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TECHNICAL NOTE 3968

THE APPLICATION OF MATRIX METHODS TO COORDINATE TRANSFORMATIONS OCCURRING IN SYSTEMS STUDIES INVOLVING LARGE MOTIONS OF AIRCRAFT

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THE APPLICATION OF MATRIX METHODS TO COORDINATE TRANSFORMATIONS OCCURRING IN SYSTEMS STUDIES INVOLVING LARGE MOTIONS OF AIRCRAFT

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SUMMARY

The purpose of this paper is to show the method and advantages of matrix algebra in setting up the geometric aspects of problems of airplane motion. Such aspects arise particularly when studies of systems which include aircraft are being made. The geometry is formulated by fixing quantities whose relative motions are to be studied, each in a coordinate system of its own. The various coordinate systems are related to each other by orthogonal transformations in matrix form, and the parameters defining the transformations are found in terms of the dynamical variables of the problem with the help of the transformation matrices. The compact notation of matrix algebra permits a clear view of the geometry involved. Use of matrix algebra provides a routine procedure for computing the detailed expressions required in a particular problem. The first part of the paper discusses those aspects of matrix algebra required for use in orthogonal transformations. The second part shows how to use orthogonal transformations in matrix form by applying them in several examples.

INTRODUCTION

There are many problems currently under study that concern the motion of one or more aircraft over relatively long periods of time. For example, long trajectories are involved in some studies of very high-altitude, high-speed aircraft. Another example is provided by current fire-control systems studies where the relative motion of two aircraft is involved. The elements of interest in such problems, in one case the position and attitude of the aircraft with respect to a nonflat earth, in another the relative positions and rates of the two aircraft, are referred to coordinate systems that undergo large changes in orientation. The formulation of these problems for study, say, on an electronic analog computer, requires the expression of these large orientation changes by means of orthogonal transformations. These transformations refer the coordinate systems, in which the elements of interest are imbedded, to an initial coordinate system whose orientation is fixed.

The equations that express the transformed components of a quantity can be obtained in a simple fashion if the orthogonal transformations of the coordinate systems involved are written in matrix form and the rules of matrix algebra followed. The matrices that perform orthogonal transformations are rotation matrices. A description of their properties and the algebra they satisfy - that is, the rules by which such operations as addition and multiplication are performed - can be found in many places (see, e.g., refs. 1, 2, 3, and 4). Their application to certain aspects of problems concerning aircraft is illustrated in references 1, 5, and 8. One purpose of the present paper is to present the matrix method of performing orthogonal transformations in as simple a manner as possible, to describe its algebra, and to illustrate the algebra in use by several examples.

The examples in this paper will serve a second purpose, namely to show that the complete geometry of the problem under study can be expressed in matrix form. By imbedding any element of interest in a coordinate system, and by following this coordinate system by means of matrices as it rotates with respect to all other elements of the problem which are fixed in rotating coordinates of their own, a single matrix expression for the over-all geometry is obtained. From such an expression the expression relating any two quantities of interest can be derived in a routine manner. Furthermore, since in any problem there are a number of ways of choosing an appropriate set of rotating coordinate systems, the compact notation of matrices which provides a clear view of the geometry of the problem is especially valuable in permitting the examination of various possible choices of sets of coordinates, thus facilitating selection of one set that may simplify the representation of the geometry on a computing machine. The matrix notation permits this examination to proceed without detailed computation having to be performed.

Once the choice of coordinates has been made, the parameters that specify them, such as the angles φ, θ, and ψ that express the orientation of an airplane, can be determined in terms of quantities that occur in the problem solution, such as the airplane rotation rates p, q, and r. The examples in the present paper also show how the determination of these parameters, or their rates of change, can be obtained by matrix methods. Thus all the geometric expressions required in these problems can be obtained by the regular and routine application of matrix methods.

A question of notation arises from problems such as those described in this paper. Not only are there several quantities of interest in each problem, but some of these quantities have to be expressed in several coordinate systems. The notation adopted in this paper to satisfy the requirements of this type of problem is described briefly in the text and more fully in the Appendix.

DISCUSSION OF BASIC PROPERTIES OF ROTATION MATRICES

The first part of the present paper is intended for those readers who either are not at all familiar with the algebra of matrices or are not accustomed to thinking of matrices in terms of their properties of performing rotations. The fundamental matrix algebra, such as the conventions governing addition and multiplication, will be presented in a general manner. Orthogonal matrices (which perform only rotations, and therefore will be called rotation matrices) form a special group of matrices and have properties, not possessed by other matrices, which simplify their manipulation. Such properties will be pointed out and subsequently often used.

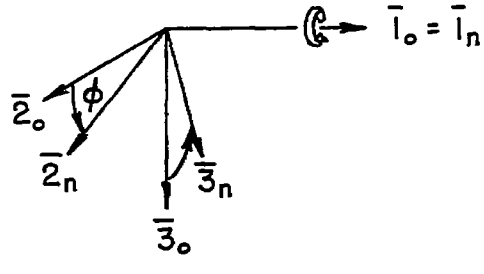
A glance at the examples in later parts of this paper will show that the matrices required in a single problem may form a long chain performing rotations in succession. The discussion of the basic properties of rotation matrices will begin with a consideration of matrices performing a single rotation. Then the properties of matrices of successive rotations will be discussed.

Single Rotations

The rotation matrix performs an orthogonal transformation on some quantity. This quantity may be a vector or a set of vectors that define a particular coordinate system. The coordinate systems considered are three-dimensional and Cartesian. Three vectors are required to define such a coordinate system. In this paper, a triad of vectors defining a coordinate system (or set of coordinates) are of unit length, at right angles to each other (mutually orthogonal), and oriented with respect to each other in such a way as to form a "right-handed" system.

Rotation matrices perform the operation of rotation on these quantities; that is, the quantity obtained by transformation differs only in orientation from the original quantity. Not only does it retain its size, but, if a coordinate system is the rotated quantity, the unit vectors of the new coordinate system are mutually orthogonal in the same way as were the unit vectors of the old.

Special forms of rotation matrices.- The matrices of simplest form are those that use one or another of the three unit vectors as an axis of rotation. In the sketch adjacent, a new coordinate system is obtained from the old by a rotation through the angle φ about the $\bar{1}$ direction. Several



Sketch (a)

items can be noted in the sketch. The three unit vectors are labeled $\bar{1}$, $\bar{2}$, $\bar{3}$. The original set has the subscript o; the new, the subscript n. The rotation with the $\bar{1}$ vector as an axis is said to have a positive sense if it has the sense of rotation of a "right-hand screw" which is advancing along the positive $\bar{1}$ direction.

If an old and a new set of coordinates are related as shown in sketch (a), the components of a vector (or the coordinates of a point) in the new set are related to its components in the old by the following set of equations:

$$\left.\begin{aligned} x_{1n} &= x_{1o} \\ x_{2n} &= \quad x_{2o}\cos\varphi + x_{3o}\sin\varphi \\ x_{3n} &= \quad -x_{2o}\sin\varphi + x_{3o}\cos\varphi \end{aligned}\right\} \tag{1a}$$

which is the same relation obtaining between the new unit vectors and the old

$$\left.\begin{aligned} \bar{1}_n &= \bar{1}_o \\ \bar{2}_n &= \quad \bar{2}_o\cos\varphi + \bar{3}_o\sin\varphi \\ \bar{3}_n &= \quad -\bar{2}_o\sin\varphi + \bar{3}_o\cos\varphi \end{aligned}\right\} \tag{1b}$$

This system can be put into the following tabular form from which the shape of the matrix emerges:

	x_{1o}	x_{2o}	x_{3o}
x_{1n}	1	0	0
x_{2n}	0	$\cos\varphi$	$\sin\varphi$
x_{3n}	0	$-\sin\varphi$	$\cos\varphi$

(2a)

How to get equations (1) from the table is clear: one reads along a row, multiplying each member of a row by its column heading. The arrangement can be changed to read like an equation if proper conventions are adopted to multiply the elements of the table by the correct quantity to give equations (1):

$$\begin{bmatrix} x_{1n} \\ x_{2n} \\ x_{3n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} x_{1o} \\ x_{2o} \\ x_{3o} \end{bmatrix} \tag{2b}$$

With the proper conventions adopted, this table is an equation which can be symbolized thus:

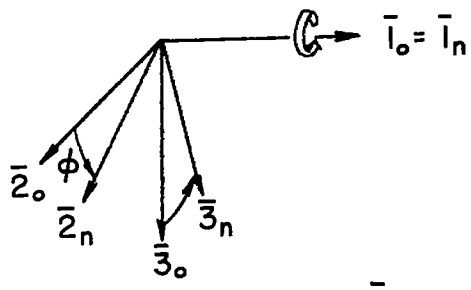
$$[\text{new components}] = (\varphi)[\text{old components}] \tag{2c}$$

where (φ) is the matrix with elements:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}$$

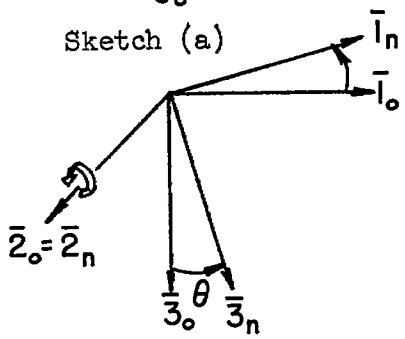
Equality indicated between two matrices holds if and only if corresponding elements are equal. Corresponding elements of two matrices are the elements in the same row and column in the two matrices. The convention used in multiplying matrices can be determined by comparing equations (1a) and (2b). In equation (2b) there are three matrices whose symbols are [new components], (φ), and [old components]. The first and third, column matrices, have three rows of elements all forming a single column. The (φ) matrix has nine elements grouped into three rows and three columns. Equations (1) indicate that the product (2b) or (2c) stands for three equations. To obtain the first of equations (1), the elements of a given row of (φ) are multiplied by each one of the elements of the following column matrix, and the products added. For example, for the first row in (φ), the element of column 1 is multiplied by the element of row 1 in [old components]; the element of column 2 in (φ) is multiplied by the element of row 2 in [old components]; element of column 3 in (φ) is multiplied by the element of row 3 in [old components]. The sum of these products is equal to the element of the first row in [new components]. The procedure is repeated for each row of (φ) until all three rows of [new components] are obtained.

The three simplest rotation matrices together with a sketch of the operation each performs are shown below:



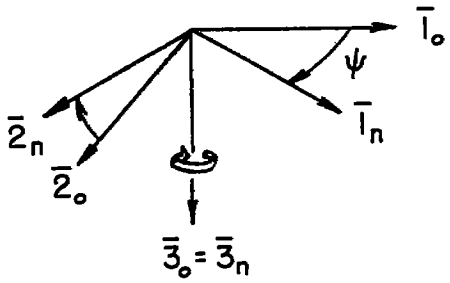
Sketch (a)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} = (\varphi)$$



Sketch (b)

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = (\theta) \qquad (3)$$



Sketch (c)

$$\begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = (\psi)$$

The three special forms of rotation matrices given next to the sketches are the operators most commonly employed because they are simple in form and easy to visualize. It is not necessary for one to know their exact forms offhand, since they are easily derived.

