. The first of these conditions yields the relation

$$AD(h) = BU(h) \tag{8}$$

The second source condition may be obtained by specifying the source output. To do this requires that

$$\int u_n \, dS = 4\pi \tag{9}$$

where u_n is the component of u perpendicular to the surface element dS; the integration is carried out over a small surface enclosing the source. Now, from momentum equation (2) there follows

$$-i\rho_o c(k - \alpha M)u_j - \rho_o c u_3 M' \delta_{j1} = \frac{\partial p}{\partial x_j}$$

on making use of equations (4). The three components of the particle velocity are then

$$\left.\begin{aligned} u_1 &= a \frac{\partial p}{\partial x_1} + b \frac{\partial p}{\partial x_3} \\ u_2 &= a \frac{\partial p}{\partial x_2} \\ u_3 &= a \frac{\partial p}{\partial x_3} \end{aligned}\right\} \tag{10}$$

where

$$\left.\begin{aligned} a^{-1} &= i\rho_o c(k - \alpha M) \\ b^{-1} &= \frac{\rho_o c}{M'}(k - \alpha M)^2 \end{aligned}\right\} \tag{11}$$

Thus

$$\int u_n \, dS = \int a(\nabla p)_n \, dS + \int b \frac{\partial p}{\partial x_3} \, dx_2 \, dx_3$$

The last integral is over the two surfaces $x_1 = 0^-$ and $x_1 = 0^+$, and it vanishes since p and its derivatives are continuous along x_1. However, because of the discontinuity in the vertical component of ∇p, the first integral on the right does not vanish but becomes

$$\int a \frac{\partial p}{\partial x_3} dx_1 \, dx_2$$

the integration being taken over the two surfaces $x_3 = h^-$ and $x_3 = h^+$. Thus, introducing an expression of the form given by equation (6) for p_o and reverting to polar coordinates r, z gives

$$2\pi \int a e^{i\kappa r \cos \gamma} \left[BU'(h) - AD'(h)\right] \kappa \, d\kappa \, d\gamma \, r \, dr$$

which must be equated to 4π. Here equations (7) have been used to express F_o, and the primes denote differentiation with respect to z.

From the well-known relations

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\kappa r \cos \gamma} \, d\gamma = J_o(\kappa r)$$

$$\int_0^{\infty} J_o(\kappa r)\kappa \, d\kappa = \delta(r) = 0 \qquad (r \neq 0)$$

$$\int \delta(r) r \, dr = 1$$

There follows immediately

$$a\left[BU'(h) - AD'(h)\right] = \frac{1}{\pi} \tag{12}$$

where $a_1 = a(h)$.

The two equations (8) and (12) determine A and B and, hence, also the unbounded solution p_o, which becomes above the source $(z > h)$

$$p_o = \frac{1}{\pi}\int_0^\infty d\kappa \int_{-\pi}^{\pi} d\gamma\, e^{i\kappa r \cos\gamma} \frac{\kappa}{W} D(h)U(z) \tag{13}$$

Here

$$W = i\rho_o c\left[k - \alpha M(h)\right](U'D - D'U) \tag{14}$$

and is independent of z. A similar expression holds below the source but with z and h interchanged.

The general solution $p_o + p_1$ can now be written

$$p = \frac{1}{\pi}\int_0^\infty \int_{-\pi}^{\pi} e^{i\kappa r \cos\gamma} \frac{1}{W}\left[D(h) + F_1\right]U(z)\kappa\, d\kappa\, d\gamma \tag{15}$$

where F_1 is determined by requiring equation (15) to satisfy the normal impedance boundary condition $\frac{p}{\text{grad } p} = -\frac{Z}{i\omega\rho_o}$ at $z = 0$. It turns out to be

$$F_1 = -\frac{kD(0) - i\zeta D'(0)}{kU(0) - i\zeta U'(0)} U(h) \tag{16}$$

where $\zeta = \frac{Z}{\rho_o c}$ and the primes again denote differentiation with respect to z.

Approximate Solutions Valid Near Ground

Up to now no restrictions have been placed on the Mach number M beyond requiring that it vary in only one direction. In the general case approximate solutions to equation (5) can be readily obtained by a modified W.K.B. method (cf., e.g., the work of Langer, ref. 2). For the present application it will be sufficient to assume that both M^2 and M' are so small that the terms involving these quantities in equation (5)

can be neglected. It will also be assumed that M increases linearly with altitude from the value zero at the ground, $z = 0$, so that

$$M = sz \tag{17}$$

where s is a constant.

