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THE PRINCIPLES OF TURBULENT HEAT TRANSFER

By H. Reichardt

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THE PRINCIPLES OF TURBULENT HEAT TRANSFER*

By H. Reichardt

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1. INTRODUCTION

The literature on turbulent heat transfer has in the course of years attained a considerable volume. Since this very complicated problem has not as yet found a complete solution, further studies in this field may be expected. The heat engineer must therefore accommodate himself to a constantly increasing number of theories and formulas. Since the theories generally start from hypothetical assumptions, and since they contain true and false assertions, verified knowledge and pure suppositions often being intermingled in a manner difficult to tell them apart, the specialist has difficulty in forming a correct evaluation of the individual studies.

The need therefore arises for a presentation of the problem of turbulent heat transfer which is not initially bound by hypothetical assumptions and in which the already known and that which is still uninvestigated can be clearly distinguished from each other. Such a presentation will be given in the present treatment.

The following brief remarks may be made with regard to the development of the theory of local heat transfer.[1] The first to recognize the intimate relation between heat transfer and flow resistance was O. Reynolds (ref. 17). The considerations of Reynolds hold, however, only for fluids with special properties (according to present terminology, they are fluids whose Prandtl number is 1).

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* "Die Grundlagen des Turbulenten Wärmeüberganges." Archiv f. die gesamte Wärmetechnik, no. 6/7, 1951, pp. 129-142.

[1]This is taken to mean the direction of investigation which has for its object the derivation of the law of heat transfer from local flow processes. In contrast to this is the semiempirical method of similarity considerations as developed by W. Nusselt. This model investigation dispenses with the knowledge of the individual processes. However, for this reason it is in the position to supply simple practical formulas even in complicated cases.

The first practical useful formula for turbulent heat transfer was derived in 1910 by L. Prandtl (ref. 12). Prandtl, like Reynolds, started from the assumption that heat and momentum are transferred by the same turbulence mechanism. For simplifying the computation, the friction layer was divided into two sections: the turbulent boundary layer itself, and a thin layer close to the wall, whose flow was assumed as completely laminar.

The Prandtl formula was found quite reliable for the representation of heat transfer of fluids with small Prandtl numbers. On the other hand, it is impossible to represent the measured heat-transfer coefficients of fluids with high Prandtl numbers by the Prandtl formula (a correction is assumed which is adjusted to the experimental data). 3871

The reason for this discrepancy for high Prandtl numbers lies in the too greatly idealized flow relations in the friction layer. Actually there is, of course, no completely turbulent boundary layer and also no completely laminar wall layer, but a continuous transition from the turbulent flow to the viscous flow in the immediate neighborhood of the wall.

By dividing the friction layer into three sections (namely, a turbulent region, a laminar-turbulent transition zone, and a laminar wall layer), as was done by Th. v. Kármán (ref. 7) and H. Reichardt (ref. 14), a considerable improvement in the theory could therefore be attained. There still remained, however, certain contradictions which resulted from the conception of a completely laminar layer at the wall.

In order to remove these discrepancies, the author, in a later paper (ref. 15), assumed an entirely continuous decrease of the turbulent exchange to zero near the wall. The results of this computation are contained in the present paper. In addition, still more general formulas are given that form the basis for a later investigation of the effect of the temperature dependence of the constants of the material on the heat transfer. Also considered in this paper is the turbulent heat transfer for extremely low Prandtl numbers.[2]

2. DERIVATION OF GENERAL FORMULAS

(a) Definitions and Assumed Expressions

The considerations will be restricted to those cases for which no unsteady changes of temperature occur at the wall to which the heat (or cold) is transferred from the flow. The wall is furthermore assumed as smooth.

[2]At the suggestion of Mr. B. Koch.

If the temperature of the wall is constant or its spatial changes are only slight, the heat flow and momentum flow at each point x have practically the same direction as y (or $-y$) perpendicular to the wall.

The transfer of the momentum and heat is, on the one hand, effected through the molecular motion, and on the other hand, through turbulent exchange. The parts determined by molecular transfer will be denoted by the subscript m and those produced by turbulence denoted by the subscript t. For the flow densities of the momentum τ and the heat q we therefore write

$$\tau = \tau_m + \tau_t \equiv \tau_m(1 + \tau_t/\tau_m) \tag{1}$$

$$q = q_m + q_t \equiv q_m(1 + q_t/q_m) \tag{2}$$

The following equations apply for the individual partial flows:

$$\tau_m = \mu \, du/dy \tag{3}$$

$$\tau_0 = \mu_0(du/dy)_0 \tag{3a}$$

$$q_m = \lambda \, dT/dy \tag{4}$$

$$q_0 = \lambda_0(dT/dy)_0 \tag{4a}$$

and

$$\tau_t = A_\tau \, du/dy \tag{5}$$

$$q_t = c_p A_q \, dT/dy \tag{6}$$

Through these formulas the coefficient of viscosity μ and of heat conduction λ, as well as the exchange magnitudes[3] for the momentum A_τ and for the heat A_q, are defined (u and T denoting, respectively, the time mean value of the flow velocity and the temperature at the distance y from the wall; c_p is the specific heat).

The subscript 0 refers to the wall. Since in the immediate neighborhood of the wall only molecular transfers are possible, τ_0 is identical with $(\tau_m)_0$ and q_0 is identical with $(q_m)_0$, while $(\tau_t)_0$ and $(q_t)_0$ do not exist.

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[3]The exchange magnitude A_τ can be denoted as the turbulent friction coefficient, the magnitude $c_p A_q$ as the turbulent heat-transfer coefficient.

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The following equations are obtained from equations (5) and (3) and from equations (6) and (4), respectively:

$$\tau_t/\tau_m = A_\tau/\mu \tag{7}$$

$$q_t/q_m = Pr^* \tau_t/\tau_m = Pr^* A_\tau/\mu \tag{8}$$

where $Pr^* = Pr\ A_q/A_\tau$ denotes the "generalized Prandtl number" (ref. 14). Further, from equations (2) and (8) there is obtained

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$$\frac{q_m}{q} = \frac{1}{1 + Pr^* A_\tau/\mu} \tag{9}$$

In order to be able to represent the temperature distribution and the heat transfer as functions of the flow magnitudes, still another relation is required that connects the dimensionless heat flow q/q_0 with the dimensionless momentum flow τ/τ_0. Since these two flows do not deviate strongly from each other in the neighborhood of the wall (where the main part of the heat transfer occurs) - at the wall itself we actually have $\tau/\tau_0 = q/q_0 = 1$ - the following expression suggests itself:

$$q/q_0 = (1 + k)\tau/\tau_0 \tag{10}$$

This equation defines a magnitude k which is small in the neighborhood of the wall and may therefore, to a first approximation, be neglected.

The expression (10) was found to give reliable values for computations involving medium and high Prandtl numbers (refs. 14 and 15). It can, however, as will appear below, be successfully employed also for small Prandtl numbers (for which k is no longer small as compared with 1).

From the above definitions and assumed expressions there will now be derived the required equations for the temperature distribution and the heat transfer. The flow magnitudes A_τ/μ, Pr^*, and τ/τ_0 are here to be considered as "given magnitudes." In order to assure the general validity of the formulas to be derived, no special assumptions in regard to the flow magnitudes will be introduced in this section.

(b) Formulas for Temperature Distribution

From equations (2), (4), (4a), and (8) there follows

$$\frac{q}{q_0} \lambda_0 \left(\frac{dT}{dy}\right)_0 = \lambda \frac{dT}{dy}(1 + Pr^* A_\tau/\mu) \tag{11}$$

It is of advantage to introduce a dimensionless temperature difference

$$\vartheta = \frac{T - T_0}{\Theta}$$

which is referred to the maximum temperature difference Θ at the distance $y = r$ (hence, $0 < \vartheta < 1$ for $0 < y/r < 1$). For the temperature gradient $\frac{d\vartheta}{d(y/r)}$, there is obtained from equation (11)

$$\frac{d\vartheta}{d(y/r)} = \left(\frac{d\vartheta}{d(y/r)}\right)_0 \frac{\lambda_0}{\lambda} \frac{q/q_0}{1 + Pr^* A_\tau/\mu} \tag{12}$$

If equation (10) is used, there is further obtained from the above

$$\vartheta = \left(\frac{d\vartheta}{d(y/r)}\right)_0 \int_0^{y/r} \frac{\lambda_0}{\lambda} \frac{(1 + k)\tau/\tau_0}{1 + Pr^* A_\tau/\mu} d(y/r) \tag{13}$$

This equation represents the dimensionless temperature as a function of the dimensionless distance from the wall y/r. The factor of proportionality $\left(\frac{d\vartheta}{d(y/r)}\right)_0$ is obtained from the condition that for $y/r = 1$ we have $\vartheta = 1$.

A first approximation of ϑ is obtained by setting k equal to zero, λ equal to λ_0, and A_q equal to A_τ. Under the integral there then appear only the magnitudes τ/τ_0, A_τ/μ, and Pr, considered as initially given.

For computing a second approximation of ϑ, an equation is required for the heat flow appearing in the numerator of equation (13). This equation is obtained from the continuity condition for the heat in the case under consideration. For example, in the case of the fully developed flows in a pipe or in a plane rectangular channel, the following relations hold:

$$\frac{q}{q_0}\left(1 - \frac{y}{r}\right) = 1 - \int_0^{y/r} \vartheta\varphi\left(1 - \frac{y}{r}\right) d(y/r) \Bigg/ \int_0^1 \vartheta\varphi\left(1 - \frac{y}{r}\right) d(y/r) \tag{14}$$

(pipe)

$$\frac{q}{q_0} = 1 - \int_0^{y/r} \vartheta\varphi \; d(y/r) \Bigg/ \int_0^1 \vartheta\varphi \; d(y/r) \quad \text{(channel)} \tag{15}$$

where $\varphi = u/U$ (ref. 14, p. 316).

Substituted here will be the value of ϑ of the first approximation of equation (13). The approximation of q/q_0, thereby obtained, yields a sufficiently accurate computation of the temperature distribution and of the heat transfer.

Formula (13) is particularly adapted for computations with small Prandtl numbers. For high Prandtl numbers for which the integrand becomes very small at even small distances y/r, it is more convenient to represent the temperature (at least as a first step) as a function of the velocity.

For such representation there is required the velocity as a function of the distance from the wall. This relation is to be obtained from equations (1), (3), (3a), and (7):

$$\frac{\tau}{\tau_0} \mu_0 \left(\frac{du}{dy}\right)_0 = \mu \frac{du}{dy} (1 + A_\tau/\mu) \tag{16}$$

After introducing a dimensionless velocity $\varphi = u/U$, which is referred to the maximum velocity at the edge of the boundary layer (at the distance y/r), the following equation is obtained for the velocity gradient:

$$\frac{d\varphi}{d(y/r)} = \left(\frac{d\varphi}{dy/r}\right)_0 \frac{\mu_0}{\mu} \frac{\tau/\tau_0}{1 + A_\tau/\mu} \tag{17}$$

Combining equations (12) and (17) and employing equations (10) yield

$$\vartheta = \left(\frac{d\vartheta}{d\varphi}\right)_0 \int_0^{\varphi} \frac{Pr \; c_{p_0}}{Pr_0 \; c_p} \frac{(1 + A_\tau/\mu)(1 + k)}{1 + Pr^* \; A_\tau/\mu} \, d\varphi \tag{18}$$

The proportionality factor $(d\vartheta/d\varphi)_0$ is obtained from the condition that for $\varphi = 1$ we likewise have $\vartheta = 1$. As regards the practical carrying out of the computations, the procedure with equation (18) is exactly the same as with equation (13)(see above).

