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TECHNICAL NOTE 4399

## RELATIVE MOTION IN THE TERMINAL PHASE OF INTERCEPTION OF A SATELLITE OR A BALLISTIC MISSILE

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# RELATIVE MOTION IN THE TERMINAL PHASE OF INTERCEPTION OF A SATELLITE OR A BALLISTIC MISSILE

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## SUMMARY

The interception of a satellite or a ballistic missile essentially outside the earth's atmosphere is considered with emphasis on the terminal phase of the attack vehicle's flight. For the discussion of the relative motion, a coordinate system is selected which has its origin at the target and axes always parallel to lines fixed in an inertial frame. The resulting vector equation of motion contains, in addition to the thrust term, an apparent gravitational term. Bounds on the magnitude of this apparent acceleration of the interceptor are obtained which permit gaging its importance in relation to thrust acceleration in the terminal phase of interception. Since the apparent gravitational acceleration approaches zero as the interceptor approaches the target, preliminary analyses of the terminal phase of interception can generally be carried out by neglecting gravity altogether. Two types of interception are discussed mathematically and a numerical example is worked out for each type. The type of interception for which the terminal relative approach speed is small compared with the target's speed is, within the limits of the present study, found to be feasible. The practicability of head-on interception with mass dispersal prior to impact is more difficult to assess; in the example considered, the mass required to be dispersed is found to be of the order of magnitude of the mass of the target.

## INTRODUCTION

The use of pilotless aircraft in the destructive interception of an airplane is presently a well-developed field of engineering both in theory and in practice. The corresponding literature is extensive and no attempt need be made to review it here. (See, for example, ref. 1.)

The interception of satellites and ballistic missiles, on the other hand, remains practically unexplored. This case is characterized by speeds so high as to approach escape speed, propulsion and control by rockets, and a shift of emphasis, particularly in the terminal phase of

flight, from aerodynamic forces to gravitational forces. In addition, the motion of a satellite or a ballistic missile can be accurately predicted over large distances from knowledge of its earlier motion.

In this paper attention is centered on the terminal phase of the interceptor's flight. Equally important problems of initial detection, tracking, launching, initial guidance and propulsion, and stabilization, for example, are not considered to an appreciable extent in the present study.

## SYMBOLS

| | |
|---|---|
| $\vec{a}$ | position vector of attack vehicle in a coordinate system with origin at target's center of mass and axes always parallel to lines fixed in an inertial frame |
| $A_f$ | target's frontal area |
| $b$ | radius of sphere over which dispersed mass is distributed |
| $c$ | navigational correction coefficient |
| $\vec{F}$ | thrust vector |
| $\vec{F}_p$ | vector sum of perturbing forces at $\vec{r}$ |
| $\vec{F}_p{}'$ | vector sum of perturbing forces at $\vec{\rho}$ |
| $G$ | Newton's universal gravitational constant, $6.670 \times 10^{-11}$ newton-meter$^2$/kg$^2$ = $3.438 \times 10^{-8}$ lb-ft$^2$/slug$^2$ |
| $m$ | mass of attack vehicle |
| $m'$ | mass of target |
| $M$ | mass of earth |
| $m_s$ | mass which strikes target |
| $m_{md}$ | minimum destructive mass |
| $\vec{r}$ | position vector of attack vehicle in a coordinate system with origin at earth's center and axes always parallel to lines fixed in an inertial frame |

| | |
|---|---|
| $t$ | time |
| $U$ | potential function for tide-generating acceleration, or for $\vec{\Gamma}$ |
| $v_1$ | magnitude of velocity vector $d\vec{a}/dt$ at time $t_1$ |
| $v_{ex}$ | exit speed of propulsive exhaust gases |
| $w_b$ | radial speed of dispersed mass |
| $\vec{\Gamma}$ | apparent gravitational acceleration acting on attack vehicle |
| $\epsilon$ | maximum error (maximum distance of center of mass of dispersed matter from target's flight path at interception) |
| $\theta_1$ | angle which velocity vector $d\vec{a}/dt$ makes with negative of position vector $\vec{a}$ at time $t_1$ |
| $\theta_{a\rho}$ | angle between positive directions of vectors $\vec{a}$ and $\vec{\rho}$ |
| $\theta_{a\Gamma}$ | angle between positive directions of vectors $\vec{a}$ and $\vec{\Gamma}$ |
| $\theta_{\rho\Gamma}$ | angle between positive directions of vectors $\vec{\rho}$ and $\vec{\Gamma}$ |
| $\vec{\rho}$ | position vector of target in a coordinate system with origin at earth's center and axes always parallel to lines fixed in an inertial frame |
| $\rho_o$ | radius of earth, 6,371 km = 3,959 miles |
| $\sigma$ | dimensionless correction constant, $c/mv_1$ |
| $\varphi$ | longitude in a spherical coordinate system |