General form of rotation matrix.- Each of the three rotation matrices in equation (3) is a special case of the general form of rotation matrix. The derivation of this general form follows easily from certain considerations concerning vectors. (The notation used below and in subsequent sections of this paper is explained in the Appendix.) A vector is a geometric or dynamic quantity, containing a magnitude and a sense, which exists independently of any description of it in a particular coordinate system. Thus, if $\bar{v}$ is some vector, say, a velocity, then

$$\bar{v} = v_{1n}\bar{1}^n + v_{2n}\bar{2}^n + v_{3n}\bar{3}^n = v_{1o}\bar{1}^o + v_{2o}\bar{2}^o + v_{3o}\bar{3}^o$$

where the subscript and superscript n's and o's refer to new and old coordinates, respectively (the new having been obtained from the old coordinates by a rotation), and the subscript numbers indicate the component of the vector. The dot product of $\bar{v}$ with $\bar{1}^n$, $\bar{2}^n$, and $\bar{3}^n$ in turn gives

$$\bar{v}\cdot\bar{1}^n = v_{1n} = v_{1o}\bar{1}^o\cdot\bar{1}^n+v_{2o}\bar{2}^o\cdot\bar{1}^n+v_{3o}\bar{3}^o\cdot 1^n$$

$$\bar{v}\cdot\bar{2}^n = v_{2n} = v_{1o}\bar{1}^o\cdot\bar{2}^n+v_{2o}\bar{2}^o\cdot\bar{2}^n+v_{3o}\bar{3}^o\cdot\bar{2}^n$$

$$\bar{v}\cdot\bar{3}^n = v_{3n} = v_{1o}\bar{1}^o\cdot\bar{3}^n+v_{2o}\bar{2}^o\cdot\bar{3}^n+v_{3o}\bar{3}^o\cdot\bar{3}^n$$

since $\bar{1}^n\cdot\bar{1}^n = \bar{2}^n\cdot\bar{2}^n = \bar{3}^n\cdot\bar{3}^n = 1$ and $\bar{1}^n\cdot\bar{2}^n = \bar{1}^n\cdot\bar{3}^n = \bar{2}^n\cdot\bar{3}^n = 0$. This set of equations can be represented in the matrix form

$$\begin{bmatrix} v_{1n} \\ v_{2n} \\ v_{3n} \end{bmatrix} = \begin{bmatrix} \bar{1}^o\cdot\bar{1}^n & \bar{2}^o\cdot\bar{1}^n & \bar{3}^o\cdot\bar{1}^n \\ \bar{1}^o\cdot\bar{2}^n & \bar{2}^o\cdot\bar{2}^n & \bar{3}^o\cdot\bar{2}^n \\ \bar{1}^o\cdot\bar{3}^n & \bar{2}^o\cdot\bar{3}^n & \bar{3}^o\cdot\bar{3}^n \end{bmatrix} \begin{bmatrix} v_{1o} \\ v_{2o} \\ v_{3o} \end{bmatrix} \tag{4}$$

The 3×3 matrix in (4) is the desired general expression. Since the vectors in the matrix are of unit magnitude, the elements in the matrix are simply cosines of angles. Each of the three rotation matrices in (3) is a special case of that in (4) obtained by making one of the unit vectors of the new coordinate set coincide with one of those of the old. The sines of angles in (3) are the cosines of 90° plus or minus those angles.

Successive Rotations

Matrix multiplication.- In writing out the terms resulting from the multiplication of a three row, three column matrix (called a 3×3 matrix) by another 3×3 matrix, one follows the same procedure as that described in the previous section. The product is a 3×3 matrix, rather than the 3×1 matrix obtained above. The procedure described furnishes the elements of one column of the product. For example, let (C) = (A)(B):

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}$$

The subscripts on the various elements indicate the position of the elements in the matrix. The first subscript indicates the row, the second, the column the element occupies. Since the elements of two matrices are equal if the matrices are equal, nine equations are contained in (C) = (A)(B). Two of the nine are:

$$c_{11} = a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} \; ; \qquad c_{12} = a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32}$$

All nine equations are contained in the expression $c_{ik} = \sum_{j=1}^{3} a_{ij}b_{jk}$.

This expression also describes the method of multiplication, for, to obtain the element in a given row (i = 1, 2, or 3) and column (k = 1, 2, or 3) in (C), each of the j column elements (j = 1, 2, and 3 in turn) of row i in (A) are multiplied by each corresponding row element j (the same j = 1, 2, and 3 in turn) of column k in (B) and the resulting terms are added.

This multiplication procedure is general.[1] It therefore applies to two special multipliers, the unit matrix and a constant. The unit matrix corresponds to the number 1 in arithmetic which has the property that the product of the multiplication of any number by 1 is identical to the original number. This property can be stated operationally in the following

[1]The multiplication procedure is general provided the matrices are conformable (ref. 1, p.6). Conformability means that in the product (C) = (A)(B) or $c_{ik} = \sum_{j=1}^{3} a_{ij}b_{jk}$ there must be as many elements with index j in b_{jk} as there are in a_{ij}.

way: Unity is that multiplier which operates on a multiplicand in such a fashion that the resulting product is the original multiplicand. The matrix with this property is the unit matrix, which has the form:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

It can be written symbolically as (I), and is the "diagonal matrix" all of whose main-diagonal elements are unity.

The product of a constant times a matrix is a matrix each element of which is the constant times the corresponding element of the multiplicand matrix. Thus, if (C) = k(B), then $c_{ij} = kb_{ij}$. The equation can also be written

$$k \begin{bmatrix} b_{11} b_{12} b_{13} \\ b_{21} b_{22} b_{23} \\ b_{31} b_{32} b_{33} \end{bmatrix} = \begin{bmatrix} k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{bmatrix} \begin{bmatrix} b_{11} b_{12} b_{13} \\ b_{21} b_{22} b_{23} \\ b_{31} b_{32} b_{33} \end{bmatrix}$$

A constant can therefore be called a diagonal matrix with equal and constant elements.

Matrix addition.- The sum of two matrices is defined as the matrix whose elements are the sums of the corresponding elements of the added matrices: let

$$(A) = \begin{bmatrix} a_{11} a_{12} a_{13} \\ a_{21} a_{22} a_{23} \\ a_{31} a_{32} a_{33} \end{bmatrix} \quad \text{and} \quad (B) = \begin{bmatrix} b_{11} b_{12} b_{13} \\ b_{21} b_{22} b_{23} \\ b_{31} b_{32} b_{33} \end{bmatrix}$$

Then (A) + (B) is the matrix:

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

This identity can be written in the form (A)+(B) = (A+B).

Geometric interpretation of rotation matrices.- The rotation matrix not only partakes of all the properties of matrices given above, but also may be interpreted geometrically as an operator that rotates a coordinate system through some angle about some axis given in the coordinate system. Multiplication of two such operators corresponds to two successive rotations. If the operation depicted in sketch (a) is followed by another identical operation, the total operation must consist in a similar rotation of twice the magnitude of either one. That the algebra agrees with the geometry is seen by multiplying the rotation matrices together:

$$(\varphi_2)(\varphi_1) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_2 & \sin\varphi_2 \\ 0 & -\sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_1 & \sin\varphi_1 \\ 0 & -\sin\varphi_1 & \cos\varphi_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_2\cos\varphi_1-\sin\varphi_2\sin\varphi_1 & \cos\varphi_2\sin\varphi_1+\sin\varphi_2\cos\varphi_1 \\ 0 & -\sin\varphi_2\cos\varphi_1-\cos\varphi_2\sin\varphi_1 & -\sin\varphi_2\sin\varphi_1+\cos\varphi_2\cos\varphi_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\varphi_2+\varphi_1) & \sin(\varphi_2+\varphi_1) \\ 0 & -\sin(\varphi_2+\varphi_1) & \cos(\varphi_2+\varphi_1) \end{bmatrix}$$

The order of the matrices implies that the operation (φ_1) is first to be performed on whatever is to its right, followed by operating by (φ_2) on everything to its right. The order in which two matrices are written is usually important. (When a number of finite rotations are performed about the same fixed axis, the order is not significant.) The following examples of rotation about two different axes illustrate that the order in which the rotations are performed is important.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi & \sin\varphi\cos\theta \\ \cos\varphi\sin\theta & -\sin\varphi & \cos\varphi\cos\theta \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta\sin\varphi & -\sin\theta\cos\varphi \\ 0 & \cos\varphi & \sin\varphi \\ \sin\theta & -\cos\theta\sin\varphi & \cos\theta\cos\varphi \end{bmatrix}$$

If the φ's and θ's are respectively equal in the two products, then the product matrices are not equal.

Matrix inversion.- After a new coordinate system has been obtained by rotation from an old, it is clearly possible to regain the old by another rotation. The second rotation is said to be the inverse rotation. If the matrix of the first rotation is (φ), that of the inverse is denoted by $(\varphi)^{-1}$. In symbols, if

$$[\text{new components}] = (\varphi)[\text{old components}]$$

then

$$[\text{old components}] = (\varphi)^{-1}[\text{new components}]$$

It follows, by multiplying the first equation by $(\varphi)^{-1}$ and the second by (φ) that $(\varphi)(\varphi)^{-1} = (I) = (\varphi)^{-1}(\varphi)$ where (I) is the unit matrix. Written out, the matrix (φ) and its inverse have the following form:

$$(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \qquad (\varphi)^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{bmatrix}$$

Upon examining the two matrices, one sees that the inverse matrix can be obtained from the (φ) matrix by exchanging the angle $-\varphi$ for every φ. This property of rotation matrices, which is not possessed by other matrices, again confirms that the results of the matrix algebra agree with the geometry, for the inverse rotation geometrically is the same as a rotation in a negative sense.

Inversion of a rotation matrix can also be achieved by transposing its elements, where the transposed matrix is obtained by exchanging elements across the main diagonal of the original matrix; that is, if (B) is the transpose of (A), then $b_{ij} = a_{ji}$, for example, $b_{12} = a_{21}$. In (φ), since this operation replaces $\sin\varphi$ by $-\sin\varphi$ and vice versa, it is equivalent to replacing φ by $-\varphi$.