For the particular case $Pr^* = 1$, equation (18) to a first approximation gives the equality of ϑ and φ.

(c) Formulas for Heat Transfer

It has become customary to express the heat transfer by dimensionless parameters. Generally used is the so-called Nusselt number, which is defined by the formula

$$Nu = \frac{q_0 d}{\lambda_0 \Delta T_u} \tag{19}$$

where d denotes a characteristic length, (for example, the diameter of a pipe) and ΔT_u is the temperature difference between the wall and a so-called mean flow temperature,

$$\Delta T_u = \frac{1}{f u_m} \int_0^f \Delta T u df$$

which refers to the quantity of heat f flowing through the flow cross section $\left(u_m = \frac{1}{f} \int u df\right)$.

Since in the present paper all temperature differences are referred to the maximum temperature difference Θ at the distance $y = r$, it is convenient to introduce a dimensionless temperature difference,

$$\vartheta_u = \Delta T_u / \Theta$$

Setting further $d = 2r$ and $q_0 = \lambda_0 (dT/dy)_0$, there follows from equation (19)

$$Nu \vartheta_u = 2\left(\frac{d\vartheta}{dy/r}\right)_0 \tag{20}$$

$\left(\frac{d\vartheta}{dy/r}\right)_0$ follows from equation (13) for $\vartheta = y/r = 1$. The following formula is then obtained for the Nusselt number:

$$Nu \vartheta_u = 2 \Big/ \int_0^1 \frac{\lambda_0}{\lambda} \frac{(1 + k)\tau/\tau_0}{1 + Pr^* A_\tau/\mu} d(y/r) \tag{21}$$

This formula holds for an arbitrary boundary-layer flow. In the case of laminar flow, $A_\tau = 0$, so that the denominator of the integrand is equal to 1. For turbulent flow also, this denominator can at

least be set approximately equal to 1 if the Prandtl number is extremely low and A_τ/μ is not too large (small Re number). The quantities q/q_0 and k are naturally different for the laminar flow than for the turbulent flow for vanishing Pr^*A_τ/μ (fig. 5).

The applicability of formula (21) is problematical, however, since the variation of the heat flow q/q_0, for the case under consideration, must be known; here again a knowledge of the temperature distribution is required. Practically, therefore, the determination of $Nu\vartheta_u$ is based on the somewhat inconvenient computation of the temperature distribution which was presented in a formal manner in the preceding section.

In order to be able to compute the heat transfer at the wall directly, a formula is therefore required in which only known or easily determinable magnitudes appear. Such a formula could be obtained through a transformation of equation (21) if it were possible to introduce the factor Pr^* into the numerator of the integrand. This, however, is unfortunately impossible, because Pr^* must be considered as a function of the distance from the wall through the magnitude A_q/A_τ contained in it. Still another formula for Nu will therefore be derived in which the desired factor Pr^* appears in the numerator of the integrand.

For this purpose, equation (12) is multiplied by $\frac{c_p}{c_{p_0}}\frac{A_q}{A_\tau}\,d\,\frac{y}{r}$; by substituting equation (10) and integrating, there is then obtained

$$\int_0^{\vartheta} \frac{c_p A_q}{c_{p_0} A_\tau}\,d\vartheta = \left(\frac{d\vartheta}{dy/r}\right)_0 \int_0^{y/r} \frac{\mu_0 Pr^*}{\mu Pr_0}\,\frac{(1+k)\tau/\tau_0}{1+Pr^*A_\tau/\mu}\,d(y/r) \tag{22}$$

For $\vartheta = y/r = 1$ this is a determining equation for the temperature gradient $\left(\frac{d\vartheta}{dy/r}\right)_0$ to be substituted in equation (20). Furthermore, using the notation[4]

$$\left(\frac{A_q}{A_\tau}\right)_m = \int_0^1 \frac{c_p A_q}{c_{p_0} A_\tau}\,d\vartheta \tag{23}$$

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[4]Considered here is the mean value $\frac{1}{\vartheta}\int_0^{\vartheta} \frac{c_p A_q}{c_{p_0} A_\tau}\,d\vartheta$ for the total-temperature range $0 < \vartheta < 1$.

yields

$$\mathrm{Nu}\vartheta_u = 2\left(\frac{A_q}{A_\tau}\right)_m \mathrm{Pr}_0 \Bigg/ \int_0^1 \frac{\mathrm{Pr}^*(1 + k)\tau/\tau_0}{1 + \mathrm{Pr}^* A_\tau/\mu} \frac{\mu_0}{\mu} \, d(y/r) \tag{21a}$$

This equation, in spite of its different form, agrees in content with equation (21); if A_q/A_τ is considered as a constant, equation (21a) transforms directly into equation (21).

The required transformation can now be made in equation (21a) by making the following substitution:

$$\frac{\mathrm{Pr}^*}{1 + \mathrm{Pr}^* A_\tau/\mu} = \frac{1}{1 + A_\tau/\mu} + \frac{\mathrm{Pr}^* - 1}{(1 + \mathrm{Pr}^* A_\tau/\mu)(1 + A_\tau/\mu)} \tag{24a}$$

Making use of equation (17) and setting[5]

$$\varepsilon = \int_0^1 \frac{\mathrm{Pr}^*(1 + A_\tau/\mu)}{1 + \mathrm{Pr}^* A_\tau/\mu} \, k d\varphi \approx \int_0^1 k d\varphi \tag{25}$$

yields, from equation (21a),

$$\mathrm{Nu}\vartheta_u = 2\left(\frac{A_q}{A_\tau}\right)_m \mathrm{Pr}_0 \left(\frac{d\varphi}{dy/r}\right)_0 \Bigg/ \left(1 + \int_0^1 \frac{(\mathrm{Pr}^* - 1)d\varphi}{1 + \mathrm{Pr}^* A_\tau/\mu} + \varepsilon\right) \tag{26}$$

The form of this equation may further be improved by collecting factors together. Since

$$\frac{\mathrm{Nu}\vartheta_u}{2r\mathrm{Pr}_0(d\varphi/dy)_0} = \frac{q_0 U}{\rho_0 c_{p0} \Theta u^{*2}} \tag{27}$$

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[5]For not too small distances from the wall for not too small Pr numbers $1 \ll A_\tau/\mu < \mathrm{Pr}^* A_\tau/\mu$, the integrand of equation (20) becomes approximately equal to k. Since in the immediate neighborhood of the wall, or small values of φ, $k = 0$ and the integrand thereby vanishes, the latter can be replaced by k practically for the entire region of the friction layer (hence, the approximation formula $\varepsilon = \int_0^1 k d\varphi$).

it is convenient to introduce a special heat-transfer parameter. A further question to be considered here, however, is whether the new dimensionless parameter is to be formed with u^* or with U.

The employment of U has the advantage that a good connection is made with older formulas. There will therefore be defined a "generalized heat-transfer coefficient" α^* through the equation

$$\alpha^* = \frac{q_0}{\rho_0 c_{p_0} \Theta U} \tag{28}$$

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whereby the heat flow at the wall is referred to the maximum value of the temperature difference and of the velocity in the boundary layer.

In place of equation (28) we can also write

$$\alpha^* = \frac{1}{Pr_0}\left(\frac{d\vartheta}{d\varphi}\right)_0 \left(\frac{u^*}{U}\right)^2 \tag{28a}$$

or

$$\alpha^* = \frac{Nu\vartheta_u\varphi_m}{Pr_0 Re} \tag{28b}$$

The expression $Nu/(Pr\ Re)$ is frequently used in technical literature. Since ϑ_u and φ_m lie in the neighborhood of 1 (figs. 7 and 9), α^* is of the same order of magnitude as $Nu/(Pr\ Re)$.

Using equation (28) gives, from equation (26),

$$\alpha^* = \left(\frac{A_q}{A_\tau}\right)_m \left(\frac{u^*}{U}\right)^2 \Bigg/ \left(1 + \int_0^1 \frac{(Pr^* - 1)d\varphi}{1 + Pr^* A_\tau/\mu} + \varepsilon\right) \tag{29}$$

This formula differs from equation (21), among other respects, in that here $1/(1 + Pr^* A_\tau/\mu) = q_m/q$ is integrated over φ, whereas in equation (21) the corresponding integration is made over y/r. Since

$$\int \frac{q_m}{q}\, d\varphi = \int \frac{q_m}{q} \frac{d\varphi}{d(y/r)}\, d(y/r)$$

and, on the other hand, $\frac{d\varphi}{d(y/r)}$ decreases with the distance from the

wall - for logarithmic velocity distribution $\frac{d\varphi}{d(y/r)}$ is proportional to $1/y$ - it follows that the integrand of equation (29) decreases at a considerably greater rate with distance from the wall, than the corresponding integrand of formula (21).

For this reason, in using formula (29) only the relations in the immediate neighborhood of the wall need be known if the Prandtl number is not too small. It is, therefore, a particular advantage of formula (29) that the knowledge of the results of the universal wall friction law, in general, is sufficient for its application. A further advantage is that formula (31) for $Pr^* = 1$ becomes particularly simple.

The magnitude $\varepsilon = \int_0^1 k d\varphi$ occurring in formula (29) is only a small magnitude with regard to the main region of application of this formula (medium and high Prandtl numbers). Hence, in spite of the existing gaps in our knowledge, the ε-term can accurately be computed if the flow is sufficiently well known. For the case of the fully developed flows in a pipe and in a plane rectangular channel, ε has been determined by the author for various Pr and Re numbers (these values are given in fig. 8).

Since the integrand k of the expression $\varepsilon = \int_0^1 k d\varphi$ differs appreciably from zero only for the parts of the boundary layer that are at a large distance from the wall, the individuality of the friction layer under consideration is expressed precisely in the ε-term. As may be seen from the smallness of the ε-values, the individuality of the flow in turbulent heat transfer plays only a small part if the Pr and Re numbers are not too small.[6]

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[6]The term $\varepsilon = \int_0^1 k d\varphi$ depends on the ratio $(q/q_0)/(\tau/\tau_0)$ (see eq. (10)). This ratio is to a large extent determined by the boundary conditions at the wall $(dq/dy)_0$, or $(d\tau/dy)_0$. In the case of pipe flow, $(q/q_0)/(\tau/\tau_0)$, and, therefore, k and ε, is relatively large, since on account of the boundary conditions $(d\tau/dy)_0 < 0$ and $(dq/dy)_0 > 0$ the distributions q/q_0 and τ/τ_0 deviate relatively widely. In the case of the plane channel flow, q/q_0 and τ/τ_0 no (Continued on next page.)

