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REFLECTION AND REFRACTION OF ACOUSTIC WAVES BY A SHOCK WAVE

By J. Brillouin

Translation of "Réflexion et réfraction d'ondes acoustiques par une onde de choc."
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REFLECTION AND REFRACTION OF ACOUSTIC WAVES BY A SHOCK WAVE*

By J. Brillouin

SUMMARY

The presence of sound waves in one or the other of the fluid regions on either side of a shock wave is made apparent, in the region under superpressure, by acoustic waves (reflected or refracted according to whether the incident waves lie in the region of superpressure or of subpressure) and by thermal waves. The characteristics of these waves are calculated for a plane, progressive, and uniform incident wave. In the case of refraction, the refracted acoustic wave can, according to the incidence, be plane, progressive, and uniform or take the form of an "accompanying wave" which remains attached to the front of the shock while sliding parallel to it. In all cases, geometrical constructions permit determination of the kinematic characteristics of the reflected or refracted acoustic waves. The dynamic relationships show that the amplitude of the reflected wave is always less than that of the incident wave. The amplitude of the refracted wave, whatever its type, may in certain cases be greater than that of the incident wave.

1. BASIC CONSIDERATIONS

1.0. Generalities

1.0.0 Subject of the report.- The reflection and the refraction of sound waves by a shock wave are of interest not only in acoustics. These phenomena may also be utilized in aerodynamics to detect or measure the development of shocks.

The present paper tests the theory by application to a relatively simple case: that of a plane and uniform shock wave separating two regions in both of which the fluid is in pressure and temperature equilibrium, except for the acoustic phenomena; the incident acoustic wave is plane, progressive, and uniform.

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*"Réflexion et réfraction d'ondes acoustiques par une onde de choc." Acustica, vol. 5, no. 3, 1955, pp. 149-163.

This problem has already been treated in several publications. Burgers (ref. 1) and Blokhintzev (ref. 2), on the one hand, have studied analytically the case of normal incidence in a fluid without viscosity. They show that, in order to satisfy the boundary conditions, it is necessary to introduce, in addition to the reflected or refracted acoustic wave, a so-called "entropy" wave. This nonprogressive wave involves temperature fluctuations. It is the asymptotic form of a thermal wave in the expression of which the damping terms have been eliminated. The results regarding this wave thus obtained apply therefore only in the neighborhood of the shock wave.

Sauer (ref. 3), on the other hand, indicated the conclusions to be drawn from Huyghens' construction and demonstrated the existence of incidences for which no refracted wave of a usual type (that is, a purely longitudinal wave) can exist.

The purpose of the present report is to examine the points left undecided by the preceding authors, namely: the behavior of the thermal wave at large distance, and the nature of the phenomenon if there is no refracted wave of longitudinal type present. We have thus been led to treat the problem in a more general form: we made an analytic study, taking into account the viscosity and the caloric conductivity, for arbitrary incidence.

1.0.1 Basic hypotheses and approximations.- (a) In the calculation we shall consider the shock wave as a discontinuity. The thickness and the duration of the development of the shock are actually very small compared to the wave lengths and the lengths of the acoustic periods.

(b) The equations governing the shock wave in a steady state remain valid in the presence of acoustic waves. The slowness and the small amplitude of the acoustic phenomena compared to the shock phenomena justify this hypothesis, at least in the first approximation.

(c) In the expression of the boundary conditions, we neglect the small oscillations of the front of the shock wave about its mean position. This is an approximation which is justified in all acoustic problems. Due to the fact that the acoustic displacements are very small with respect to the wave lengths, the errors thus committed are of the second order.[1]

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[1]This hypothesis is implicitly contained in Burgers' calculation. This author calculates the motion of the shock wave by a method which is, incidentally, not applicable to the case of oblique waves; then, eliminating in the boundary conditions the terms of the second order, he achieves a simplification which corresponds to our hypothesis.

The boundary conditions will then be expressed in point form, that is, that the relationships expressing them contain only the values of the quantities concerned taken from both sides of the boundary (front of the shock wave) at one point of that boundary.

1.0.2 Irreversibility of the shock wave and its consequences.- The development of a shock wave is irreversible. Thus it is evident beforehand that it will not be possible to satisfy the boundary conditions if in the two regions separated by the shock wave only acoustic phenomena, that is, adiabatic phenomena, are introduced.

As a result, the small motions produced by the meeting between a shock wave and an acoustic wave will be expressed in their general form as it results from the work of Stokes and Kirchhoff, that is, taking into account the viscosity and the thermal conductivity of the gases.

1.0.3 Separation of the boundary conditions.- Since these conditions are in point form, one can divide them into two groups.

(a) Kinematic conditions which govern the nature and form of the waves produced by reflection or refraction: When all the waves concerned are plane, progressive, and uniform, these conditions may be put in the geometric form of Huyghens' construction. In the other cases it is necessary to resort to calculation in order to determine the nature and the form of the waves produced. This is what is done for instance in optics in order to treat the problem of total reflection and to obtain the Cauchy wave.

The kinematic conditions simply express the fact that on the boundary surface, the development in space and time of the phenomena associated with the reflection or the refraction is the same. Thus it is not necessary to use, for expressing them, the theoretical relationships which govern the shock wave.

(b) Dynamic conditions which come into consideration only for calculating the amplitude of the phenomena whose nature has been determined by the kinematic conditions.

We shall study these two groups of conditions separately.

1.1. Characteristics of the Shock Wave.

Boundary Conditions

1.1.0 Definitions, notation.- In designating the two regions of the fluid separated by the shock wave we shall avoid using the terms upstream

and downstream which, in aerodynamics as well as in hydraulics, give rise to ambiguity. If the shock wave is motionless (wind tunnel, hydraulic jump), the region downstream is under superpressure. If the wave is propagated in a fluid at rest (shock tube, tidal wave), the upstream region is under superpressure.

Thus our notation will be

E_0 = region under subpressure, extending from S toward the positive x

E_1 = region under superpressure, extending from S toward the negative x

S = front (plane) of the shock wave separating the two regions, perpendicular to the axis of the x

We shall use three systems of axes:

ox_0yz fixed with respect to the fluid in the region E_0

ox_1yz fixed with respect to the fluid in the region E_1

$oXyz$ fixed with respect to the front of the shock wave S

With

$a_0 > 0$ being the velocity of propagation of the front S in the fluid contained in E_0

$a_1 > 0$ the velocity of propagation of S in the fluid contained in E_1

U the difference between the flow velocity of the fluid in E_1 and its flow velocity in E_0

We have:

$$x_1 = x_0 - Ut \qquad X = x_0 - a_0t = x_1 - a_1t \tag{1.1.1}$$

and

$$a_0 - a_1 = U \tag{1.1.2}$$

Finally we define:

	Region E_0	Region E_1	
Pressure	P_0	P_1	
Specific mass	ρ_0	ρ_1	(1.1.3)
Absolute temperature	T_0	T_1	
Velocity of sound	c_0	c_1	

We shall take as the independent variable defining the amplitude of the shock wave the ratio:

$$\xi = P_0/P_1 \tag{1.1.4}$$

ξ decreases from 1 to 0 in proportion as the amplitude of the shock increases.

We shall use dimensionless functions for describing the phenomenon. We give them below with their expressions as functions of ξ derived from the classical theory of shock waves.

Putting

$$\mu = \frac{\gamma + 1}{\gamma - 1}$$

$$\Gamma = \frac{c_0}{c_1} = \left(\frac{T_0}{T_1}\right)^{1/2} = \left[\frac{\xi(\mu + \xi)}{1 + \mu\xi}\right]^{1/2} \tag{1.1.5}$$

$$R = \frac{\rho_0}{\rho_1} = \frac{1 + \mu\xi}{\mu + \xi} = \frac{\xi}{\Gamma^2} \tag{1.1.6}$$

$$Y = \frac{U}{c_0} = \frac{(\mu - 1)(1 - \xi)}{\left[(\mu + 1)\xi(\mu + \xi)\right]^{1/2}} \tag{1.1.7}$$

$$A_0 = \frac{a_0}{c_0} = \left[\frac{\mu + \xi}{(\mu + 1)\xi}\right]^{1/2} \qquad A_0 > 1 \tag{1.1.8}$$

$$A_1 = \frac{a_1}{c_1} = \left[\frac{1 + \mu\xi}{\mu + 1}\right]^{1/2} \qquad A_1 < 1 \tag{1.1.9}$$

Finally, we shall have to use the following functions in which appear the derivatives R' and Y' of R and Y with respect to ξ:

$$\xi Y' = -\frac{(\mu - 1)\left[\mu + (\mu + 2)\xi\right]}{2(\mu + \xi)\left[(\mu + 1)\xi(\mu + \xi)\right]^{1/2}} \tag{1.1.10}$$

$$J = -\left(\frac{\mu - 1}{\mu + 1}\right)^2\left(\frac{\mu + 1}{\mu - 1}\,\xi\,\frac{R'}{R} - 1\right)$$

$$= -\left(\frac{\mu - 1}{\mu + 1}\right)^2\left(\frac{(\mu + 1)^2\xi}{(1 + \mu\xi)(\mu + \xi)} - 1\right) \tag{1.1.11}$$

$$\frac{1}{R\Gamma} = \frac{\Gamma}{\xi} = \left[\frac{\mu + \xi}{\xi(1 + \mu\xi)}\right]^{1/2} \tag{1.1.12}$$

$$G = -\frac{\mu + 1}{\mu - 1}\,\xi\Gamma Y'$$

$$= \frac{\mu + (\mu + 2)\xi}{2(\mu + \xi)}\left(\frac{\mu + 1}{1 + \mu\xi}\right)^{1/2} \tag{1.1.13}$$

$$H = -\frac{\mu + 1}{\mu - 1}\left(\xi Y' - \frac{Y}{\mu - 1}\right)$$

$$= \frac{1}{2}\left[\frac{\mu + (\mu + 2)\xi}{\mu + \xi} - 2\,\frac{1 - \xi}{\mu + 1}\right]\left[\frac{\mu + 1}{\xi(\mu + \xi)}\right]^{1/2} \qquad (1.1.14)$$

Tables I and II give the values of these various functions, most frequently calculated according to rule which gives sufficient accuracy for our present purpose.