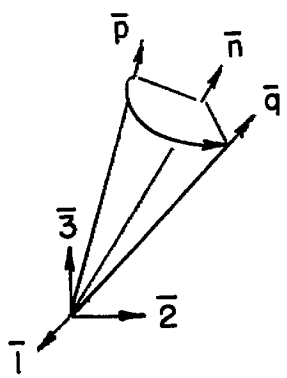
Reversal property.- Suppose there has been a sequence of rotations, for example, a (φ) followed by a (θ) rotation. A consideration of the geometry indicates that to return to the old coordinates from the new, one must rotate first through $(-\theta)$ or $(\theta)^{-1}$, then through $(-\varphi)$ or $(\varphi)^{-1}$. It is easy to show that again the algebra and the geometry agree. First of all the product operation $(\theta)(\varphi)$ is itself an orthogonal rotation operator. Hence $[(\theta)(\varphi)]^{-1}[(\theta)(\varphi)] = (I)$. But since $(\theta)^{-1}(\theta) = (I)$ and $(\varphi)^{-1}(\varphi) = (I)$, it must be that $(\varphi)^{-1}(\theta)^{-1}(\theta)(\varphi) = (I)$. As a result, $(\varphi)^{-1}(\theta)^{-1} = [(\theta)(\varphi)]^{-1}$. In words, the inverse of a product of matrices

is the product of the inverses of the matrices taken in reversed order. It is worth repeating that the property $(-\varphi) = (\varphi)^{-1}$ is not a general property of matrices, but belongs to those matrices that perform only a rotation.

EXAMPLE APPLICATIONS

Rotation of a Vector About an Arbitrary Axis

For a first example, the form of single rotation matrices will be generalized. The rotation will no longer be performed about one of the old coordinate directions, but about an arbitrary axis, that is about an axis $\bar{n}$ of unit length with components n_1, n_2, and n_3 in the given coordinate system. Another difference from the previous discussion is that in this example a vector $\bar{p}$ is to be rotated about the axis $\bar{n}$ through some arbitrary angle φ into a new position $\bar{q}$. Heretofore, the coordinate system was rotated, and to transform a vector meant to express the same vector in two coordinate systems with the same origin but with one system rotated with respect to the other. This difference, however, reduces to a matter of sign, for to rotate a coordinate system through a positive angle is entirely equivalent to rotating a vector in the (now fixed) coordinate system through the same angle but in the opposite sense.



Sketch (d)

The quantities given in the problem are illustrated in sketch (d). A unit vector $\bar{n}$ has the components n_1, n_2, and n_3 in the coordinate system formed by the triad of unit vectors $\bar{1}$, $\bar{2}$, and $\bar{3}$. Therefore $\bar{n} = n_1\bar{1}+n_2\bar{2}+n_3\bar{3}$ and $n_1^2+n_2^2+n_3^2 = 1$. It is desired to rotate another vector $\bar{p}$ (with given components p_1, p_2, and p_3) through some given angle φ into the new position shown in the sketch as the vector $\bar{q}$. The problem is to find the new components q_1, q_2, and q_3 in terms of φ and the components of $\bar{p}$ and $\bar{n}$.

This problem has been solved both by matrix methods (ref. 1, p. 253; and ref. 3, p. 96) and by pure vector methods (ref. 2, p. 168). The method here will use only the matrices already discussed, and will illustrate the approach adopted in succeeding examples in this paper. The general procedure is as follows: The vector $\bar{n}$ will be thought of as one of a triad of unit vectors forming a coordinate system. The vector $\bar{p}$ will be expressed in this coordinate system which will then be rotated in a negative sense through the desired angle φ. The resulting new components of $\bar{p}$ will then be returned to the fixed coordinate system by an inverse transformation. These will be the desired components of $\bar{q}$.

Let $[(\theta)(\psi)]^{-1}$ be the matrix transforming $\bar{p}$ from the fixed into the n coordinate system. Then

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = [(\theta)(\psi)](-\varphi)[(\theta)(\psi)]^{-1} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{5}$$

where

$$(\theta)(\psi) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix}$$

and

$$(\varphi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}$$

The angles ψ and θ can be eliminated by means of the components of $\bar{n}$ in the fixed coordinate system. Since

$$\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = [(\theta)(\psi)]\bar{n} = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ -\sin\psi & \cos\psi & 0 \\ \sin\theta\cos\psi & \sin\theta\sin\psi & \cos\theta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

$n_1 = \cos\theta\cos\psi$; $n_2 = -\sin\psi$; $n_3 = \sin\theta\cos\psi$. From the latter expressions one gets: $\cos\psi = \sqrt{n_1^2+n_3^2}$; $\cos\theta = n_1/\sqrt{n_1^2+n_3^2}$; $\sin\theta = n_3/\sqrt{n_1^2+n_3^2}$. Therefore,

$$[(\theta)(\psi)] = \begin{bmatrix} n_1 & \dfrac{-n_1n_2}{\sqrt{n_1^2+n_3^2}} & \dfrac{-n_3}{\sqrt{n_1^2+n_3^2}} \\ n_2 & \sqrt{n_1^2+n_3^2} & 0 \\ n_3 & \dfrac{-n_2n_3}{\sqrt{n_1^2+n_3^2}} & \dfrac{n_1}{\sqrt{n_1^2+n_3^2}} \end{bmatrix}$$

and

$$[(\theta)(\psi)]^{-1} = \begin{bmatrix} n_1 & n_2 & n_3 \\ \dfrac{-n_1n_2}{\sqrt{n_1^2+n_3^2}} & \sqrt{n_1^2+n_3^2} & \dfrac{-n_2n_3}{\sqrt{n_1^2+n_3^2}} \\ \dfrac{-n_3}{\sqrt{n_1^2+n_3^2}} & 0 & \dfrac{n_1}{\sqrt{n_1^2+n_3^2}} \end{bmatrix}$$

Performing the operations indicated in equation (5) one obtains a matrix with the following elements:

<u>The elements of the first column</u>

$$n_1^2 + \frac{n_1^2n_2^2\cos\varphi}{(n_1^2+n_3^2)} + \frac{n_3^2\cos\varphi}{(n_1^2+n_3^2)}$$

$$= n_1^2 + \frac{n_1^2n_2^2\cos\varphi + n_3^2\cos\varphi + n_1^2(n_1^2+n_3^2)\cos\varphi}{(n_1^2+n_3^2)} - n_1^2\cos\varphi$$

$$= n_1n_1(1-\cos\varphi)+\cos\varphi$$

$n_1n_2-n_1n_2\cos\varphi+n_3\sin\varphi$

$$n_1n_3+\frac{n_1n_3n_2^2\cos\varphi-n_1n_3\cos\varphi}{n_1^2+n_3^2}-\frac{n_2n_3^2\sin\varphi+n_2n_1^2\sin\varphi}{n_1^2+n_3^2}$$

$$=n_1n_3+\frac{n_1n_3(n_2^2-1)\cos\varphi}{n_1^2+n_3^2}-n_2\sin\varphi$$

$$=n_1n_3(1-\cos\varphi)-n_2\sin\varphi$$

<u>The elements of the second column</u>

$n_1n_2(1-\cos\varphi)-n_3\sin\varphi$

$n_2^2+(n_1^2+n_3^2)\cos\varphi=n_2^2(1-\cos\varphi)+\cos\varphi$

$n_2n_3(1-\cos\varphi)+n_1\sin\varphi$

<u>The elements of the third column</u>

$$n_1n_3+\frac{n_1n_3n_2^2\cos\varphi-n_1n_3\cos\varphi}{n_1^2+n_3^2}+\frac{n_2n_3^2\sin\varphi+n_2n_1^2\sin\varphi}{n_1^2+n_3^2}$$

$$=n_1n_3(1-\cos\varphi)+n_2\sin\varphi$$

$n_2n_3(1-\cos\varphi)-n_1\sin\varphi$

$$n_3^2+\frac{n_2^2n_3^2\cos\varphi+n_1^2\cos\varphi}{n_1^2+n_3^2}$$

$$=n_3^2(1-\cos\varphi)+\frac{n_2^2n_3^2\cos\varphi+n_1^2\cos\varphi+n_3^2(n_1^2+n_3^2)\cos\varphi}{n_1^2+n_3^2}$$

$$=n_3n_3(1-\cos\varphi)+\cos\varphi$$

With these results, equation (5) can be written in the form

$$\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} n_1n_1(1-\cos\varphi)+\cos\varphi & n_1n_2(1-\cos\varphi)-n_3\sin\varphi & n_1n_3(1-\cos\varphi)+n_2\sin\varphi \\ n_1n_2(1-\cos\varphi)+n_3\sin\varphi & n_2n_2(1-\cos\varphi)+\cos\varphi & n_2n_3(1-\cos\varphi)-n_1\sin\varphi \\ n_1n_3(1-\cos\varphi)-n_2\sin\varphi & n_2n_3(1-\cos\varphi)+n_1\sin\varphi & n_3n_3(1-\cos\varphi)+\cos\varphi \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

$$= \left\{(1-\cos\varphi)\begin{bmatrix} n_1n_1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2n_2 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3n_3 \end{bmatrix} + \cos\varphi(I)+(\sin\varphi)\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}\right\}\begin{bmatrix} p_2 \\ p_2 \\ p_3 \end{bmatrix} \quad (6)$$

where (I) is the unit matrix. To see that expressions (3) are contained in equation (6) one must recall that whereas equation (6) expresses the rotation of a vector in a fixed coordinate system, expressions (3) rotate the coordinate system with the vector fixed. Since the two operations are opposite in sense, the angles in the matrices will be opposite in sign. With this in mind, after letting $\overline{n}$ equal first $n_1\overline{1}$, then $n_2\overline{2}$, then $n_3\overline{3}$, and letting φ equal first $-\varphi$, then $-\theta$, then $-\psi$, one obtains the expressions (3).