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The principal problem of the turbulent heat transfer lies in the integral expression $\int_0^1 \frac{(Pr^* - 1)}{1 + Pr^* A_\tau/\mu} d\varphi$ of formula (29). It is therefore justifiable to introduce a special notation for this expression. It would be of advantage if the factor $(Pr^* - 1)$ could first be taken outside the integral. Because of the local variability of A_q/A_τ and of the temperature dependent Pr, this unfortunately cannot, in general, be done.

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Accordingly, $(Pr^* - 1)$ can be compensated only under the integral sign, and for this purpose it will be divided by the factor $(Pr_g - 1)$, where Pr_g denotes a mean value of the Prandtl number for the boundary layer near the wall. With regard to a later application of the universal wall friction law (see below) it is convenient, in addition, to replace the velocity φ by the universal velocity u/u^*, where $u^* = \sqrt{\tau_0/\rho_0}$ and is the so-called shear stress velocity. The following magnitude b will therefore be defined as follows:[7a]

$$b \equiv \int_0^{U/u^*} \frac{(Pr^* - 1)/(Pr_g - 1)}{1 + Pr^* A_\tau/\mu} d(u/u^*) \tag{30}$$

When this expression is used, equation (31) is obtained from equation (29):

$$\alpha^* = \frac{(A_q/A_\tau)_m (u^*/U)^2}{1 + (Pr_g - 1)b(u^*/U) + \varepsilon} \tag{31}$$

longer differ so greatly, since in this case $(d\tau/dy)_0 < 0$ and $(dq/dy)_0 = 0$. Hence, the ε-values for the channel are smaller than the corresponding values for the pipe, as may be seen from figure 8.

In the flow at the flat plate, the distributions of q/q_0 and τ/τ_0 coincide approximately over the greater part of the φ-region, because here we have $(d\tau/dy)_0 = (dq/dy)_0$. From this it follows that the ε-values for the case of the plate are still considerably smaller than the corresponding values of the plane channel flow. Hence, a general neglect of the ε-term for the flow along the plate should be permissible.

[7a]This magnitude, which in previous papers was denoted by a, has been redenoted by b in order to avoid confusion with the notation prescribed by the standard regulations (DIN 1341) for thermal diffusibility α.

In this representation of the turbulent heat transfer, the basic form of the old Prandtl equation (see eq. (41)) is expressed clearly.[7b]

In order to obtain this customary form of the representation, the heat-transfer coefficient α^* would be formed with the maximum velocity U (see eq. (28)). There is still the possibility, however, of referring the heat-transfer coefficient to the shear-stress velocity u^*, hence, of forming a dimensionless $q_0/(\rho_0 c_{p_0} \Theta u^*)$ (as was previously indicated). We shall, therefore, consider briefly this heat-transfer coefficient.

Since the identity relation

$$\frac{q_0}{\rho_0 c_{p_0} \Theta u^*} \equiv \frac{Nu \vartheta_u}{2Pr_0 \eta_r} \tag{32}$$

holds, we may write

$$\frac{Nu \vartheta_u}{2Pr_0 \eta_r} = \alpha^* \frac{U}{u^*} \tag{33}$$

in place of equation (28). With the aid of formulas (31) and (33), the Nusselt number can be directly determined without computing the temperature distribution. The practical application possibility of these formulas is, however, connected with the condition that the product $Pr\ \eta_r$ does not exceed a certain limiting value (more details on this are given in the section on the problem of the heat-conducting layer.

The validity of the formulas derived in the present section is dependent only on the single assumption that the heat flow and the momentum flow have approximately the same direction at right angles to the wall. This assumption is practically satisfied in all setups for which discontinuous changes of the wall temperature are avoided.

In order to apply the general formulas, specific assumptions must in addition be made on the magnitudes A_q/A_τ, A_τ/μ, and τ/τ_0, considered as "given". This will be done in the following section.

[7b]In previous papers there is found in place of $(u^*/U)^2$ the so-called resistance coefficient ζ, which is connected with u^*/U by the following equation: $(u^*/U)^2 = \tau_0/(\rho U^2) = 0.125\, \varphi_m^2 \zeta$.

3. INTRODUCTION OF SPECIAL ASSUMPTIONS

(a) Ratio of Exchange Magnitudes A_q/A_τ

Up to the year 1932, in which Fage and Falkner (ref. 5) observed in the wake of a heated cylinder the various possibilities of the propagation of heat and momentum, the general idea has been held that the turbulence mechanism for the momentum transport and for the heat transport are identical. The tests carried out by Fage and Falkner at the instigation of Taylor showed, however, that the heat is more strongly propagated than the momentum. This result supported the so-called vorticity transfer theory of Taylor (ref. 19), in which different mechanisms are postulated for the mass or heat transport and for the momentum propagation. 3871

The stronger propagating ability of heat as compared with the momentum transport was also confirmed later in free jets. According to the investigations so far available, the ratio of the two exchange magnitudes in free turbulence is given by $A_q/A_\tau \approx 2$.

These results had as yet no effect on the theory of heat transfer in turbulent friction layers. In this connection, there were considered the velocity and temperature measurements of F. Elias in a turbulent airstream along a heated plate (ref. 3, 1930). From these tests a far reaching congruence was obtained of the temperature and velocity profiles in friction layers. This result was looked upon as a confirmation of the previously held conception of the identity between momentum exchange and heat transport.

This conclusion from the measurement results of Elias was, however, in error, as was shown by the author in the year 1940. From the agreement in the dimensionless φ and ϑ profiles (or from $(d\vartheta/d\varphi)_0 \approx 1$), the agreement of A_q and A_τ holds only under the assumption that $Pr = 1$. In the tests of Elias, who used air as the flow medium, Pr was approximately 0.72. For this Prandtl number and for $(d\vartheta/d\varphi)_0 = 1$, it follows from equation (18), however, that $A_q/A_\tau \approx 1/0.72 \approx 1.40$, that is, a value considerably greater than 1.

Tests of Lorenz and Friedrichs (ref. 9), who measured the distributions of the velocity and temperature of an air stream in a heated pipe, likewise led to a similar result. (Here the author obtained the value $A_q/A_\tau \approx 1.5$ from the experimentally found value $(d\vartheta/d\varphi)_0 \approx 0.97$.)

These tests show, therefore, that the heat exchange in friction layers is also greater than the momentum exchange. The ratio $A_q/A_\tau \approx 2$

observed in free turbulence is, however, not attained in friction layers. For this case we apparently have

$$1 < (A_q/A_\tau)_m < A_q/A_\tau < 2 \tag{34}$$

This is supported by the following considerations:

The fact that $(A_q/A_\tau)_m$ in friction layers is smaller than in free turbulence permits concluding that the lowering of this ratio is a consequence of the effect of the wall. Such an effect is naturally stronger in the immediate neighborhood of the wall than at a point at a large distance from the wall. If, therefore, the ratio of the exchange magnitudes is lowered through the effect of a wall, it follows that A_q/A_τ decreases with decreasing distance from the wall (or increases with increasing distance from the wall).

The value of the magnitude that A_q/A_τ apparently approaches at a large distance from the wall is 2, which is observed in free turbulence. On the minimum value of A_q/A_τ, in the immediate neighborhood of the wall, no experimental observations are as yet available for this region; therefore, a hypothesis must be assumed. The most probable assumption with regard to the neighborhood of the wall would be that only a single exchange mechanism holds at the wall, at which, therefore, $(A_q/A_\tau)_0 = 1$. This leads us back to the old hypothesis of the equality of the exchange mechanisms, but with the restriction that this identity of heat exchange and momentum exchange is assumed only for the immediate neighborhood of the wall.

There will now be considered the consequences that follow from this hypothesis. The magnitude A_q/A_τ appears in equations (31) and (30), where it is contained in the magnitude $Pr^* = A_q/A_\tau \, Pr$. Equation (30) will be considered first. Under the assumption of an explicit wall friction law (for large Pr and Re numbers), the integrand of equation (30) exists only for the immediate neighborhood of the wall, for which there is to be set $A_q = A_\tau$. In equation (30), therefore, Pr^* can be replaced by Pr. This is of great use in the further considerations.

The factor $(A_q/A_\tau)_m$, appearing in the numerator of expression (31), is represented by equation (23). This formula may be simplified by setting approximately $c_p = c_{p0}$ (on account of the weak temperature dependence of the specific heats, this simplification should always be permissible). In place of equation (23) we can, therefore, write approximately

$$\left(\frac{A_q}{A_\tau}\right)_m = \int_0^1 \frac{A_q}{A_\tau} \, d\vartheta = \int_0^1 \frac{A_q}{A_\tau} \frac{d\vartheta}{d(y/r)} \, d\,\frac{y}{r} \tag{23a}$$

The factor $(A_q/A_\tau)_m$, therefore, represents a mean value of the ratio of the exchange magnitudes over the entire region of the friction layer. This mean value is to be formed directly over ϑ or indirectly over y/r; the variable y/r is introduced because A_q/A_τ does not depend directly on ϑ, but must be regarded as a function of y/r (or of $\eta_r y/r$). (For a preassigned Reynolds number, $\eta_r = ru^*/\nu$ is a constant, as will be seen later.)

Since the variation of $\frac{A_q}{A_\tau}(y/r)$ is as yet unknown, only statements of a general character can be made at first with regard to the mean value of this ratio. For example, from equation (23) we may derive the result that the condition (34) is satisfied if A_q/A_τ has the limiting value 1 at the wall and the value 2 at the edge of the friction layer.

A further result is that the Prandtl number plays a part since ϑ and $d\vartheta/d(y/r)$ depend on this number. There may first be considered the relations for a large Pr where the temperature as is known is sufficiently well balanced over the flow cross section (fig. 4). The temperature gradient is, therefore, approximately zero over the cross section. In the layer near the wall, however, $d\vartheta/d(y/r)$ reaches considerable values. On account of the small thickness of this layer, however, A_q/A_τ may be treated as a constant, and according to the assumption may be set equal to 1. There follows, therefore, from equation (23a)

$$(A_q/A_\tau)_m \rightarrow 1 \quad \text{for} \quad Pr \rightarrow \infty$$

The value $(A_q/A_\tau)_m$ thus attains the assumed minimum value at high Prandtl numbers. On the other hand, from equation (23a) it is seen that as this mean value deviates more strongly from 1 the sharper the temperature profile at relatively large distances from the wall where A_q/A_τ approaches the assumed maximum value 2. According to our hypothesis, therefore, the smaller the Prandtl number, the greater the values of $(A_q/A_\tau)_m$ to be expected.

An explanation of the questions raised here, also in particular the question of the effect of the Reynolds number on $(A_q/A_\tau)_m$, can be provided only by experiment. Of the test results thus far available, the measurements of the temperature and velocity distributions undoubtedly favor the previously presented view on the effect of the wall on the ratio of the exchange magnitudes.

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Under the assumption that in the denominator of equation (18) we can set $Pr^* = Pr$ and with account taken of a new determination of ε (ref. 15), there follows both from the test results of Elias on the flat plate and from the results of Lorenz and Friedrichs (ref. 9) in the pipe according to equation (18):

$$(A_q/A_\tau)_m \approx 1.3 \quad \text{for} \quad Pr \approx 0.72$$

3871 According to the above considerations, this mean value, on account of the relatively small Prandtl number of the air, appears plausible as regards its order of magnitude.