With these restrictions equation (5) becomes

$$\frac{d^2F}{dz^2} + \left(k^2 - \alpha^2 - \beta^2 - 2k\alpha sz\right)F = 0$$

or

$$\frac{d^2F}{dz^2} + \left[k^2 - \kappa^2 - 2k\kappa sz \cos(\gamma + \phi)\right]F = 0 \tag{18}$$

The two restrictions on the range of validity of this equation can be expressed as

$$s/k \ll 1 \tag{19}$$

and

$$sz \ll 1 \tag{20}$$

The first is a high-frequency requirement, while the second is a restriction to low heights; in practical cases in atmospheric acoustics neither restriction seems to be very stringent.

The general solution to equation (18) is of the form

$$F(\kappa,\gamma,z) = u^{1/2}Z_{1/3}\left(\frac{2}{3}u^{3/2}\right) \tag{21}$$

where

$$u = \left[k^2 - \kappa^2 - 2k\kappa sz \cos(\gamma + \phi)\right]\left[2\epsilon k\kappa s \cos(\gamma + \phi)\right]^{-2/4} \tag{22}$$

and $Z_{1/3}$ represents a linear combination of one-third-order Bessel functions. The quantity $\epsilon = \pm 1$ will be specified later. The function $U(z)$ can be written provisionally in terms of a Hankel function of the μth kind as

$$U(z) = u^{1/2} H_{1/3}^{(\mu)}\left(\frac{2}{3} u^{3/2}\right) \tag{23}$$

where $\mu = 1$ or 2 depending on the value of ϵ and will be specified later. The function $D(z)$ is then defined similarly in terms of the other Hankel function.

Transformation of Integral in Complex Plane

It is of interest to notice that if the function $F(\kappa,\gamma,z) = \left[D(h) + F_1\right] U(z)$ in equation (15) did not depend on γ then the integration over γ could be carried out directly and would yield $J_0(\kappa r)$, which was the starting point in the problem of a sound source in a temperature gradient (ref. 3). In that case it proved advantageous to divide up the integral into the sum of two integrals by writing $2J_0 = H_0^{(1)} + H_0^{(1)}$. Guided by these considerations the γ contour in equation (15) is deformed into the sum of two Hankel function contours as shown in figure 2. This is allowed since the integrand involves γ only in the form $\cos\gamma$ and $\cos(\gamma + \phi)$, which are unchanged if γ is increased by 2π, so that clearly the contributions from the infinite branches of the contours cancel each other. As in the case of the integral representation of the Hankel functions the integral converges provided that

$$-\eta < \arg \kappa r < \pi - \eta$$

where η lies between 0 and π. The solution now has the form

$$p = \frac{1}{\pi}\int_{C_1} d\gamma \int_0^\infty e^{i\kappa r \cos\gamma} \frac{k}{W} F(\kappa,\gamma,z)d\kappa + \frac{1}{\pi}\int_{C_2} d\gamma \int_0^\infty e^{i\kappa r \cos\gamma} \frac{\kappa}{W} F(\kappa,\gamma,z)d\kappa \tag{24}$$

The convergence of these integrals in the complex κ plane is next examined with a view to deforming the κ integration path into closed contours, as was done in the temperature problem. In other words, for the κ integration in the first integral in equation (24), a contour might be taken enclosing the first quadrant in the clockwise direction, and, similarly, a contour enclosing the fourth quadrant in the counterclockwise direction for the second integral as indicated in figure 3. To justify this it must be shown that the contributions from the infinite arcs vanish, that there are no branch points in the first and fourth quadrants of the κ plane, and that the integrals along the imaginary axes cancel each other.

The absence of branch points follows directly from the fact that the functions $u^{1/2}H_{1/3}^{(\mu)}\left(\frac{2}{3}u^{3/2}\right)$ are integral transcendental functions of u, that is, they are single valued and analytic for all finite values of u. (They are the solutions to $\left(d^2F/du^2\right) + uF = 0$, which is regular for all finite values of u.) Also, $u(k)$ is analytic in the first and fourth quadrants of the complex κ plane, and, hence, $e^{i\kappa r \cos\gamma} F(\kappa,\gamma,z)$ is free of branch points in this region.