A valuable by-product of equation (6) is obtained from considering the vector $\overline{q}-\overline{p}$ generated during a small time interval by a rotation through the small angle φ. If, in the limit as $\Delta t \rightarrow 0$, $\varphi/\Delta t \rightarrow \dot{\varphi} = \overline{\omega}$, then

$$\lim_{\Delta t \rightarrow 0} \frac{(1-\cos\varphi)}{\Delta t} = 0 \qquad \text{and} \qquad \lim_{\Delta t \rightarrow 0} \frac{\overline{q}-\overline{p}}{\Delta t} = \dot{\overline{p}}$$

where $\dot{\overline{p}}$ is the velocity of the point p as the vector $\overline{p}$ rotates about $\overline{n}$ with angular speed ω. Equation (6) becomes

$$\lim_{\Delta t \rightarrow 0} \frac{\overline{q}-\overline{p}}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{(1-\cos\varphi)}{\Delta t}\begin{bmatrix} n_1n_1-1 & n_1n_2 & n_1n_3 \\ n_2n_1 & n_2n_2-1 & n_2n_3 \\ n_3n_1 & n_3n_2 & n_3n_3-1 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} + \lim_{\Delta t \rightarrow 0} \frac{\sin\varphi}{\Delta t}\begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

or

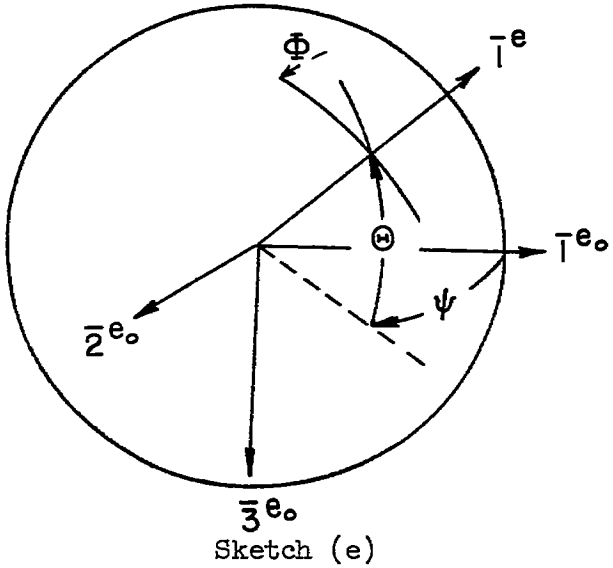
$$\dot{\bar{p}} = \dot{\varphi} \begin{bmatrix} 0 & -n_3 & n_2 \\ n_3 & 0 & -n_1 \\ -n_2 & n_1 & 0 \end{bmatrix} \bar{p} = \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & 1 & 0 \end{bmatrix} \bar{p} = \bar{\omega} \times \bar{p} \qquad (7)$$

where ω_1, ω_2, and ω_3 are the components of the angular velocity vector $\bar{\omega}$ and $\bar{\omega} \times \bar{p}$ is the vector or cross product of the vectors $\bar{\omega}$ and $\bar{p}$.

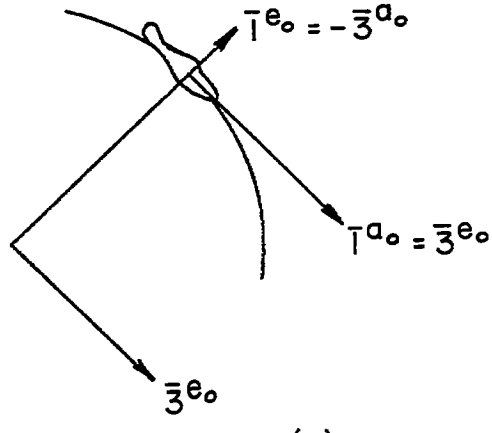
Location of an Airplane on the Surface of the Earth

One problem of some current interest concerns the study of the motion of an airplane over a significantly large sector of the surface of the earth. With a simplifying assumption, the analytical expressions of the geometry required in this problem provide a good example of the use of matrices in orthogonal transformations. The reason for the need of geometry here is that the airplane equations of motion require specification of the gravity vector direction which changes as the airplane moves over the earth. For the example, the gravity vector is assumed to be directed toward the origin of a set of coordinates which is located at the center of the earth (called earth coordinates).

The expression of the direction of the gravity vector with respect to the airplane requires knowledge of the position and orientation (that is, heading) of the airplane with respect to the earth coordinates. This knowledge is determined from the time history of airplane velocity, and the specifications of the airplane's initial position and velocity. The geometry is set up by establishing a set of earth coordinates which rotate in such a way that the moving airplane is located on one of the coordinate directions. In the sketches below, which are intended to



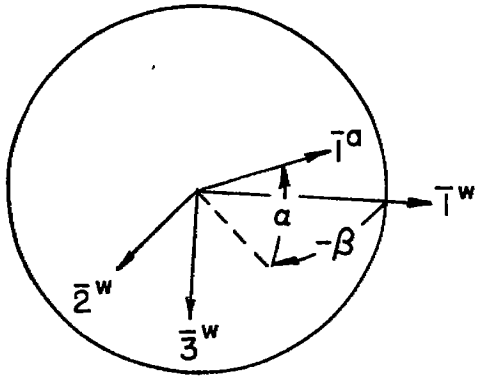
Sketch (e)



Sketch (f)

define the parameters used in the geometry, earth coordinates are labeled e, and airplane body coordinates are labeled a.

Sketch (e) shows how "present earth coordinates" are derived from an original set of earth coordinates: the sequence of rotations, each of which is made in a positive sense, is first through the angle Ψ, then through the angle Θ, finally through the angle Φ. Each rotation has the matrix associated with it which is given by the expression with the corresponding lower case Greek letter in expressions (3). Three similar rotations through angles ψ, θ, and φ serve to determine the present airplane coordinate orientation with respect to the airplane's original coordinate system. Sketch (f) shows the relation between the original earth and original airplane coordinate systems. The original airplane coordinates are derived from the original earth coordinates by a θ-like rotation about the $\bar{2}^{e_o}$ axis through 90° in a negative sense. Two other angles must be specified, which relate the airplane's velocity vector to airplane coordinates. These angles are shown in sketch (g).



Sketch (g)

The sketch shows that the airplane coordinate system is obtained from the airplane velocity coordinate system (airplane wind coordinates) first by a positive rotation through the angle $-\beta$ (β is the angle of sideslip), then by a positive rotation through the angle of attack α. These angular designations are conventional.

With the geometry so set up, the gravity vector is $\bar{g} = -g\bar{1}^e$; the airplane velocity vector is $\bar{V} = v\bar{1}^w$; and the airplane position vector $\bar{R} = R\bar{1}^e$. There remains the need to express $\bar{g}$ in aircraft coordinates and, having expressed $\bar{V}$ in earth coordinates, to determine the angular rates $\dot{\Psi}$, $\dot{\Theta}$, and $\dot{\Phi}$.

The equations connecting the various coordinate systems can be expressed as follows:

$$[\text{present airplane coordinates}] = (\varphi)(\theta)(\psi)[\text{initial airplane coordinates}]$$

$$= (\varphi)(\theta)(\psi)(-90^\circ)[\text{initial earth coordinates}]$$

$$[\text{present airplane coordinates}] = (\alpha)(-\beta)[\text{present wind coordinates}]$$

$$[\text{present earth coordinates}] = (\Phi)(\Theta)(\Psi)[\text{initial earth coordinates}]$$

From these equations one has

$$\left.\begin{aligned}
&[\text{present airplane coordinates}] \\
&\qquad = (\varphi)(\theta)(\psi)(-90^\circ)[(\Phi)(\Theta)(\Psi)]^{-1}[\text{present earth coordinates}] \\
&\text{or} \\
&\qquad\qquad \bar{g}_a = (\varphi)(\theta)(\psi)(-90^\circ)[(\Phi)(\Theta)(\Psi)]^{-1}\,\bar{g}_e
\end{aligned}\right\}\quad(8)$$

and

$$\left.\begin{aligned}
&[\text{present earth coordinates}] \\
&= \left\{(\varphi)(\theta)(\psi)(-90^\circ)[(\Phi)(\Theta)(\Psi)]^{-1}\right\}^{-1}(\alpha)(-\beta)[\text{present wind coordinates}] \\
&\text{or} \\
&\qquad \bar{v}_e = \left\{(\varphi)(\theta)(\psi)(-90^\circ)[(\Phi)(\Theta)(\Psi)]^{-1}\right\}^{-1}(\alpha)(-\beta)\bar{v}_w
\end{aligned}\right\}\quad(9)$$

where the subscripts on the vectors imply that the components of the vectors are written in the coordinate system indicated by the subscript. The superscript -1 on the product matrix of equation (8) in brackets means that the inverse of the product $[(\Phi)(\Theta)(\Psi)]$ is desired. It will be recalled that $[(\Phi)(\Theta)(\Psi)]^{-1} = (\Psi)^{-1}(\Theta)^{-1}(\Phi)^{-1} = (-\Psi)(-\Theta)(-\Phi)$. Since the whole product matrix of equation (8) is used in inverse form in equation (9), it will be convenient to obtain it or its inverse as a unit first and then form the product with $(\alpha)(-\beta)$.

Equations (8) and (9) can be simplified by setting the angle $\Phi = 0$, or the matrix $(\Phi) = (I)$. Only two parameters are necessary to specify the orientation of a vector in each of the two equations. Again because only two parameters are needed, (α) and $(-\beta)$ are sufficient to express the airplane velocity vector in airplane coordinates.

Equation (9) is:

$$\begin{bmatrix} v_{1e} \\ v_{2e} \\ v_{3e} \end{bmatrix} = \begin{bmatrix} c\theta & 0 & -s\theta \\ 0 & 1 & 0 \\ s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} c\Psi & s\Psi & 0 \\ -s\Psi & c\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} c\psi & -s\psi & 0 \\ s\psi & c\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\varphi & -s\varphi \\ 0 & s\varphi & c\varphi \end{bmatrix} \begin{bmatrix} c\alpha & 0 & -s\alpha \\ 0 & 1 & 0 \\ s\alpha & 0 & c\alpha \end{bmatrix} \begin{bmatrix} c\beta & -s\beta & 0 \\ s\beta & c\beta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix}$$

where c stands for cosine and s stands for sine, and where

$$(-90^\circ)^{-1} = \begin{bmatrix} c(-90^\circ) & 0 & s(-90^\circ) \\ 0 & 1 & 0 \\ -s(-90^\circ) & 0 & c(-90^\circ) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} v_{1e} \\ v_{2e} \\ v_{3e} \end{bmatrix} = \begin{bmatrix} \begin{matrix} -s\Theta c\theta c\psi \\ +c\Theta s\Psi c\theta s\psi \\ +c\Theta c\Psi s\theta \end{matrix} & \begin{matrix} -s\Theta(-s\psi c\varphi+s\varphi s\theta c\psi) \\ +c\Theta s\Psi(c\varphi c\psi+s\varphi s\theta s\psi) \\ -c\Theta c\Psi s\varphi c\theta \end{matrix} & \begin{matrix} -s\Theta(s\varphi s\psi+c\varphi s\theta c\psi) \\ +c\Theta s\Psi(-s\varphi c\psi+c\varphi s\theta s\psi) \\ -c\Theta c\Psi c\varphi c\theta \end{matrix} \\ \begin{matrix} c\Psi c\theta s\psi \\ -s\Psi s\theta \end{matrix} & \begin{matrix} c\Psi(c\varphi c\psi+s\varphi s\theta s\psi) \\ +s\Psi s\varphi c\theta \end{matrix} & \begin{matrix} c\Psi(-s\varphi c\psi+c\varphi s\theta s\psi) \\ +s\Psi(c\varphi c\theta) \end{matrix} \\ \begin{matrix} c\Theta c\theta c\psi \\ +s\Theta s\Psi c\theta s\psi \\ +s\Theta c\Psi s\theta \end{matrix} & \begin{matrix} c\Theta(-s\psi c\varphi+s\varphi s\theta c\psi) \\ +s\Theta s\Psi(c\varphi c\psi+s\varphi s\theta s\psi) \\ -s\Theta c\Psi s\varphi c\theta \end{matrix} & \begin{matrix} c\Theta(s\varphi s\psi+c\varphi s\theta c\psi) \\ +s\Theta s\Psi(-s\varphi c\psi+c\varphi s\theta s\psi) \\ -s\Theta c\Psi c\varphi c\theta \end{matrix} \end{bmatrix} \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta & -s\alpha \\ s\beta & c\beta & 0 \\ s\alpha c\beta & -s\beta s\alpha & c\alpha \end{bmatrix} \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix} \tag{10}$$