Subscripts:

| | |
|---|---|
| 1 | refers to start of terminal phase of interception |
| i | refers to start of mass dispersal (type II) |
| 2 | refers to time at which target is hit |

## GENERAL CONSIDERATIONS

The problem of intercepting a satellite vehicle or a ballistic missile essentially outside the earth's atmosphere has two features which alone are sufficient to set it apart from the interception problems encountered in utilizing a guided missile to destroy a conventional aircraft. These features are:

(1) The predictability of the target's flight path over large distances. The accurate measurement of the position and velocity of the target at some point in time makes possible the computation of its position and velocity at future times by the methods of celestial mechanics. This means that the interception program can be primarily predetermined, the subsequent corrections required in flight being comparatively small. For that portion of the attack vehicle's flight path which lies in the atmosphere, aerodynamic controls can be employed, whereas outside the atmosphere propulsive reaction devices must be used for control.

(2) The target's extremely high speed. Because of this, the choice of interception program is more critically dependent upon the capabilities of the components of the interception system concerned with initial detection, tracking, guidance, propulsion, stabilization, and destruction. In particular, for an interception system requiring a direct hit, the terminal approach speed, relative to the target, must be low enough to permit the system to overcome its limitations in attack-vehicle maneuverability and in component accuracy and response time. An important case is the extreme case in which the terminal relative approach speed is small, that is, when the attack vehicle's velocity vector closely approximates the target's velocity vector immediately prior to the hit. Interception programs with this characteristic will be called type I. On the other hand, the target's extremely high speed can clearly be used to advantage in destroying it by a head-on attack. In this case the demands placed upon the interception system probably preclude scoring a direct hit. However, the timed dispersal of all or a part of the attack vehicle's mass prior to interception can be utilized to reduce the accuracy requirement. A simple and important case is that in which the center of mass of the attack vehicle, during and somewhat before dispersal, is in the projected orbit of the target. At contact, then, the velocities are equal in magnitude and oppositely directed. Head-on interception programs of this nature will be called type II.

The attack vehicle can be launched from the ground or from another vehicle such as an aircraft, a satellite, or a satelloid. Flight paths of ground-launched interceptors of the two types described are shown schematically in figure 1. The chief differences between the two flight paths are the following:

(1) The terminal portion of type I will generally be extended because of the small relative approach speed. Thus, the time in flight tends to be greater than in the case of type II.

(2) The launching point of type I will, in general, lie nearer to the point of initial detection. This suggests, for example, a single location for the equipment for detection, tracking, launching, and so forth, in the case of type I.

(3) Because of the difference in flight-path orientation, the earth's rotational motion may be usable to better advantage in one case than in the other.

(4) If considerable variation in point of destruction is allowable, the time at which the interceptor is launched need not, in general, be as precise for type II as for type I.

(5) If the eccentricity of the target's elliptical orbit is large and if the point of destruction is near the minor axis of the ellipse, then, depending upon which way the target's velocity vector is inclined to the horizontal, the time in flight tends to differ considerably for the two types.

These differences, of course, become less important as the overall performance of the interception system increases.