1.1.1 Wave magnitudes.- The presence of waves with small amplitude which have an acoustic origin causes in the regions E_0 and E_1 small variations of the pressure, the specific mass, and the temperature about their mean values.

In order to represent them we set:

$\bar{\omega} = \delta P/P$ = relative excess of pressure

$S = \delta\rho/\rho$ = relative excess of specific mass, or condensation (1.1.15)

$\theta = \delta T/T$ = relative excess of temperature

In the case of usual acoustic waves, $\bar{\omega}$, s, and θ are very small and may be considered in the calculation as infinitesimals of the first order.

With regard to the vibratory velocities, we shall have to consider only the component u along ox.

1.1.2 Boundary conditions.- According to our hypotheses, the boundary conditions must express that the relationships (1.1.4) to (1.1.7) which govern the shock wave are maintained in the presence of oscillations of acoustic origin.

By differentiation of (1.1.4), (1.1.5), and (1.1.6) we obtain

$$\bar{\omega}_1 - \bar{\omega}_0 = -\delta\xi/\xi \qquad (1.1.16)$$

$$\theta_1 - \theta_0 = -2\,\frac{\Gamma'}{\Gamma}\,\delta\xi = 2\xi\,\frac{\Gamma'}{\Gamma}\left(\bar{\omega}_1 - \bar{\omega}_0\right) \tag{1.1.17}$$

$$s_1 - s_0 = -\,\frac{R'}{R}\,\delta\xi = \xi\,\frac{R'}{R}\left(\bar{\omega}_1 - \bar{\omega}_0\right) \tag{1.1.18}$$

In order to obtain, starting from (1.1.7), a correct expression one must take some precautions. One could be tempted - as we were - to assume, as it is usual in acoustics that the presence of the acoustic waves does not modify the sonic velocities c_0 and c_1.

This approximation remains correct in the interior of the regions E_0 and E_1 but at their boundaries it cannot be assumed; the relationship (1.1.5) contradicts it. Thus one must differentiate (1.1.7) with the assumption that c_0 is variable. One then obtains

$$\delta U = c_0\delta Y + Y\delta c_0 \tag{1.1.19}$$

with $\delta U = u_1 - u_0$, the difference between the components u_1 and u_0 of the vibratory velocities being given by

$$\delta Y = Y'\delta\xi = -\xi Y'\left(\bar{\omega}_1 - \bar{\omega}_0\right)$$

On the other hand, c_0 is given by the classical relationship

$$c_0^2 = \gamma P_0/\rho_0$$

whence

$$2\delta c_0/c_0 = \delta P_0/P_0 - \delta\rho_0/\rho_0 = \bar{\omega}_0 - s_0$$

As we shall show later on in our problem, only incident acoustic waves can exist in the region E_0. The oscillation phenomenon is therefore adiabatic which permits us to write $s_0 = \bar{\omega}_0/\gamma$; hence

$$\delta c_0/c_0 = \bar{\omega}_0/(\mu + 1)$$

The relationship (1.1.19) may then be written

$$\frac{u_1 - u_0}{c_0} = \left(\frac{Y}{\mu + 1} + \xi Y'\right)\bar{\omega}_0 - \xi Y\bar{\omega}_1 \tag{1.1.20}$$

It will be convenient to use this equation in the form

$$c_1 u_1 - G\frac{c_1^2}{\gamma}\bar{\omega}_1 = \frac{1}{\Gamma}\left(c_0 u_0 - H\frac{c_0^2}{\gamma}\bar{\omega}_0\right) \tag{1.1.21}$$

where Γ, G, H are the functions defined, respectively, by the relationships (1.1.5), (1.1.13), (1.1.14).

1.2. Waves of Small Amplitudes in Gases

1.2.0 Mathematical representation.- The mathematical representation of this problem is a result of the work of Stokes and Kirchhoff. An outline for it can be found in Lord Rayleigh's book (Theory of Sound, vol. II., section 247). We recapitulate it with our notation.

u, v, w	components of the velocity of a particle
ρ	specific mass of the gas at rest
$s = \delta\rho/\rho$	condensation
P	pressure of the gas at rest
$\bar{\omega} = \delta P/P$	relative excess of pressure
T	absolute temperature of the gas at rest
$\theta = \delta T/T$	relative excess of temperature
γ	ratio of the specific heats
ν	coefficient of kinematic viscosity
k	coefficient of thermal conductivity, according to Kirchhoff equal to $5\nu/2$
c	sonic velocity

Equation of continuity

$$\partial s/\partial t + \partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0 \tag{1.2.1}$$

Dynamic equations

$$\frac{\partial u}{\partial t} + \frac{c^2}{\gamma}\frac{\partial \bar{\omega}}{\partial x} = \nu\left(\Delta^2 u - \frac{1}{3}\frac{\partial^2 s}{\partial t\ \partial x}\right) \quad \text{and similarly in } y,z \tag{1.2.2}$$

The logarithmic differentiation of the equation $P = \rho rT$ yields the equation of state

$$\bar{\omega} = s + \theta \tag{1.2.3}$$

Since the motions are very small, the terms of the second order are neglected. The total relative temperature excess θ is the sum of the excess θ_a, resulting from the adiabatic compression, and the excess θ_t, stemming from the conductivity. Thus we have

$$\theta_a = (\gamma - 1)s \qquad \partial\theta_t/\partial t = k\Delta^2\theta$$

Hence we obtain finally the thermal equation

$$\partial\theta/\partial t = (\gamma - 1)\partial s/\partial t + k\Delta^2\theta \tag{1.2.4}$$

The calculation then is performed as follows: We eliminate $\bar{\omega}$ utilizing (1.2.3). We then assume that the different variables depend on the time only through the factor e^{ht} (h can be complex). The factor h therefore replaces the sign of differentiation with respect to time.

Equations (1.2.1), (1.2.2), and (1.2.4) then are written

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z + hs = 0 \tag{1.2.1'}$$

$$hu - \nu\Delta^2 u = -\partial M/\partial x, \text{ and similarly in } y,z \tag{1.2.2'}$$

$$h\theta - k\Delta^2\theta = (\gamma - 1)hs \tag{1.2.4'}$$

where we have set

$$M = \left(c^2/\gamma + h\nu/3\right)s + \left(c^2/\gamma\right)\theta \tag{1.2.5}$$

We eliminate s between (1.2.1') and (1.2.4') which gives

$$\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = -(h\theta - k\Delta^2\theta)/(\gamma - 1) \tag{1.2.6}$$

and M is written

$$M = \frac{c^2}{\gamma - 1}\left[\left(1 + \frac{1}{3}\frac{h\nu}{c^2}\right)\theta - \frac{k}{h\gamma}\left(1 + \frac{\gamma}{3}\frac{h\nu}{c^2}\right)\Delta^2\theta\right] \tag{1.2.5'}$$

Let us differentiate equations (1.2.2'), respectively, with respect to x, y, z, and add term by term. We take (1.2.6) into account and eliminate M, using (1.2.5'); there remains

$$\frac{h^2\theta}{c^2} - \left(1 + \frac{hk}{c^2} + \frac{4}{3}\frac{h\nu}{c^2}\right)\Delta^2\theta + \frac{k}{h\gamma}\left(1 + \frac{4\gamma}{3}\frac{h\nu}{c^2}\right)\Delta^4\theta = 0 \tag{1.2.7}$$

a particular solution of which is given by the form

$$\theta = B_1Q_1 + B_2Q_2 \tag{1.2.8}$$

where B_1, B_2 are constants, and Q_1 and Q_2 functions of x, y, z which satisfy the relationships

$$\Delta^2Q_1 = m_1^2Q_1 \qquad \Delta^2Q_2 = m_2^2Q_2 \tag{1.2.9}$$

in which m_1^2 and m_2^2 are determined by the equation

$$\left(\frac{h}{mc}\right)^4 - \left[1 + \left(1 + \frac{4\nu}{3k}\right)\sigma\right]\left(\frac{h}{mc}\right)^2 + \sigma\left(\frac{1}{\gamma} + \frac{4\nu}{3k}\sigma\right) = 0 \tag{1.2.10}$$

with σ being the dimensionless number

$$\sigma = hk/c^2 \tag{1.2.11}$$

The components u, v, w of the vibratory velocity then derive, except for the common factor e^{ht}, from the potential $Q_1 + Q_2$. The constants B_1 and B_2 are taken from (1.2.6) which gives

$$B_i = (\gamma - 1)\frac{m_i^2}{km_i^2 - h} \quad (i = 1,2) \tag{1.2.12}$$

The condensation s_i corresponding to Q_i is

$$s_i = -\left(m_i^2/h\right)\varphi_i e^{ht} \tag{1.2.13}$$

The relative excess of pressure $\bar{\omega}_i$ is obtained by combination of (1.2.3) and (1.2.13)

$$\bar{\omega}_i = -\frac{m_i^2}{h}\frac{km_i^2 - \gamma h}{km_i^2 - h} Q_i e^{ht} \tag{1.2.14}$$