Because of the presence of only one element in the column matrix on the right, only the first column of the product of the two 3×3 matrices is required. Performing the multiplication, one gets:

$$\frac{v_{1e}}{V} = c\alpha c\beta[-s\Theta c\theta c\psi+c\Theta s\Psi s\psi c\theta+c\Theta c\Psi s\theta]+s\beta[-s\Theta(-s\psi c\varphi+s\varphi s\theta c\psi)+c\Theta s\Psi(c\varphi c\psi+s\varphi s\theta s\psi)-$$
$$c\Theta c\Psi s\varphi c\theta]+s\alpha c\beta[-s\Theta(s\varphi s\psi+c\varphi s\theta c\psi)+c\Theta s\Psi(-s\varphi c\psi+c\varphi s\theta s\psi)-c\Theta c\Psi c\varphi c\theta]$$

$$\frac{v_{2e}}{V} = c\alpha c\beta[c\Psi c\theta s\psi-s\Psi s\theta]+s\beta[c\Psi(c\varphi c\psi+s\varphi s\theta s\psi)+s\Psi s\varphi c\theta]+s\alpha c\beta[c\Psi(-s\varphi c\psi+c\varphi s\theta s\psi)+$$
$$s\Psi c\varphi c\theta]$$

$$\frac{v_{3e}}{V} = c\alpha c\beta[c\Theta c\theta c\psi+s\Theta s\Psi c\theta s\psi+s\Theta c\Psi s\theta]+s\beta[c\Theta(-s\psi c\varphi+s\varphi s\theta c\psi)+s\Theta s\Psi(c\varphi c\psi+s\varphi s\theta s\psi)-$$
$$s\Theta c\Psi s\varphi c\theta]+s\alpha c\beta[c\Theta(s\varphi s\psi+c\varphi s\theta c\psi)+s\Theta s\Psi(-s\varphi c\psi+c\varphi s\theta s\psi)-s\Theta c\Psi c\varphi c\theta]$$

To express the gravity vector in airplane coordinates, the inverse of the first (3×3) matrix on the right side of equation (10) is wanted. This inversion is performed by transposing the matrix. Since $\bar{g} = -g\bar{1}^e$, only the elements of the first row of the matrix in equation (10) are needed. The resulting equations are

$$g_{1a} = (-g)[-s\Theta c\theta c\psi+c\Theta s\Psi c\theta s\psi+c\Theta c\Psi s\theta]$$

$$g_{2a} = (-g)[-s\Theta(-s\psi c\varphi+s\varphi s\theta c\psi)+c\Theta s\Psi(c\varphi c\psi+s\varphi s\theta s\psi)-c\Theta c\Psi s\varphi c\theta]$$

$$g_{3a} = (-g)[-s\Theta(s\varphi s\psi+c\varphi s\theta c\psi)+c\Theta s\Psi(-s\varphi c\psi+c\varphi s\theta s\psi)-c\Theta c\Psi c\varphi c\theta]$$

The rotation parameters φ, θ, ψ, Θ, Ψ, which are continuous functions of time, remain to be determined. It is assumed that the rate of rotation of the airplane, $\bar{\Omega}^a = p\bar{1}^a+q\bar{2}^a+r\bar{3}^a$, is know continuously from airplane equations of motion. The letters p, q, r are the conventional terms for the components of $\bar{\Omega}^a$ in airplane coordinates. The rates $\dot{\varphi}$, $\dot{\theta}$, $\dot{\psi}$ will be expressed in terms of p, q, r. In the same way, $\dot{\Theta}$ and $\dot{\Psi}$ will be expressed in terms of the components of angular rate about the earth axes, Ω^e_{1e}, Ω^e_{2e}, and Ω^e_{3e}. In turn, the angular rate of rotation of Ω^e is obtained from the knowledge of the airplane velocities in earth coordinates which are subject to the two conditions that $\bar{R} = R\bar{1}^e$, that is, that the coordinates so rotate that the coordinate vector $\bar{1}^e$ remains pointing at the airplane; and that $\Phi = \dot{\Phi} = 0$.

One convenient method for obtaining the expressions for $\dot{\varphi}$, $\dot{\theta}$, and $\dot{\psi}$. is given below. Other methods can be found in references 2 and 4. The angular velocity vector can be expressed in the forms

$$\bar{\Omega}^a = p\bar{1}^a+q\bar{2}^a+r\bar{3}^a = \dot{\varphi}\bar{1}^a+\dot{\theta}\bar{2}^{a'}+\dot{\psi}\bar{3}^{a_o'} \tag{11}$$

where the coordinate systems indicated by primes are intermediate coordinates defined by

$$\begin{bmatrix}\bar{1}^a\\ \bar{2}^a\\ \bar{3}^a\end{bmatrix} = (\varphi)\begin{bmatrix}\bar{1}^{a'}\\ \bar{2}^{a'}\\ \bar{3}^{a'}\end{bmatrix} = (\varphi)(\theta)\begin{bmatrix}\bar{1}^{a_o'}\\ \bar{2}^{a_o'}\\ \bar{3}^{a_o'}\end{bmatrix} = (\varphi)(\theta)(\psi)\begin{bmatrix}\bar{1}^{a_o}\\ \bar{2}^{a_o}\\ \bar{3}^{a_o}\end{bmatrix} \tag{12}$$

Because of the form of the various rotation matrices,

$$\bar{1}^{a}=\bar{1}^{a'} \; ; \qquad \bar{2}^{a'}=\bar{2}^{a_o'} \; ; \qquad \bar{3}^{a_o'}=\bar{3}^{a_o} \tag{13}$$

From an examination of equations (11) and (12), one can realize that the matrices that will be needed are (φ), (θ), and $(\varphi)(\theta)$ which are proposed below in tabular form

$$(\varphi) \; : \quad \begin{array}{c|ccc} & \bar{1}^{a'} & \bar{2}^{a'} & \bar{3}^{a'} \\ \hline \bar{1}^{a} & 1 & 0 & 0 \\ \bar{2}^{a} & 0 & c\varphi & s\varphi \\ \bar{3}^{a} & 0 & -s\varphi & c\varphi \end{array} = \begin{array}{|ccc} \hline \bar{1}^{a}\cdot\bar{1}^{a'} & \bar{1}^{a}\cdot\bar{2}^{a'} & \bar{1}^{a}\cdot\bar{3}^{a'} \\ \bar{2}^{a}\cdot\bar{1}^{a'} & \bar{2}^{a}\cdot\bar{2}^{a'} & \bar{2}^{a}\cdot\bar{3}^{a'} \\ \bar{3}^{a}\cdot\bar{1}^{a'} & \bar{3}^{a}\cdot\bar{2}^{a'} & \bar{3}^{a}\cdot\bar{3}^{a'} \end{array}$$

$$(\theta) \; : \quad \begin{array}{c|ccc} & \bar{1}^{a_o'} & \bar{2}^{a_o'} & \bar{3}^{a_o'} \\ \hline \bar{1}^{a'} & c\theta & 0 & -s\theta \\ \bar{2}^{a'} & 0 & 1 & 0 \\ \bar{3}^{a'} & s\theta & 0 & c\theta \end{array} = \begin{array}{|ccc} \hline \bar{1}^{a'}\cdot\bar{1}^{a_o'} & \bar{1}^{a'}\cdot\bar{2}^{a_o'} & \bar{1}^{a'}\cdot\bar{3}^{a_o'} \\ \bar{2}^{a'}\cdot\bar{1}^{a_o'} & \bar{2}^{a'}\cdot\bar{2}^{a_o'} & \bar{2}^{a'}\cdot\bar{3}^{a_o'} \\ \bar{3}^{a'}\cdot\bar{1}^{a_o'} & \bar{3}^{a'}\cdot\bar{2}^{a_o'} & \bar{3}^{a'}\cdot\bar{3}^{a_o'} \end{array}$$

$$(\varphi)(\theta) \; : \quad \begin{array}{c|ccc} & \bar{1}^{a_o'} & \bar{2}^{a_o'} & \bar{3}^{a_o'} \\ \hline \bar{1}^{a} & c\theta & 0 & -s\theta \\ \bar{2}^{a} & s\varphi s\theta & c\varphi & s\varphi c\theta \\ \bar{3}^{a} & c\varphi s\theta & -s\varphi & c\varphi c\theta \end{array} = \begin{array}{|ccc} \hline \bar{1}^{a}\cdot\bar{1}^{a_o'} & \bar{1}^{a}\cdot\bar{2}^{a_o'} & \bar{1}^{a}\cdot\bar{3}^{a_o'} \\ \bar{2}^{a}\cdot\bar{1}^{a_o'} & \bar{2}^{a}\cdot\bar{2}^{a_o'} & \bar{2}^{a}\cdot\bar{3}^{a_o'} \\ \bar{3}^{a}\cdot\bar{1}^{a_o'} & \bar{3}^{a}\cdot\bar{2}^{a_o'} & \bar{3}^{a}\cdot\bar{3}^{a_o'} \end{array} \tag{14}$$

Now, returning to equation (11) one can solve for $\dot{\varphi}$, $\dot{\theta}$, and $\dot{\psi}$ by forming the appropriate dot products, then consulting the tables in (14):

$$\left.\begin{aligned} \dot{\varphi} &= p\bar{1}^{a}\cdot\bar{1}^{a}+q\bar{2}^{a}\cdot\bar{1}^{a}+r\bar{3}^{a}\cdot\bar{1}^{a}-\dot{\theta}\bar{2}^{a'}\cdot\bar{1}^{a}-\dot{\psi}\bar{3}^{a_o'}\cdot\bar{1}^{a} \\ &= p+\dot{\psi}s\theta \end{aligned}\right\} \tag{15a}$$

$$\left.\begin{aligned}\dot{\theta} &= p\bar{1}^a\cdot\bar{2}^{a'}+q\bar{2}^a\cdot\bar{2}^{a'}+r\bar{3}^a\cdot\bar{2}^{a'}-\dot{\varphi}\bar{1}^a\cdot\bar{2}^{a'}-\dot{\psi}\bar{3}^a\cdot\bar{2}^{a'}\\ &= qc\varphi-rs\varphi\end{aligned}\right\}\quad(15b)$$

$$\left.\begin{aligned}\dot{\psi} &= p\bar{1}^a\cdot\bar{3}^{ao'}+q\bar{2}^a\cdot\bar{3}^{ao'}+r\bar{3}^a\cdot\bar{3}^{ao'}-\dot{\varphi}\bar{1}^a\cdot\bar{3}^{ao}-\dot{\theta}\bar{2}^{a'}\cdot\bar{3}^{ao'}\\ &= -(p-\dot{\varphi})s\theta+qs\varphi c\theta+rc\varphi c\theta\\ &= \dot{\psi}s^2\theta+qs\varphi c\theta+rc\varphi c\theta\\ \dot{\psi}(1-s^2\theta) &= \dot{\psi}c^2\theta = (qs\varphi+rc\varphi)c\theta\\ \dot{\psi}c\theta &= qs\varphi+rc\varphi\end{aligned}\right\}\quad(15c)$$

The term $(p - \dot{\varphi})$ in equation (15c) was eliminated by using equation (15a).