The behavior of the integrands on the infinite arcs is next studied. First of all it is noticed that the exponential term $e^{i\kappa r \cos\gamma}$ tends to zero as κr tends to infinity in the first quadrant for values of γ lying on the contour C_1, and similarly this exponential tends to zero as κr tends to infinity in the fourth quadrant for values of γ lying on the contour C_2. It must also be shown that $F(\kappa,\gamma,z) \to 0$ as $\kappa r \to \infty$ when $\mathrm{Re}\,\kappa > 0$ (Re indicates real values). In the limit of large values of κ, u becomes

$$u \to -\frac{\kappa^2\left[1 + 2\frac{k}{\kappa} sz \cos(\gamma + \phi)\right]}{\left[2\epsilon k\kappa s \cos(\gamma + \phi)\right]^{2/3}}$$

whence

$$\frac{2}{3}u^{3/2} \to (\pm i)\epsilon\left[\frac{\kappa^2}{3ks \cos(\gamma + \phi)} + \kappa z\right]$$

If the following convention is adopted

$$\left[k^2 - \kappa^2 - 2k\kappa sz \cos(\gamma + \phi)\right]^{1/2} = i\left[\kappa^2 - k^2 + 2k\kappa sz \cos(\gamma + \phi)\right]^{1/2}$$

for $\kappa^2 > k^2 - 2k\kappa sz \cos(\gamma + \phi)$, then it is necessary to choose the minus sign. This requires that $\mu = 1$ if $\epsilon = -1$ and $\mu = 2$ if $\epsilon = 1$. It is then easily found from the asymptotic forms of equation (23) that

$$D(h)U(z) \rightarrow \frac{3}{\pi}(uu_1)^{-1/4} e^{-i\epsilon\frac{2}{3}\left(u^{3/2}-u_1{}^{3/2}\right)}$$

$$\rightarrow -\frac{3}{\pi}\, i\left[2\epsilon\, \frac{ks}{\kappa^2} \cos(\gamma + \phi)\right]^{1/3} e^{-\kappa(z-h)}$$

where $u_1 = u(h)$. Similarly,

$$\frac{kD(0) - i\zeta D'(0)}{kU(0) - i\zeta U'(0)} \rightarrow e^{-i\epsilon\pi/6} \exp\left[\frac{-2\kappa^2}{3ks \cos(\gamma + \phi)}\right]$$

$$U(z)U(h) \rightarrow \frac{3}{\pi}\, i\left[2\epsilon\, \frac{ks}{\kappa^2} \cos(\gamma + \phi)\right]^{1/3} e^{-2\kappa^2/3ks \cos(\gamma+\phi)} e^{-\kappa(z+h)} e^{5\pi i\epsilon/12}$$

whence

$$F(\kappa,\gamma,z) \rightarrow \frac{3}{i\pi}\left[2\epsilon\, \frac{ks}{\kappa^2} \cos(\gamma + \phi)\right]^{1/3} \left[e^{-\kappa(z-h)} + e^{-\kappa(z+h)}\right]$$

and so has the required behavior.

It can now be proved that the two integrals taken along the imaginary axes cancel each other, that is, that

$$\int_0^{i\infty}\int_{C_1} e^{i\kappa r \cos\gamma} F(\kappa,\gamma)\kappa\, d\kappa\, d\gamma + \int_0^{-i\infty}\int_{C_2} e^{i\kappa r \cos\gamma} F(\kappa,\gamma)\kappa\, d\kappa\, d\gamma = 0 \tag{25}$$

for

$$-\eta < \arg \kappa r < \pi - \eta$$

where η lies between 0 and π. To do this κr is replaced by $\kappa r e^{i\pi}$ in the second integral in equation (25) to obtain

$$\int_0^{i\infty}\int_{C_2'} e^{-i\kappa r \cos \gamma} F(-\kappa,\gamma)\kappa \, d\kappa \, d\gamma$$

with the requirement $-\eta < \arg \kappa r e^{i\pi} < \pi - \eta$ or $-\eta - \pi < \arg \kappa r < -\eta$. The integration path C_2' (shown by the heavy dashed line in fig. 2) is now the same as C_1 taken in the opposite sense and displaced by π. Accordingly γ is replaced by $\gamma + \pi$, so that this integral takes the form

$$-\int_0^{i\infty}\int_{C_1} e^{i\kappa r \cos \gamma} F\left[-\kappa,(\gamma + \pi)\right]\kappa \, d\kappa \, d\gamma$$

with

$$-\eta < \arg \kappa r < \pi - \eta$$

For equation (25) to hold it is then necessary for

$$F(\kappa,\gamma) = F\left[-\kappa,(\gamma + \pi)\right]$$

Reference to equation (22) shows that this condition is fulfilled in the present case and, hence, that equation (25) is valid.

The general expression for the pressure field p has now been reduced to the sum of two contour integrals taken around the first and fourth quadrants of the κ plane. These integrals can be expressed as a sum of their residues taken at the poles of $F(\kappa,\gamma,z)$ in this plane. Finally, the integrations over γ can be evaluated approximately by a saddle-point integration through the points of stationary phase of the

integrands; namely, $\gamma = 0$ for the C_1 contour and $\gamma = \pi$ for the C_2 contour.

