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TECHNICAL NOTE 4154

APPROXIMATE CALCULATION OF THE COMPRESSIBLE TURBULENT BOUNDARY LAYER WITH HEAT TRANSFER AND ARBITRARY PRESSURE GRADIENT

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APPROXIMATE CALCULATION OF THE COMPRESSIBLE TURBULENT BOUNDARY LAYER WITH HEAT TRANSFER AND ARBITRARY PRESSURE GRADIENT

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SUMMARY

An approximate method for the calculation of compressible turbulent boundary layer with heat transfer and arbitrary pressure gradient is presented. The method involves the momentum integral and moment-of-momentum equations as simplified by using Stewartson's transformation. The Ludwieg-Tillmann skin-friction relation is used in these equations in a form suitable for compressible flow with heat transfer through application of the reference enthalpy concept. A tentative extension of Reynolds analogy is suggested for estimating heat-transfer effects.

The method, as applied to insulated surfaces, is quite well founded but, for noninsulated isothermal surfaces, depends on a number of speculative assumptions. These assumptions are qualitatively proper, and it is hoped that they will yield reasonable quantitative results. The detailed application of the method for practical calculations is described.

INTRODUCTION

In the absence of any rigorous method, various semiempirical procedures have been developed for incompressible turbulent-boundary-layer calculations. An excellent description of these procedures is given in reference 1. In brief, for flows with zero pressure gradient, the Kármán momentum integral equation is utilized together with an assumed boundary-layer-velocity profile, usually a power-law profile, and one of several empirical skin-friction relations. For flows with pressure gradient, an additional or auxiliary equation is required to account for the effect of pressure gradient on the boundary-layer-velocity profile. A skin-friction relation compatible with the streamwise pressure gradient should be employed.

Although some procedures use an empirical auxiliary equation, Tetervin and Lin (ref. 2) have suggested the moment-of-momentum equation as an auxiliary equation. The moment-of-momentum equation is obtained by multiplying the integrand of the momentum integral equation by a distance normal to the surface and then integrating with respect to that distance.

These calculation methods generally require the simultaneous solution of two differential equations, the Kármán momentum equation and the auxiliary equation. Maskell (ref. 3) has developed a simpler method in which the momentum equation is replaced by an empirically determined approximation which is directly integrable and thus determines the momentum thickness. The profile shape parameter is obtained from an empirical auxiliary differential equation. The Ludwieg-Tillmann skin-friction formula (ref. 4) is used to calculate the skin-friction distribution and to determine an approximate separation point for flows with adverse pressure gradient.

The present report describes an approximate method for the calculation of compressible turbulent boundary layer with heat transfer and pressure gradient. The momentum and moment-of-momentum integral equations for compressible flow are first simplified by using a form of the Stewartson transformation (e.g., see ref. 5). The possibility of using the Stewartson transformation for turbulent flow was pointed out by Van Le (ref. 6). In fact, Englert (ref. 7) and Mager (ref. 8) have respectively applied the Stewartson transformation to the Truckenbrodt (ref. 9) and Maskell (ref. 3) methods, which enables the calculation of compressible turbulent boundary layers over insulated surfaces. In the present report, the Ludwieg-Tillmann skin-friction relation is used in a form suitable for compressible flow with heat transfer through application of the reference enthalpy concept (ref. 10). An approximation obtained from reference 3 for the shear-stress distribution through the boundary layer and the power-law velocity profile are used to simplify further the moment-of-momentum equation. The location of turbulent separation is identified as that where the skin friction, when sensibly extrapolated, becomes zero. Maskell has shown this procedure to be in good agreement with experiment for flows over airfoils where the form factor increases rapidly near separation and thus causes a quick fall of skin friction in a small increment of longitudinal distance. Where separation does not occur so rapidly, the proposed method may yield large errors in separation location.

A speculative extension of Reynolds analogy is used to estimate heat-transfer effects for isothermal surfaces. This extension was suggested from the results obtained in reference 11 for laminar-boundary-layer flows with heat transfer and pressure gradient.

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BOUNDARY-LAYER INTEGRAL EQUATIONS

A brief description of the essential steps and assumptions involved in the proposed calculation method will now be presented. The bulk of the development is for a Prandtl number of 1.0. However, in the section entitled Heat Transfer, a modification is suggested to include the effect of a Prandtl number different from 1.0. All symbols are listed in appendix A.

The transformed momentum and moment-of-momentum integral boundary-layer equations, respectively, are:

$$\frac{d\theta_{tr}}{dX} + \frac{\theta_{tr}}{U_e}\frac{dU_e}{dX}\left[2 + H_i + \frac{\int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) dY}{\theta_{tr}}\right] + \frac{\theta_{tr}}{R}\frac{dR}{dX} = \frac{\tau_w}{\rho_0 \frac{p_e}{p_0}\left(\frac{a_e}{a_0}\right)^2 U_e^2} \tag{1}$$

$$\begin{aligned}\frac{dH_i}{dX} = &- \frac{1}{U_e}\frac{dU_e}{dX}\left[\frac{H_i(H_i+1)^2(H_i-1)}{2}\right]\left[1 + \frac{2}{(H_i+1)\theta_{tr}}\int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) dY\right. \\ &\left. - \frac{2(H_i - 1)}{H_i^2(H_i+1)\theta_{tr}^2}\int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) Y\, dY\right] \\ &+ \frac{\tau_w(H_i^2 - 1)}{\rho_0 \frac{p_e}{p_0}\left(\frac{a_e}{a_0}\right)^2 U_e^2\theta_{tr}}\left[H_i - (H_i + 1)\int_0^1 \frac{\tau}{\tau_w}\, d\left(\frac{Y}{\Delta}\right)\right]\end{aligned} \tag{2}$$

The development of these equations, which apply to flows with axial symmetry, is outlined in appendix B. For two-dimensional plane flows, the term $\frac{\theta_{tr}}{R}\frac{dR}{dX}$ in equation (1) vanishes. Equations (1) and (2) describe the variation of a "transformed" momentum thickness defined as

$$\theta_{tr} = \int_0^{\Delta} \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) dY \tag{3}$$

and an "incompressible" form factor

$$H_i = \frac{\int_0^{\Delta}\left(1 - \frac{U}{U_e}\right) dY}{\theta_{tr}} \tag{4}$$

with the transformed longitudinal variable X for a specified external velocity distribution $U_e(X)$. A distinction is made between the transformed and incompressible form factors since the transformed form factor for a compressible boundary layer with heat transfer is (obtained from appendix C, ref. 12)

$$H_{tr} = \frac{\int_0^{\Delta}\left(1 - \frac{U}{U_e}\right) dY + \int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) dY}{\theta_{tr}}$$

$$= H_i + \frac{\int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) dY}{\theta_{tr}} \tag{5}$$

Physically, H_{tr} is the form factor of a noninsulated boundary layer at Mach number zero and includes the variation of density with temperature. For boundary layers over insulated surfaces, the transformed and incompressible form factors are identical.

In order to solve equations (1) and (2) a skin-friction relation must first be selected. The Ludwieg-Tillmann relation (ref. 4) has been chosen because of its applicability to flows with pressure gradients. This relation, however, does not allow the skin-friction coefficient to equal zero and, therefore, is in error near separation. This deficiency can be overcome in a practical calculation as shown in the section entitled Skin Friction. As extended to compressible flow (see appendix C), the Ludwieg-Tillmann relation in terms of transformed quantities is

$$C_f \doteq \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} = 0.246\ e^{-1.561H_i}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{-0.268}\left(\frac{t_e}{t_{ref}}\right)^{0.732}\left(\frac{t_e}{t_0}\right)^{0.268} \tag{6}$$

This analysis assumes that the fluid Prandtl number is 1.0. The definition of reference enthalpy as obtained from reference 10 is

$$\frac{h_{ref}}{h_e} = \frac{1}{2}\left(\frac{h_w}{h_e} + 1\right) + 0.22\left(\frac{h_{aw}}{h_e} - 1\right) \tag{7}$$

where for $Pr = 1.0$, $h_{aw} = h_0$. The temperature t_{ref} is evaluated at the reference enthalpy. Before the solution of equations (1) and (2) is possible, some assumptions are still required concerning the shearing stress and temperature profiles within the boundary layer.