## EQUATIONS OF MOTION

The equations of motion for the two types are basically the same and, outside the earth's atmosphere, may be expressed in vector form as

$$\frac{d^2\vec{r}}{dt^2} = -G\frac{M}{r^3}\vec{r} + \frac{1}{m}\vec{F} + \frac{1}{m}\vec{F}_p \tag{1}$$

where $\vec{r}$ is the position vector[1] of the center of mass of the attack vehicle at time $t$, $G$ is the universal gravitational constant, $M$ is the earth's mass, $m$ is the instantaneous mass (nonincreasing with time), $\vec{F}$ is the thrust, and $\vec{F}_p$ is the vector sum of perturbing forces such as those due to the earth's oblateness and motion around the sun, the moon's

[1]In a coordinate system with origin at the earth's center and axes always parallel to lines which are at rest with respect to the average positions of the fixed stars, that is, with respect to an inertial frame; see, for example, ref. 2, ch. I, or ref. 3, ch. X.

gravitational field, and so forth. Let $\vec{\rho}$ denote the position vector of the target's center of mass. Then,

$$\vec{\rho} = \vec{r} - \vec{a} \tag{2}$$

where $\vec{a}$ is the position vector of the attack vehicle relative to the target.

Using equation (2) to replace $\vec{r}$ in equation (1) leads to

$$\frac{d^2\vec{a}}{dt^2} - \frac{1}{m}\vec{F} + \frac{GM}{r^3}\vec{a} = -\frac{d^2\vec{\rho}}{dt^2} + \frac{GM}{r^3}\vec{\rho} + \frac{1}{m}\vec{F}_p \tag{3}$$

This differential equation is somewhat deceiving. The first two terms on the right side are both large in absolute value, but for $a \ll \rho$ (the most important case), the magnitude of their sum is comparatively small. Hence, for calculations, it is desirable to replace the entire sum on the right side of equation (3) by its approximate value derived as follows.

Since, by equation (2),

$$r = \left|\vec{\rho} + \vec{a}\right|$$

it follows that

$$r^{-3} = \left(\rho^2 + 2\vec{\rho} \cdot \vec{a} + a^2\right)^{-3/2} \tag{4}$$

Consequently, for $a \ll \rho$, the approximate expression

$$r^{-3} = \rho^{-3}\left(1 - 3\rho^{-2}\vec{\rho} \cdot \vec{a}\right) \tag{5}$$

is obtained by discarding terms of higher order in the series expansion of the right side of equation (4). Another approximation which will be needed is the following. The last term on the right side of equation (3) represents the vector sum of the perturbing accelerations at the position $\vec{r}$ of the attack vehicle and at time $t$. For $a \ll \rho$, it will be sufficiently accurate for present purposes to employ the approximation

$$\frac{1}{m}\vec{F}_p = \frac{1}{m'}\vec{F}_p' \tag{6}$$

where the primed quantities refer to the target (position $\vec{\rho}$). Since the exact equation of motion of the target's center of mass is

$$\frac{d^2\vec{\rho}}{dt^2} = -\frac{GM}{\rho^3}\vec{\rho} + \frac{1}{m'}\vec{F}_p' \tag{7}$$

equations (5), (6), and (7) permit approximating the vector sum on the right side of equation (3) according to

$$-\frac{d^2\vec{\rho}}{dt^2} - \frac{GM}{r^3}\vec{\rho} + \frac{1}{m}\vec{F}_p = 3\frac{GM}{\rho^3}\frac{\vec{\rho}\cdot\vec{a}}{\rho^2}\vec{\rho} \tag{8}$$

In view of the order of the approximation in equation (8), the third term on the left side of equation (3) is adequately approximated by

$$\frac{GM}{r^3}\vec{a} = \frac{GM}{\rho^3}\vec{a} \tag{9}$$

(Recall eq. (5).) With the aid of the expressions (8) and (9), equation (3) can be reduced to the approximate form

$$\frac{d^2\vec{a}}{dt^2} = \frac{1}{m}\vec{F} - \frac{GM}{\rho^3}\left(\vec{a} - 3\frac{\vec{\rho}\cdot\vec{a}}{\rho^2}\vec{\rho}\right) \tag{10}$$

Equation (10) is, then, the vector equation of relative motion which applies accurately when the attack vehicle is near the target. Even in the case of zero thrust, the task of finding the general solution of equation (10) appears to be extremely difficult if for no other reason than that $\rho$ cannot be expressed as a closed-form function of time when the target's orbit is noncircular.