1.2.1 Simplifications, separation of the wave types (acoustic and thermal).- In the case where the origin of the waves is acoustic, we shall have $h = i\omega - \delta$. In the relationship (1.2.10) which determines the ratios m/h, h appears only in the dimensionless number $\sigma = kh/c^2$. When this number is very small, the roots of (1.2.10) assume very simple approximate values. In air, in cgs units, $\nu = 0.146$; thus $k = 5\nu/2 = 0.365$. At the usual temperatures c is of the order of 34×10^3 cm/s. Finally, if we restrict ourselves to the upper limit of an - already very ultransonic - frequency of 80,000 cycles per second, the circular frequencies ω are smaller than $0.5 \times 10^6 s^{-1}$. Under these conditions, $k\omega/c^2$ is less than 1.6×10^{-4}.

We are therefore justified in assuming that the number σ will remain very small as long as the damping factor δ is not very large.

We shall therefore make the corresponding simplifications, except for verifying - when the occasion arises - that the condition $\sigma \ll 1$ is always satisfied.

The roots of (1.2.10) assume then the approximate values

$$m_1^2 \approx (h/c)^2 \qquad m_2^2 \approx \gamma h/k \qquad (1.2.15), (1.2.16)$$

The first gives us a velocity potential for an acoustic wave in its usual form. The second gives us the potential for waves of thermal conductivity.

In the case of plane waves, with the coordinate z eliminated by a suitable choice of axes, we shall be able to write

$$Q_i = e^{\pm m_i(\alpha x+\beta y)} \qquad (1.2.17)$$

with $\alpha^2 + \beta^2 = 1$, in order to satisfy (1.2.9).

We then obtain the table of the following values:

	Acoustic wave	Thermal wave
Q	$e^{\pm h(\alpha x+\beta y)/c}$	$e^{\pm\sqrt{\gamma h/k}(\alpha x+\beta y)}$
s	$-(h/c^2)Qe^{ht}$	$-(\gamma/k)Qe^{ht}$
θ	$-(\gamma - 1)s$	$-s$
$\bar{\omega}$	γs	0
u	$\pm c\alpha s$	$\mp\sqrt{kh/\gamma}\alpha s$
v	$\pm c\beta s$	$\mp\sqrt{kh/\gamma}\beta s$

(1.2.18)

1.3. Numerical Tables of the Different Functions Introduced in the Calculation

Table I

ξ	Γ	R	R'	Y	$-\xi Y'$	A_0	A_1
1	1	1	0.714	0	0.714	1	1
0.9	0.985	0.93	0.735	0.0756	0.726	1.046	0.956
0.8	0.97	0.85	0.76	0.160	0.737	1.10	0.910
0.7	0.95	0.78	0.78	0.26	0.756	1.17	0.86
0.6	0.93	0.70	0.80	0.39	0.78	1.25	0.81
0.5	0.90	0.62	0.83	0.53	0.81	1.36	0.76
0.4	0.87	0.53	0.85	0.72	0.85	1.51	0.70
0.3	0.82	0.45	0.88	0.96	0.92	1.73	0.63
0.2	0.75	0.36	0.91	1.38	1.04	2.10	0.56
0.1	0.62	0.26	0.94	2.18	1.35	2.95	0.48
0.05	0.48	0.214	0.956	3.27	1.82	4.15	0.431
0.025	0.36	0.191	0.964	4.75	3.18	5.86	0.405
0	0	0.1667	0.973	∞	∞	∞	0.378

Table II

ξ	J	$1/R\Gamma$	H	G
1	0	1	1	1
0.9	0.0007	1.094	1.00035	1.00035
0.8	0.0031	1.21	1.00168	1.00165
0.7	0.0079	1.36	1.0052	1.0044
0.6	0.016	1.55	0.012	1.009
0.5	0.029	1.80	1.025	1.018
0.4	0.051	2.17	1.046	1.030
0.3	0.085	2.74	1.090	1.054
0.2	0.144	3.76	1.185	1.092
0.1	0.254	6.18	1.45	1.166
0.05	0.351	9.65	1.89	1.228
0.025	0.420	11.42	2.56	1.270
0	0.510	∞	∞	1.323

2. THE KINEMATIC CONDITIONS

2.0 Statement of the Problem and Formulas

2.0.0 General forms - apparent circular frequency Ω and phase velocity w at the front of the shock wave.- Let us consider in one of the regions E_0 or E_1 an acoustic field incident on S which we shall represent by the velocity potential

$$\Phi(r,y,z,t) \tag{2.0.1}$$

with Φ and x having the subscripts (0 or 1) corresponding to the region concerned.

This field produces at the front S a phenomenon which with respect to the axes $oXyz$ fixed to S will be governed by the form

$$\psi(y,z,t) = \Phi(at,y,z,t) \tag{2.0.2}$$

where a has the same subscript as Φ.

As a result of the fact that our boundary conditions are in point form, there must appear at S for every field associated with the incident field by reflection or refraction a corresponding phenomenon whose spatial distribution and development with time are governed by the form ψ. This condition of kinematic order is independent of the dynamic conditions which derive from a particular choice of the equations connecting the values of $\bar{\omega}$, s, u, θ, taken from both sides of S.

Since we limit ourselves to the case where the incident wave is plane, progressive, and uniform, we have

$$\Phi = e^{i\omega[t-(x\cos\varphi+y\sin\varphi)/c]} \tag{2.0.3}$$

with φ being the angle of incidence.

The function ψ which represents the phenomenon at S then assumes the form

$$\psi = e^{i\Omega(t-y/w)} \tag{2.0.4}$$

with

$$\frac{\Omega}{\omega} = 1 - A \cos \varphi = \frac{1 - A + (1 + A)\tau^2}{1 + \tau^2} \tag{2.0.5}$$

$$\frac{w}{c} = \frac{\Omega}{\omega \sin \varphi} = \frac{1 - A \cos \varphi}{\sin \varphi} = \frac{1 - A + (1 + A)\tau^2}{2\tau} \tag{2.0.6}$$

where

$$\tau = \tan \frac{1}{2} \varphi$$

c, A, ω, φ, τ should have the subscript of the region which contains the wave.

2.0.1 Condition necessary for a plane, progressive, and uniform acoustic wave to be an incident wave.- Let us consider a mobile particle M carried along by the waves and advancing on a sound ray. If M approaches the S plane with increasing t, the wave is incident. If it recedes from it, the wave is reflected or refracted.

With respect to the axes fixed to the fluid in the region considered, the velocity at which M is displaced toward increasing x is $c \cos \varphi$.

In the region under subpressure E_0, the distance of M from S is, except for a constant, $(c_0 \cos \varphi_0 - a_0)t$. It decreases therefore constantly regardless of what φ_0 may be since we have $a_0 > c_0$. Consequently, any plane, progressive, and uniform wave present in E_0 is an incident wave.

In the region under superpressure E_1, the distance from M to S is, except for a constant, $(a_1 - c_1 \cos \varphi_1)t$.

Since a_1 is less than c_1, two cases are possible

(a) $$a_1 - c_1 \cos \varphi_1 < 0 \quad \text{where} \quad |\varphi_1| < \arccos A_1 \tag{2.0.7}$$

$$\text{or} \quad |\tau_1| < \sqrt{(1 - A_1)/(1 + A_1)}$$

The wave is incident.

(b) $$a_1 - c_1 \cos \varphi_1 > 0 \quad \text{where} \quad \arccos A_1 < |\varphi_1| < \pi \qquad (2.0.8)$$

$$\text{or} \quad |\tau_1| > \sqrt{(1 - A_1)/(1 + A_1)}$$

The wave is produced by reflection or refraction.