For zero pressure gradient, dH_i/dX is known to be negative. This requires from equation (2) that

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$$\int_0^1 \frac{\tau}{\tau_w} d\left(\frac{Y}{\Delta}\right) > \frac{H_i}{H_i + 1}$$

The present analysis uses the following simple relation, valid for $1.25 < H_i < 1.40$:

$$\int_0^1 \frac{\tau}{\tau_w} d\left(\frac{Y}{\Delta}\right) = 1.03 \frac{H_i}{H_i + 1} \tag{8}$$

This relation was obtained by matching with the equivalent numerical values of Maskell (ref. 3). Equation (2) appears usable even for flows with large pressure gradients inasmuch as the pressure gradient term in equation (2) should then far outweigh the friction term (eq. (8)). This is somewhat verified by the agreement between experiment and theory in reference 12 where the pressure gradient term alone is used to describe shock-induced separation. Equation (8) itself is, of course, incorrect near separation where $\tau_w \to 0$.

The terms in equation (2) involving enthalpy integrals are evaluated by drawing an analogy to their values for laminar flow. The results of reference 5 show that (from fig. 7 of ref. 5) a zero transformed form factor is obtained for a flat plate at absolute zero surface temperature, which implies a zero displacement thickness for Mach number zero flow with a surface temperature of absolute zero. This is a consequence of the similarity between velocity and temperature profiles for the laminar flat plate. Also to be noted from figure 7 of reference 5 is that the curves of transformed form factor for insulated and cooled surfaces are almost parallel to each other when plotted against a pressure gradient parameter. For turbulent flow, it is assumed that the qualitative behavior is similar to that for laminar flow. This results in the following expression for transformed form factor:

$$H_{tr} = \frac{\int_0^\Delta \left(1 - \frac{U}{U_e}\right) dY + \int_0^\Delta \left(\frac{h_s}{h_0} - 1\right) dY}{\theta_{tr}} \approx H_i + \left(\frac{h_w}{h_0} - 1\right) H_{i,fp} \tag{9}$$

where $H_{i,fp}$ is the incompressible form factor over a flat plate. Thus,

$$\int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) dY \approx \left(\frac{h_w}{h_0} - 1\right) H_{i,fp}\theta_{tr} \tag{10}$$

Using equation (10) and manipulating algebraically with the aid of the power-law velocity profile $\frac{U}{U_e} = \left(\frac{Y}{\Delta}\right)^N$ yield the following identity:

$$\int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) Y\, dY = \Delta^2\left(\frac{h_w}{h_0} - 1\right) \frac{H_{i,fp} - 1}{2(H_{i,fp} + 3)} \tag{11}$$

By using relations (6), (8), (10), (11), and (B4), equations (1) and (2) can be rewritten in terms of the physical longitudinal distance and Mach number in the following manner:

$$\frac{d\theta_{tr}}{dx} + \frac{\theta_{tr}}{M_e}\frac{dM_e}{dx}\left[2 + H_i + \left(\frac{h_w}{h_0} - 1\right)H_{i,fp}\right] + \frac{\theta_{tr}}{R}\frac{dR}{dx} = A \tag{12}$$

$$\frac{dH_i}{dx} = -\frac{1}{M_e}\frac{dM_e}{dx}\left[\frac{H_i(H_i+1)^2(H_i-1)}{2}\right]\left\{1 + \left(\frac{h_w}{h_0} - 1\right)\left[\frac{2H_{i,fp}}{H_i + 1} - \frac{1}{2}\left(\frac{H_i+1}{H_i-1}\right)\left(\frac{H_{i,fp}-1}{H_{i,fp}+3}\right)\right]\right\}$$
$$- 0.03\, H_i(H_i^2 - 1)\frac{A}{\theta_{tr}} \tag{13}$$

where

$$A = 0.123\, e^{-1.561H_i}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{-0.268}\left(\frac{t_e}{t_{ref}}\right)^{0.732}\left(\frac{t_e}{t_0}\right)^{3.268} \tag{14}$$

For incompressible flow, equations (12) to (14) become

$$\frac{d\theta_i}{dx} + \frac{\theta_i}{U_e}\frac{dU_e}{dx}(2 + H_i) + \frac{\theta_i}{R}\frac{dR}{dx} = 0.123\, e^{-1.561H_i}\left(\frac{U_e\theta_i}{\nu_0}\right)^{-0.268} \tag{15}$$

and

$$\frac{dH_i}{dx} = -0.03\, H_i(H_i^2 - 1)\frac{0.123\, e^{-1.561H_i}}{\theta_i}\left(\frac{U_e\theta_i}{\nu_0}\right)^{-0.268} - \frac{1}{U_e}\frac{dU_e}{dx}\left[\frac{H_i(H_i+1)(H_i^2-1)}{2}\right] \tag{16}$$

Equation (15) is the usual momentum integral equation and is identical to that of Maskell (ref. 3) except for the present inclusion of axial symmetry. The moment-of-momentum equation (16) differs only slightly from that of Maskell which is

$$\frac{dH_i}{dX} = \frac{f(H_i)}{\theta_i\left(\frac{U_e\theta_i}{\nu_0}\right)^{0.268}} + \frac{e^{1.561H_i}}{U_e}\frac{dU_e}{dX}\,g(H_i)$$

The first term on the right side of equation (16) has been matched with Maskell's values for $1.25 < H_i < 1.40$, while the agreement between the last term of equation (16) and Maskell's empirical relation is seen in figure 1 to be quite good. Thus, it is felt that the present method is in agreement with that of Maskell for incompressible flow, and, therefore, it is also expected to agree with the method of Mager (ref. 8) for compressible flow over insulated surfaces.

The additional assumptions (eqs. (9) to (11)) made to include noninsulated surfaces are of a speculative nature. However, since they are qualitatively proper, it is hoped that they will also yield reasonable quantitative estimates of the turbulent-boundary-layer characteristics over noninsulated surfaces.

CALCULATION PROCEDURE

The quantities generally desired in a boundary-layer calculation are the boundary-layer integral thicknesses, skin friction, and heat transfer. In the present procedure, it is first necessary to calculate the transformed-momentum-thickness distribution. Anyone interested only in estimating heat-transfer rates can proceed directly to the section entitled Heat Transfer after first calculating the transformed-momentum-thickness distribution.

Transformed Momentum Thickness

The following form of equation (12) (derived in appendix D) is used for the calculation of transformed momentum thickness:

$$\left[\theta_{tr}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{0.2155} M_e^B R^{1.2155}\right]_x - \left[\theta_{tr}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{0.2155} M_e^B R^{1.2155}\right]_{x_1}$$

$$= 0.01173 \int_{x_1}^{x} M_e^B \frac{\left(\frac{t_e}{t_{ref}}\right)^{0.732} R^{1.2155}}{\left(\frac{t_0}{t_e}\right)^{3.268}} dx \tag{17}$$

where

$$B = 4.2 + 1.2155\, H_{i,fp}\left(\frac{h_w}{h_0} - 1\right) \tag{18}$$

$$\frac{h_{ref}}{h_e} = \frac{1}{2}\left(\frac{h_w}{h_e} + 1\right) + 0.22\left(\frac{h_{aw}}{h_e} - 1\right) \tag{7}$$

and

$$\frac{h_{aw}}{h_e} = 1 + \frac{\gamma - 1}{2} M_e^2 \left(Pr\right)_{ref}^{1/3} \tag{19}$$

The quantities a_0 and ν_0 are, respectively, the velocity of sound and kinematic viscosity evaluated at the stagnation conditions of the local external stream. The temperature t_{ref} is evaluated at the reference enthalpy h_{ref}. The local external Mach number M_e and radius of axisymmetric body R should be known as functions of x in order to perform the calculation indicated in equation (17). For two-dimensional flow, the radius R is a constant. Thus, the quantities $R^{1.2155}$ inside and outside the integral in equation (17) cancel. The flat-plate incompressible form factor $H_{i,fp}$ may, for most purposes, be taken as 1.3, approximately that of a 1/7 power velocity profile.