## APPARENT GRAVITATIONAL ACCELERATION

It is of interest to consider the effect of the gravitational term in equation (10) for extreme cases:

For case 1,

$$\vec{\rho}\cdot\vec{a} = 0 \qquad \left(\vec{a} \text{ perpendicular to } \vec{\rho}\right)$$

In this case equation (10) can be written

$$\frac{d^2\vec{a}}{dt^2} = \frac{1}{m}\vec{F} - \frac{GM}{\rho^3}\vec{a}$$

That is, the effect of gravity is to tend to reduce the distance $a$ separating the attack vehicle and the target; moreover, this tendency

approaches zero as the separation $a$ approaches zero. The analogy to the linear restoring force in the case of simple harmonic motion may also be noted.

For cases 2,

$$\vec{\rho} \cdot \vec{a} = \pm\rho a \qquad \left(\vec{a} \begin{Bmatrix} \text{same direction, same sense} \\ \text{same direction, opposite sense} \end{Bmatrix} \text{ as } \vec{\rho}\right)$$

In these cases,

$$\vec{\rho} = \pm\rho \frac{\vec{a}}{a}$$

and

$$\frac{\vec{\rho} \cdot \vec{a}}{\rho^2} \vec{\rho} = \pm \frac{\rho a}{\rho^2}\left(\pm\rho \frac{\vec{a}}{a}\right) = \vec{a}$$

Equation (10) then reduces to

$$\frac{d^2\vec{a}}{dt^2} = \frac{1}{m} \vec{F} + 2 \frac{GM}{\rho^3} \vec{a}$$

That is, the effect of gravity is to tend to increase the distance $a$ separating the attack vehicle and the target; this tendency approaches zero as the separation $a$ approaches zero.

Let $\vec{\Gamma}$ denote the gravitational acceleration vector in equation (10). In the general case its magnitude is

$$\Gamma = \left| -\frac{GM}{\rho^3}\left(\vec{a} - 3 \frac{\vec{\rho} \cdot \vec{a}}{\rho^2} \vec{\rho}\right)\right|$$

$$= \frac{GM}{\rho^2} \frac{a}{\rho} \sqrt{1 + 3\left(\frac{\vec{\rho} \cdot \vec{a}}{\rho a}\right)^2} \tag{11}$$

Consequently,

$$\frac{GM}{\rho^2} \frac{a}{\rho} \leqq \Gamma \leqq 2 \frac{GM}{\rho^2} \frac{a}{\rho} \tag{12}$$

which shows the range of the magnitude of the gravitational acceleration in equation (10); moreover, it shows that case 1 and cases 2 are extremes.

The inequality on the right in expression (12) shows that when the attack vehicle is within a certain distance, say $a_o$, of the target, the magnitude $\Gamma$ of the gravitational acceleration term in the equation of relative motion (10) is less than $2\left(GM/\rho^2\right)a_o/\rho$. This upper bound on $\Gamma$ is a particularly useful check when a simplified analysis is performed with the effect of gravity neglected.

The expression for the vector $\vec{\Gamma}$ shows that it lies in the plane of the two vectors $\vec{a}$ and $\vec{\rho}$. (The special cases in which $\vec{\Gamma}$ and $\vec{a}$ are collinear are case 1 and cases 2, discussed previously.) Having obtained the expression (11) for the magnitude of $\vec{\Gamma}$, it is desirable to have expressions for the angles $\theta_{a\Gamma}$ and $\theta_{\rho\Gamma}$ which $\vec{\Gamma}$ makes with $\vec{a}$ and $\vec{\rho}$, respectively. Since

$$\frac{\vec{\rho} \cdot \vec{a}}{\rho a} = \cos \theta_{a\rho}$$

where $\theta_{a\rho}$ is the angle which $\vec{a}$ makes with $\vec{\rho}$, it follows that

$$\cos \theta_{a\Gamma} = \frac{\vec{a} \cdot \vec{\Gamma}}{a\Gamma} = \frac{3 \cos^2\theta_{a\rho} - 1}{\sqrt{1 + 3 \cos^2\theta_{a\rho}}} \tag{13}$$

and

$$\cos \theta_{\rho\Gamma} = \frac{\vec{\rho} \cdot \vec{\Gamma}}{\rho\Gamma} = \frac{2 \cos \theta_{a\rho}}{\sqrt{1 + 3 \cos^2\theta_{a\rho}}} \tag{14}$$