2.0.2 General relations.- The plane waves, whether acoustic or thermal, are governed by the form

$$e^{ht-m(\alpha x+\beta y)} \qquad (2.0.9)$$

with

$$\alpha^2 + \beta^2 = 1$$

and

$$\begin{aligned} m &= \pm h/c \quad \text{acoustic waves} \\ m &= \pm\sqrt{\gamma h/k} \quad \text{thermal waves} \end{aligned} \qquad (2.0.10)$$

On the S-plane which is displaced at the velocity a in the region considered, the exponent of (2.0.9) must assume the form $i\Omega(t - y/w)$; hence the relationships

$$i\Omega = h - \alpha m a \qquad i\Omega/w = \beta m \qquad (2.0.11)$$

that is

$$\alpha = (h - i\Omega)/am \qquad \beta = i\Omega/wm \qquad (2.0.12)$$

$\alpha^2 + \beta^2 = 1$ gives

$$(am/\Omega)^2 + (a/w)^2 - (h/\Omega - i)^2 = 0 \qquad (2.0.13)$$

Replacing, in this equation, m by its value taken from one of the relationships (2.0.10), we obtain an equation which will determine h as a function of Ω and w. We then take m, α, and β from (2.0.10) and (2.0.12).

2.1 The Reflected or Refracted Thermal Wave

Inserting into (2.0.13) the value $m^2 = \gamma h/k$ gives

$$(h/\Omega)^2 - 2\left(i + \gamma a^2/2k\Omega\right)h/\Omega - \left(1 - a^2/w^2\right) = 0 \qquad (2.1.1)$$

whence

$$\frac{h}{\Omega} = i + \frac{\gamma a^2}{2k\Omega}\left[1 \pm \sqrt{1 + \left(\frac{2k\Omega}{\gamma aw}\right)^2 + 2i\,\frac{2k\Omega}{\gamma a^2}}\right] \qquad (2.1.2)$$

In order to be suitable for our purpose, the root must have a negative real part: a wave which vanishes when t increases indefinitely.

As we have seen, the dimensionless number $k\Omega/a^2$ is very small. On the other hand, (2.0.6) gives $\Omega/w = (\omega/c) \sin \varphi$ (where ω, c, and φ have the subscripts of the region which contains the incident wave).

Consequently, $k\Omega/aw = (kw/ac) \sin \varphi$ is the same order of magnitude as $k\Omega/a^2$, that is, very small.

If we then take the approximate value of the radical of the formula (2.1.2), we see that the suitable root is the one where this radical has the negative sign.

In order to obtain an approximate value of h/Ω, we must expand the radical up to the terms of the second order; the terms of the first order disappear and we obtain

$$h = -\frac{k\Omega^2}{\gamma}\,\frac{a^2 + w^2}{a^2w^2} \qquad (2.1.3)$$

One can see easily that

$$\frac{kh}{c^2} = \frac{1}{\gamma c^2}\,\frac{k^2\Omega^2\left(a^2 + w^2\right)}{a^2w^2}$$

remains always very small. The approximation made in order to separate the thermal wave from the acoustic wave is therefore valid.

On the other hand, (2.1.3) shows that h is real and very small. Consequently, the thermal wave is not perceptibly oscillating and its damping is slow.

From the value of h we derive

$$m = i\Omega\sqrt{a^2 + w^2}/aw \tag{2.1.4}$$

$$\alpha = -\frac{w}{\sqrt{a^2 + w^2}} = \cos\varphi_t$$

$$\beta = \frac{a}{\sqrt{a^2 + w^2}} = \sin\varphi_t \tag{2.1.5}$$

The spatial distribution of the temperature excesses and of the condensations is therefore sinusoidal. The wave is oblique. It is not progressive and dies out on the spot.

As a result, this wave can originate only behind the S-plane (the movement of which it cannot precede). Thus it can be present only in the region under superpressure E_1.

The absence of any thermal wave in the E_0 region justifies therefore the method by which we have established the boundary condition (1.1.20).

The potential $Q = e^{-m(\alpha x+\beta y)}$ is then written

$$Q = e^{i\Omega(x/a-y/w)} \tag{2.1.6}$$

The quantities fixed to the wave then are (see 1.2.13)

$$s_t = -\frac{\gamma}{k} Qe^{ht} \qquad \theta_t = -s_t \qquad \bar{\omega}_t = 0$$

and the velocity components

$$u_t = -i\frac{k\Omega}{\gamma a} s_t \qquad v_t = i\frac{k\Omega}{\gamma w} s_t \tag{2.1.7}$$

These values will be useful for simplification of the dynamic conditions.

2.2. The Reflected Acoustical Wave

In this case, the incident wave is, like the reflected wave, situated in the region under superpressure E_1. The relationships (2.0.5) and (2.0.6), connecting the kinematic characteristics ω and φ (or τ) of the acoustic wave with the circular frequency Ω and the phase velocity w which can be observed at the front of the shock wave, are simultaneously valid for the incident wave $\left(\omega_1, \varphi_1, \tau_1\right)$ and the reflected wave $\left(\omega_1', \varphi_1', \tau_1'\right)$.

Eliminating Ω and w, we obtain

$$\tau_1\tau_1' = \left(1 - A_1\right)/\left(1 + A_1\right) \tag{2.2.1}$$

$$\frac{\omega_1'}{\omega_1} = \frac{\sin \varphi_1}{\sin \varphi_1'} = \frac{\left(1 - A_1\right)^2 + \left(1 + A_1^2\right)\tau_1^2}{\left(1 - A_1\right)\left(1 + A_1\right)\left(1 + \tau_1^2\right)} \tag{2.2.2}$$

(2.2.1) shows that to the incident wave $\left(\tau_1 < \sqrt{\left(1 - A_1\right)/\left(1 + A_1\right)}\right)$ there corresponds indeed a reflected wave $\left[\tau_1' > \sqrt{\left(1 - A_1\right)/\left(1 + A_1\right)}\right]$.

(2.2.2) gives the ratio of the circular frequencies (perceived by an observer carried along by the fluid) of the incident wave and of the reflected wave.

When φ_1 increases from zero (normal incidence) to $\arccos A_1$ (limiting angle of incidence), τ_1 increases from 0 to $\sqrt{\left(1 - A_1\right)/\left(1 + A_1\right)}$, ω_1'/ω_1 increases from $\left(1 - A_1\right)/\left(1 + A_1\right)$ to 1.

Thus we have

$$\frac{1 - A_1}{1 + A_1} < \frac{\omega_1'}{\omega_1} < 1 \qquad \text{and} \qquad 1 < \frac{\sin \varphi_1'}{\sin \varphi_1} < \frac{1 + A_1}{1 - A_1}$$

Due to the fact that any plane, progressive, and uniform incident wave produces a reflected wave of the same type, Huyghens' construction is valid.

2.3. The Refracted Acoustic Wave

2.3.0 Generalities - separation of the two types of refracted waves.- In order for refraction to occur, the incident wave must be situated in the region under subpressure E_0. On the other hand, any plane, progressive, and uniform wave situated in E_0 is an incident wave and will give rise to the production of refracted waves.

The phase velocity w of the phenomenon produced on S by the incident wave is given by (2.0.6)

$$w = \frac{(A_0 + 1)\tau_0^2 - (A_0 - 1)}{2\tau_0} c_0 \tag{2.3.1}$$

where

$$A_0 > 1$$

When the angle of incidence φ_0 varies from $-\pi$ to $+\pi$, τ_0 varies from $-\infty$ to $+\infty$. w increases constantly and its variation is given by the following table:

φ_0	$-\pi$	$-\text{arc cos } 1/A_0$	0	$\text{arc cos } 1/A_0$	$+\pi$
τ_0	$-\infty$	$-\sqrt{\frac{A_0 - 1}{A_0 + 1}}$	0	$+\sqrt{\frac{A_0 - 1}{A_0 + 1}}$	$+\infty$
w	$-\infty$	0	$\overbrace{+\infty \quad -\infty}$	0	$+\infty$

The same phase velocity w can therefore be produced by two waves; the incidence of the first lies between $-\pi$ and 0, that of the second between 0 and $+\pi$, with the first being larger in absolute value than $\text{arc cos } 1/A_0$, the second smaller than that value. The corresponding values of τ, τ_0, and τ_0', relative to these two waves, are connected by the relationship

$$\tau_0\tau_0' = -(A_0 - 1)/(A_0 + 1) \tag{2.3.2}$$

Finally, (2.0.6) shows us that the circular frequencies of the two incident waves which produce on S the same phenomenon (equality of w and of Ω) are in the ratio

$$\omega_0'/\omega_0 = |\sin \varphi_0/\sin \varphi_0'|$$

Thus, the acoustic phenomenon observed on S, and all the more in the interior of the E_1 region, may be produced indifferently by one or the other, or any linear combination, of the two incident waves described above. Not a single observation made in E_1 can serve to differentiate.

The character of the refracted wave depends on the absolute value of the phase velocity w. If $|w|$ is greater than $\sqrt{c_1^2 - a_1^2}$, the refracted wave is plane, progressive, and uniform. If $|w|$ is less than $\sqrt{c_1^2 - a_1^2}$, one cannot make any plane, progressive, and uniform wave in E_1 correspond to it. The refracted wave then assumes an exponential structure and we shall call it accompanying wave.