In equation (17) the distance x_1 is the starting point of the calculation. If x_1 is taken to be the transition location, then the transition value of θ_{tr} would be obtained from a laminar-boundary-layer calculation (by using, for example, the method of ref. 5). For cases where the flow is predominantly turbulent, it is usually sufficient to ignore the laminar region and take the origin of boundary-layer development at the leading edge ($x_1 = 0$). For this latter situation, equation (17) becomes

$$\theta_{tr}^{1.2155} = \frac{0.01173}{\left(\frac{a_0}{\nu_0}\right)^{0.2155} M_e^{B+0.2155} R^{1.2155}} \int_0^x M_e^B \frac{\left(\frac{t_e}{t_{ref}}\right)^{0.732}}{\left(\frac{t_0}{t_e}\right)^{3.268}} R^{1.2155}\, dx \tag{20}$$

Again, for two-dimensional flow, the quantities $R^{1.2155}$ in equation (20) should be omitted.

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Transformed Form Factor

After the transformed momentum thickness is obtained, the quantity H_i can be calculated by numerical integration of equation (13):

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$$\frac{dH_i}{dx} = -\frac{1}{M_e}\frac{dM_e}{dx}\left[\frac{H_i(H_i+1)^2(H_i-1)}{2}\right]\left\{1+\left(\frac{h_w}{h_0}-1\right)\left[\frac{2H_{i,fp}}{H_i+1}-\frac{1}{2}\left(\frac{H_i+1}{H_i-1}\right)\left(\frac{H_{i,fp}-1}{H_{i,fp}+3}\right)\right]\right\}$$
$$-0.03\, H_i(H_i^2-1)\frac{A}{\theta_{tr}} \tag{13}$$

where

$$A = 0.123\, e^{-1.561H_i}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{-0.268}\left(\frac{t_e}{t_{ref}}\right)^{0.732}\left(\frac{t_e}{t_0}\right)^{3.268} \tag{14}$$

For flow over an insulated surface, equation (13) can be alternately written as

$$f(H_i) = f(H_{i,1})\frac{M_e}{M_{e,1}}\, e^{\int_{x_1}^{x}\frac{0.060}{H_i+1}\frac{A}{\theta_{tr}}\,dx} \tag{21}$$

where the function

$$f(H_i) = \frac{H_i^2}{(H_i^2-1)^{1/2}(H_i+1)}\, e^{1/(H_i+1)} \tag{22}$$

is that used in reference 12 and for convenience is plotted in figure 2 herein. If friction effects are omitted in equation (21), there results the relation applied in reference 12 to shock-induced separation.

The initial value of H_i for flow over sharp or pointed bodies may be taken as $H_{i,fp}$ (suggested as 1.3). For blunt bodies, the value of H_i at the stagnation point is somewhere between 1.0 and 1.3. A value of 1.1 is suggested. Sample calculations in references 3 and 7 show that a poor guess of the initial value of H_i is inconsequential, since the form factor will tend to reach its proper value in the first few steps of calculation. An error greater than 0.1 in the initial value of H_i should be avoided.

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Compressible Momentum Thickness

As shown in appendix C of reference 12, the compressible momentum thickness is obtained by using the relation

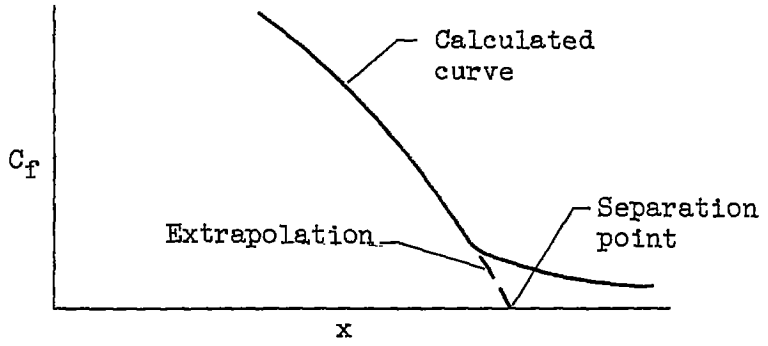
$$\theta = \theta_{tr}\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^3 \tag{23}$$

Skin Friction

From equation (6),

$$C_f = 2A\left(1 + \frac{\gamma - 1}{2} M_e^2\right)^3 \tag{24}$$

where the quantity A is given in equation (14). In an adverse pressure gradient flow, the point of turbulent separation is obtained by sensible extrapolation to zero of the skin-friction coefficients calculated from equation (24). This is shown in the following sketch:



Since the Ludwieg-Tillmann formula cannot give $C_f = 0$, it is probably in error for values of C_f close to zero. Maskell (ref. 3) suggests that the extrapolation be started where C_f is falling rapidly and be continued to $C_f = 0$ without an inflection point as shown in the sketch.

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Compressible Form Factor

As shown in appendix C of reference 12, the compressible form factor can be expressed as

$$H = H_{tr}\left(1 + \frac{\gamma - 1}{2} M_e^2\right) + \frac{\gamma - 1}{2} M_e^2 \tag{25}$$

where

$$H_{tr} = H_i + \left(\frac{h_w}{h_0} - 1\right) H_{i,fp} \tag{9}$$

Equation (25) represents well the experimental data of references 13 and 14 as shown in figure 3. It is also in good agreement with the curve of reference 15 for a 1/7 power profile.

Heat Transfer

In reference 11, it is shown for laminar flow that the ratio of a skin-friction parameter to a Reynolds analogy parameter, which appears in the relation for heat transfer, is almost invariant with pressure gradient. This ratio was then evaluated for zero pressure gradient. For laminar flow, this simplification yields good results except for flows over heated and moderately cooled surfaces with large favorable pressure gradient. It is assumed herein that the corresponding ratio of skin-friction parameter to Reynolds analogy parameter for turbulent flow $e^{-1.561H_i}/[C_f/(St)(Pr)]$ behaves similarly; that is, it is essentially invariant with pressure gradient and is well approximated by its zero pressure gradient value of $e^{-1.561H_{i,fp}}/2(Pr)^{1/3}$. The last expression makes use of the Colburn flat-plate analogy $C_f = (St/2)\left[(Pr)^{-2/3}\right]$. The Reynolds analogy for turbulent flow is then written:

$$St = \frac{C_f}{2} (Pr)^{-2/3} e^{1.561(H_i - H_{i,fp})} \tag{26}$$

The variation of Stanton number with Prandtl number of $(Pr)^{-2/3}$ is proper for zero pressure gradient. Although for laminar flow the exponent of the Prandtl number varies somewhat with pressure gradient, it is felt that for turbulent flow the exponent would vary less. In the absence of more precise information, the -2/3 exponent will be used with pressure gradient. From equations (6) and (26) the expression for Stanton number for a compressible turbulent boundary layer is then

$$St = \frac{0.123\ e^{-1.561H_{i,fp}} \left(\frac{t_e}{t_{ref}}\right)^{0.732}}{\left(\frac{t_0}{t_e}\right)^{0.268} \left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{0.268}} (Pr)_{ref}^{-2/3} \tag{27}$$

Since the skin friction has been eliminated from equation (27), the heat transfer can be directly estimated once the transformed momentum thickness is known.

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The heat-transfer rate to the wall is obtained by using the following relation:

$$q = (St)\rho_e u_e (h_{aw} - h_w) \tag{28}$$

where

$$h_{aw} = h_e\left(1 + (Pr)_{ref}^{1/3} \frac{\gamma - 1}{2} M_e^2\right) \tag{29}$$

SPECIAL CASES

The results of the present method for zero pressure gradient flow and stagnation-point flow will be described in detail. Both the two-dimensional and axisymmetric cases are considered. The value of $H_{i,fp}$ for these cases is taken as 1.3.