Equation (13) shows, for example, that $\vec{\Gamma}$ is perpendicular to $\vec{a}$ for angles $\theta_{a\rho} = \cos^{-1}\left(\pm\sqrt{1/3}\right)$, that is, $54^\circ 44'$ and $125^\circ 16'$. For both cases, the value of $\Gamma$ is, by equation (11), $\sqrt{2}\left(GM/\rho^2\right)a/\rho$. The corresponding angles $\theta_{\rho\Gamma}$ are, by equation (14), acute and obtuse, respectively. These cases, together with cases 1 and 2 described previously, are illustrated schematically in figure 2. The acceleration field shown in figure 2 may be thought of as being rigidly attached to the head of the changing vector $\vec{\rho}$; the intensity of this attached field varies, of course, as $1/\rho^3$, as well as varying directly as the distance $a$. Even for a target in a circular orbit, in which case $\rho = \left|\vec{\rho}\right| =$ Constant, the acceleration field shown in figure 2 rotates about the target in any coordinate

system whose origin is at the target and whose axes remain parallel to lines fixed in an inertial frame (cf. discussion that follows regarding proper representation of $\vec{a}$). The preceding theoretical discussion is amplified in part by figures 3 to 6. Figures 7 and 8 have been added to facilitate numerical estimates. The figures are self-explanatory.

In the application of equation (10) it must be realized that any coordinate system to which the position vector $\vec{a}$ may properly be referred must not only have its origin at the target's center of mass, but also have its axes always parallel to lines fixed in an inertial frame. This is necessary in order to be consistent with equations (1), (2), and (3) and the original definition of the vector $\vec{r}$. If $\vec{r}$ had been referred to a coordinate system fixed in the earth, then terms corresponding to the centrifugal and Coriolis accelerations would have entered equation (1) because of the earth's rotation; the subsequent analysis would have been slightly more complicated. Equation (10) applies to the terminal phase of interception; consequently, the more complicated analysis is clearly undesirable when the tracking and guidance mechanisms for the terminal phase are totally contained in the attack vehicle.

The apparent gravitational acceleration $\vec{\Gamma}$ is mathematically identical with a tide-generating acceleration and with the disturbing effect of a third body in celestial mechanics. This and some related matters are discussed in the appendix of this paper.

## TERMINAL GUIDANCE

Guidance in the terminal phase of flight must be based at least upon the vector $\vec{a}$ or its variation with time, or both. A method which will be considered here is somewhat similar to the familiar proportional navigation course (see fig. 9) and is defined by the equation

$$\vec{F} = -c \frac{d}{dt}\left(\frac{1}{a}\vec{a}\right) \qquad (c > 0) \qquad (15)$$

where $c$ is the correction coefficient. Equation (15) states that the thrust $\vec{F}$ applied is to be perpendicular to $\vec{a}$, in the direction tending to reduce the rate at which $\vec{a}$ is turning, and proportional to the angular rate at which $\vec{a}$ is turning. For this method of interception, equations (10) and (15) give

$$\frac{d^2\vec{a}}{dt^2} + \frac{c}{m}\frac{d}{dt}\left(\frac{\vec{a}}{a}\right) + \frac{GM}{\rho^3}\left(\vec{a} - 3\frac{\vec{\rho}\cdot\vec{a}}{\rho^2}\vec{\rho}\right) = 0 \qquad (16)$$

as the equation of motion of the attack vehicle relative to the target in the terminal phase of flight.

If, because of rocket operation, the mass $m$ of the attack vehicle decreases appreciably during the terminal phase of flight, an additional equation must be solved simultaneously with equation (16). Let the exit speed of the propulsive exhaust gases, relative to the attack vehicle, be denoted by $v_{ex}$; then

$$\frac{dm}{dt} = -\frac{F}{v_{ex}} \tag{17}$$

where $F$ is the magnitude of the thrust vector $\vec{F}$. The additional equation required follows from equations (15) and (17); that is,

$$\frac{dm}{dt} = -\frac{c}{v_{ex}}\left|\frac{d}{dt}\left(\frac{\vec{a}}{a}\right)\right| \tag{18}$$

This must be solved simultaneously with equation (16). If the correction coefficient $c$ is a function of one or more variables, another equation expressing this dependence must be included. However, cases in which both $m$ and $c$ are practically constant are clearly of first importance and consideration will be restricted to these in the following discussion.