First of all, on carrying out the approximate integration over γ, equation (24) becomes

$$p = \sqrt{\frac{2}{\pi r}}\left[\int_{\text{(1st quadrant)}} \frac{1}{\sqrt{\kappa}} e^{i\left[\kappa r-(\pi/4)\right]} \frac{\kappa}{W} F(\kappa,0,z)\, d\kappa + \int_{\text{(4th quadrant)}} \frac{1}{\sqrt{\kappa}} e^{-i\left[\kappa r-(\pi/4)\right]} \frac{\kappa}{W} F(\kappa,\pi,z)\, d\kappa\right] \tag{26}$$

At this point the analysis is restricted to the case of a perfectly hard ground by letting $\zeta \to \infty$ in equation (16). The more general case can be treated by the methods used in the temperature problem (ref. 4). For a hard ground the poles of F are given by the roots of $U'(0) = 0$, which becomes

$$H^{(\mu)}_{-2/3}\left(\frac{2}{3} u^{3/2}\right) = 0$$

whence

$$\frac{2}{3} u_o^{3/2} = A_n e^{i\epsilon\pi}$$

where the A_n components are constants, $A_1 = 0.686$, $A_2 = 3.90$, $A_3 = 7.05$, . . . , $A_n = \left[n - (3/4)\right]\pi$ (where n is large).

Substituting for u_o gives

$$\frac{\left(k^2 - \kappa_n^2\right)^{3/2}}{2\epsilon k\kappa s \cos(\gamma + \phi)} = \frac{3}{2} A_n e^{\epsilon i\pi}$$

or

$$\kappa_n^2 = k^2 - \left[3\epsilon\kappa_n skA_n \cos(\gamma + \phi)\ e^{\epsilon i\pi}\right]^{2/3}$$

As a result of the restriction $s/k \ll 1$ (eq. (19)), this equation can be writen approximately

$$\kappa_n \approx k \left\{1 - \frac{1}{2}\left[3\epsilon \frac{s}{k} A_n \cos(\gamma + \phi)\ e^{\epsilon i\pi}\right]^{2/3}\right\} \tag{27}$$

It is now required that $\mathrm{Im}\kappa_n > 0$ (where Im indicates imaginary values) for $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$ (damped outgoing propagation on upwind side of source). This leads to the result that $\epsilon = -1$ and $\mu = 1$.

Since the values of κ_n all lie in the first quadrant, the contour integral enclosing the fourth quadrant must vanish; this leaves the integral enclosing the first quadrant in the κ plane, which can be evaluated as a sum of residues in the form

$$p = 2\pi i \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi\kappa_n r}}\ e^{i\left[\kappa_n r - (\pi/4)\right]} U(\kappa_n, z) \left[\frac{\kappa_n}{W} \frac{D'(\kappa_n, 0)}{\left(\frac{\partial}{\partial\kappa}\right) U'(\kappa_n, 0)}\right]_{\gamma=0} \tag{28}$$

By using the Wronskian relationship $D'U - DU' = u_1 W$, which gives $D' = a_1 W/U$ at $\kappa = \kappa_n$, equation (28) can be rewritten in terms of U only as follows

$$p = \sum_{n=1}^{\infty} \sqrt{\frac{8\pi\kappa_n}{r}}\ e^{i\left[\kappa_n r + (\pi/4)\right]} \left[\frac{a_1 U(h) U(z)}{U(0)(\partial/\partial\kappa) U'(0)}\right] \tag{29}$$

where

$$U = u^{1/2} H_{1/3}^{(1)}\left(\frac{2}{3} u^{3/2}\right)$$

$$u = \left(k^2 - \kappa_n{}^2 - 2k\kappa_n sz \cos \phi\right)(-2k\kappa s \cos \phi)^{-2/3} \tag{30}$$

$$\kappa_n \approx k\left[1 - \frac{1}{2}\left(3 \frac{s}{k} A_n \cos \phi\right)^{2/3}\right] \tag{31}$$

$$H_{-2/3}{}^{(1)}\left(A_n e^{-i\pi}\right) = 0$$

$$a_1{}^{-1} = i\rho_o c(k - \kappa sh \cos \phi)$$

High-Frequency Behavior of Solution

As in the corresponding temperature problem, the high-frequency behavior of the solution (eq. (29)) can be studied in the shadow zone by replacing the functions U by their asymptotic forms, which are valid for large values of u. It is then found, on reintroducing the time factor $e^{-i\omega t}$, that

$$p = \sum_{n=1} B_n r^{-1/2} \exp \left\{i\left[\kappa_n r + \frac{2}{3}\left(u^{3/2} + u_1{}^{3/2}\right) - \omega t\right]\right\} \tag{32}$$

where

$$B_n = \sqrt{\frac{24k}{i\pi}}\left[(uu_1)^{1/4} u_o{}^2 a_1{}^{-1} \frac{\partial u_o}{\partial z} \frac{\partial u_o}{\partial \kappa}\right]^{-1}\left[H_{1/3}{}^{(1)}\left(A_n e^{-i\pi}\right)\right]^{-2}$$