Zero Pressure Gradient Flow

Flat plate. - For flow over an isothermal flat plate (from eq. (20)),

$$\theta_{tr}^{1.2155} = \frac{0.01173\left(\frac{t_e}{t_{ref}}\right)^{0.732} x}{\left(\frac{a_0}{\nu_0}\right)^{0.2155} M_e^{0.2155} \left(\frac{t_0}{t_e}\right)^{3.268}} \tag{30}$$

While the transformed form factor is more strictly described by the relation

$$\frac{dH_i}{dx} = -0.03\ \frac{H_i(H_i^2 - 1)A}{\theta_{tr}} \tag{31}$$

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it usually suffices to assume $H_i = H_{i,fp}$ = constant. From equations (23) and (30),

$$\theta = \frac{0.0259\left(\frac{t_e}{t_{ref}}\right)^{0.602}\left(\frac{t_0}{t_e}\right)^{0.311} x^{0.823}}{\left(\frac{M_e a_0}{\nu_0}\right)^{0.177}} \tag{32}$$

and from equations (14), (24), and (30),

$$C_f = 0.086\left(\frac{M_e a_0 x}{\nu_0}\right)^{-0.22}\left(\frac{t_e}{t_{ref}}\right)^{0.571}\left(\frac{t_0}{t_e}\right)^{0.452} \tag{33}$$

The Stanton number relation from equations (27) and (30) is

$$St = 0.043\left(\frac{M_e a_0 x}{\nu_0}\right)^{-0.22}\left(\frac{t_e}{t_{ref}}\right)^{0.571}\left(\frac{t_0}{t_e}\right)^{0.452}(Pr)_{ref}^{-2/3} \tag{34}$$

Cone surface. - The thicknesses and skin-friction and heat-transfer parameters will be written out in complete form and will also be compared to the flat-plate values. With $R = x \sin \theta_c$ the transformed momentum thickness is given from equation (20) by

$$\theta_{tr}^{1.2155} = \frac{0.01173\left(\frac{t_e}{t_{ref}}\right)^{0.732} x}{2.2155\left(\frac{a_0}{\nu_0}\right)^{0.2155} M_e^{0.2155}\left(\frac{t_0}{t_e}\right)^{3.268}} \tag{35}$$

The variation of transformed form factor is more correctly given by equation (31) but may be taken as constant and equal to $H_{i,fp}$. The compressible momentum thickness is, from equations (23) and (35),

$$\theta = \frac{0.0135\left(\frac{t_e}{t_{ref}}\right)^{0.602}\left(\frac{t_0}{t_e}\right)^{0.311} x^{0.823}}{\left(\frac{M_e a_0}{\nu_0}\right)^{0.177}} \tag{36}$$

or, alternately, for the same external Mach number and stagnation conditions,

$$\frac{\theta_{cone}}{\theta_{fp}} = \left(\frac{1}{2.2155}\right)^{\frac{1}{1.2155}} = 0.520 \tag{37}$$

The skin-friction relation for a cone is, from equations (14), (24), and (35),

$$C_f = 0.102\left(\frac{M_e a_0 x}{\nu_0}\right)^{-0.22}\left(\frac{t_e}{t_{ref}}\right)^{0.571}\left(\frac{t_0}{t_e}\right)^{0.452} \tag{38}$$

and

$$\frac{C_{f,cone}}{C_{f,fp}} = (0.520)^{-0.268} = 1.192 \tag{39}$$

From equations (27) and (35),

$$St = 0.051\left(\frac{M_e a_0 x}{\nu_0}\right)^{-0.22}\left(\frac{t_e}{t_{ref}}\right)^{0.571}\left(\frac{t_0}{t_e}\right)^{0.452}(Pr)_{ref}^{-2/3} \tag{40}$$

and

$$\frac{(St)_{cone}}{(St)_{fp}} = \frac{C_{f,cone}}{C_{f,fp}} = 1.192 \tag{41}$$

These results for the ratios of cone to flat-plate quantities agree very well with Gazley's value of 1.176 (ref. 16) and Van Driest's value of about 1.15 (ref. 17).

Stagnation-Point Flow

The quantities of interest in stagnation-point flow are momentum thickness and heat transfer. For stagnation-point flow it is assumed that the velocity along the surface varies linearly with the distance from the stagnation point and that static and total temperatures are equal for the region of interest. This treatment of turbulent stagnation flows is certainly somewhat fictional. Turbulent flow will be obtained only after the transition point and therefore not at the stagnation point itself. Also, because of the assumption of equal static and total temperatures in the inviscid flow, only a portion of the subsonic part of

the stagnation region can be treated by using the simple formulas that follow. These relations are included, therefore, only as a guide to what might be expected from turbulent flow over blunt bodies.

Two dimensional. - With the previously mentioned assumptions, equation (20) yields for momentum thickness:

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$$\theta = \frac{\left(\frac{0.01173}{B + 1}\right)^{0.823}\left(\frac{t_e}{t_{ref}}\right)^{0.602} x^{0.644}}{\left(\frac{\frac{du_e}{dx}}{\nu_0}\right)^{0.177}} \tag{42}$$

while the Stanton number, from equations (27) and (42), is

$$St = \frac{0.044[\nu_0(B + 1)]^{0.22}\left(\frac{t_e}{t_{ref}}\right)^{0.571}}{\left(\frac{du_e}{dx}\right)^{0.22} x^{0.44}} (Pr)_{ref}^{-2/3} \tag{43}$$

At the stagnation point, the Stanton number is seen to be infinite. It is perhaps more instructive to examine the heat-transfer coefficient, since the latter is more directly involved in calculating the heat-transfer rates:

$$\frac{q}{h_{aw} - h_w} = (St)\rho_e u_e = 0.044\rho_e(Pr)_{ref}^{-2/3}[\nu_0(B+ 1)]^{0.22}\left(\frac{t_e}{t_{ref}}\right)^{0.571}\left(\frac{du_e}{dx}\right)^{0.78} x^{0.56} \tag{44}$$

The heat-transfer coefficient and, therefore, the heat-transfer rate at a turbulent stagnation point are zero. This is a consequence of using the Ludwieg-Tillmann skin-friction relation at zero velocity where it is certainly inapplicable.

Axisymmetric. - For momentum thickness with $R = x$, equation (20) becomes

$$\theta = \frac{\left(\frac{0.01173}{B + 2.2155}\right)^{0.823}\left(\frac{t_e}{t_{ref}}\right)^{0.602} x^{0.644}}{\left(\frac{\frac{du_e}{dx}}{\nu_0}\right)^{0.177}} \tag{45}$$

and the heat-transfer coefficient from equations (27) and (45) is

$$\frac{q}{h_{aw} - h_w} = 0.044\,\rho_e(Pr)_{ref}^{-2/3}\left[\nu_0(B + 2.2155)\right]^{0.22}\left(\frac{t_e}{t_{ref}}\right)^{0.571}\left(\frac{du_e}{dx}\right)^{0.78}x^{0.56} \tag{46}$$

These stagnation region results, both two dimensional and axisymmetric, are in very close agreement with the results of Van Driest (ref. 18). The velocity gradient du_e/dx that appears in equations (43) to (46) is evaluated for supersonic flow with $M_\infty > 2$ from the relation

$$\frac{du_e}{dx} = \frac{a_0}{R_{sp}}\sqrt{\frac{2}{\gamma}\left(1 - \frac{p_\infty}{p_{sp}}\right)} \tag{47}$$

which was derived for bodies having finite nose curvature based on the assumption of a modified Newtonian flow in the neighborhood of the stagnation point.

CONCLUDING REMARKS

An approximate method for the calculation of compressible turbulent boundary layer with heat transfer and arbitrary pressure gradient has been presented. The method as applied to insulated surfaces is quite well founded, but for noninsulated isothermal surfaces it depends on a number of speculative assumptions. These assumptions, however, are qualitatively proper, and it is hoped that they will also yield reasonable quantitative results. The proposed method of estimating the location of separation for flow with adverse pressure gradients should be adequate for flows where the separation is characterized by a rapid drop of the skin friction over a short longitudinal distance.