The angles of incidence φ_0 corresponding to the value of w, at which one passes from one type of refracted wave to the other, are given by

$$\tau_0 = \frac{\pm\sqrt{c_1^2 - a_1^2} \pm \sqrt{c_1^2 - a_1^2 + a_0^2 - c_0^2}}{a_0 + c_0} \tag{2.3.3}$$

When $|\tau_0|$ is outside the interval defined by the above boundaries, the refracted wave is plane, progressive, and uniform. In the opposite case we obtain the accompanying wave.

Expressed as functions of the theoretical relationships which govern the shock wave, the two limiting values of τ_0 are written

$$\tau_0 = \sqrt{\frac{\mu(1 - \xi)}{\mu + \xi}}\,\frac{\pm\sqrt{1 + \mu\xi} \pm \sqrt{(\mu + 1)(1 + \xi)}}{\sqrt{\mu + \xi} + \sqrt{(\mu + 1)\xi}} \tag{2.3.4}$$

The values of the corresponding limiting angles are given in figure 1.

2.3.1 Plane, progressive, and uniform refracted wave: $w^2 > c_1^2 - a_1^2$.- The equation (2.0.6) applied to the two regions E_0 and E_1 which contain the incident wave and the refracted wave, respectively, gives the relationship which connects τ_0 and τ_1

$$\frac{c_1 - a_1 + (c_1 + a_1)\tau_1^2}{\tau_1} = \frac{c_0 - a_0 + (c_0 + a_0)\tau_0^2}{\tau_0} \tag{2.3.5}$$

Since τ_0 is governed by the incidence φ_0, τ_1 is determined by an equation of the second degree of which only the root, which is larger than $\sqrt{(c_1 - a_1)/(c_1 + a_1)}$, corresponds to the refracted wave. The other root gives in E_1 the incident wave which could, by reflection, produce the refracted wave.

Thus we have

$$\tau_1 = \frac{c_0 - a_0 + (c_0 + a_0)\tau_0^2 + \sqrt{\left[c_0 - a_0 + (c_0 + a_0)\tau_0^2\right]^2 - 4\left(c_1^2 - a_1^2\right)\tau_0^2}}{2\left(c_1 + a_1\right)\tau_0} \tag{2.3.6}$$

which we may write, using the dimensionless functions A_0, A_1, Γ,

$$\tau_1 = \frac{1 - A_0 + (1 + A_0)\tau_0^2 + \sqrt{\left[(1 - A_0) + (1 + A_0)\tau_0^2\right]^2 - \frac{4}{\Gamma^2}\left(1 - A_1^2\right)\tau_0^2}}{2\,\frac{1 + A_1}{\Gamma}\,\tau_0} \tag{2.3.6'}$$

The value of the circular frequency ω_1 of the refracted wave, perceived by an observer who is carried along with the fluid in the region E_1, is then given by (2.0.6) whence we derive

$$\frac{\omega_1}{\omega_0} = \frac{1}{\Gamma}\frac{\sin\varphi_0}{\sin\varphi_1} \tag{2.3.7}$$

For an observer fixed to the axes, carried along with the fluid situated in E_0, there occurs a shift in frequency and the circular frequency perceived will be ω_1'.

$$\left.\begin{aligned}\omega_1' &= \omega_1\left(1 + \frac{U}{c_1}\cos\varphi_1\right)\\ &= \left(1 + \frac{Y}{\Gamma}\cos\varphi_1\right)\frac{\sin\varphi_0}{\sin\varphi_1}\frac{\omega_0}{\Gamma}\end{aligned}\right\} \tag{2.3.8}$$

2.3.2 <u>The accompanying acoustic wave $w^2 < c_1^2 - a_1^2$.</u>- We shall show that the refracted wave may then be represented by a velocity potential of the form

$$\Phi_1 = e^{\left(i\omega_1 - \delta_1\right)t + \left[(i\eta+\zeta)x + (i\eta'+\zeta')y\right]/c_1} \tag{2.3.9}$$

Φ_1 must satisfy the general equation of sound $\Delta^2\Phi_1 = \left(1/c_1^2\right)\partial^2\Phi_1/\partial t^2$ which involves the relationships

$$\zeta^2 - \eta^2 + \zeta'^2 - \eta'^2 = \delta_1^2 - \omega_1^2 \qquad \zeta\eta + \zeta'\eta' = -\delta_1\omega_1 \tag{2.3.10}$$

On the other hand, on the front S, that is, for $x = a_1 t$, the exponent of Φ_1 must take the form $i\Omega(t - y/w)$ whence

$$i\omega_1 - \delta_1 + \frac{a_1}{c_1}(i\eta + \zeta) = i\Omega \qquad \frac{i\eta' + \zeta'}{c_1} = -i\frac{\Omega}{w} \tag{2.3.11}$$

From (2.3.10) and (2.3.11) we then take

$$w_1/\Omega = 1\Big/\left(1 - A_1^2\right) \tag{2.3.12}$$

$$\frac{\delta}{\Omega} = \frac{A_1}{1 - A_1^2}\sqrt{\left(\frac{c_1}{w}\right)^2\left(1 - A_1^2\right) - 1} \tag{2.3.13}$$

$$\eta/\Omega = -A_1\Big/\left(1 - A_1^2\right) = -A_1\omega_1/\Omega \tag{2.3.14}$$

$$\frac{\zeta}{\Omega} = \frac{1}{A_1}\frac{\delta}{\Omega} \tag{2.3.15}$$

$$\eta'/\Omega = -c_1/w \tag{2.3.16}$$

$$\zeta' = 0 \tag{2.3.17}$$

The equation (2.3.13) shows that this type of wave exists only if the phase velocity w is smaller than $c_1\sqrt{1 - A_1^2} = \sqrt{c_1^2 - a_1^2}$; consequently, this wave appears only in the case studied in the present section. On the other hand, this condition of existence implies that we have $c_1 > a_1$. No wave of this type can therefore appear in the E_0 region where c_0 is less than a_0.

Before describing the accompanying wave, we must verify that the condition $\sigma \ll 1$ is satisfied, that is, that we have always $k\delta/c_1^2 \ll 1$. Replacing Ω in (2.3.13) by its value taken from (2.0.6), we obtain

$$\frac{k\delta}{c_1^2} = A_1\Gamma\frac{\sqrt{1 - A_1^2 - (w/c_1)^2}}{1 - A_1^2}\sin\varphi_0\frac{k\omega}{c_0^2} \tag{2.3.18}$$

$k\omega/c_0^2$ is very small; A_1 and Γ lie between 0 and 1; thus we have a very small $k\delta/c_1^2$.

In order to make the characteristics of the wave evident, we shall take the axes oXY whose origin o is carried along with the S-plane and is displaced toward increasing y at the velocity w. We therefore set

$$X = x_1 - a_1t \qquad Y = y - wt \tag{2.3.19}$$

The potential Φ_1 then takes the form

$$\Phi_1 = e^{\left[(i\eta+\zeta)X+i\eta'Y\right]/c_1} \tag{2.3.20}$$

which is independent of t.

The wave is therefore motionless with respect to the axes oXY. Its amplitude decreases as $e^{\zeta X/c_1}$ if one moves away from the shock wave (toward the negative x). This wave constitutes therefore a phenomenon which runs behind the shock wave while sliding at the same time transversely at the velocity w. These characteristics justify the name we gave it: accompanying wave.

Let us define its structure in detail. Toward increasing Y the phenomenon is spatially periodic and its wave length is

$$\lambda_Y = 2\pi c_1/\eta' = 2\pi w/\Omega \tag{2.3.21}$$

which was, besides, evident in view of its origin. Toward decreasing X the phenomenon is spatially of a damped pseudoperiodic nature. Its wave length is

$$\lambda_X = 2\pi c_1/\eta = 2\pi\left(c_1^2 - a_1^2\right)\Big/a_1\Omega \tag{2.3.22}$$

The planes of equal phase, or pseudo-wave planes are given by: $\eta X + \eta' Y = \text{const.}$ They form, therefore, with the S-plane of the shock wave the angle χ given by

$$\tan \chi = -\eta'/\eta = -\left(c_1^2 - a_1^2\right)\Big/a_1 w \tag{2.3.23}$$

This formula permits a very simple geometric construction of that angle.

The rate of damping of the wave toward the negative X is

$$\frac{\zeta}{c_1} = \frac{c_1\Omega}{c_1^2 - a_1^2}\sqrt{\frac{c_1^2 - a_1^2}{w^2} - 1} \tag{2.3.24}$$

It is zero for the limiting values $w = \pm\sqrt{c_1^2 - a_1^2}$. It increases when w decreases.