(b) Ratio of Friction Coefficients A_τ/μ

If we disregard the region of the boundary layer near the wall, the ratio of the friction coefficients A_τ/μ is an individual function of the distance from the wall for the case under consideration.

CK-3 For the fully developed flows through pipes and channels, A_τ/μ may be represented approximately by the following function (ref. 16):

$$\frac{A_\tau}{\mu\eta_r} = \frac{\varkappa}{3}\left[0.5 + (z/r)^2\right]\left[1 - (z/r)^2\right] \tag{35}$$

where $z/r = 1 - y/r$ is the distance from the axis of the pipe or channel (A_τ/μ must be symmetrical with respect to the axis), and $\varkappa$ is the universal constant known from the mixing-length theory[8] which has an approximate value of 0.4. The effect of Reynolds number is expressed through the factor $\eta_r = ru^*/\nu$. The function (35) is plotted in figure 1.

As may be deduced from formula (35) or seen from figure 1, $A_\tau/(\mu\eta_r)$ increases near the wall approximately linearly with the distance from the wall with the slope

$$\frac{A_\tau/(\mu\eta_r)}{y/r} = \varkappa$$

On account of $\eta_r = ru^*/\nu$, it follows that for A_τ/μ

$$A_\tau/\mu = \varkappa\eta \tag{36}$$

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[8]See L. Prandtl: Fuhrer durch die Stromungslehre. Braunschweig, 1949, pp. 118-119.

Near the wall, therefore, A_τ/μ is determined by the dimensionless distance from the wall $\eta = yu^*/\nu$, and the distance y/r no longer appears. It should be borne in mind, however, that formula (36) holds only for the "fully turbulent region"; that is, where A_τ/μ is sufficiently large. The turbulent exchange vanishes, however, not only at $\eta = 0$, but is actually already very small at a certain distance η.

With regard to these relations in the immediate neighborhood of the wall, the author has set up the following expression[9] (ref. 15): 3871

$$A_\tau/\mu = \varkappa\left[\eta - \eta_n \operatorname{Tang}(\eta/\eta_n)\right] \tag{37}$$

where η_n is a measure for the thickness of the viscous wall layer. The corresponding curve is shown in figure 2 for the values of the constants $\varkappa = 0.4$ and $\eta_n = 11$, derived from flow measurements. The distance for which $A_\tau/\mu = 1$ is $\eta \cong \eta_1 = 10.8$ (thus approximately $\eta_1 \approx \eta_n$).

Substituting expression (37) into the relation obtained from expression (17)

$$\frac{du/u^*}{d\eta} = \frac{\tau/\tau_0}{1 + A_\tau/\mu} \tag{38}$$

there is obtained for τ/τ_0[10] the universal velocity distribution u/u^*, which is likewise represented in figure 2. As may be seen from the position of the plotted test points of the author (ref. 15), this computed velocity variation is experimentally confirmed at least for not too small η-values (for very small η-distances the experiments are too inaccurate to serve as a test of the theory).

For extreme η-values equation (37) goes over into the following function:

$$A_\tau/\mu = \varkappa(\eta - \eta_n) \quad (\text{for} \quad \eta \gg \eta_n) \tag{37a}$$

$$A_\tau/\mu = (\varkappa/3\eta_n^2)\eta^3 \quad (\text{for} \quad \eta \ll \eta_n) \tag{37b}$$

Equation (37a) differs from (36) only in that the surface of the wall appears displaced toward the fluid by the thickness η_n of the viscous layer.

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[9]Where $\operatorname{Tang} X \equiv \dfrac{e^X - e^{-X}}{e^X + e^{-X}}$

[10]In the neighborhood of the wall where only the η-distance (and not y/r) is in question, we can set $\tau = \tau_0$.

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NACA Technical Note 4067

By Richard R. Heldenfels and Louis F. Vosteen
August 1957

Page 7: Equation (10) should be corrected as follows:

$$I_4 = \iint \left[\left(\frac{\partial^2 f_2}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f_2}{\partial y^2} \right)^2 + 2\mu \frac{\partial^2 f_2}{\partial x^2} \frac{\partial^2 f_2}{\partial y^2} + 2(1 - \mu) \left(\frac{\partial^2 f_2}{\partial x \, \partial y} \right)^2 \right] dx \, dy$$

Since from equation (36) or (37a) there follows the well-known logarithmic velocity profile (44), which was always observed for large η, the expression (37) corresponds to the experimental data for these values of η.

With regard to the variation of A_τ/μ for small η, only suppositions can be made. The turbulent longitudinal oscillations u' are to a first approximation proportional to the distance η. From this the result may be derived, for a two-dimensional oscillatory motion, with the aid of the continuity condition, that the transverse oscillations v' are proportional to η^2. Since $\tau_\tau = \rho\overline{u'v'}$, it follows from this consideration, if u' and v' are correlated, that $A_\tau \sim \eta^3$.[11] The expression (37) should, therefore, be physically questionable also with regard to its variation for small η (eq. (37b)).

In the application of equation (37), however, at least for very high Prandtl numbers, heat-transfer coefficients that were too high were computed (see more detailed discussion, sec. 5). From this it follows that the expression (37) for very small η-distances gives too high values of A_τ/μ. Probably the assumed correlation of the longitudinal and transverse fluctuations does not exist in the immediate neighborhood of the wall, and the power series for A_τ/μ first begins with a term of the fifth degree. The variation of A_τ/μ for the region $\eta < 6$ should, therefore, be corrected and written

$$(A_\tau/\mu)_\chi = 2.7 \cdot 10^{-5}\eta^5 \tag{39}$$

From $\eta > 9$ on the original variation according to expression (37) is maintained (see table I). In figure 2 the corrected variation of A_τ/μ is shown by the dotted curve.

This change of A_τ/μ has as a consequence a velocity profile (shown dotted) which hardly differs from the original profile (the deviation is at any rate considerably smaller than the scatter region of the test points). Although the change made in A_τ/μ for very small η-values hardly has any hydrodynamic effect, the thermal effect of this correction at high Prandtl numbers is quite considerable, as will be seen from the considerations in section 5.*

[11]The third power of η is the lowest power that can be considered. The expressions of K. Elser (ref. 4) and R. G. Deissler (ref. 2) which contradict it cannot, therefore, be corrected for the immediate neighborhood of the wall (here they give exchange values that are too high).

*Remark made during the proof correction:

In further investigations a new expression (Continued on next page.)

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(c) Computation of Temperature Distributions and Nusselt Numbers

The formulas given above served for the computation of the temperature profiles of turbulent flow in a pipe. This computation was made on the basis of equation (18) for different Prandtl numbers Pr (between 0 and 1000), which were locally invariable and for the Reynolds number $Re = 3 \times 10^4$ (or $\eta_r = 800$). For all Prandtl numbers there was thus assumed a definite velocity profile φ, namely, the profile of the fully developed flow for which $\tau/\tau_0 = 1 - y/r$.[12] The constants of the material Pr and c_p were assumed as constant.

The magnitude A_τ/μ occurring in equation (18) was determined for $\eta < 6$ from equation (39), for $\eta < 30$ from equation (37), and for $\eta > 50$ from equation (35). The temperature distributions of the first approximation ($k = 0$) determined by graphical integration served for the computation of the first approximations of q/q_0 according to equation (14). These distributions of the heat flux q/q_0, similarly obtained by graphical integration, are plotted in figure 5.[13]

From q/q_0 there is obtained $1 + k = \frac{q/q_0}{1 - y/r}$, and with the aid of this relation the ϑ-profiles can be computed to a second approximation. These second approximations of ϑ, which should not deviate much from the exact solutions, are plotted in figures 3 and 4.

By the same numerical approximation procedure, there was also computed the temperature distribution of the laminar flow with the aid of equation (13) (the parabolic velocity profile was assumed here). The profile obtained in this manner, shown dotted in figure 4, does not appreciably differ from that of L. Graetz (ref. 6) and W. Nusselt (ref. 11) computed by an exact method.

$$A_\tau/\mu = \varkappa\eta_n[\eta/\eta_n - \mathrm{Tang}(\eta/\eta_n) - 1/3\ \mathrm{Tang}^3(\eta/\eta_n)] \tag{39a}$$

was found to give good results for $\varkappa = 0.4$ and $\eta_n = 7.15$. The distance from the wall for which $A_\tau/\mu = 1$ is here found to be $\eta_1 = 10.7$. For very small η-values expression (39a) goes over asymptotically into

$$A_\tau/\mu = 0.2\ \varkappa\eta_n(\eta/\eta_n)^5 \approx 3\times10^{-5}\eta^5 \tag{39b}$$

[12]The numerical values of $\varphi(y/r)$ for different Re numbers are given in reference 15.

[13]The heat flux first increases with y/r because the cross section $2\pi(r-y)\Delta x$, through which the radial heat flows, decreases more strongly with y/r than the radial heat flow itself.

With the temperature distributions there were simultaneously obtained also the temperature gradients at the wall $\left(\frac{d\vartheta}{dy/r}\right)_0$, which according to formula (20) are equal to 0.5 $Nu\vartheta_u$. Also ϑ_u is obtained from the temperature computation (fig. 9). Table 2 gives the values of $Nu\vartheta_u$, ϑ_u, and Nu, and in figure 6 log Nu is plotted as a function of log Pr. As may be seen from this curve, the Nusselt number, with decreasing Prandtl number, asympotically approaches its limiting value for Pr = 0, which is indicated by a horizontal line (this value is about 5.22). The dotted line below it corresponds to the value Nu ≈ 3.77 of the laminar flow (according to the exact computation of Graetz and Nusselt, Nu = 3.66).

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Further, with the aid of equation (13) the Nusselt numbers were computed for vanishing Pr for various Re numbers (table 3). The differences in the Nusselt numbers are conditioned only by the difference in the velocity profiles.**

4. THE PROBLEM OF HEAT-CONDUCTING LAYER

As regards the computation of b according to equation (30), fundamental difficulties arise through the fact that the Prandtl number (particularly for viscous fluids) depends on the temperature which greatly varies precisely in the neighborhood of the wall. It is thus necessary to compute with a Prandtl number Pr* that depends on the distance η even if it is assumed that A_q and A_τ are identical.

Here we shall not, however, enter further into these complicated questions. The basic assumption of the complicated theory for variable "constants" of the material is the solution of equation (30) for unchanging values of these constants. In what follows, this case will now be considered.

**Added remark during proof correction:

At the conclusion of the above investigations a paper by R. N. Lyon appeared on the heat transfer of liquid metals (ref. 20). Lyon computed the heat transfer on the basis of continuity considerations, the assumption, among others, being made that the temperature gradient in the flow direction dT/dx is independent of the distance from the wall. The derived formula is applied to the region of small Prandtl numbers, which are characteristic of liquid metals. The Nu values of Lyon's theory are larger throughout than the corresponding values of the present study. For example, for the viscous flow, Lyon computes the value Nu = 4.36, whereas in the above computation the value 3.77 was obtained (and the exact value is 3.66). The heat-transfer coefficients obtained by Lyon are, on the average, smaller than his theoretical values.