Lewis Flight Propulsion Laboratory
National Advisory Committee for Aeronautics
Cleveland, Ohio, September 12, 1957

APPENDIX A

SYMBOLS

A	skin-friction parameter, eq. (14)
a	velocity of sound
B	exponent of Mach number in eq. (17)
C_f	local skin-friction coefficient
$f(H_i)$, $g(H_i)$	functions in Maskell's moment-of-momentum equation (ref. 3)
H	form factor, δ^*/θ
h	enthalpy
M	Mach number
M_e	local external Mach number
N	power of power-law profile
Pr	Prandtl number
p	pressure
q	heat-transfer rate
R	radius of axisymmetric body
R_{sp}	radius of curvature at stagnation point of blunt body
St	Stanton number, $\frac{q}{\rho_e u_e (h_{aw} - h_w)}$
t	temperature
U	transformed longitudinal velocity
u	longitudinal velocity
V	transformed normal velocity
v	normal velocity

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X transformed coordinate along surface

x coordinate along surface

Y transformed normal coordinate

y normal coordinate

γ ratio of specific heats

Δ upper limit of integration in momentum of moment-of-momentum integral equations

δ^* displacement thickness

θ momentum thickness

θ_c cone half-angle, deg

ν kinematic viscosity

ρ mass density

τ shear stress in boundary layer

Ψ stream function

Subscripts:

aw adiabatic wall

e local flow outside boundary layer (external)

fp flat-plate value

i incompressible

ref evaluated at reference condition

s local stagnation value

sp stagnation-point value

tr transformed

w wall or surface value

0 free-stream stagnation value

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1 initial condition

∞ free-stream (static) value

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APPENDIX B

DERIVATION OF TRANSFORMED BOUNDARY-LAYER INTEGRAL EQUATIONS

For steady compressible flow with axial symmetry and a Prandtl number of 1.0, the time-averaged continuity and momentum equations may be written.

Continuity:

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\rho u}{R}\frac{dR}{dx} = 0 \tag{B1}$$

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Momentum:

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial y} \tag{B2}$$

The radius of the body R is a function of x alone. For a two-dimensional flow, the R term in the continuity equation is not involved.

Stewartson's Transformation

The velocities in equations (B1) and (B2) can be replaced through the definition of a stream function:

$$\left.\begin{aligned} \Psi_y &= \frac{\rho u R}{\rho_0} \\ \Psi_x &= -\frac{\rho v R}{\rho_0} \end{aligned}\right\} \tag{B3}$$

The coordinates x and y are transformed by the following form of Stewartson's transformation:

$$\left.\begin{aligned} X &= \int_0^x \frac{p_e}{p_0}\frac{a_e}{a_0}\,dx \\ Y &= \frac{a_e}{a_0}\int_0^y \frac{\rho}{\rho_0}\,dy \end{aligned}\right\} \tag{B4}$$

The transformed coordinates are now represented by capital letters (X and Y), and the subscript e refers to local conditions at the outer edge of the boundary layer (external). The subscript 0 refers to free-stream stagnation values.

Applying equations (B3) and (B4) to equations (B1) and (B2) results in the following system:

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$$U_X + V_Y + \frac{U}{R}\frac{dR}{dX} = 0 \tag{B5}$$

$$UU_X + VU_Y = U_e U_{e_X}\frac{h_s}{h_0} + \frac{1}{\rho_0 \frac{p_e}{p_0}\left(\frac{a_e}{a_0}\right)^2}\frac{\partial\tau}{\partial Y} \tag{B6}$$

where the stream function has been replaced by the transformed velocities

$$\left.\begin{aligned} UR &\equiv \Psi_Y \\ VR &\equiv -\Psi_X \end{aligned}\right\} \tag{B7}$$

The relation between physical and transformed longitudinal velocities is

$$U = u\frac{a_0}{a_e}$$

The integrand of the momentum integral equation is obtained by subtracting equation (B6) from the product of equation (B5) and the quantity $U_e - U$. This results in

$$\left[U(U_e - U)\right]_X + \left[V(U_e - U)\right]_Y + U_{e_X}\left[U_e - U + U_e\left(\frac{h_s}{h_0} - 1\right)\right] +$$

$$(U_e - U)\frac{U}{R}\frac{dR}{dX} = -\frac{1}{\rho_0 \frac{p_e}{p_0}\left(\frac{a_e}{a_0}\right)^2}\frac{\partial\tau}{\partial Y} \tag{B8}$$

If equation (B8) is integrated with respect to Y between the limits $Y = 0$ and $Y = \Delta$ where Δ is a constant distance normal to the transformed surface sufficiently large so that the conditions $U = U_e$ and $h_s = h_0$ can both be satisfied, there results

$$\frac{d\theta_{tr}}{dX} + \frac{\theta_{tr}}{U_e}\frac{dU_e}{dX}\left[2 + H_i + \frac{\int_0^{\Delta}\left(\frac{h_s}{h_0} - 1\right) dY}{\theta_{tr}}\right] + \frac{\theta_{tr}}{R}\frac{dR}{dX} = \frac{\tau_w}{\rho_0\left(\frac{p_e}{p_0}\right)\left(\frac{a_e}{a_0}\right)^2 U_e^2} \tag{1}$$

where

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$$\theta_{tr} = \int_0^{\Delta} \frac{U}{U_e}\left(1 - \frac{U}{U_e}\right) dY \tag{3}$$

and

$$H_i = \frac{\int_0^{\Delta}\left(1 - \frac{U}{U_e}\right) dY}{\theta_{tr}} \tag{4}$$

The quantity H_i is the transformed form factor for the boundary layer over an insulated surface. For a noninsulated surface, the transformed displacement thickness is written:

$$\delta_{tr}^* = \int_0^{\Delta}\left[\left(1 - \frac{U}{U_e}\right) + \left(\frac{h_s}{h_0} - 1\right)\right] dY \tag{B9}$$

while the transformed momentum thickness is still given by equation (3) (see appendix C, ref. 12).

The moment-of-momentum equation is obtained by multiplying equation (B10) by Y and then integrating with respect to Y between $Y = 0$ and $Y = \Delta$:

$$\frac{\partial}{\partial X}\int_0^{\Delta} YU_e^2\,\frac{U}{U_e}\left(1-\frac{U}{U_e}\right)dY + U_e\int_0^{\Delta} Y\,\frac{\partial}{\partial Y}\left[V\left(1-\frac{U}{U_e}\right)\right]dY$$

$$+\frac{U_e^2}{R}\,\frac{dR}{dX}\int_0^{\Delta} Y\,\frac{U}{U_e}\left(1-\frac{U}{U_e}\right)dY + U_e\,\frac{dU_e}{dX}\left[\int_0^{\Delta} Y\left(1-\frac{U}{U_e}\right)dY\right.$$

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$$\left.+\int_0^{\Delta} Y\left(\frac{h_s}{h_0}-1\right)dY\right] = -\frac{1}{\rho_0\,\frac{p_e}{p_0}\left(\frac{a_e}{a_0}\right)^2}\int_0^{\Delta} Y\,\frac{\partial\tau}{\partial Y}\,dY \qquad \text{(B10)}$$

The procedure of reducing this equation to usable form is that of Tetervin and Lin (ref. 2) except for the present consideration of noninsulated surfaces. This involves evaluating the integrals in equation (B10) with the aid of the power-law assumption $U/U_e = (Y/\Delta)^N$ and then reverting to the form factor H_i by using the power-law relation $H_i = 1 + 2N$. The result is

$$\frac{dH_i}{dX} = -\frac{1}{U_e}\,\frac{dU_e}{dX}\left[\frac{H_i(H_i+1)^2(H_i-1)}{2}\right]\left[1+\frac{2}{(H_i+1)\theta_{tr}}\int_0^{\Delta}\left(\frac{h_s}{h_0}-1\right)dY\right.$$

$$\left.-\frac{2(H_i-1)}{H_i^2(H_i+1)\theta_{tr}^2}\int_0^{\Delta}\left(\frac{h_s}{h_0}-1\right)Y\,dY\right]$$

$$+\frac{\tau_w(H_i^2-1)}{\rho_0\,\frac{p_e}{p_0}\left(\frac{a_e}{a_0}\right)^2 U_e^2\theta_{tr}}\left[H_i-(H_i+1)\int_0^1\frac{\tau}{\tau_w}\,d\left(\frac{Y}{\Delta}\right)\right] \qquad (2)$$

An important result here is that the moment-of-momentum equation is independent of radius variation so that equation (2) holds for both two-dimensional and axisymmetric flow. This was pointed out in reference 2.