The angular velocity of the earth coordinates is

$$\bar{\Omega}^e = \Omega^e_{1r}\,\bar{1}^r+\Omega^e_{2r}\,\bar{2}^r+\Omega^e_{3r}\,\bar{3}^r = \dot{\Theta}\bar{2}^{e'}+\dot{\Psi}\bar{3}^{eo'}$$

The primed coordinates here are the same sort of intermediate coordinates as those in equation (11). The procedure just completed for $\dot{\varphi}$, $\dot{\theta}$, $\dot{\psi}$ could be repeated here. But since the only changes are the names of the components and the fact that $\Phi = \dot{\Phi} = 0$, $\dot{\Theta}$ and $\dot{\Psi}$ can be written down just by comparison with equations (15):

$$\left.\begin{aligned}\dot{\varphi} &= p+\dot{\psi}s\theta & 0 &= \Omega^e_{1e}+\dot{\Psi}s\Theta\\ \dot{\theta} &= qc\varphi-rs\varphi & \dot{\Theta} &= \Omega^e_{2e}\\ \dot{\psi}c\theta &= qs\varphi+rc\varphi & \dot{\Psi}c\Theta &= \Omega^e_{3e}\end{aligned}\right\}\quad(16)$$

A final step remains in the geometry, the determination of $\bar{\Omega}^e$. This vector is obtained from the knowledge of the airplane's velocity:

$$\bar{v} = \dot{\bar{R}} = \frac{\partial\bar{R}}{\partial t}+\bar{\Omega}^e\times\bar{R}$$

The term $\dot{\bar{R}}$ expresses the total rate of change of position of the airplane, that is, its velocity. When this velocity is expressed in the rotating earth coordinates, however, it takes the explicit form shown in the expression on the right. The last term of this expression was derived in equation (7). It gives the full expression for the velocity in rotating coordinates if the vector $\bar{R}$ is constant in length. The other term in the expression above, $\frac{\partial \bar{R}}{\partial t}$, accounts for changes in length of $\bar{R}$. Since $\bar{R} = R\bar{1}^e$

$$\bar{v} = v_{1e}\bar{1}^e + v_{2e}\bar{2}^e + v_{3e}\bar{3}^e = \bar{1}^e \frac{\partial R}{\partial t} + R\bar{\Omega}^e \times \bar{1}^e$$

$$= \bar{1}^e \frac{\partial R}{\partial t} + R(\Omega_{3e}\bar{2}^e - \Omega_{2e}\bar{3}^e)$$

or

$$v_{1e} = \frac{\partial R}{\partial t}$$

$$v_{2e} = R\Omega_{3e} = R\dot{\Psi}c\Theta$$

$$v_{3e} = -R\Omega_{2e} = -R\dot{\Theta}$$

These expressions complete the geometry.

Although some of the transformations obtained along the way could have been obtained more easily without the use of the matrix method, this method is very valuable in providing a simple way of getting the complete expressions required in the problem. Furthermore, once a person gets the feel of the matrix method he appreciates its dual advantages of being a straightforward method of getting the correct expressions he wants and of providing a simple and clear view of the transformations required in the problem. The summary below, which actually was the analysis of the problem used before any computations were made, shows in simple fashion the various relationships involved in specifying the geometry:

[present airplane coordinates]

$$= (\varphi)(\theta)(\psi)[\text{initial airplane coordinates}]$$

$$= (\varphi)(\theta)(\psi)(-90^\circ)[\text{initial earth coordinates}]$$

$$= (\alpha)(-\beta)[\text{present wind coordinates}]$$

[present earth coordinates] $= (\Theta)(\Psi)$[initial earth coordinates]

To express $\overline{V} = v\overline{1}^w$ in earth coordinates:

$$\overline{v}_e = (\Theta)(\Psi)[(\varphi)(\theta)(\psi)(-90^\circ)]^{-1}(\alpha)(-\beta)\overline{v}_w$$

The subscripts on the vectors indicate the coordinate system in which their components are to be expressed. To express $\overline{g} = (-g)\overline{1}^e$ in airplane coordinates:

$$\overline{g}_a = \left\{(\Theta)(\Psi)[(\varphi)(\theta)(\psi)(-90^\circ)]^{-1}\right\}^{-1}\overline{g}_e$$

To determine the rotation of the earth coordinates:

$$\overline{v} = \overline{1}^e \frac{\partial R}{\partial t} + R\overline{\Omega}^e \times \overline{1}^e$$

Position of a Bomber Relative to an Attacking Airplane

The determination of the position of a bomber relative to an attacking airplane is a geometry problem that has to be set up in order to study weapon fire-control systems by means of an analog computer. The final result sought in the geometry is the orientation of the line of sight and the range of the bomber from the attacker. As in the first example, with the proper precautions, a coordinate system can be assigned so as to make the line of sight correspond in direction to a unit vector of the coordinate system. With this assignment made, one seeks the rate of rotation of the line-of-sight coordinates required to make $\overline{R} = R\overline{1}^l$. Since the line-of-sight orientation relative to the airplane is desired, its rate of rotation will be expressed as the airplane rate plus a rotation rate relative to the airplane. Furthermore, since the study of automatic fire-control systems involves the use of a tracking radar, it will be convenient to express the relative position of the line of sight by means of the same kind of angles by which the relative orientation of the tracking

line or heading of the radar antenna is expressed. The precaution mentioned above concerning the assignment of a coordinate system to the line of sight can be specified in a way to bring about the correspondence between the tracking line and line-of-sight systems. The condition will be: since the antenna does not bank relative to the airplane, neither will the line-of-sight coordinates.

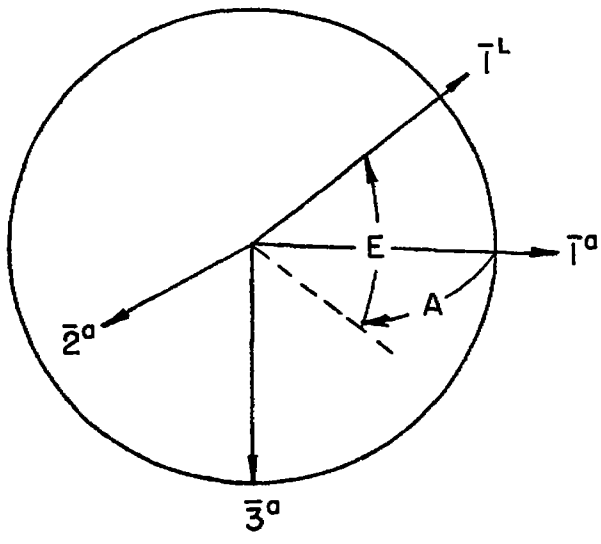
The velocity of the bomber, $\bar{v}^b = v^b\bar{1}^b$, is assumed known. The airplane velocity given in wind axes is $\bar{v}^a = v^a\bar{1}^w$; angle of attack, α, sideslip, β, and the airplane angular velocity, $\bar{\Omega}^a = p\bar{1}^a+q\bar{2}^a+r\bar{3}^a$, are assumed to be obtainable from the solution of the airplane equations of motion. One can obtain the angular velocity of the line of sight, $\bar{\Omega}^l$, by finding its components from the equation

$$\bar{v}_l = \bar{v}_l^b - \bar{v}_l^a = \bar{1}^l \frac{\partial R}{\partial t} + R\bar{\Omega}_l^l \times \bar{1}^l \tag{17}$$

As before, the subscripts indicate the coordinate system in which the vectors are to be expressed. Equation (17) implies that the bomber and airplane velocities are to be expressed in line-of-sight coordinates.

The transformation from wind coordinates (in which $\bar{v}^a = v^a\bar{1}^w$) to line-of-sight coordinates can be achieved by

$$\begin{aligned}[\text{line-of-sight coordinates}] &= (E)(A)[\text{airplane coordinates}] \\ &= (E)(A)(\alpha)(-\beta)[\text{wind coordinates}]\end{aligned} \tag{18}$$



Sketch (h)

where (A) performs a rotation about $\bar{3}^a$ and (E) performs a rotation about $\bar{2}^l$ as shown in sketch (h). The nomenclature and order of rotation correspond to usage in some present fire-control systems. The magnitude of these angles E and A specifying the relative orientation of the line of sight is the final desideratum of the geometry.

If the bomber velocity coordinates are first transformed into wind coordinates, the rotation matrix in equation (18) will complete the

transformation to line-of-sight coordinates. It is convenient to transform the bomber velocity coordinates first into the initial attacker wind coordinates, then transform them into present wind coordinates. The transformations can follow the scheme:

$$[\text{bomber coordinates}] = (\Theta^b)(\Psi^b)[\text{initial wind coordinates}]$$

$$[\text{present wind coordinates}] = (\Phi^w)(\Theta^w)(\Psi^w)[\text{initial wind coordinates}]$$

which gives the transformation

$$[\text{present wind coordinates}]$$

$$= (\Phi^w)(\Theta^w)(\Psi^w)(-\Psi^b)(-\Theta^b)[\text{bomber coordinates}] \tag{19}$$

The property of rotation matrices that $(\psi^b)^{-1} = (-\psi^b)$ has been used in equation (19). Now, since (Ψ^w) and $(-\Psi^b)$ are rotations about the same axis, $(\Psi^w)(-\Psi^b)$ can be written (Ψ) where the angle $\Psi = \Psi^w - \Psi^b$.