## SIMPLIFIED ANALYSIS NEGLECTING GRAVITY

As in the case of equation (10), the gravitational term in equation (16) will generally make it necessary to resort to numerical methods of integration in precise calculations. Fortunately, however, the gravitational term approaches zero as the attack vehicle approaches the target; consequently, much can be learned by examining the simpler differential equation

$$\frac{d^2\vec{a}}{dt^2} + \frac{c}{m}\frac{d}{dt}\frac{\vec{a}}{a} = 0 \tag{19}$$

which results upon setting $\vec{\Gamma}$ equal to zero. Integrating equation (19) once gives

$$\frac{d\vec{a}}{dt} + \frac{c}{m}\frac{\vec{a}}{a} = \left(\frac{d\vec{a}}{dt}\right)_1 + \frac{c}{m}\left(\frac{\vec{a}}{a}\right)_1 \tag{20}$$

where the subscript 1 refers to values at the start of the terminal phase of flight. Since the vector on the right of equation (20) remains

constant in magnitude and direction, and since the vector

$$\frac{c}{m}\frac{\vec{a}}{a} \qquad (c > 0)$$

has the constant magnitude $c/m$, the attack vehicle is assured of hitting the target provided

$$\frac{c}{m} > \left|\left(\frac{d\vec{a}}{dt}\right)_1 + \frac{c}{m}\left(\frac{\vec{a}}{a}\right)_1\right| \tag{21}$$

which is easily seen from the vector diagram (fig. 10) corresponding to equation (20). Let $v_1$ denote the magnitude of the velocity vector $\left(d\vec{a}/dt\right)_1$ and let $\theta_1$ be the angle which the latter makes with the negative of the vector $\vec{a}$. Then the inequality (21) is equivalent to

$$\left(\frac{c}{m}\right)^2 > \left(\frac{c}{m} - v_1 \cos \theta_1\right)^2 + \left(v_1 \sin \theta_1\right)^2$$

or

$$\frac{c}{m} > \frac{1}{2}\frac{v_1}{\cos \theta_1} \qquad \left(-\frac{\pi}{2} < \theta_1 < \frac{\pi}{2}\right) \tag{22}$$

(The inequality would be reversed for $\cos \theta_1 < 0$, corresponding to $c < 0$, but this does not result in a hit.) This, then, is a sufficient condition for the attack vehicle to hit the target when gravity can be neglected and when the method of terminal-phase flight is that described analytically by equation (15).

The relative motion corresponding to equation (19), or the equivalent equation (20), is confined to the plane of the vectors $\left(d\vec{a}/dt\right)_1$ and $\left(\vec{a}/a\right)_1$ or, in the trivial case for which these are collinear, the motion is confined to the line of the unit vector $(\vec{a}/a)_1$. Since, in accordance with equation (15), the thrust has zero component along $\vec{a}$, it follows from analytical mechanics (see, for example, ref. 2, ch. I, sec. 19, or ref. 3, ch. V, sec. 1) that the radial component of the acceleration vanishes; that is,

$$\frac{d^2a}{dt^2} - a\left|\frac{d}{dt}\frac{\vec{a}}{a}\right|^2 = 0 \tag{23}$$

Since $a \gtreqless 0$, equation (23) yields the relation

$$\frac{d}{dt}\left(-\frac{da}{dt}\right) \leqq 0 \tag{24}$$

The equality applies in the trivial case of straight-line motion. Assume now that relation (22) is satisfied, so that a hit will occur. Then, relation (24) shows that the magnitude $-da/dt$ of the radial velocity is nonincreasing with time. This result is used in the following analysis of the variation of thrust magnitude during the terminal phase of flight.