On substituting for κ_n from equation (31) into equation (30)

$$u = \left(-2\frac{s}{k} \cos \phi\right)^{1/3} kz + \left(\frac{3}{2} A_n e^{-i\pi}\right)^{2/3}$$

is obtained which gives approximately, for large values of kz,

$$\frac{2}{3}u^{3/2} \approx k\left[\frac{2}{3} \sigma^{1/2} z^{3/2} + 2\delta_n (z/\sigma)^{1/2}\right]$$

where

$$\sigma = -2s \cos \phi$$

$$\delta_n = \frac{1}{2}\left(-3\frac{s}{k} A_n \cos \phi\right)^{2/3} e^{-2i\pi/3}$$

The terms in the exponent of the asymptotic solution (32) become then

$$\kappa_n r + \frac{2}{3}\left[u(z)\right]^{3/2} + \frac{2}{3}\left[u(h)\right]^{3/2} - \omega t = \kappa_n r + k\left[\frac{2}{3}\sigma^{1/2}\left(z^{3/2} + h^{3/2}\right) + 2\delta_n(z/\sigma)^{1/2} + (h/\sigma)^{1/2} - \omega t\right]$$

$$= k(r - r_o)(1 - \delta_n) - c(t - t_o)$$

Here, these quantities have been identified with r_o, the horizontal distance from source to shadow boundary, and with t_o, the corresponding travel along the limiting ray, which have been derived from ray acoustics in the appendix.

In terms of these quantities the high-frequency form of the solution is

$$p \approx \sum_{n=1} (\text{Constant})_n \frac{(s/k)^{1/6}}{(zh)^{1/4}} r^{-1/2} e^{-i\omega\left[t - t_o - \frac{r - r_o}{c}(1 - \delta_n)\right]} \tag{33}$$

Within the shadow zone the solution is adequately expressed by the first mode alone because of the rapid attenuation of the higher modes. To this approximation the sound pressure is damped at the rate of

$$8.68 \operatorname{Im} k\delta_1 \approx 6.1(-s \cos \phi)^{2/3} k^{1/3} \text{db/unit distance} \tag{34}$$

within the shadow $\left(\frac{\pi}{2} < \phi < \frac{3\pi}{2},\ r > r_o\right)$. This is in addition to the damping due to cylindrical divergence, which is expressed by the factor $r^{-1/2}$.

The combined effects of a wind and a temperature gradient can be accounted for by allowing k to vary with z. In the high-frequency approximation this results in the factor $-s \cos \phi$ in equation (34) being replaced by $-s \cos \phi - \frac{1}{c}\frac{dc}{dz}$, where $\frac{dc}{dz}$ is positive for a temperature inversion. A shadow will form in this case within that sector for which the factor $\left(-s \cos \phi - \frac{1}{c}\frac{dc}{dz}\right)^{2/3}$ is real. By replacing s by $\frac{1}{c}\frac{dV}{dz}$, the result is obtained that a shadow will form within the sector defined by

$$|\phi| < \left(\frac{dc}{dz}\Big/\frac{dV}{dz}\right)$$

provided that ϕ is now measured from the 180° line (-x-axis) and $\frac{dv}{dz} > 0$. If $\left|\frac{dc}{dz}\right| > \frac{dV}{dz}$ then the temperature gradient predominates over the velocity gradient. In this case there will be a shadow in all directions if $\frac{dc}{dz} < 0$ and no shadow at all if $\frac{dc}{dz} > 0$.

Within the normal or illuminated zone the high-frequency (ray acoustics) approximation to the solution can be readily obtained by integrating equation (24) around the points of stationary phase of the integrand in the κ plane (ref. 5).

Massachusetts Institute of Technology,
Cambridge, Mass., April 6, 1956.

APPENDIX A

RAY ACOUSTICS IN A WIND GRADIENT

The acoustic-ray paths are determined by Fermat's principle, which requires that they be such that the integral

$$\int \frac{ds}{c + V \sin\theta \cos\phi} \tag{A1}$$

is stationary. Here ds is an increment of path length along the ray, V is the wind speed, θ is the angle of inclination of the ray from the vertical, and ϕ is the constant polar angle between the ray and the wind direction.