The complete problem now has the form:

$$\begin{bmatrix} v_{1l} \\ v_{2l} \\ v_{3l} \end{bmatrix} = \begin{bmatrix} \partial R/\partial t \\ R\Omega^l_{2l} \\ -R\Omega^l_{3l} \end{bmatrix} = (E)(A)(\alpha)(-\beta)\left\{(\Phi)(\Theta)(\Psi)(-\Theta^b)\begin{bmatrix} v^b \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} v^a \\ 0 \\ 0 \end{bmatrix}\right\} \tag{20}$$

The superscripts in Φ^w and Θ^w can be dropped without ambiguity. The quantities $\dot{\Phi}$, $\dot{\Theta}$, $\dot{\Psi}^w$ are to be obtained from $\bar{\Omega}^a$. The angular rates, $\dot{\Theta}^b$, $\dot{\Psi}^b$, specifying the bomber motions are assumed known; $\dot{E}$ and $\dot{A}$ are to be obtained from the relative angular velocity $\bar{\Omega} = \bar{\Omega}^l - \bar{\Omega}^a$. The matrix $(\alpha)(-\beta)$ is given in equation (10). The matrix $(E)(A)$ is obtained from $(\alpha)(-\beta)$ by substituting E for α and A for $-\beta$. Then

$$(E)(A)(\alpha)(-\beta) = \begin{bmatrix} cEcA & cEsA & -sE \\ -sA & cA & 0 \\ sEcA & sEsA & cE \end{bmatrix} \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta & -s\alpha \\ s\beta & c\beta & 0 \\ s\alpha c\beta & -s\beta s\alpha & c\alpha \end{bmatrix}$$

$$= \begin{bmatrix} cEcAc\alpha c\beta+cEsAs\beta-sEs\alpha c\beta & -cEcAc\alpha s\beta+cEsAc\beta+sEs\beta s\alpha & -cEcAs\alpha-sEc\alpha \\ -sAc\alpha c\beta+cAs\beta & sAc\alpha s\beta+cAc\beta & sAs\alpha \\ sEcAc\alpha c\beta+sEsAs\beta+cEs\alpha c\beta & -sEcAc\alpha s\beta+sEsAc\beta-cEs\beta s\alpha & -sEcAs\alpha+cEc\alpha \end{bmatrix}$$

The matrix $(\Theta)(\Psi)$ can be obtained from the matrix $(E)(A)$ by the appropriate literal substitutions. Then

$$(\Phi)(\Theta)(\Psi)(-\Theta^b) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\Phi & s\Phi \\ 0 & -s\Phi & c\Phi \end{bmatrix} \begin{bmatrix} c\Theta c\Psi & c\Theta s\Psi & -s\Theta \\ -s\Psi & c\Psi & 0 \\ s\Theta c\Psi & s\Theta s\Psi & c\Theta \end{bmatrix} \begin{bmatrix} c\Theta^b & 0 & s\Theta^b \\ 0 & 1 & 0 \\ -s\Theta^b & 0 & c\Theta^b \end{bmatrix}$$

$$= \begin{bmatrix} c\Theta c\Psi c\Theta^b+s\Theta s\Theta^b & c\Theta s\Psi & c\Theta c\Psi s\Theta^b-s\Theta c\Theta^b \\ -c\Phi s\Psi c\Theta^b+s\Phi(s\Theta c\Psi c\Theta^b-c\Theta s\Theta^b) & c\Phi c\Psi+s\Phi s\Theta s\Psi & -c\Phi s\Psi s\Theta^b+s\Phi(s\Theta c\Psi s\Theta^b+c\Theta c\Theta^b) \\ s\Phi s\Psi c\Theta^b+c\Phi(s\Theta c\Psi c\Theta^b-c\Theta s\Theta^b) & -s\Phi c\Psi+c\Phi s\Theta s\Psi & s\Phi s\Psi s\Theta^b+c\Phi(s\Theta c\Psi s\Theta^b+c\Theta c\Theta^b) \end{bmatrix}$$

The quantities $\dot{\Phi}$, $\dot{\Theta}$, $\dot{\Psi}^w$, $\dot{E}$, and $\dot{A}$ remain to be determined; Θ^b and $\dot{\Psi}^b$ are assumed given. The three angular rates $\dot{\Phi}$, $\dot{\Theta}$, and $\dot{\Psi}^w$, by means of which the airplane wind coordinates are specified, are entirely analogous to $\dot{\phi}$, $\dot{\theta}$ and $\dot{\psi}$ in equations (11) and (15). Instead of p, q, and r, however, the wind coordinate rates $\Omega^w{}_{1w}$, $\Omega^w{}_{2w}$, and $\Omega^w{}_{3w}$ must be used. These components can be found as follows. Let $\bar{\Omega}^{a-w}$ be the angular velocity of the airplane with respect to the wind axes. Then

$$\bar{\Omega}^{a-w} = \dot{\alpha}\bar{2}^a-\dot{\beta}\bar{3}^w = \Omega^{a-w}_{1a}\bar{1}^a+\Omega^{a-w}_{2a}\bar{2}^a+\Omega^{a-w}_{3a}\bar{3}^a$$

This equation is again equivalent to equation (11) for the condition $\dot{\phi} = \varphi = 0$. The individual expressions can be obtained from those on the right in (16) after the appropriate literal substitutions

$$\Omega_{1a}^{a-w} = \dot{\beta}s\alpha$$

$$\Omega_{2a}^{a-w} = \dot{\alpha}$$

$$\Omega_{3a}^{a-w} = \dot{\beta}c\alpha$$

The wind axis angular velocity, $\overline{\Omega}^{w} = \overline{\Omega}^{a} - \overline{\Omega}^{a-w}$ has, therefore, the components in airplane coordinates $\Omega_{1a}^{w} = p - \dot{\beta} \sin \alpha$; $\Omega_{2a}^{w} = q - \dot{\alpha}$; $\Omega_{3a}^{w} = r + \beta \cos \alpha$. These components are transformed into components in the wind axis system through the transformation inverse to $(\alpha)(-\beta)$. With that transformation done, $\dot{\Phi}$, $\dot{\Theta}$, and $\dot{\Psi}^{w}$ are expressed in terms of p, q, r, $\dot{\alpha}$, and $\dot{\beta}$.

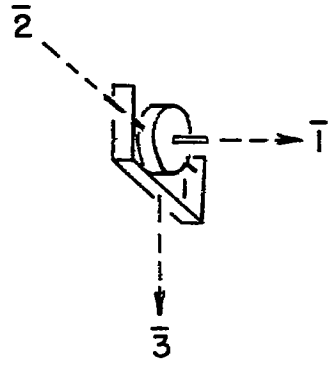
The quantities $\dot{A}$ and $\dot{E}$ are found in similar fashion, being analogous to $-\dot{\beta}$ and $\dot{\alpha}$, respectively. The relative velocity of the line of sight, $\overline{\Omega}$, equals $\overline{\Omega}^{l} - \overline{\Omega}^{a}$. The two components Ω_{2l}^{l} and Ω_{3l}^{l} are found from equation (20); $\overline{\Omega}^{a}$ is expressed in line-of-sight coordinates by means of the transformation: $\overline{\Omega}_{l}^{a} = (E)(A)\overline{\Omega}_{a}^{a}$. Then $\dot{A}$ and $\dot{E}$ are given by

$$\Omega_{2l} = \dot{E} = \Omega_{2l}^{l} + p \sin A - q \cos A$$

$$\Omega_{3l} = \dot{A} \cos E = \Omega_{3l}^{l} - \sin E(p \cos A + q \sin A) - r \cos E$$

Rotation Matrix Method and Gimbaled Mechanisms

The matrix method gives an extremely simple method of analyzing gimbaled mechanisms because the angles in the matrices can be chosen to correspond to the gimbal angles of the mechanism. Consider the antenna depicted in sketch (i). The 3 axis is free to rotate in bearings mounted on a base and carries a frame which can be called an outer gimbal. Call $-\Gamma_0$ the angle made by the rotation of the axis relative to the base. The 2 axis is free to rotate in bearings mounted on the outer gimbal, and carries the radar antenna with it. Call $-\Gamma_1$ the angle made by the 2 axis in turning with respect to the outer gimbal. These physical angles, each measuring a rotation



Sketch (i)

about an axis of the mechanism, are the same as the analytical angles, each of which describes a rotation about an analytic axis. The analysis' angle A is measured by $-\Gamma_o$, and E by $-\Gamma_i$.

An orientation gyro is another gimbaled mechanism. Sketch (i) represents a gyro if the 2 axis, instead of carrying an antenna, carries another gimbal that supports the gyro rotor, and if this rotor has its spin axis along axis 1. An orientation gyro has the property that its spin axis tends to maintain its orientation in space no matter what the motion of the base. Then the orientation of coordinates in the base is related to axes fixed in the inner gimbal by (ref. 5):

$$[\text{base coordinates}] = (\Gamma_o)(\Gamma_i)[\text{inner gimbal coordinates}]$$

Customarily in the analysis of airplane motion, the present airplane orientation is described relative to an initial orientation by the angles ψ(yaw), θ, and φ. Hence

$$\begin{aligned}[\text{present airplane coordinates}] &= (\varphi)(\theta)(\psi)[\text{initial coordinates}] \\ &= (\varphi)(\theta)[\text{yawed coordinates}]\end{aligned}$$

By fixing a gyro to an airplane appropriately, Γ_o and Γ_i can measure φ and θ directly. The correct arrangement for the measurement of φ and θ requires the 3 axis aligned with the negative airplane 1 axis, 2 with the airplane 2 axis, and 1 with the airplane 3 axis under the null conditions $\varphi = \theta = 0$ and (sketch(i)) $\Gamma_o = \Gamma_i = 0$. Airplane yaw is not measured. This gyro arrangement corresponds to the standard vertical gyro arrangement.

Transformation of Moment of Inertia

Because they offer a good example of the use of rotation matrices, the equations of transformation of moment and products of inertia and of the force and moment derivatives of airplane dynamics will now be derived. The special cases involving a transformation from one body axis to another through a constant θ-like angle are well known (e.g., ref. 6). But since transformation equations, if their derivation is not understood, can be misused, it is desirable to derive the transformation in general, and illustrate the general method by a particular and common example.

The fundamental dynamical law containing the moment of inertia is

$$\overline{p} = (J)\overline{m}$$

where $\overline{p}$ is the angular momentum vector, $\overline{\omega}$ is the angular velocity vector, and (J) is used here to represent the moment of inertia matrix

$$(J) = \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix}$$

$$J_{xx} = \int (y^2+z^2)\rho \, d\tau$$
$$J_{xy} = -\int xy\rho \, d\tau, \text{ etc}$$

where ρ is the mass density of the body at location x, y, z, and $d\tau$ is an element of volume.