Under the simplifying assumption that $Pr_g = Pr^* = \text{const}$, equation (30) with account taken of equation (9) goes over into

$$b = \int_0^{U/u^*} \frac{du/u^*}{1 + Pr\ A_\tau/\mu} = \int_0^{U/u^*} \frac{q_m}{q}\, du/u^* \tag{30a}$$

A rough approximation for b is obtained by starting from the earlier customary division of the friction layer into a "laminar" wall layer and a "fully turbulent" principal layer. In the laminar layer, which extends up to the distance η_a, $A_\tau/\mu = 0$, whereas in the turbulent region $(\eta > \eta_b)$ we have $A_\tau/\mu \to \infty$. In this idealization the integrand of equation (30a) is 1 for the laminar layer and 0 for the turbulent region. There is therefore obtained from equation (30a)

$$b = u_a/u^* = \varphi_a U/u^* \tag{41}$$

where $\varphi_a = u_a/U$ is the dimensionless flow velocity at the bounding edge of the laminar layer.

If this expression for b is substituted in equation (31), A_q/A_τ set equal to 1, and the ε-term neglected, there is obtained

$$\alpha^* = \frac{(u^*/U)^2}{1 + (Pr - 1)\varphi_a} \quad \text{(Prandtl equation)} \tag{41}$$

This, according to its physical content, is the old equation of L. Prandtl (ref. 12) for the turbulent heat transfer.

A better approximation for b is obtained by dividing the friction layer into three zones: the viscous layer at the wall, the laminar-turbulent transition region, and the fully turbulent region, already discussed in the introduction (ref. 14). We shall not, however, repeat the discussion here, but in what follows we shall apply expressions (37) and (39), which represent A_τ/μ as a continuous function of the distance η. In actuality, there exists no completely laminar layer, some turbulent exchange always exists even in the immediate neighborhood of the wall. This exchange, hydrodynamically considered, may be very small; but this does not mean that the minimum exchange has only a slight effect on the heat transfer.

At high Prandtl numbers, in spite of "practically" viscous flow in the immediate neighborhood of the wall, there takes place a predominantly turbulent heat transfer, as may be deduced from equation (8). If, for example, we substitute $A_\tau/\mu = 0.01$ and $Pr^* = 1000$ in this equation,

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there is obtained $q_t/q_m = 10$. In this case, therefore, the turbulent heat transfer is ten times as large as the molecular heat transfer, although the turbulent exchange A_τ is negligibly small as compared with the viscosity μ. From this example it is seen what errors are possible if the turbulent exchange in the immediate neighborhood of the wall is set exactly equal to zero.

This state of affairs makes it appear advisable to differentiate between a "predominantly viscous layer" and a "predominantly heat conducting layer". The viscosity is predominant in the region near the wall $0 < \eta < \eta_1$ if the distance η_1 is defined by the condition that τ_t and τ_m are equal there. On the other hand, the heat conduction is predominant in the region $0 < \eta < \eta_2$ if the distance η_2 is characterized by the fact that here $q_t = q_m$, or $Pr^* A_\tau/\mu = 1$ (see eq. (8)). The defining equations for η_1 and η_2 are therefore

$$\frac{A_\tau}{\mu}(\eta_1) = 1 \tag{42}$$

$$\frac{A_\tau}{\mu}(\eta_2) = 1/Pr^* \tag{43}$$

From these equations it follows that the layer thicknesses η_1 and η_2 are equal only for $Pr^* = 1$, and that $(A_\tau/\mu)_2$, and, therefore, η_2 itself, decreases with increasing Prandtl number. At very high Prandtl numbers η_2 lies deep within the predominantly viscous region near the wall. But with decreasing Prandtl number, the η_2-boundary is displaced far into the predominantly turbulent region.[14] At extremely low Prandtl numbers, the heat conduction exceeds the turbulent heat transfer in the entire region of the friction layer.

For the numerical determination of the magnitude b according to equation (30a), it is important to know at which point y_2/r the limit η_2 lies, so that it can then be decided which function for A_τ/μ to substitute in equation (30a). Since $y_2/r = \eta_2/\eta_r$, y_2/r depends on η_r or on the Re number.

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[14]If, in addition, the law for the wall friction holds in this region, equation (43) may with the aid of equation (37a) be transformed into

$$\eta_2 = \frac{1}{\varkappa Pr^*} + \eta_n \tag{43a}$$

This equation holds, however, only for sufficiently high Reynolds numbers.

Universal values of b, that is, not influenced by the Re number but determined only by the Prandtl number, are obtained if y_2/r is so small (smaller than 0.05) that within the limits of the heat conducting wall layer A_τ/μ or u/u^* depends practically only on η. For very small Prandtl numbers, for which η_2 according to equation (43) is very large, this condition can be satisfied, however, only for very large η_r values or extremely high Reynolds numbers.

This universal b was now computed for different Prandtl numbers, with application of formula (37), which represents A_τ/μ as a function only of η. This function serves to determine $q_m/q = 1/(1 + Pr\ A_\tau/\mu)$; q_m/q was plotted, however, not against η but against u/u^* (fig. 10). By graphical integration, $b = \int \frac{q_m}{q}\, du/u^*$ was then determined. These values of b are presented in table 4 and plotted in figure 11 as a function of log Pr.

Before discussing the result of this computation, a more detailed consideration will be given to figure 10, in which the q_m/q curves are shown. In this figure is also drawn a dotted horizontal line $q_m/q = 1/2$. Where this line intersects the q_m/q curves of the individual Prandtl lines, we therefore have $q_m = q/2$ or $q_m = q_t$. These points of intersection therefore characterize the respective limits of the region near the wall with predominant heat conduction. The abscissas of the intersection points represent the velocity u_2/u^* at the bounding edge of the heat conducting layer for the individual Prandtl numbers. As may be seen, u_2/u^* increases with decreasing Pr.

For large values of η for which the logarithmic velocity law

$$u/u^* = \frac{1}{\varkappa}\, ln\ \eta + const \tag{44}$$

holds, q_m/q may be computed in a simple fashion. From equations (30a), (36), and (44) there follows:

$$q_m/q = 0.5 + 0.5\ Tang\left[\frac{\varkappa}{2}\left(\frac{u_2}{u^*} - \frac{u}{u^*}\right)\right] \tag{45}$$

This q_m/q distribution, holding for very high Re numbers and small Prandtl numbers,[15] lies symmetrical to the point ($q_m/q = 0.5$; $u/u^* = u_2/u^*$),

[15]For example, for Pr = 0.01 we have $u_2/u^* = 19.4$ (table 4). For this value there follows from equation (43a) or (Continued on next page.)

as may also be seen from figure 10. Hence, for large Re and small Pr

$$b = \int_0^{U/u^*} \frac{q_m}{q}\, du/u^* \approx \frac{u_2}{u^*} \tag{46}$$

From this it follows further, if equation (44) is applied to u_2/u^*, equation (43a) (footnote 14) used for η_2, and the empirical values $\varkappa = 0.4$, const = 5.5 and $\eta_n = 11$ are substituted:

$$b \approx u_2/u^* = 5.75 \log (11 + 2.5/Pr) + 5.5 \quad (Pr < 0.1) \tag{47}$$

This formula shows that for very small Prandtl numbers, u_2/u^* or b is approximately a linear function of -log Pr. This is also seen from figure 11.

For medium and small Prandtl numbers, the relations are more complicated. The q_m/q distribution no longer lies symmetrical to the point considered, and a certain difference arises between b and u_2/u^*. This difference is not large, however, (as is seen from table 4 and fig. 11) so that, for high Prandtl numbers also, we may set approximately $b \approx u_2/u^*$.

Since for small u/u^* values $u/u^* \approx \eta$, we may also write for high Pr values:

$$b \approx u_2/u^* \approx \eta_2$$

For high Prandtl numbers, therefore, b is approximately equal to the thickness η_2 of the heat conducting layer.

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equation (44) (if we set $\varkappa = 0.4$) $\eta_2 = 261$. This position of η should lie very near the wall, that is, y_2/r should be small since the universal wall friction law here applied holds only under this assumption. If, in order to satisfy this condition, we set $y_2/r = 0.05$, then there is obtained $\eta_r = 5220$. To this value of η_r corresponds $U/u^* = 27.8$ and the Reynolds number $Re = 2\eta_r\varphi_m U/u^* = 2.5\times10^5$. Furthermore, $\varphi_2 = u_2/U = 0.70$. For $Pr = 0.001$ the corresponding values are $u_2/u^* = 25$ (compare table 4), $\eta_2 = 2510$, $\eta_r = 50{,}200$, $U/u^* = 33.5$, $Re = 3\times10^6$, and $\varphi_2 = 0.75$.

If we substitute the approximation formula $b = u_2/u^*$ in equation (31), the denominator of this equation becomes

$$1 + (Pr_g - 1)\varphi_2 + \varepsilon$$

where $\varphi_2 = u_2/U$ denotes the dimensionless velocity at the outer edge of the heat conducting layer. By comparing this expression with the denominator of the Prandtl equation $1 + (Pr - 1)\varphi_a$ (eq. (41)) where φ_a is the velocity at the boundary of the laminar layer, the following result is obtained: 3871

The new theory differs from the classical theory, among other respects, in the fact that in place of the velocity φ_a at the boundary of the laminar layer there enters a velocity bu^*/U, which is approximately equal to the velocity φ_2 at the boundary of the predominantly heat conducting layer at the wall.

Whereas the velocity φ_a of the classical theory represents a purely hydrodynamic magnitude, the velocity $bu^*/U \approx \varphi_2$ depends also on the Prandtl number, which is contained in the magnitude $b \approx u_2/u^*$. The magnitude b is, however, determined exclusively through the Prandtl number (that is, is independent of the Re number), only if the Re number is so large that the heat conducting layer lies in the immediate neighborhood of the wall, where the universal friction law holds (i.e., all magnitudes depend only on η). This condition, which will be numerically formulated below, is to be observed particularly for small Prandtl numbers.

Since at small Prandtl numbers and high Re numbers $Pr \approx 1/\varkappa\eta_2$ (see footnote 15), we can under these conditions for the factor $Pr\ \eta_r$ appearing in equation (33) write

$$Pr\ \eta_r \approx \frac{1}{\varkappa y_2/r} \tag{48}$$

If we now assume that the universal friction law still holds with sufficient accuracy in the region and in the neighborhood of $y_2 < 0.05\ r$, there follows from equation (48) with $\varkappa = 0.4$:

$$Pr\ \eta_r > 50 \tag{49}$$

If this condition is satisfied, the given b-values may be substituted in equation (31). The factor $bu^*/U \approx \varphi_2$ for such small Pr (or such large b) then also remains always smaller than 1.

If in equation (49) η_r is further replaced by Re by means of the identity relation $Re = 2\eta_r\varphi_m U/u^*$, there is obtained as condition for Pr Re

$$Pr\ Re > 100\ \varphi_m U/u^*$$

Using the numerical values in footnote 16, there is then obtained

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$$Pr\ Re > 2500 \tag{50}$$

Accordingly, the universal b may be applied if, for example, for $Pr = 0.1$; $Re > 2.5\times10^4$, or for $Pr = 0.01$; $Re > 2.5\times10^5$.