For $w = 0$, we have on the one hand $\Omega = 0$, on the other

$$\Omega/w = \omega_0 \sin \varphi_0/c_0 \quad \text{and} \quad \cos \varphi_0 = c_0/a_0$$

ζ_1/c_1 then assumes the value

$$\left(\frac{\zeta_1}{c_1}\right)_{w=0} = \frac{c_1}{\sqrt{c_1^2 - a_1^2}} \frac{\sqrt{a_0^2 - c_0^2}}{a_0} \frac{\omega_0}{c_0}$$

which may be expressed as a function of the variable ξ which defines the amplitude of the shock wave, and of the wave length $\lambda_0 = 2\pi c_0/\omega_0$ of the incident wave in the region E_0

$$\left(\frac{\zeta_1}{c_1}\right)_{w=0} = \frac{2\pi}{\lambda_0}\sqrt{\frac{\mu + 1}{\mu + \xi}} \tag{2.3.25}$$

In order to calculate the characteristics of the accompanying wave from those of the incident wave, it suffices to replace Ω and w in the relations (2.3.12) to (2.3.16) by their values taken from (2.0.5) and (2.0.6), namely

$$\left.\begin{aligned} \Omega &= \omega_0(c_0 - a_0 \cos \varphi_0)/c_0 \\ w &= (c_0 - a_0 \cos \varphi_0)/\sin \varphi_0 \end{aligned}\right\}$$

The formulas obtained are complicated. It is unnecessary to write them here explicitly.

2.4. Geometric Constructions

2.4.0 <u>Huyghens' construction (fig. 2)</u>.- With the plane of the figure being motionless with respect to the region under subpressure E_0, S and S' are the positions of the shock-wave front at the times $t = 0$ and $t = 1$. The distance SS' is a_0:

Let us consider in the region E_0, to the right of S, a plane, progressive, and uniform wave which intersects S at the point O_0 at the time $t = 0$. At the time $t = 1$, this wave will intersect S' at a point D and its potential prolongation will be tangent to a circle C_0, with the radius c_0 and the center O_0, entirely situated to the left of S' since $c_0 < a_0$.

Assume B to be the foot of a line through O_0 perpendicular to S'. BD will be equal to the phase velocity w observed on the shock-wave front.

We can always make two tangents to the circle C_0 pass through D. The figure therefore confirms that in the region E_0 two waves I_0, I_0' exist to which the same phase velocity w corresponds on S. These two waves are incident because the point of contact of the tangents drawn from D to C_0 is to the left of S: that is the region where E_0 is virtual.

Let us now consider the region E_1. The fluid flows there toward the right, with the velocity U. A plane wave passing at the time $t = 0$ through the point O_0, will, at the time $t = 1$, be tangent to a circle C_1 with the radius c_1, whose center O_1 is at the distance $O_0O_1 = U$ from O_0.

O_1B is therefore the velocity a_1 of the shock-wave front with respect to the fluid filling up the region E_1. Since a_1 is less than c_1, the circle C_1 intersects the straight line S' at two points, F and F'.

The incident waves I_1 are those whose point of contact with C_1 is to the right of S' (there where E_1 is virtual); the reflected or refracted waves touch C_1 to the left of S'.

Let us now assume D to be a point on S' situated outside the segment FF'. From this point we may draw two tangents to C_1, one of which corresponds to an incident wave I_1 while the other corresponds to a reflected or refracted wave R_1.

Thus, if we assume a wave R_1 in E_1, this wave determines the point D, and we see that it can have been produced either by reflection of the wave I_1 (situated in E_1) or by refraction of one or the other of the waves I_0 and I_0' situated in E_0.

Finally, it appears that when the point D lies between F and F', there correspond to it two more possible incident waves, I_0 and I_0' in E_0, but none in E_1 since D lies inside the circle C_1.

The cross-hatched sectors of the figure show the angles of incidence φ_0 for which this condition is realized. Huyghens' construction thus does not permit us to know what is happening in the region E_1.

2.4.1 Construction of the accompanying wave (fig. 2).- Formula (2.3.23) gives the angle χ which is formed by the pseudo-wave plane (plane of equal phase) of the accompanying wave and the S-plane. It lends itself to a simple construction.

Let us draw the tangent from the point F' to the circle C_1. It intersects the straight line O_1B at G. $O_1G = \overline{O_1F'}^2/O_1B = c_1^2/a_1$. Since O_1B is equal to a_1, we shall have

$$BG = \frac{c_1^2}{a_1} - a_1$$

Since, on the other hand, D_1, situated between F and F', is the point where the incident wave intersects S' at the time $t = 1$, we have $BD_1 = w$. Consequently

$$-\frac{BG}{BD_1} = -\frac{c_1^2/a_1 - a_1}{w} = \tan \chi$$

Thus, the angle BD_1G is the desired angle χ. The prolongation of the straight line GD_1, to the left of S', gives the pseudo-wave plane of the accompanying wave.

The construction thus shows very clearly the variation in direction of this pseudo-wave plane as a function of w, that is, of the position of the point D_1.

Figure 2 shows these different constructions for a whole series of incidences. The arrows placed on the traces of the various waves indicate the direction of their propagation. The sketch corresponds to the case where the shock wave involves an abrupt doubling of the pressure, that is, where one has

$$P_0/P_1 = \xi = 0.5$$

2.5 Use of the Acoustic Phenomena for the Study of the Shock Waves

The kinematic conditions are expressed by formulas into which enter only velocity ratios: sonic velocity, velocity of the wave front, flow velocity. They do not stipulate the validity of the theoretical relationships which express these velocities as functions of the shock amplitude.

Moreover, in the case where the shock-wave front is preceded or followed by compression or expansion phenomena which entail variations in temperature and in the flow velocity of the fluid, the acoustic waves undergo continuous refractions. These phenomena, incidentally, are well known.

Consequently, the experimental study of the kinematics of the acoustic waves reflected or refracted by a shock-wave front furnishes a research method which is probably not without interest for aerodynamicists. For undertaking it, we may consider photographic, even motion-picture methods which show the form and the deformations of wave fronts, or microphonic methods which detect the variations of frequency.

One will note that a large amount of information can be collected solely by the study of the reflection of waves, that is, by means used exclusively in the region under superpressure E_1.

3. DYNAMIC CONDITIONS

3.0. Elimination of Thermal Waves

Let us start from the relationships established in section 1.1.2, namely

$$\theta_1 - \theta_0 = 2\xi \frac{\Gamma'}{\Gamma}\left(\overline{\omega}_1 - \overline{\omega}_0\right) \tag{1.1.17}$$

$$s_1 - s_0 = \xi \frac{R'}{R}\left(\overline{\omega}_1 - \overline{\omega}_0\right) \tag{1.1.18}$$

$$c_1 u_1 - G\,\frac{c_1^2\bar{\omega}_1}{\gamma} = \frac{1}{\Gamma}\left(c_0 u_0 - H\,\frac{c_0^2\bar{\omega}_0}{\gamma}\right) \tag{1.1.21}$$

We have seen that in our problem the region under subpressure E_0 can contain only incident waves, representable by the velocity potential

$$\Phi = e^{i\omega_0\left[t-\left(x_0\cos\varphi_0+y_0\sin\varphi_0\right)/c_0\right]} \tag{3.0.1}$$

Thus we have

$$\left.\begin{aligned}
s_0 &= -i\left(\omega_0/c_0^2\right)\Phi_0 \\
\bar{\omega}_0 &= \gamma s_0 = -i\gamma\left(\omega_0/c_0^2\right)\Phi_0 \\
\theta_0 &= (\gamma - 1)s_0 = -i(\gamma - 1)\left(\omega_0/c_0^2\right)\Phi_0 \\
u_0 &= -i\left(\omega_0/c_0\right)\cos\varphi_0\Phi_0 = c_0\cos\varphi_0 s
\end{aligned}\right\} \tag{3.0.2}$$

In the region E_1 there is a superposition of an acoustic phenomenon (letters with the subscript 1a) and a thermal phenomenon (subscript 1t).

The formulas (2.1.7) give us for the thermal phenomenon

$$\bar{\omega}_{1t} = 0 \qquad \theta_{1t} = -s_{1t} \qquad u_{1t} = -i\left(k\Omega/\gamma a_1\right)s_{1t} \tag{3.0.3}$$

In the acoustic phenomenon we have

$$\bar{\omega}_{1a} = \gamma s_{1a} \tag{3.0.4}$$

and in all

$$\bar{\omega}_1 = \bar{\omega}_{1t} + \bar{\omega}_{1a} = \bar{\omega}_{1a} \qquad \theta_1 = \theta_{1t} + \theta_{1a} \qquad s_1 = s_{1t} + s_{1a} \qquad u_1 = u_{1t} + u_{1a}$$

The equation (1.1.18) thus yields

$$s_{1t} = -\gamma J\left(\bar{\omega}_1 - \bar{\omega}_0\right) \tag{3.0.5}$$

where J is the function defined by the formula (1.1.11).

This relationship will give us the thermal wave as soon as we have calculated the acoustic wave.

Introducing this value s_{1t} into the third relation (3.0.3), we have

$$u_{1t} = i\left(k\Omega/a_1\right)J\left(\bar{\omega}_1 - \bar{\omega}_0\right) \tag{3.0.6}$$

This we insert into (1.1.21) which becomes

$$c_1u_{1a} - \left(G - i\,\frac{k\Omega}{a_1c_1}\,\gamma J\right)\frac{c_1^2\bar{\omega}_1}{\gamma} = \frac{1}{\Gamma}\left[c_0u_0 - \left(H - i\,\frac{k\Omega}{a_1c_0}\,\gamma J\right)\frac{c_0^2\bar{\omega}_0}{\gamma}\right] \tag{3.0.7}$$

We know that the numbers $k\Omega/a_1c_1$ and $k\Omega/a_1c_0$ are very small. On the other hand, table II in section 1 shows us that the function J is always smaller than the functions G and H. Consequently, we can neglect in equation (3.0.7) the imaginary terms, that is, the terms stemming from the thermal wave.