APPENDIX C

ADAPTATION OF LUDWIEG-TILLMANN TURBULENT SKIN-FRICTION RELATION TO COMPRESSIBLE FLOW

The Ludwieg-Tillmann skin-friction relation for incompressible flow is from reference 4:

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$$\frac{\tau_w}{\frac{1}{2}\rho u_e^2} = 0.246\ e^{-1.561H}\ Re_\theta^{-0.268} \tag{C1}$$

where $Re_\theta = u_e\theta/\nu$. The adaptation of this relation to compressible flow follows the procedure used by Eckert (ref. 10) for the Schultz-Grunow relation. With the skin friction expressed in terms of a Reynolds number based on x, the fluid properties in the skin-friction relation are evaluated at a reference enthalpy. This reference enthalpy of reference 10 was selected so as to give good agreement with the skin-friction results of many investigations over a wide range of flow conditions. The first step then is the conversion of equation (C1) to x dependence. This will be done by using the momentum integral equation with zero pressure gradient. The reference enthalpy method is then applied, and the resultant relation is reconverted to a Reynolds number based on θ, again by using the momentum integral equation with zero pressure gradient. The zero pressure gradient momentum integral is

$$\frac{d\theta}{dx} = \frac{C_f}{2} = 0.123\ e^{-1.561H}\left(\frac{u_e\theta}{\nu}\right)^{-0.268} \tag{C2}$$

Integrating equation (C2) with constant H yields

$$\theta = \left[(1.268)(0.123)e^{-1.561H}\left(\frac{u_e}{\nu}\right)^{-0.268} x\right]^{1/1.268} \tag{C3}$$

Substituting equation (C3) in equation (C1) then yields

$$\frac{\tau_w}{\frac{1}{2}\rho u_e^2} = 0.364\ e^{-1.23H}\left(\frac{u_e x}{\nu}\right)^{-0.211} \tag{C4}$$

By following the procedure of Eckert (ref. 10),

$$\frac{\tau_w}{\frac{1}{2}\rho_{ref}u_e^2} = 0.364\ e^{-1.23H_i}\left(\frac{u_e x}{\nu_{ref}}\right)^{-0.211} \tag{C5}$$

or

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$$\frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} = 0.364\ e^{-1.23H_i}\left(\frac{u_e x}{\nu_{ref}}\right)^{-0.211}\frac{t_e}{t_{ref}} \tag{C6}$$

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The subscript ref indicates that the quantities should be evaluated at the reference enthalpy defined by the relation

$$\frac{h_{ref}}{h_e} = \frac{1}{2}\left(\frac{h_w}{h_e} + 1\right) + 0.22\left(\frac{h_{aw}}{h_e} - 1\right) \tag{C7}$$

It is customary to demonstrate the adequacy of a compressible skin-friction adaptation by comparing the resultant ratio $C_f/C_{f,i}$ with experimental variations. The quantity $C_{f,i}$ is the value of skin-friction coefficient that would have been obtained without the compressibility adaptation. Thus, from equations (C4) and (C6),

$$\frac{C_f}{C_{f,i}} = \left(\frac{\nu_e}{\nu_{ref}}\right)^{-0.211}\frac{t_e}{t_{ref}} \tag{C8}$$

For flow over an insulated surface with the adiabatic wall temperature equal to stagnation temperature (Prandtl number, 1.0) with viscosity proportional to temperature and $\gamma = 1.4$, equation (C8) becomes

$$\frac{C_f}{C_{f,i}} = \left(1 + 0.144\ M_e^2\right)^{-0.578} \tag{C9}$$

This relation agrees very well with the relations of Eckert (ref. 10) and Tucker (ref. 15), as seen in figure 4, which in turn are in excellent agreement with available experimental data.

For use in the present method, equation (C6), which now includes compressibility effects, is reconverted to a Reynolds number based on momentum thickness. The resultant relation is

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} = 0.246\ e^{-1.561H_i}\left(\frac{u_e\theta}{\nu_{ref}}\right)^{-0.268}\left(\frac{t_e}{t_{ref}}\right)^{1.268} \tag{C10}$$

Since, in the present method, the equations are solved in a transformed plane (as transformed by the Stewartson transformation), equation (C10) is transformed similarly with the assumption of a linear viscosity-temperature relation. The result is

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$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho_e u_e^2} = 0.246\ e^{-1.561H_i}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{-0.268}\left(\frac{t_e}{t_{ref}}\right)^{0.732}\left(\frac{t_e}{t_0}\right)^{0.268} \tag{6}$$

While the assumptions made in the adaptation described here are many, it is felt that the result is an adequate description of skin-friction coefficient for a turbulent boundary layer with pressure gradient.

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APPENDIX D

FURTHER INTEGRATION OF MOMENTUM INTEGRAL EQUATION

The momentum integral equation (12) is

$$\frac{d\theta_{tr}}{dx} + \frac{\theta_{tr}}{M_e}\frac{dM_e}{dx}\left(2 + H_{tr}\right) + \frac{\theta_{tr}}{R}\frac{dR}{dx} = A \tag{D1}$$

where

$$A = 0.123\, e^{-1.561H_i}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{-0.268}\left(\frac{t_e}{t_{ref}}\right)^{0.732}\left(\frac{t_e}{t_0}\right)^{3.268} \tag{14}$$

and where

$$H_{tr} = H_i + \left(\frac{h_w}{h_0} - 1\right) H_{i,fp} \tag{9}$$

By following Maskell (ref. 3), let

$$\left.\begin{aligned} \Theta &= \theta_{tr}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^k \\ &\text{and} \\ \zeta &= A\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^k \end{aligned}\right\} \tag{D2}$$

The substitution of equations (D2) into (D1) yields

$$\frac{d\Theta}{dx} + (1 + k)\frac{\Theta}{R}\frac{dR}{dx} = (1 + k)\left[\zeta - \frac{\Theta}{M_e}\frac{dM_e}{dx}\left(H_{tr} + \frac{2 + k}{1 + k}\right)\right] \tag{D3}$$

Maskell found that, for incompressible flow, the right side of equation (D3) could be represented by

$$0.01173 - 4.2\frac{\Theta}{U_e}\frac{dU_e}{dx} \tag{D4}$$

with k taken as 0.2155. Since, for incompressible flow, $H_{tr} = H_i$, the second term of equation (D4) implies that

$$(1 + k)\left(H_i + \frac{2 + k}{1 + k}\right) = 4.2 \tag{D5}$$

It then follows that

$$(1 + k)\left[H_i + \left(\frac{h_w}{h_0} - 1\right) H_{i,fp} + \frac{2 + k}{1 + k}\right] = 4.2 - 1.2155\left(\frac{h_w}{h_0} - 1\right) H_{i,fp} \tag{D6}$$

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so that equation (D3) may be written as

$$\frac{d\Theta}{dx} + 1.2155\,\frac{\Theta}{R}\,\frac{dR}{dx} = 0.01173\left(\frac{t_e}{t_{ref}}\right)^{0.732}\left(\frac{t_e}{t_0}\right)^{3.268} - B\,\frac{\Theta}{M_e}\,\frac{dM_e}{dx} \tag{D7}$$

where

$$B = 4.2 + 1.2155\,H_{i,fp}\left(\frac{h_w}{h_0} - 1\right) \tag{D8}$$

The values of B given by equation (D8) with $H_{i,fp} = 1.3$ are in very good agreement with those for laminar flow (ref. 5) and those estimated by Skopets (ref. 19) for turbulent flow.