5. COMPARISON OF THEORY WITH EMPIRICAL RESULTS

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A good part of the extensive experimental material on turbulent heat transfer has been evaluated by the Prandtl formula (41). It was immediately recognized that classical theory does not correctly represent the experimental facts, in that the factor φ_a (which properly should be a function only of the Re number) is also dependent on the Prandtl number. It was therefore considered as an essential object of experimental investigation to determine the dependence of the magnitude φ_a on the parameters Pr and Re from tests.

The results of investigations in this direction by various authors, in connection with turbulent pipe flow, have recently been evaluated by B. Koch (ref. 8). Almost all the authors make use of the original (Prandtl) formulation that φ_a is proportional to $Re^{-1.8}$. M. ten Bosch (ref. 1), however, assumes $\varphi_a \approx Re^{-0.1}$. This formulation is undoubtedly the better one, because according to the very careful tests of J. Nikuradse (ref. 10) u^*/U can be represented for the usual region of Reynolds numbers by the equation

$$u^*/U = 0.125\ Re^{-0.1} \quad (10^4 < Re < 10^6) \tag{51}$$

From the series of empirical formulas, therefore, those of ten Bosch will be chosen and used for checking the theory.

The formula of this author is [NACA note: See appendix.]

$$\varphi_a = B\ Pr^{-0.185} Re^{-0.1} \quad \text{(ten Bosch, empirical)} \tag{52}$$

where B is a coefficient which has the value 1.40 for the heating of the fluid and the value 1.12 for the cooling of the fluid. This relation is supported by numerous test results of various investigators, in particular by the very careful measurements of G. Rohonczi (ref. 18).

If we set $\varphi_a = bu^*/U$ in equation (52), there is obtained from equations (51) and (52) an empirical formula for b independent of Re:

$$b = 8\ B\ Pr^{-0.185} \quad \text{(ten Bosch, empirical)} \tag{53}$$

Since ten Bosch developed equation (52) from measurements at medium and high Prandtl numbers, formula (53) derived from equation (52) must be compared with the theoretical b-values of the medium and large Prandtl numbers. For this purpose the b-values obtained through graphical integration from table 4 will be represented by an approximation formula. This formula is

$$b = 10\ Pr^{-0.30} \quad (1 < Pr < 200) \tag{54}$$

A comparison of equations (53) and (54) shows that for Prandtl numbers in the neighborhood of 1 the theoretical b satisfactorily agrees with the experimental b (the mean value of the factor B, which lies between 1.12 and 1.40, is 1.26). For high Prandtl numbers, however, the theoretical b-values are too small (for Pr = 100, for example, by the factor 0.6), that is, the theoretical heat-transfer coefficients are too high.

This evident deficiency of the theory for high Prandtl numbers for which the heat conducting layer lies entirely within the viscous wall layer, is only to be explained by the circumstance that the expression of equation (37)(which for the neighborhood of the wall goes over into eq. (37b)), used for the computation of b, gives too high exchange values for the predominantly viscous region. For this reason, as already stated in paragraph 3, section (b), for the immediate neighborhood of the wall ($\eta < 6$) there was set up expression (39), which gives considerably smaller values of A_τ/μ (fig. 2).

If these corrected values of A_τ/μ are substituted in equation (30a) for the region $\eta < 6$, there are obtained the corrected b-values, which are denoted by b_k and which are likewise given in table 4 and figure 11. As is shown by figure 11, the dotted b_k curve lies above the b-curve. At high Prandtl numbers b_k is considerably larger than b, while the difference at medium Prandtl numbers is insignificant (for $Pr < 0.1$ we have practically $b_k = b$).

For the region of medium and high Prandtl numbers the previously given b_k values may be represented by the following approximation formula:

$$b_k = 9.12\ Pr^{-0.20} \quad (1 < Pr < 200) \tag{55}$$

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This formula agrees approximately with the empirical formula (53). At any rate, the differences between equations (53) and (55) lie within the limits of the error range of the experimental results.

A further correction of the assumed expression for the values of A_τ/μ near the wall at the present time would also have no significance since the check of any theoretical fine details, in view of the uncertainty as to A_q/A_τ and the uncertainties of the measured values, is quite impossible. In addition, it is to be borne in mind that the magnitude b (eq. (30)) is influenced by the temperature dependence of the Prandtl number (and, therefore, accurately speaking, is no universal constant)[16], and that this influence must be given theoretical treatment.

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The assumption by ten Bosch and other investigators that the factor B depends on the temperature gradient, in view of the structure of formula (30), appears entirely plausible. Whether the problem, however, can be so simplified that two B-values are sufficient (for heating and cooling the fluid, respectively) is undecided.

6. SUMMARY

1. Under the assumption of a uniformly smooth wall, general formulas are derived for the temperature distribution in turbulent friction layers and for the turbulent heat transfer. In these formulas the following problematical magnitudes occur: the ratio of the turbulent to the viscous friction (A_τ/μ) and the ratio of the exchange magnitudes for the heat and for the momentum (A_q/A_τ).

2. In order to solve practical problems, concrete expressions are required for A_q/A_τ and for A_τ/μ. As regards the ratio of the exchange magnitudes, it was found from available tests that A_q/A_τ in boundary layers is lowered as compared with its value of about 2 in free turbulence. It is then assumed that A_q/A_τ with decreasing distance from the wall decreases from the value of about 2 at the outer edge of the boundary layer to the value 1 in the immediate neighborhood of the wall.

3. For the ratio of the friction coefficients A_τ/μ in the various regions of the boundary layer, formulas are available which the author has developed in previous work on flow investigations (eqs. (35) and (37)). These formulas express the fact that the transition from the predominantly turbulent friction in the nucleus of the boundary layer to

[16]The given b-values hold, therefore, only for very small temperature differences.

the predominantly viscous friction in the immediate neighborhood of the wall takes place continuously.

With the aid of these formulas for A_τ/μ, the temperature variation ϑ over the radius of a pipe (figs. 3 and 4), the corresponding distribution of the heat flow q/q_0 (fig. 5), and the Nusselt number (fig. 6) are computed by an already tested approximation method for a wide range of Prandtl numbers.

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4. The continuous transition from the predominantly turbulent to the predominantly molecular friction corresponds to a similar continuous transition from the predominantly turbulent to the predominantly molecular heat transfer in the neighborhood of the wall. The dimensionless thickness η_2 of the wall layer with predominant heat conduction agrees, however, with the thickness η_1 of the predominantly viscous wall layer only for the Prandtl number $Pr = 1$. For high Prandtl numbers the heat conducting layer lies deeply within the approximately viscous layer; whereas the heat conducting layer for very small Prandtl numbers may extend over the entire range of the boundary layer.

5. A computation of the heat transfer at the wall according to a general formula (eq. (31)) is possible without preliminary computation of the temperature profile if the heat conducting layer is restricted to the region near the wall in which the universal wall friction law holds. This assumption is approximately satisfied if the product of the Prandtl number by the Reynolds number is greater than about 2500.

6. In the general formula for turbulent heat transfer (eq. (31)), there occurs a factor $bu^*/U = \varphi_2$ which corresponds to the factor φ_a of the Prandtl formula (41). Whereas the magnitude φ_a of the classical theory denotes the dimensionless velocity at the boundary of the fictitious "laminar layer" and thus represents a pure flow magnitude, φ_2 is approximately equal to the velocity at the boundary of the heat conduction layer; φ_2 is not a purely hydrodynamic quantity, since the factor b depends on the Prandtl number.

For very large Prandtl numbers the magnitude b is approximately equal to the dimensionless thickness η_2 of the predominantly heat conducting layer.

7. The magnitude b which is independent of the Re number, depends on the heat-flow distribution q_m/q. This distribution (fig. 10) was computed using the expression (37) for A_τ/μ. The b-values thus obtained (table 4 and fig. 11) agree for medium Pr numbers with the

experimentally obtained values as is shown by the comparison with an empirical formula of ten Bosch. For high Prandtl numbers, however, too small b-values, that is, too large heat-transfer coefficients, are obtained by computation. From this it follows that expression (37) gives too high values of A_τ/μ for small distances from the wall ($\eta < 6$).

For the region $\eta < 6$ the values of A_τ/μ were therefore corrected in the direction of a still stronger decrease of the turbulent exchange at the wall (eq. (39)). The corrected values thus obtained b_k (table 4 and fig. 11) also for high Prandtl numbers, are quite well represented by the empirical formula.

T/RC

SYMBOLS

A	turbulent exchange magnitude, kg/m h
A_τ	for momentum $\left(\tau_t = A_\tau \frac{du}{dy}\right)$
A_q	for heat $\left(q_t = c_p A_q \frac{dT}{dy}\right)$
B	factor used by M. ten Bosch
b	factor used by H. Reichardt
c_p	specific heat at constant pressure, kcal/kg (°C)
d	diameter, m
f	flow cross section, m^2
$k = q\tau_0/q_0\tau - 1$	
q	heat flux (q_0 at the wall; q_m molecular heat flow), $kcal/m^2h$
r	radius of pipe or channel (flat sides) or thickness of the friction layer, m
T	(absolute) temperature, °K
$T - T_0$	temperature difference between flowing medium and wall (Θ maximum temperature difference), °C

U	maximum value of the u-velocity, m/h
u	flow velocity, m/h
u_m	u-velocity averaged over flow cross section, m/h
$u^* = \sqrt{\tau_0/\rho_0}$	shear stress velocity, m/h
u, u', v'	flow velocities (u or $\bar{u}$ mean velocity; u' fluctuating velocity parallel to the wall; v' fluctuating velocity at right angles to wall), m/h
x, y	coordinates (x parallel to wall; y distance from wall), m
$z = r - y$	distance from pipe axis or from middle plane of channel, m
$Re = u_m d/\nu$	Reynolds number
$Pr = \mu\, c_p/\lambda$	Prandtl number
Pr_g	Prandtl number referred to a suitable mean temperature of boundary layer near wall
$Pr^* = \frac{A_q}{A_\tau} \cdot Pr$	generalized Prandtl number
$Nu = \frac{q_0 d}{\lambda_0 \Delta T_u}$	Nusselt number
$\alpha^* = \frac{q_0}{\rho_0 c_{p0} U\Theta}$	generalized heat-transfer coefficient
$\varepsilon = \int_0^1 k\,d\varphi$	coefficient
ζ	resistance coefficient
$\eta = yu^*/\nu$	dimensionless distance from wall (η_1 thickness of predominantly viscous wall layer; η_2 thickness of predominantly heating wall layer; $\eta_r = ru^*/\nu$)

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Θ	maximum temperature difference, °C
$\vartheta = (T - T_0)/\Theta$	dimensionless temperature difference (ϑ_m referred to mean temperature, ϑ_u to mean flow temperature)
$\varkappa$	constant of universal wall friction law
λ	thermal conductivity, kcal/m h C
μ	viscosity, kg/m h
$\nu = \mu/\rho$	kinematic viscosity, kg/m h
ρ	density of flow medium, kg/m^3
τ	shear stress (τ_0 at wall; $\tau_m = \mu \frac{du}{dy}$ molecular shear stress; τ_t turbulent shear stress), kp/m^2
φ	$\varphi = u/U$, $\varphi_m = u_m/U$

Subscripts:

0	wall
1	boundary of predominantly viscous layer
2	boundary of predominantly heat conducting layer
a	boundary of laminar layer
g	boundary layer (near the wall)
k	corrected magnitude
m	molecular transfer
m	mean value
n	reference length
r	pipe axis
q	heat flow
t	turbulent transfer
u	flow medium
τ	momentum flow

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APPENDIX

DISCUSSION CONCERNING A CORRECTION TO THE TEXT

The following extracts from an exchange of letters between Dr. Grigull and Dr. Reichardt are included in this translation at the request of Dr. Reichardt.[17]

Dr. Grigull to Dr. Reichardt, March 4, 1952:

I recently received your latest paper (Arch. ges. Wärmetechn. 2 (1951) 129/42) which I read with great interest. I noted, however, that the empirical values of ten Bosch which were used for comparison refer to the ratio u_a/u_m (eq. (52)), whereas you refer the Prandtl equation (41) to U. The values of ten Bosch are, therefore, larger by approximately the factor 1.2. If these values are reduced, the computed b-values will be found to lie between your original and the corrected b_k-values (table 4). This means that your expression for the exchange values is more correct than appears from your comparison.