The identity

$$\frac{d}{dt}\frac{\vec{a}}{a} = \frac{1}{a}\left(\frac{d\vec{a}}{dt} - \frac{da}{dt}\frac{\vec{a}}{a}\right)$$

and equation (19) yield

$$\frac{d^2}{dt^2}\frac{\vec{a}}{a} = -\frac{1}{a}\left(\frac{c}{m} + 2\frac{da}{dt}\right)\frac{d}{dt}\frac{\vec{a}}{a} - \frac{1}{a}\frac{d^2a}{dt^2}\frac{\vec{a}}{a} \tag{25}$$

The right side of equation (25) is the sum of two perpendicular vectors; consequently,

$$\begin{aligned}
\frac{d}{dt}\left|\frac{d}{dt}\frac{\vec{a}}{a}\right| &= \frac{d}{dt}\left[\left(\frac{d}{dt}\frac{\vec{a}}{a}\right)\cdot\left(\frac{d}{dt}\frac{\vec{a}}{a}\right)\right]^{1/2} \\
&= \frac{\left(\frac{d}{dt}\frac{\vec{a}}{a}\right)\cdot\left(\frac{d^2}{dt^2}\frac{\vec{a}}{a}\right)}{\left|\frac{d}{dt}\frac{\vec{a}}{a}\right|} \\
&= -\frac{1}{a}\left(\frac{c}{m} + 2\frac{da}{dt}\right)\left|\frac{d}{dt}\frac{\vec{a}}{a}\right|
\end{aligned}$$

Therefore, by using equation (15),

$$\frac{dF}{dt} = -\frac{1}{a}\left(\frac{c}{m} + 2\frac{da}{dt}\right)F \tag{26}$$

is obtained for the time rate of change of the thrust magnitude $F$. Assume, as in the discussion following relation (24), that relation (22) is satisfied. Then, if the relation

$$\frac{c}{m} \geqq 2v_1 \cos\theta_1 \tag{27}$$

is also satisfied, equation (26) shows that the thrust magnitude $F$ will not increase during the terminal phase of flight.

Thus, if $c/m$ satisfies the relations (22) and (27), the maximum thrust required of the attack vehicle in the terminal phase occurs at the start, and its value $F_1$ is given by

$$\frac{F_1}{m} = \frac{c}{m} \frac{v_1 \left|\sin \theta_1\right|}{a_1} \tag{28}$$

If $\theta_1 \leqq 75^\circ$, relations (22) and (27) are satisfied by $c/m = \sigma v_1$ where $\sigma \geqq 2$, in which case equation (28) becomes

$$\frac{F_1}{m} = \sigma \frac{{v_1}^2}{a_1} \left|\sin \theta_1\right| \tag{29}$$

## NUMERICAL EXAMPLE FOR TYPE I INTERCEPTION

Consider a type I vehicle for which the terminal phase of flight starts at a range of 50,000 to 100,000 feet with a relative speed of 500 to 1,000 ft/sec and a heading error $\theta_1 \leqq 20^\circ$. Assume that $\sigma = 2$ $\left(\text{that is, } c/m = 2v_1\right)$. The maximum required thrust $F_1$ given by equation (29) is

$$\frac{F_1}{m} = 2 \frac{1000^2}{50000} \sin 20^\circ = 13.7 \text{ ft/sec}^2$$

If the terminal phase of flight starts at time $t_1$ and the target is hit at time $t_2$, then (approximately)

$$t_2 - t_1 = \frac{a_1}{-\left(\frac{da}{dt}\right)_1} = \frac{a_1}{v_1 \cos \theta_1} \tag{30}$$

In the numerical example just considered, therefore, the maximum time possible in the terminal phase is approximately

$$t_2 - t_1 = \frac{100000}{500 \cos 20^\circ} = 213 \text{ sec}$$

In this time interval a target moving at 25,000 ft/sec will have traveled

$$(213)(25,000) = 5.33 \times 10^6 \text{ ft} = 1.01 \times 10^3 \text{ miles}$$

This again is maximum for the numerical example. The corresponding maximum fractional rate at which mass is expelled in propulsion of the attack vehicle is

$$-\frac{1}{m}\frac{dm}{dt} = \frac{F_1/m}{v_{ex}} = \frac{13.7}{10000} = 1.37 \times 10^{-3} \text{ sec}^{-1} \tag{31}$$

where the exit speed $v_{ex}$ of the propellant gases has been taken as 10,000 ft/sec. Thus, a substantial mass loss may occur in the extreme case. Finally, relations (12) show that the magnitude $\Gamma$ of the apparent gravitational acceleration experienced by the attack vehicle during the terminal phase lies in the range

$$0 \leqq \Gamma \leqq 2(32.2)\ \frac{20}{4000} \text{ ft/sec}^2$$

or

$$0 \leqq \Gamma \leqq 0.3 \text{ ft/sec}^2$$

Thus, the maximum value of $\Gamma$ is fairly small in comparison with the maximum value $F_1/m = 13.7$ ft/sec$^2$ and can be neglected in the present approximate calculation.