Thus there remains a condition which involves only the acoustic waves present on both sides of the shock-wave front S

$$c_1u_{1a} - G\,\frac{c_1^2\bar{\omega}_1}{\gamma} = \frac{1}{\Gamma}\left(c_0u_0 - H\,\frac{c_0^2\bar{\omega}_0}{\gamma}\right) \tag{3.0.8}$$

3.1. The Reflected Wave

We have to consider in the E_1 region two waves, one incident, the other reflected, represented by their velocity potentials

$$\Phi_1 = B_1 e^{i\omega_1\left[t-\left(x_1\cos\varphi_1+y_1\sin\varphi_1\right)/c_1\right]} \tag{3.1.1}$$

$$\Phi_1' = B_1' e^{i\omega_1'\left[t-\left(x_1\cos\varphi_1'+y_1\sin\varphi_1'\right)/c_1\right]} \tag{3.1.2}$$

The E_0 region, on the other hand, does not contain any wave. The second term of the condition (3.0.8) is therefore zero.

Assuming the kinematic conditions satisfied (3.0.8), we then have

$$B_1\omega_1\left(\cos\varphi_1 - G\right) + B_1'\omega_1'\left(\cos\varphi_1' - G\right) = 0 \tag{3.1.3}$$

The amplitudes of the waves are proportional to the factors $B\omega$. The ratio of the amplitude of the reflected wave and of the amplitude of the incident wave, that is, the amplitude-reflection coefficient, R, then is

$$R = -\left(\cos\varphi_1 - G\right)/\left(\cos\varphi_1' - G\right) \tag{3.1.4}$$

Let us refer to the kinematic relationship (2.2.1) which connects the tangents τ_1, τ_1' of the half angles $\frac{1}{2}\varphi_1$, $\frac{1}{2}\varphi_1'$; we can eliminate φ_1', and we obtain

$$R = \frac{\left(\frac{1 - A_1}{1 + A_1}\right)^2 + \tau_1^2}{1 + \tau_1^2}\,\frac{\frac{1 - G}{1 + G} - \tau_1^2}{\left(\frac{1 - A_1}{1 + A_1}\right)^2 - \frac{1 - G}{1 + G}\tau_1^2} \tag{3.1.5}$$

with $\tau_1^2 < \left(1 - A_1\right)/\left(1 + A_1\right)$ (incident wave).

For $\varphi_1 = 0$, τ_1 is equal to 0 (normal incidence), R_0 is equal to $(1 - G)/(1 + G)$, which is negative because G is greater than 1. (See table II, section 1.)

For $\varphi_1 = \arccos A_1$, $\tau_1 = \pm\sqrt{\left(1 - A_1\right)/\left(1 + A_1\right)}$ (limiting incidence), R_{lim} is equal to -1.

Since we have in this case $\varphi_1' = \varphi_1$, $\omega_1' = \omega_1$, this value signifies that in the region where the two waves are superimposed on one another (and which, incidentally, is vanishing) their sum is zero. In other words, the phenomenon involves only an incident acoustic wave which stops at S, which is actually the case.

Thus, the coefficient of amplitude reflection is constantly negative and, in absolute value, increases from $(G - 1)/(G + 1)$ to 1 when φ_1 increases from 0 to arc cos A_1.

Thus it will be possible, in practice, to observe the reflected wave at incidences close to the normal only if the shock wave is intense (G clearly larger than 1).

3.2. The Progressive Refracted Wave

Let us represent the incident wave, situated in E_0, by its velocity potential

$$\Phi_0 = B_0 e^{i\omega_0\left[t-\left(x_0\cos\varphi_0+y_0\sin\varphi_0\right)/c_0\right]}$$

and the refracted wave, situated in E_1, by

$$\Phi_1 = B_1 e^{i\omega_1\left[t-\left(x_1\cos\varphi_1+y_1\sin\varphi_1\right)/c_1\right]}$$

Starting from the instant where the kinematic conditions are satisfied, the exponents of the exponential terms are the same on S. The dynamic condition (3.0.8) then gives

$$\Gamma B_1\omega_1(\cos\varphi_1 - G) = B_0\omega_0(\cos\varphi_0 - H) \qquad (3.2.1)$$

We shall compare the amplitudes of the acoustic pressures in the two waves, that is, we shall form the ratio

$$R = P_1\bar{\omega}_1/P_0\bar{\omega}_0 = \rho_1 B_1\omega_1/\rho_0 B_0\omega_0$$

the value of which is given by (3.2.1):

$$R = \frac{1}{R\Gamma}\frac{H - \cos\varphi_0}{G - \cos\varphi_1} \tag{3.2.2}$$

where R is equal to ρ_0/ρ_1 and

$$\frac{1}{R\Gamma} = \sqrt{\frac{\mu + \xi}{\xi(1 + \mu\xi)}} = \frac{\Gamma}{\xi}$$

(See table II, section 1.)

In order to calculate R, one must refer to the kinematic conditions which give φ_1 as a function of φ_0.

We shall give below only the values R as functions of ξ for $\varphi_0 = 0$ (incident wave parallel to the shock wave and receding before the latter) and $\varphi_0 = \pi$ (incident wave parallel to the shock wave and coming to meet it).

We have in the two cases $\varphi_1 = \pi$ whence

$$\varphi_0 = 0 \qquad R_0 = \frac{1}{R\Gamma}\frac{H - 1}{G + 1}$$

$$\varphi_0 = \pi \qquad R_\pi = \frac{1}{R\Gamma}\frac{H + 1}{G + 1}$$

ξ	R for	
	$\varphi_0 = 0$	$\varphi_0 = \pi$
1	0	1
.9	.0002	1.094
.8	.0010	1.22
.7	.0035	1.36
.6	.0092	1.55
.5	.022	1.82
.4	.049	2.19
.3	.120	2.79
.2	.332	3.92
.1	1.29	7.00
.05	3.85	12.5
.025	7.85	17.9
0	∞	∞

3.3. The Accompanying Wave

With the incident wave being represented in the foregoing manner, the accompanying wave is derived from the potential

$$\Phi_1 = B_1 e^{(i\omega_1 - \delta_1)t + \left[(i\eta + \zeta)x_1 + i\eta' y\right]/c_1}$$

When the kinematic conditions are satisfied, the boundary condition (3.0.8) gives

$$i\omega_0 B_0 \frac{H - \cos\varphi_0}{c_0} = (i\omega_1 - \delta_1)B_1\left(G - \frac{i\eta + \zeta}{i\omega_1 - \delta_1}\right) \qquad (3.3.1)$$

The ratio R which interests us most is the ratio of the acoustic pressure immediately behind the shock wave $p_1 = -\rho_1(i\omega_1 - \delta_1)B_1$ and of the acoustic pressure in the incident wave $p_0 = -i\rho_0\omega_0 B_0$.

Taking the relations (2.3.12 to 2.3.17) into account, we obtain

$$R = \frac{\cos\varphi_0 - H}{\Gamma R} \frac{i - \delta_1/\omega_1}{i(A_1 - G) - (\delta_1/\omega_1)(1/A_1 - G)} \qquad (3.3.2)$$

and from the fact that δ_1/ω_1 is equal to $A_1\sqrt{\left(1 - A_1^2\right)\left(c_1/w\right)^2 - 1}$, we obtain

$$|R| = \frac{H - \cos\varphi_0}{\Gamma R}\sqrt{\frac{A_1^2 - W_1^2}{\left(1 - A_1G\right)^2 + \left(G^2 - 1\right)W_1^2}}$$

where

$$W_1 = \frac{w}{c_1} = \Gamma\,\frac{1 - A_0\cos\varphi_0}{\sin\varphi_0}$$

For illustrative purposes we have performed the calculation of $|R|$ in the extreme case where w is equal to 0, that is, $\cos\varphi_0$ is equal to $1/A_0$. We have then

$$|R|_{w=0} = \frac{1}{\Gamma R}\,\frac{H - 1/A_0}{1/A_1 - G}$$

For $\varphi_0 = \arccos 1/A_0$ we obtain

ξ = 1 0.9 0.8 0.7 0.6 0.5 0.4 0.3 0.2 0.1 0.05 0.025 0

$|R|$ = 1 1.05 1.15 1.31 1.46 1.71 2.06 2.66 3.86 7.45 14.6 22.8 ∞

We see that, as in the previous case, the acoustic pressures may be increased by a significant factor.

4. SUMMARY OF RESULTS

The presence of acoustic waves in one or the other of the regions which separate the shock-wave front is manifested in the following phenomena:

(a) In the region under subpressure E_0 in which the shock wave progresses at a supersonic velocity, no type of reflected or refracted wave is produced.

(b) In the region under superpressure E_1 the encounter of an incident acoustic wave (coming from the E_0 region or the E_1 region) with a shock wave causes the formation of two waves of different type. One is an acoustic wave (refracted or reflected depending on the origin of the incident wave); the other is a thermal wave.