I would appreciate it if you check my statements and let me know whether they are correct. I am particularly interested in this question, because I am preparing a new edition of my text book (Gröber-Erk-Grigull, Grundgesetze der Wärmeübertragung, 3rd revised edition, Springer-Verlag 1955) and would like to consider your theory in somewhat more detail since I consider it very suitable for arriving at a generally valid equation for turbulent heat transfer.

Dr. Reichardt to Dr. Grigull, March 13, 1952:

You are quite right in your criticism. I have overlooked the fact that the φ of ten Bosch refers to the mean velocity u_m, instead of the maximum velocity U which I used. In my equation (53) there must, therefore, appear the φ_m term. Thereby the deviation between the computed b-values and the experimental b-values at high Prandtl numbers becomes smaller.

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[17]Personal correspondence with T. M. Hallman, Lewis Flight Propulsion Laboratory, dated June 27, 1955.

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TABLE 1

η	A_τ/μ	$(A_\tau/\mu)k$
0.2	8.8×10^{-6}	9×10^{-8}
.4	7.1×10^{-5}	3×10^{-7}
.6	2.4×10^{-4}	2.1×10^{-6}
.8	5.6×10^{-4}	8.8×10^{-6}
1	.001	2.7×10^{-5}
2	.009	8.6×10^{-4}
3	.030	6.5×10^{-3}
4.5	.093	4.9×10^{-2}
6	.209	.193
9	.630	.630
11	1.05	1.05
12	1.30	1.30
15	2.14	2.14
18	3.12	3.12
21	4.19	4.19
24	5.31	5.31
36	10.0	10.0
48	14.8	14.8
67	22.8	22.8
100	35.6	35.6

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TABLE 2

[$Re = 3\times10^4$; $\eta_r = 800$]

Pr	$Nu\vartheta_u$	ϑ_u	Nu	log Nu
Laminar flow	2.18	0.578	3.77	0.576
0	2.50	.480	5.22	.717
.01	3.74 (4.11)	.515 (.515)	7.26 (7.97)	.861 (.902)
.1	12.8 (15.2)	.647 (.647)	19.7 (23.5)	1.30 (1.36)
1	63.2 (69.0)	.807 (.807)	78.3 (85.5)	1.89 (1.93)
10	226 (233)	.950 (.950)	238 (246)	2.38 (2.39)
100	593 (595)	.989 (.989)	600 (601)	2.78 (2.78)
1000	1291 (1327)	1.000 (1.000)	1291 (1327)	3.11 (3.12)

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TABLE 3

Re	$Nu\vartheta_u$	ϑ_u	Nu
3×10^4	2.51	0.496	5.05
3×10^4	2.52	.480	5.24
3×10^5	2.55	.470	5.43
3×10^6	2.58	.468	5.50

TABLE 4

Pr	0.001	0.01	0.1	1	10	100	1000
η_2	2510	261	35.6	10.8	4.60	2.18	1.05
η_{2_k}	2510	261	35.6	10.8	5.17	3.24	2.20
u_2/u^*	25.0	19.4	14.0	8.84	4.50	2.07	1.00
$(u_2/u^*)k$	25.0	19.4	14.0	8.81	5.05	3.25	2.06
b	25.1	19.5	14.1	9.10	5.04	2.49	1.18
b_k	25.1	19.5	14.1	9.19	5.56	3.47	2.20

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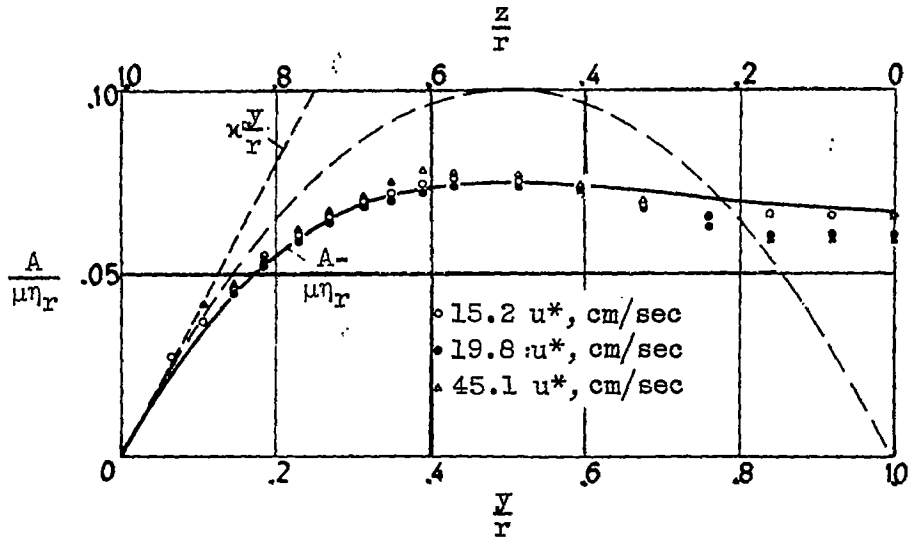

Figure 1. - Variation of dimensionless exchange magnitude $A\tau/\mu\eta\tau$ with distance from wall $y/r = 1 - z/r$.

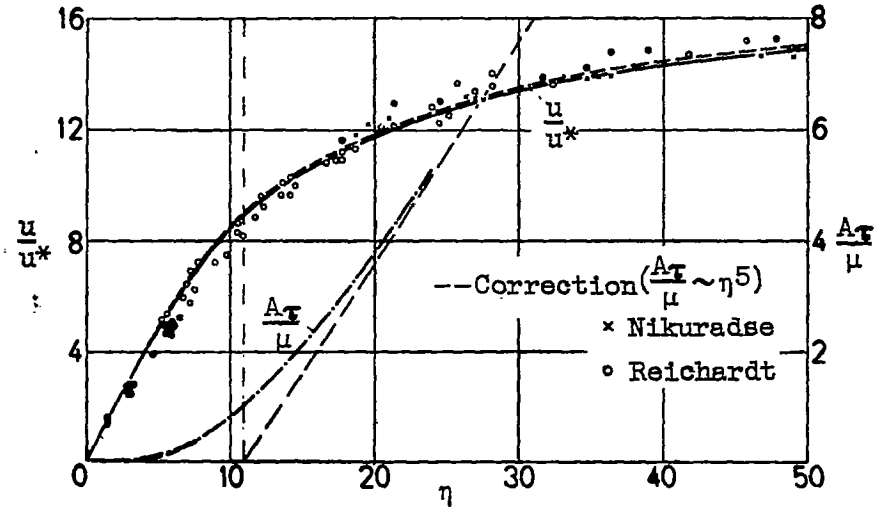

Figure 2. - Ratio of turbulence to viscous friction $A\tau/\mu$ and dimensionless velocity u/u^* as function of the dimensionless distance from the wall, $\eta = y\mu^*/\upsilon$.

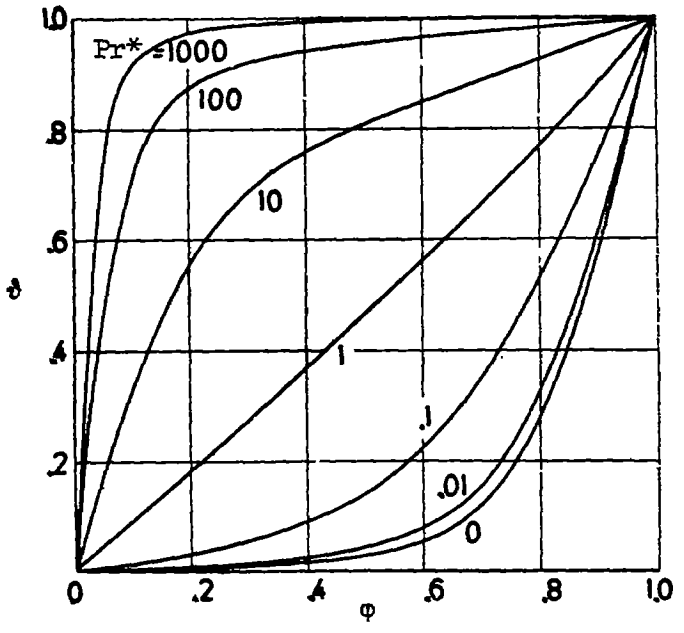

Figure 3. - Temperature distributions ϑ of turbulent flow in a pipe (second approximation) over the velocity φ for Re = 3×10^4 (η_r = 800) for various Prandtl numbers.

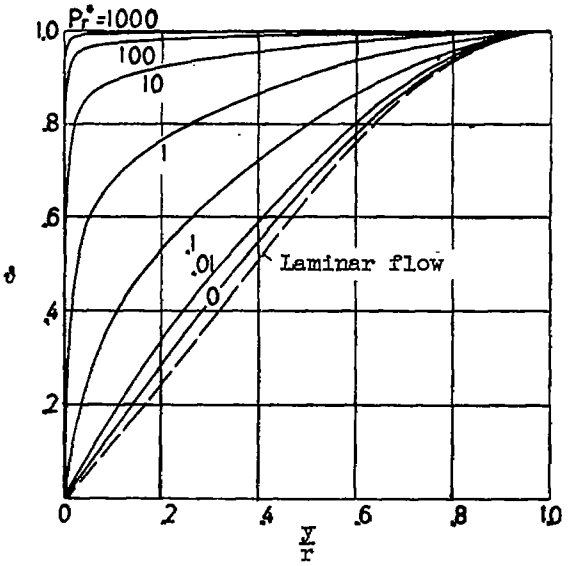

Figure 4. - Temperature distributions ϑ of the turbulent flow in a pipe over the dimensionless distance from the wall y/r. Re and Pr as in figure 3.