Equation (D7) is a first-order linear equation for Θ and can be integrated to yield

$$\left(\Theta M_e^B\,R^{1.2155}\right)_x - \left(\Theta M_e^B\,R^{1.2155}\right)_{x_1} = 0.01173\int_{x_1}^{x} M_e^B\,\frac{\left(\frac{t_e}{t_{ref}}\right)^{0.732}}{\left(\frac{t_0}{t_e}\right)^{3.268}}\,R^{1.2155}\,dx \tag{D9}$$

where

$$\Theta = \theta_{tr}\left(\frac{M_e a_0 \theta_{tr}}{\nu_0}\right)^{0.2155} \tag{D10}$$

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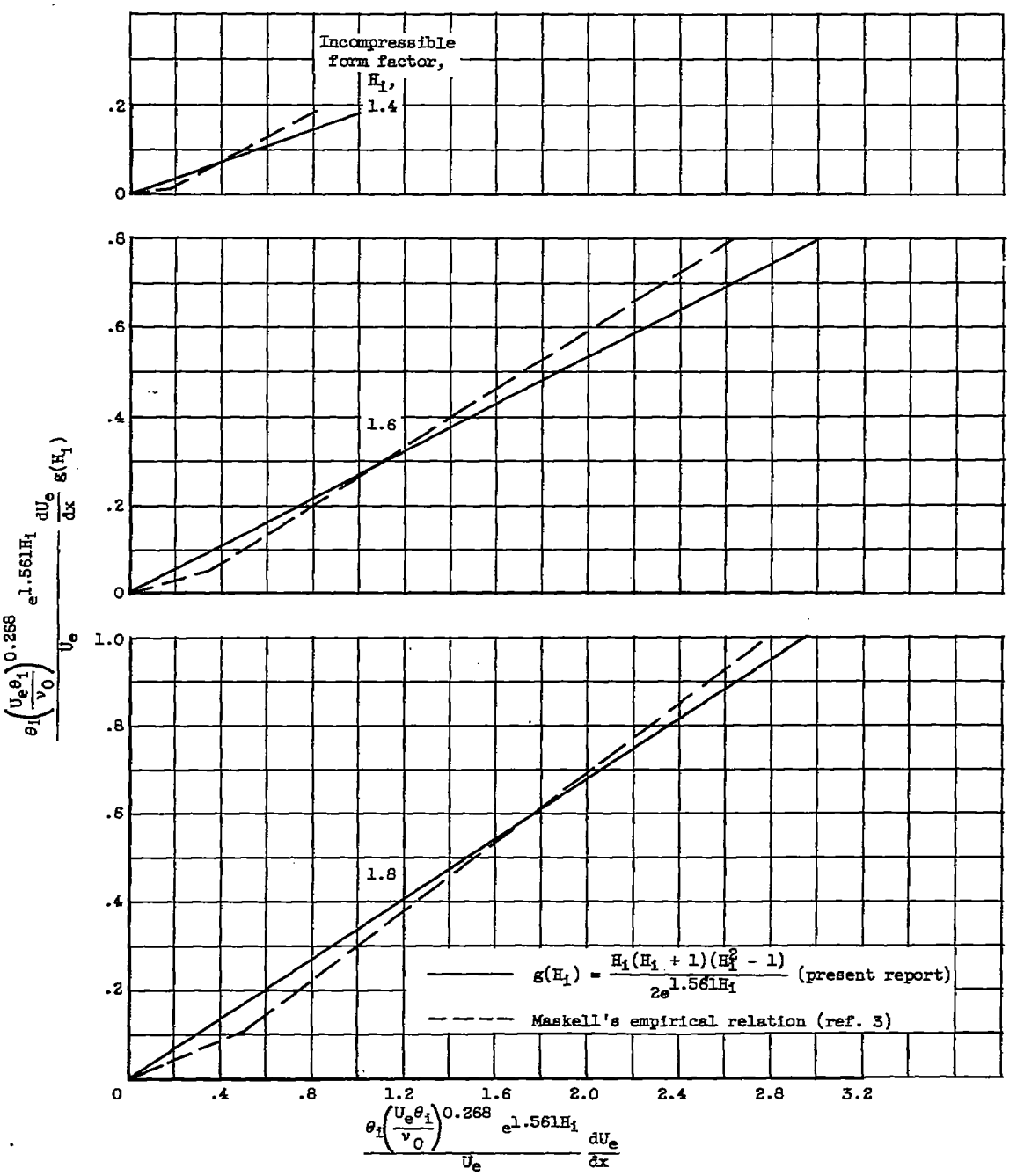

Figure 1. - Comparison of present moment-of-momentum relation with that of Maskell (ref. 3).

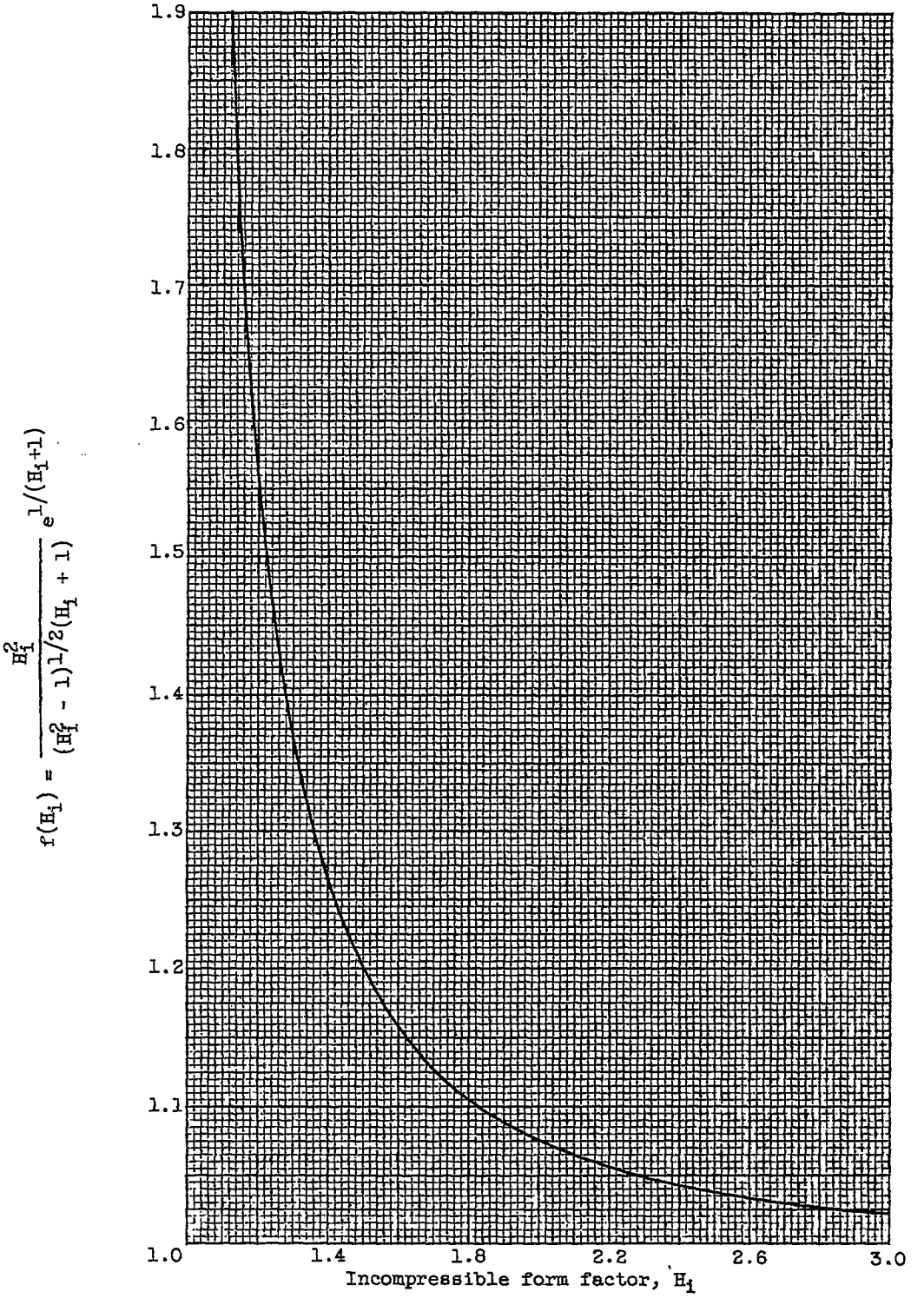

Figure 2. - Function for calculation of form-factor variation for insulated surface.