From the geometry of the problem it is seen that

$$\sin\theta = \frac{r'}{\sqrt{1 + (r')^2}}$$

where $r' = \frac{dr}{dz}$. With this substitution, and writing $ds = \sqrt{1 + (r')^2}dz$, the Euler equation is obtained for which equation (A1) is stationary in the form

$$\frac{d}{dz}\frac{\partial}{\partial r'}\left(\frac{1 + (r')^2}{\sqrt{1 + (r')^2} + Mr' \cos\phi}\right) = 0$$

or

$$\frac{\partial}{\partial r'}\frac{1 + (r')^2}{\sqrt{1 + (r')^2} + Mr' \cos\phi} = K \tag{A2}$$

where $M = V/c$ and K is a ray parameter. The limiting ray which defines the shadow boundary can be specified by requiring that $r' \rightarrow \infty$ at the ground, where the wind velocity is zero. This requirement gives $K = 1$ for the limiting ray. Carrying out the differentiating in equation (A2) and solving for r' gives

$$r' = \frac{1}{\sqrt{-2M \cos \phi}}\left(1 + \frac{25}{16} M \cos \phi + \ldots\right) \tag{A3}$$

if higher powers of M are ignored. If M is set equal to sz it is found to the first order in M that

$$r = \int_0^z \frac{dz}{\sqrt{-2sz \cos \phi}} = 2\sqrt{\frac{z}{-2s \cos \phi}}$$

The total horizontal distance between the source at height h and a point at height z on the shadow boundary is then

$$r_o = 2\left(\sqrt{\frac{z}{-2s \cos \phi}} + \sqrt{\frac{h}{-2s \cos \phi}}\right) \tag{A4}$$

The corresponding travel time t_o of the limiting ray between these two points is given by the integral in equation (A1), which becomes, on substituting for ds and $\sin \theta$ in terms of r',

$$\frac{1}{c}\int \frac{1 + (r')^2}{\sqrt{1 + (r')^2} + Mr' \cos \phi} dz$$

Equation (A3) is now used to write $r' = 1\Big/\sqrt{-2M \cos \phi}$, and M is set equal to sz. Then, to the same approximation as before,

$$ct_o = r_o + \frac{2}{3}(-2s \cos \phi)^{1/2}\left(z^{3/2} + h^{3/2}\right) \tag{A5}$$

is obtained, where r_o is given in equation (A4).

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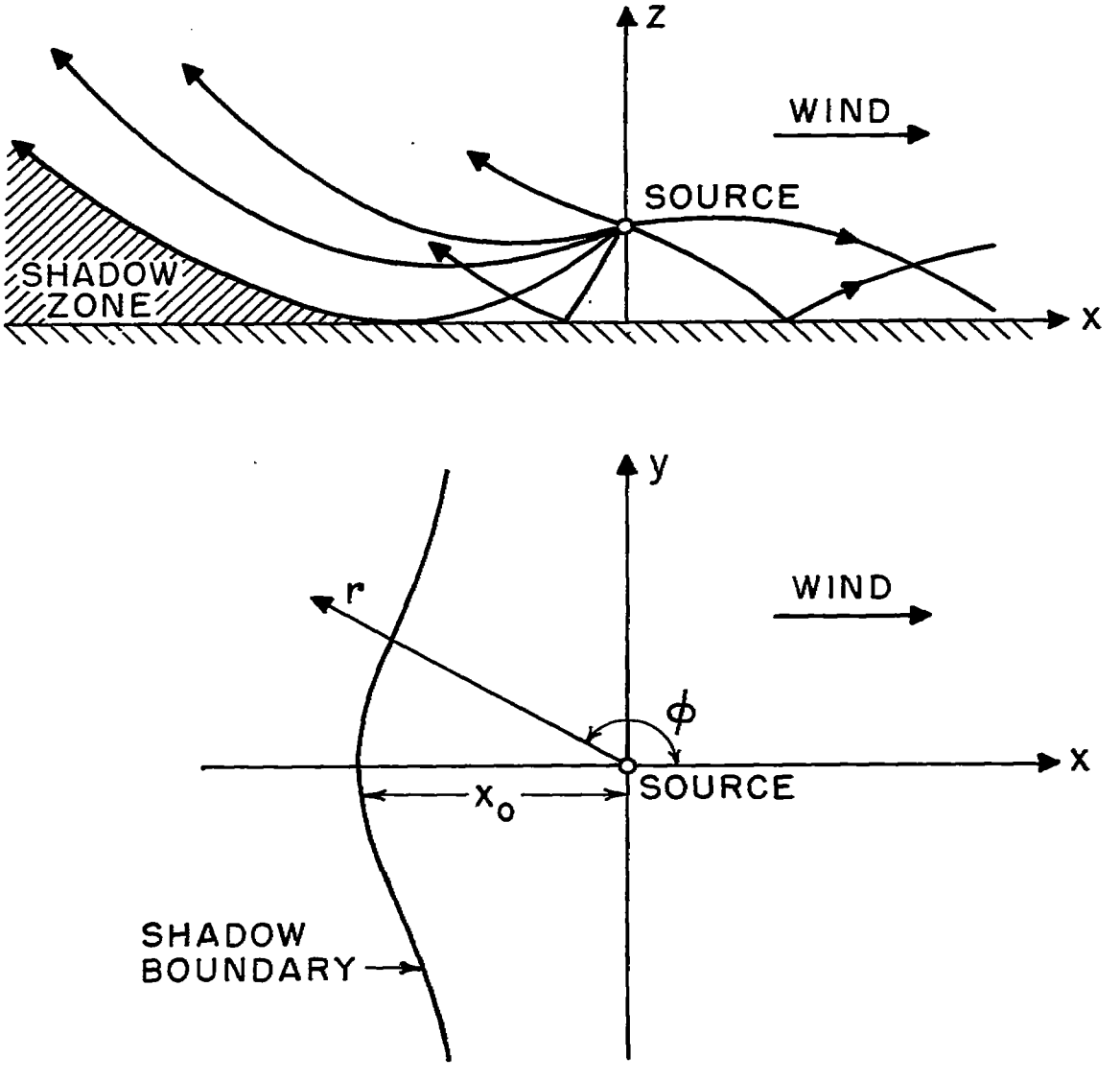