The clue to determining the law of transformation of (J) is the recognition that equation (21) is a general dynamical expression having the same form no matter in which coordinates it may happen to be expressed in any particular example. Independently of the coordinates in which they are represented, angular momentum and angular velocity are connected by equation (21):

$$\overline{p}_o = (J_o)\overline{\omega}_o \quad ; \quad \overline{p}_n = (J)\overline{\omega}_n \tag{21}$$

Now we have seen in transforming airplane and bomber velocity that the components of a vector transform in the same way as do coordinate systems. Thus, if Γ is a constant angle of rotation about some axis so that

$$[\text{new coordinates}] = (\Gamma)[\text{old coordinates}]$$

then

$$\overline{p}_n = (\Gamma)\overline{p}_o \quad ; \quad \overline{\omega}_n = (\Gamma)\overline{\omega}_o \quad \text{or} \quad \overline{\omega}_o = (\Gamma)^{-1}\overline{\omega}_n$$

then

$$\overline{p}_n = (\Gamma)\overline{p}_o = (\Gamma)(J_o)(\Gamma)^{-1}\overline{\omega}_n = (J_n)\overline{\omega}_n$$

The transformation sought has the form

$$(J_n) = (\Gamma)(J_o)(\Gamma)^{-1}$$

If (Γ) is a transformation through a θ-like angle, then

$$(J_n) = \begin{bmatrix} c\Gamma & 0 & -s\Gamma \\ 0 & 1 & 0 \\ s\Gamma & 0 & c\Gamma \end{bmatrix} \begin{bmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{xy} & J_{yy} & J_{yz} \\ J_{xz} & J_{yz} & J_{zz} \end{bmatrix} \begin{bmatrix} c\Gamma & 0 & s\Gamma \\ 0 & 1 & 0 \\ -s\Gamma & 0 & c\Gamma \end{bmatrix}$$

$$= \begin{bmatrix} J_{xx}c^2\Gamma+J_{zz}s^2\Gamma-2J_{xz}s\Gamma c\Gamma & J_{xy}c\Gamma-J_{yz}s\Gamma & (J_{xx}-J_{zz})s\Gamma c\Gamma+J_{xz}(c^2\Gamma-s^2\Gamma) \\ J_{xy}c\Gamma-J_{yz}s\Gamma & J_{yy} & J_{xy}s\Gamma+J_{yz}c\Gamma \\ (J_{xx}-J_{zz})s\Gamma c\Gamma+J_{xz}(c^2\Gamma-s^2\Gamma) & J_{xy}s\Gamma+J_{yz}c\Gamma & J_{xx}s^2\Gamma+J_{zz}c^2\Gamma+2J_{xz}s\Gamma c\Gamma \end{bmatrix} \tag{22}$$

Transformation of Force and Moment Derivatives of Airplane Equations of Motion

To determine the law of transformation of the force and moment derivatives occurring in the equations of airplane motion, it is necessary, as with the moment of inertia, to find the derivatives in an equation whose form remains constant under the transformation. If the dependent variables are the six components of linear and angular velocity, the equations can be written in vector form as follows:

$$\left.\begin{aligned} \frac{\partial}{\partial t}[\bar{h}]+(\Omega)[\bar{h}] &= (M_1)[\bar{v}]+(M_2)[\bar{\Omega}]+(M_3)[\bar{\delta}] \\ \frac{\partial}{\partial t}[\bar{p}]+(\Omega)[\bar{p}] &= (M_4)[\bar{v}]+(M_5)[\bar{\Omega}]+(M_6)[\bar{\delta}] \end{aligned}\right\} \tag{23}$$

where $\bar{h}$ is the linear momentum, $\bar{p}$ is the angular momentum, $\bar{v}$ is the linear velocity, $\bar{\Omega}$ the angular velocity of the airplane, and

$$(\Omega) = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}; \quad (M_1) = \begin{bmatrix} X_u & 0 & X_w \\ 0 & Y_v & 0 \\ Z_u & 0 & Z_w \end{bmatrix}; \quad (M_2) = \begin{bmatrix} 0 & X_q & 0 \\ Y_p & 0 & Y_r \\ 0 & Z_q & 0 \end{bmatrix}; \quad (M_3) = \begin{bmatrix} 0 & X_{\delta e} & 0 \\ 0 & 0 & Y_{\delta r} \\ 0 & Z_{\delta e} & 0 \end{bmatrix}$$

$$[\bar{\delta}] = \begin{bmatrix} \delta_a \\ \delta_e \\ \delta_r \end{bmatrix}; \quad [\bar{v}] = \begin{bmatrix} u \\ v \\ w \end{bmatrix}; \quad (M_4) = \begin{bmatrix} 0 & L_v & 0 \\ M_u & 0 & M_w \\ 0 & N_v & 0 \end{bmatrix}; \quad (M_5) = \begin{bmatrix} L_p & 0 & L_r \\ 0 & M_q & 0 \\ N_p & 0 & N_r \end{bmatrix}; \quad (M_6) = \begin{bmatrix} L_{\delta a} & 0 & L_{\delta r} \\ 0 & M_{\delta e} & 0 \\ N_{\delta a} & 0 & N_{\delta r} \end{bmatrix}$$

The transformation law of each of the (M_i) in equation (23) is identical to that of the moment of inertia, and is derived in the same way. Glauert's formulas (ref. 6) are obtained using the θ-like rotation matrix (Γ) as in equations (22). For example, replacing (J) in equations (22) by (M_1) one obtains for the new X'_u the expression:

$$X'_u = X_u \cos^2\Gamma - (X_w + Z_w) \sin\Gamma \cos\Gamma + Z_w \sin^2\Gamma$$

CONCLUDING REMARKS

The present paper has attempted to show how convenient it is to use rotation matrices when one is setting up the geometric aspects of dynamical problems. A rudimentary knowledge of certain aspects of matrix algebra provides a tool entirely adequate for the solution of these geometric problems. The compact notation of matrix algebra permits the clear view required for their straightforward solution. The detailed computation of the expressions required in a particular problem becomes a matter of routine, and can easily be checked for errors. No algebraic dexterity is required to determine the parameters involved because they are obtained by the same direct methods used in the rest of the problem. Many of the parts used in one problem can be saved and used in other problems. Finally, because planning and computing become distinct tasks, it is a simple matter for one to devise and investigate various paths to the solution of the geometric problems without performing any computations.

Ames Aeronautical Laboratory
National Advisory Committee for Aeronautics
Moffett Field, Calif., Mar. 4, 1957

APPENDIX

A DESCRIPTION OF NOTATION

Problems whose answers are currently being sought on simulators have caused a need for a uniform and consistent notation. The need flows from two sources: frequently more than one quantity of a sort is used in a single problem; and a single quantity is often required in more than one coordinate system. The person who is studying the problem is certainly familiar with the meanings of the symbols he employs. But if, as often happens, he employs a new letter for each new quantity, and still new letters for some components of these quantities, his audience must have constant recourse to a glossary of symbols. Some people, to avoid this pitfall, use a shorthand of subscripts by which they distinguish between quantities of the same sort. Unfortunately, this shorthand, which is not notation, quickly becomes too cumbersome.

The notation proposed for use in the simulation of dynamical problems has three fundamental parts. The base letter of a quantity refers to quality; it indicates the sort of thing described. Superscripts are used to distinguish two quantities of the same quality. Subscripts are used to specify the coordinate system in which a quantity is represented, and to name the components of a quantity in a particular coordinate system. For example, the symbol $\overline{v}$ designates a linear velocity; $\overline{v}^b$ is a particular linear velocity, say, bomber velocity; $\overline{v}_w^b$ signifies the representation of bomber velocity in some particular reference system, for example, in attacker wind coordinates.

When modification of a base letter by superscripts or subscripts is not required for clarity, it should not be used. Use of $\overline{\Omega}$ for the angular velocity of the line of sight relative to the attacker exemplifies the simplification possible. In this case, of all the angular velocities considered one need not have a superscript. It is convenient that the one angular velocity thus singled out be either the one most used, or the one that otherwise would have the most complicated superscript. There are many places where modification of a base letter is superfluous.

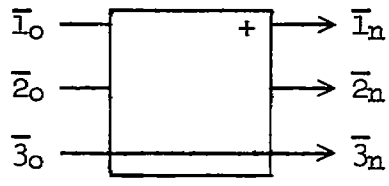
Certain letters are favored over others for certain sorts of quantities because of custom, or because of their mnemonic value. It seems to be desirable to reserve Greek letters for angles and angular velocities. The angles Φ, Θ, Ψ have been used to describe the orientation of a body with respect to a fixed coordinate system. Other letters describe orientation with respect to moving coordinates. Each of the letters Φ, Θ, Ψ refers to a particular sort of angle, that is, one generated by a rotation about a $\overline{1}$, $\overline{2}$, or $\overline{3}$ direction. It seems preferable to capitalize these letters simply for typographical reasons if they are to bear superscripts. The use of the small letters φ, θ, and ψ to indicate the orientation of an airplane body coordinate system, and p, q, and r to

represent airplane angular velocity components in body coordinates is honored by custom. It seems undesirable to change customary symbols when their use is not confusing and when they adequately fit the description needed.

Another type of notation, a schematic representation of rotation operations, provides a convenient formulation of the geometry involved in a problem. It is derived from the fact that resolvers are the physical means of performing rotations in analog equipment.

Induction resolvers (ref. 7) have the property of converting input quantities A_o and B_o to output quantities A_n and B_n where $A_n = A_o\cos\Gamma + B_o\sin\Gamma$ and $B_n = -A_o\sin\Gamma + B_o\cos\Gamma$. That is, if $A = \bar{1}$, $B = \bar{2}$, and $\Gamma = \Psi$, the matrix (Ψ) and the resolver below perform

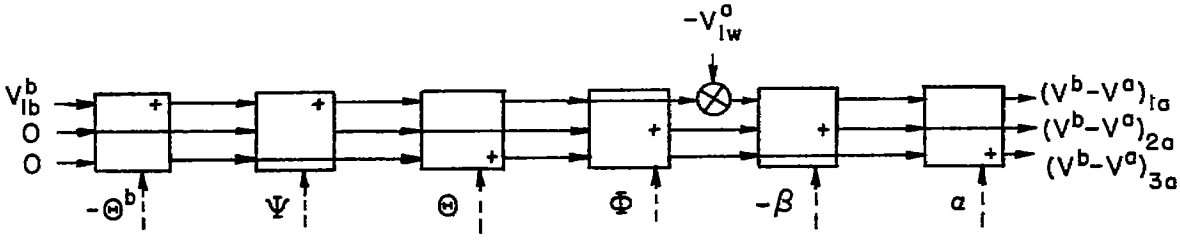
$$\begin{bmatrix} \bar{1}_n \\ \bar{2}_n \\ \bar{3}_n \end{bmatrix} = (\Psi) \begin{bmatrix} \bar{1}_o \\ \bar{2}_o \\ \bar{3}_o \end{bmatrix}$$



equivalent operations. The presence of the + sign and the short circuit in the resolver symbol should be noted. The + sign indicates which output quantity contains the $+\sin\Psi$ term. For an angle whose positive sense agrees with the positive sense of the coordinate system, the + sign follows the indicated short circuit in the cyclic order 1, 2, 3, 1, 2, ... Using this resolver symbol, one can write the matrix equation

$$(\bar{v}^b - \bar{v}^a)_a = (\alpha)(-\beta)[(\Phi)(\Theta)(\Psi)(-\Theta^b)\bar{v}_b^b - \bar{v}_w^a]$$

in the form



The latter form is more explicit than the former. It conveys enough information that a person can use it rather than the various intermediate equations to aid him in wiring a problem on a simulator.

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