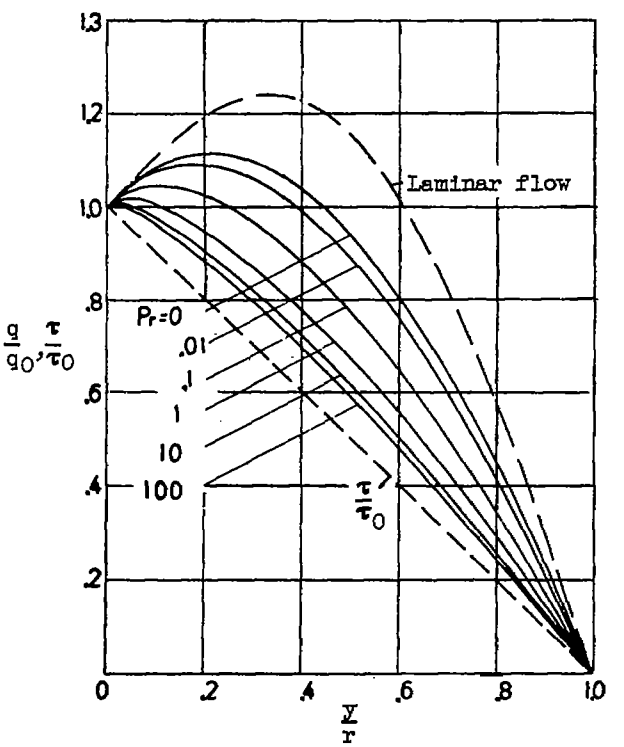

Figure 5. - Distributions of dimensionless heat flux q/q_0 and shear stress τ/τ_0 in a tube over the distance from the wall y/r (for Re, 3×10^4).

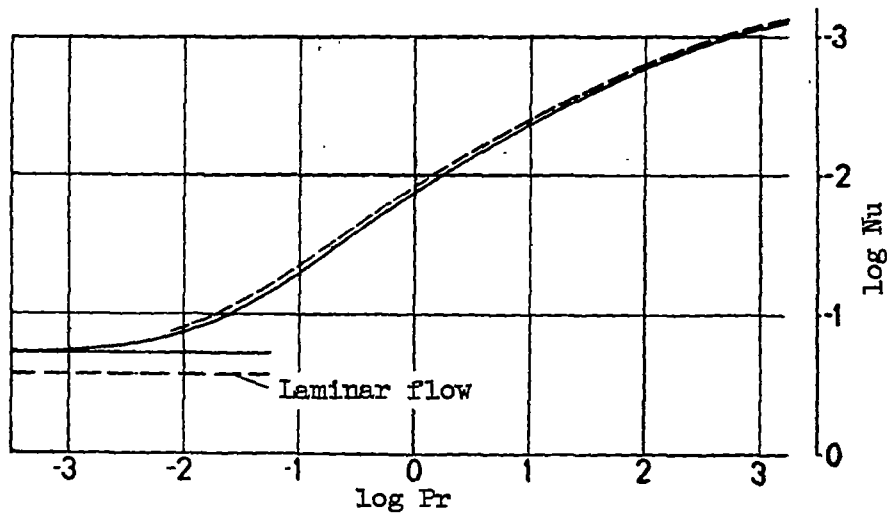

Figure 6. - The Nusselt number as a function of the Prandtl number for Re = 3×10^4. With decreasing Pr number, Nu approaches the limiting value 5.22 (solid horizontal). For laminar flow Nu = 3.77 (dotted horizontal).

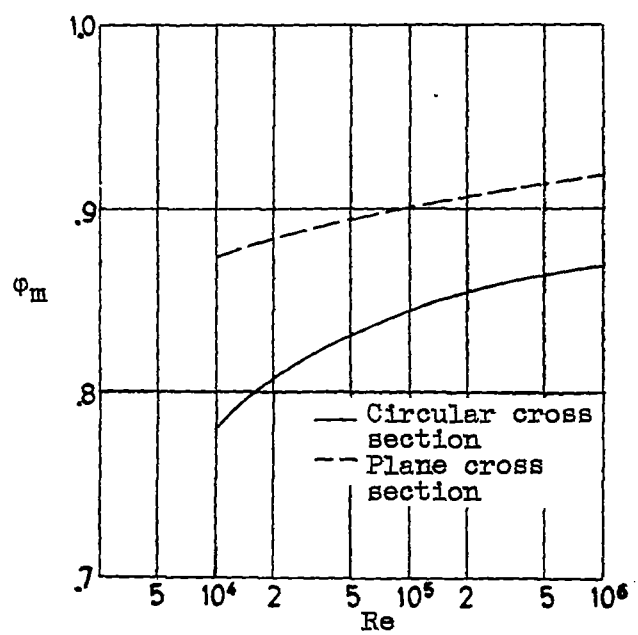

Figure 7. - Ratio φ_m of mean velocity u_m to maximum velocity U as a function of Re number.

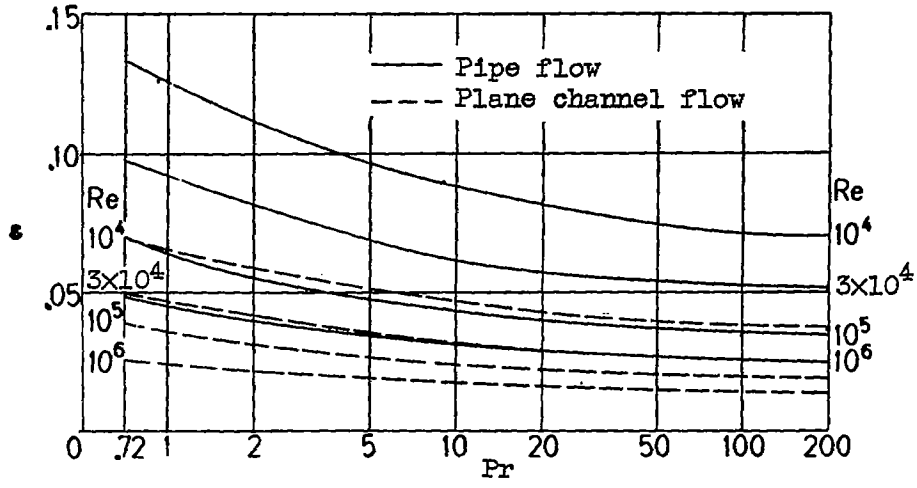

Figure 8. - Coefficient ε as a function of Prandtl number.

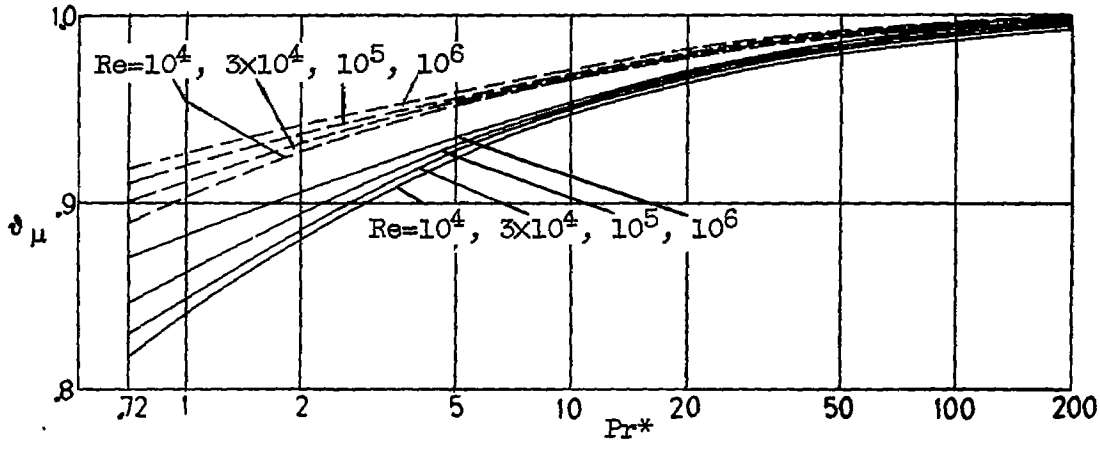

Figure 9. - Dimensionless mean flow temperature ϑ_μ as a function of the Prandtl number.

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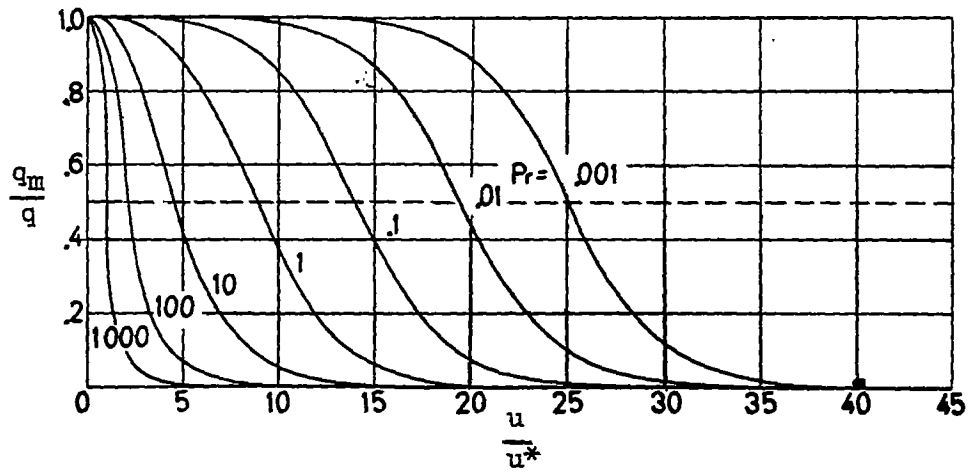

Figure 10. - Ratio of molecular heat flow q_m to total flow q as a function of dimensionless velocity u/u^* for various Prandtl numbers.

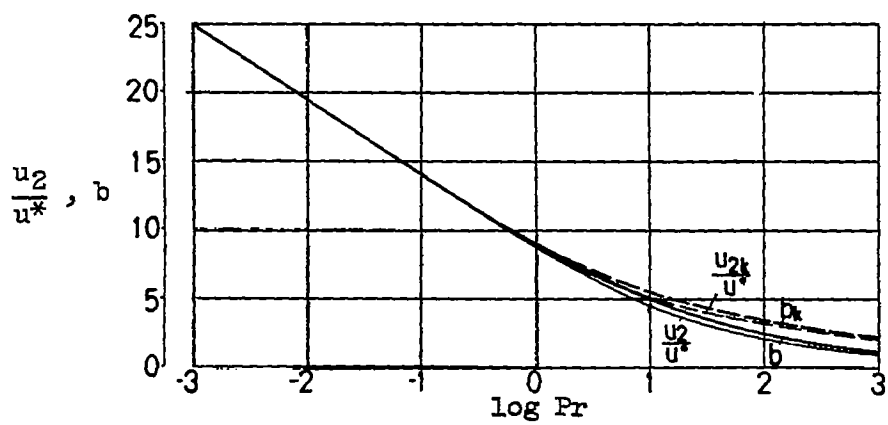

Figure 11. - Dimensionless velocity u_2/u^* at the boundary η_2 of the predominantly heat conducting layer and magnitude $b = \int_0 \frac{u}{u^*} \frac{q_m}{q} \frac{du}{u^*} \approx \frac{u_2}{u^*}$ as functions of the logarithm of the Pr number.