Figure 1.- Illustration of mechanism of shadow-zone formation.

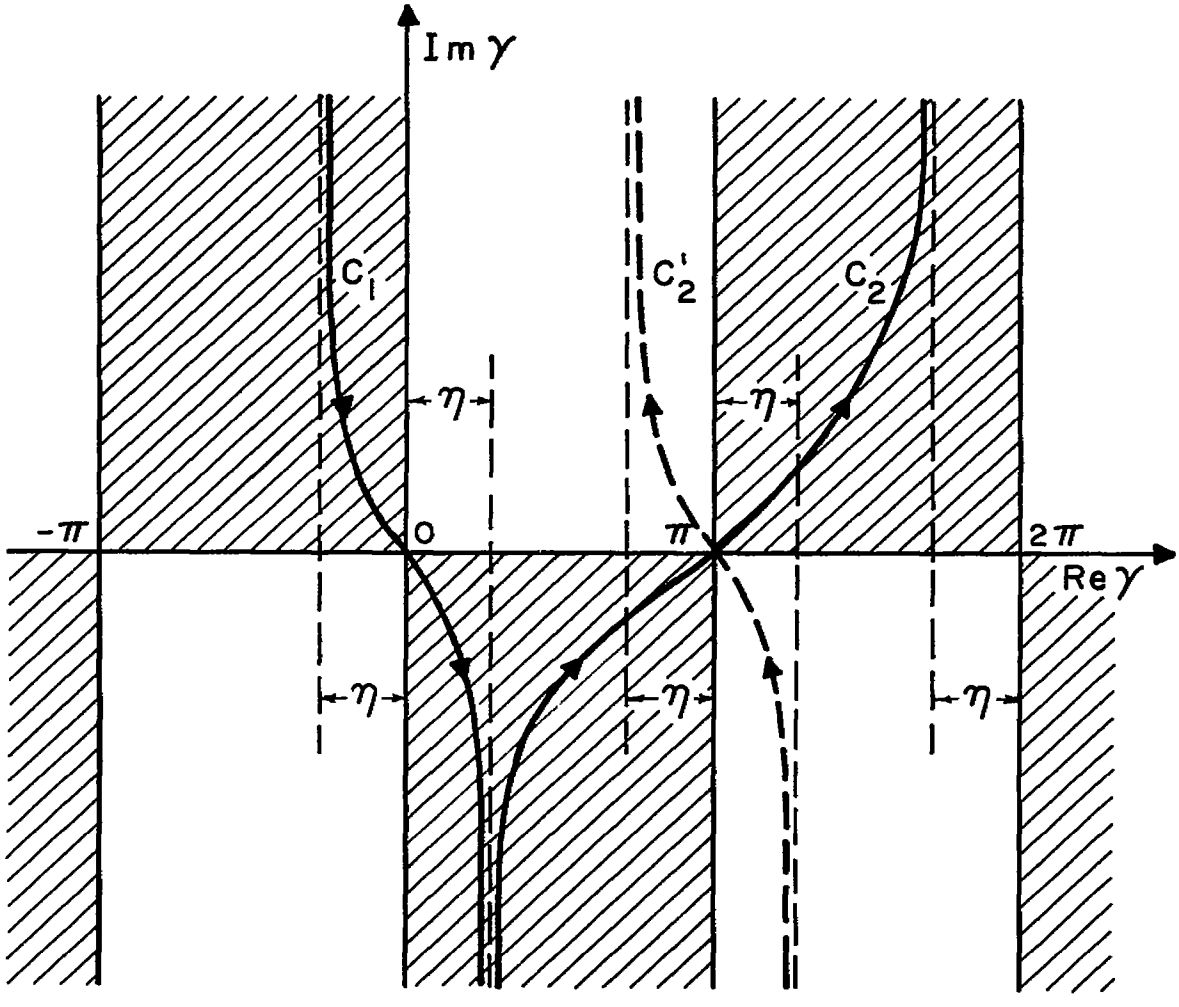

Figure 2.- Integration paths in complex γ plane.