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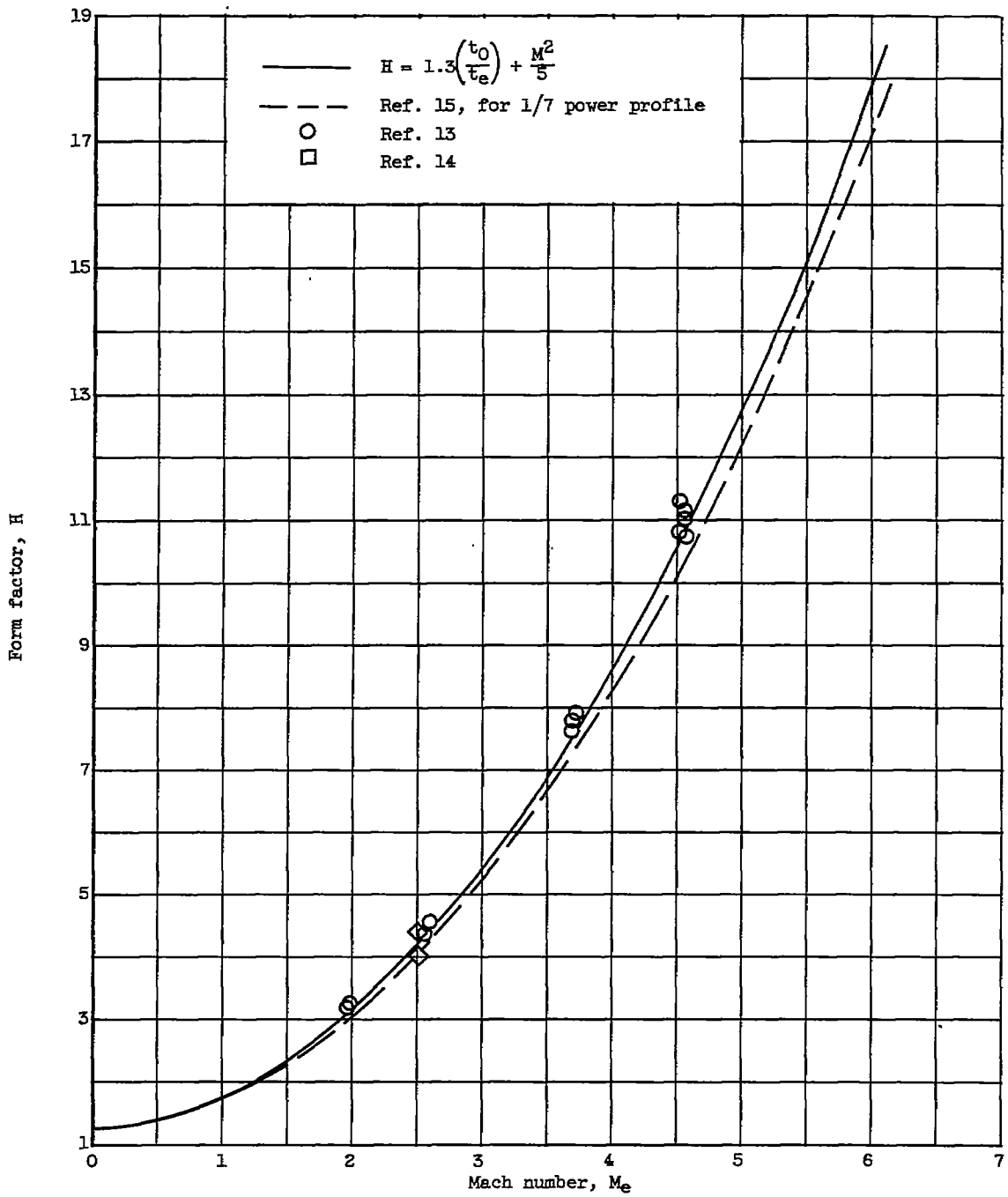

Figure 3. - Variation of flat-plate form factor with Mach number and comparison of experiment with theory.

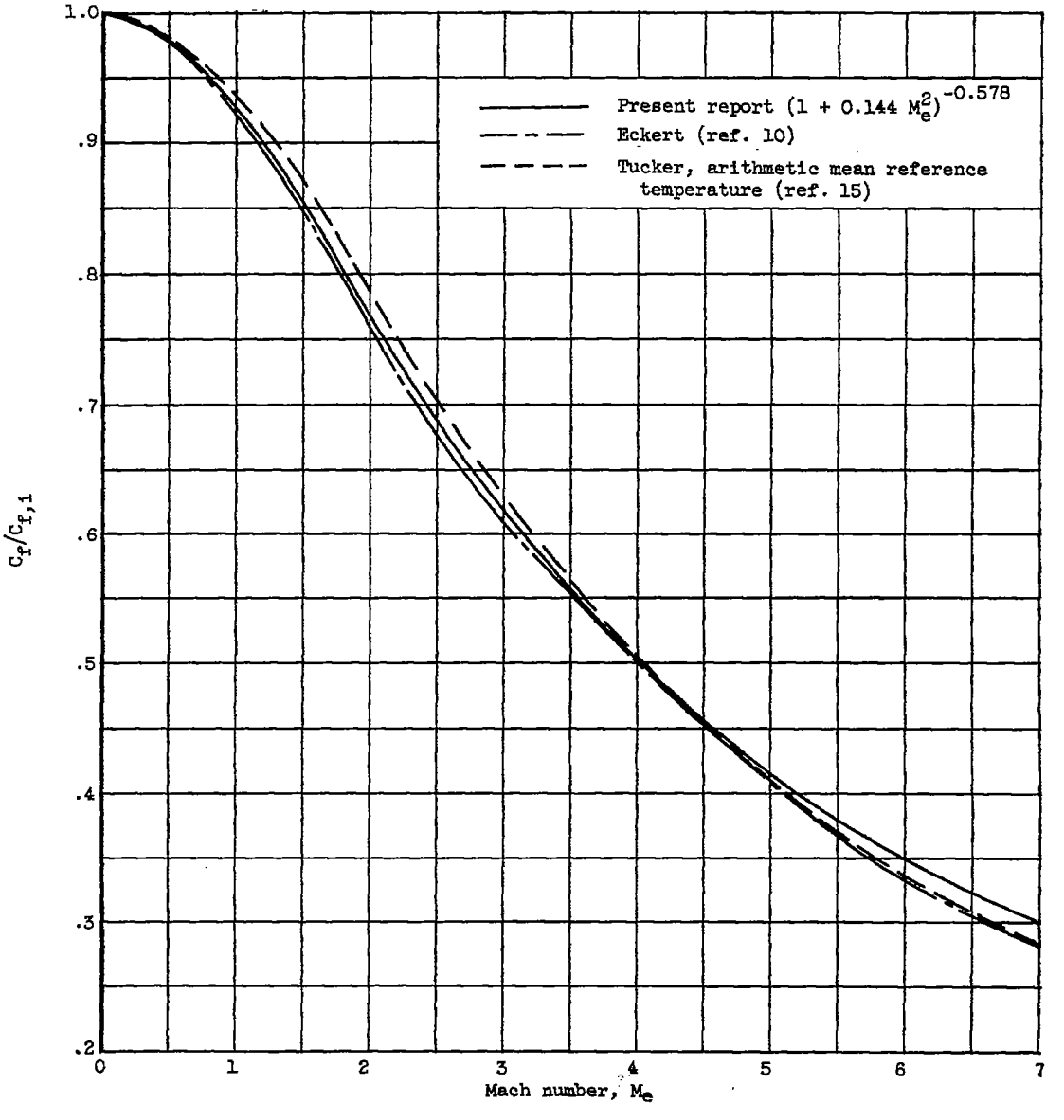

Figure 4. - Comparison of present variation of skin friction with Mach number with variations of references 10 and 15.