(c) The thermal wave is not progressive in the first approximation (that is, when the incident acoustic waves are of an ordinary amplitude). It is therefore motionless with respect to the gas contained in the region under superpressure and carried along by this gas in its motion of overall flow. It then moves away from the shock-wave front. This wave involves spatial periodic variations in temperature and in density but no variation in pressure. It is damped slowly and aperiodically.

(d) When the incident acoustic wave is situated in the region under superpressure, in that same region a reflected acoustic wave, of the same type as the incident wave, is produced; that is, in the case studied here, a plane, progressive, and uniform wave.

Huygens' construction permits obtaining its kinematic characteristics: direction of propagation, and wave length.

The amplitude of the pressure variations in the reflected wave is always smaller than the amplitude of these variations in the incident wave.

(e) When the incident acoustic wave is situated in the region under subpressure, a refracted acoustic wave is produced in the region under superpressure; the characteristics of the latter wave depend on the phase velocity w of the phenomenon produced on the shock front by the incident wave. The main characteristics of the refracted wave are the following:

(α) In the region under subpressure E_0 there exist always two incidences of plane waves which produce on the shock front the same phase velocity w and, consequently, in the region under superpressure E_1 the same refracted wave (except for the amplitude ratio).

The observation of the refracted wave in the region under superpressure E_1 therefore does not permit distinguishing whether this wave has been produced by one or by the other of the two possible incident waves, or by a combination of both.

(β) If the phase velocity w of the disturbance produced on the shock wave by the incident acoustic wave is larger than a

certain limiting value w_0 (which depends on the intensity of the shock), the refracted wave is of the same type as the incident wave: plane, progressive, and uniform. Huyghens' construction permits obtaining its kinematic characteristics.

The amplitude of the pressure variations in the refracted wave may be, depending on the case (angle of incidence, shock amplitude) smaller than, equal to, or larger than the amplitude of these variations in the incident wave. All things being equal, this amplitude increases with the shock amplitude.

The increase in amplitude may become very important in the case of intense shocks.

(γ) If the phase velocity is smaller than the limiting velocity w_0, which corresponds to incidences situated in a sector the boundaries of which are functions of the shock amplitude, the refracted wave assumes the form of an "accompanying wave." This wave remains attached to the shock wave which it accompanies while shifting parallel to that wave at the velocity w.

The wave planes (planes of equal phase but not-equal intensity) are oblique with respect to the shock. Their direction may be obtained by a simple geometric construction.

The wave amplitude decreases exponentially as one moves away from the shock-wave front. Immediately behind the shock wave, the amplitude of the pressure variations in the accompanying wave increases with increasing shock intensity. It may be larger than the amplitude of the pressure variations in the incident wave.

(ϑ) In summary, the presence of acoustic waves in the region under subpressure, in which the shock wave progresses, may manifest itself behind the shock wave, on the one hand, by a field of progressive acoustic waves which expand gradually in the entire region under superpressure; on the other hand, by the appearance of accompanying waves, that is, of oscillating disturbances which the shock wave carries along behind it in its progress.

Both may, in certain cases, involve oscillations of pressure very much larger than those that existed before the arrival of the shock wave.

Translated by Mary L. Mahler
National Advisory Committee
for Aeronautics

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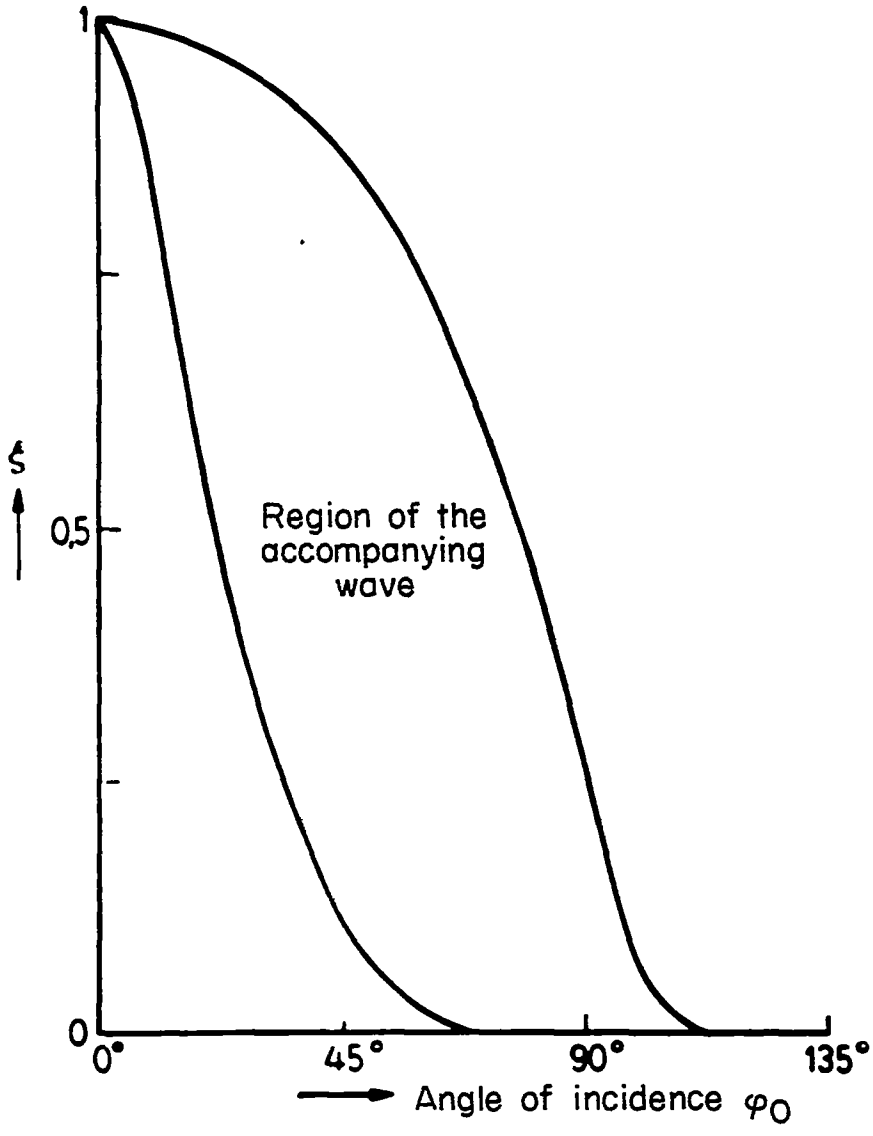
1
0,5
0
ξ
Region of the accompanying wave
0°
45°
90°
135°
Angle of incidence φ_0

Figure 1

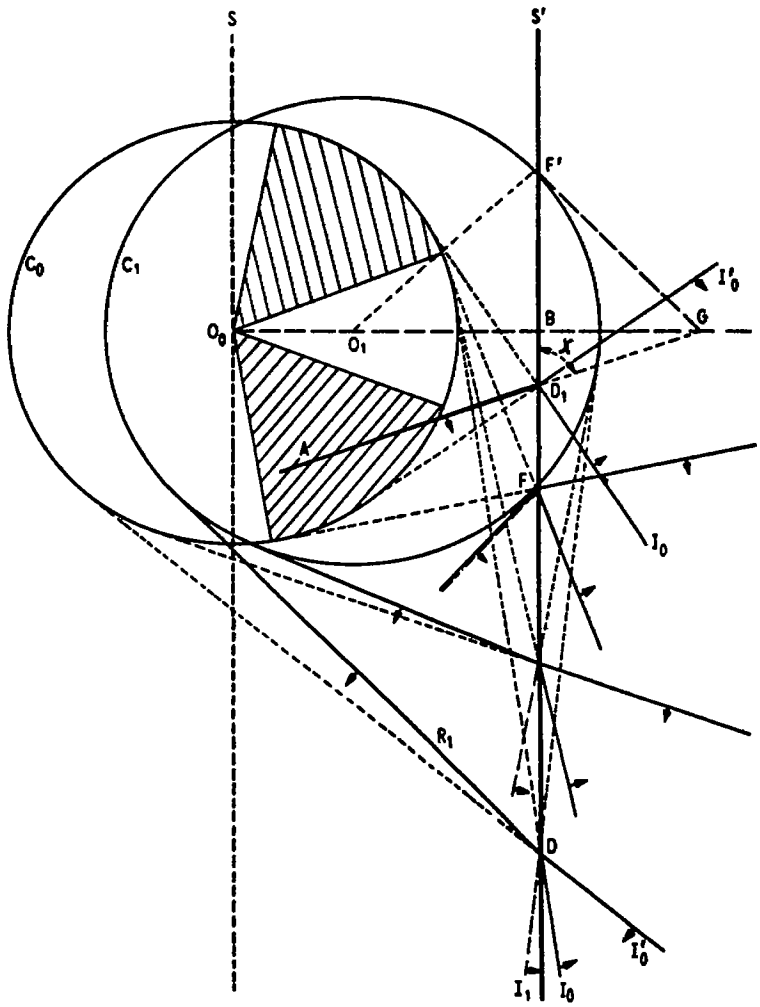

Figure 2