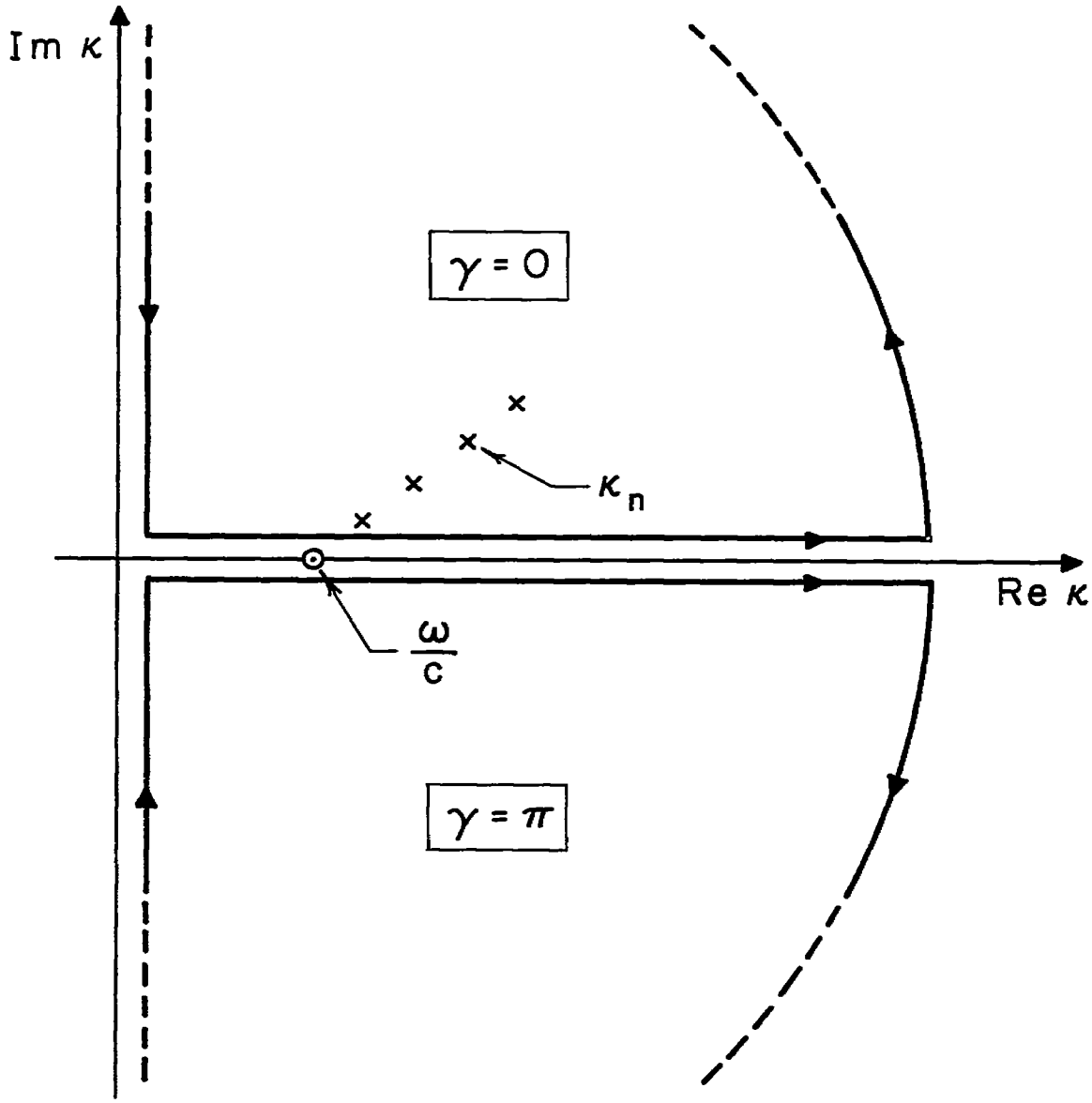

Figure 3.- Integration paths in complex κ plane.

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