Although the numerical example examined indicates that the application of an interception system of type I has attendant difficulties, considerable latitude exists for the adjustment of design parameters and none of the problems appears to be extreme in nature. Consequently, this type of interception system is considered feasible.

## NUMERICAL EXAMPLE FOR TYPE II INTERCEPTION

For interception of type II the relative speed $v_1$ is of the order of 50,000 ft/sec. If $\sigma$ is taken as 2 (as before), equation (29) shows that reasonable values of $F_1/m$ imply large separation $a_1$ and very small heading error $\theta_1$ at the start of the type II terminal phase.

For example, the values $a_1 = 500{,}000$ ft and $\theta_1 = 1^\circ$ give $F_1/m = 175$ ft/sec$^2$, $\Gamma < 1.6$ ft/sec$^2$, and $t_2 - t_1 = 10$ sec (approximately). If it is assumed further that $v_{ex} = 10{,}000$ ft/sec, the maximum fractional rate at which mass is expelled in propulsion becomes $0.0175$ sec$^{-1}$.

These numerical magnitudes serve mainly to indicate that the demands placed upon the type II interception system are severe. Even for a system which makes extensive use of ground equipment, the probability of a direct hit in this case appears small. A distance of nearest approach (miss distance) of the order of 1,000 feet will be assumed to be attainable in head-on interception. Furthermore, the timed dispersal of all or a part of the mass of the attack vehicle will be considered as a means of increasing the probability of a destructive hit to a value which is effectively unity.

## MASS DISPERSAL

The dispersal of mass can be initiated by an explosion or some other means, which in the simplest case amounts to giving an impulse to each pellet or piece of matter ejected.

For simplicity, assume that at time $t$ the mass $m$ is distributed uniformly over the surface of a sphere of radius

$$b = w_b\left(t - t_i\right) \tag{32}$$

where $w_b$ is the radial speed of each piece of matter with respect to the center of mass (center of sphere) and $t_i$ is the time at which dispersal is initiated. Let $t_2$ denote the time at which the target arrives at a point adjacent to the center of the sphere. At time $t_2$ the average surface density of mass on the sphere's surface is

$$\frac{m}{4\pi b_2^{\,2}} = \frac{m}{4\pi\left[w_b\left(t_2 - t_i\right)\right]^2}$$

Hence, the mass $m_s$ which strikes the target is such that

$$m_s \geqq 2A_f \frac{m}{4\pi b_2^{\,2}}$$

where $A_f$ is the target's frontal area. If $m_{md}$ is the minimum destructive mass, then the probability of destruction is a maximum when

$$m_{md} = \frac{A_f m}{2\pi b_2^2} \tag{33}$$

that is,

$$b_2 = \sqrt{\frac{A_f}{2\pi} \frac{m}{m_{md}}}$$

Equation (33) has the following obvious and more important interpretation, however. Suppose the accuracy of the interception system is such that the center of mass of the attack vehicle is assuredly within a distance $\epsilon$ of the target's flight path at time $t_2$. Then, if $\epsilon$ is substituted for $b_2$ in equation (33), the minimum mass $m$ required to be dispersed is

$$m = m_{md} \frac{2\pi\epsilon^2}{A_f} \tag{34}$$

Thus, the dispersal speed $w_b$ and time interval $t_2 - t_1$ are to be adjusted in accordance with

$$b_2 = \epsilon = w_b\left(t_2 - t_1\right)$$

If the attack vehicle is approaching the target at a relative speed $v_1$ at time $t_1$ and if they are separated by the distance $a_1$ at this instant, the distance at which dispersal is initiated is approximately

$$a_1 = \left(t_2 - t_1\right)v_1 = \frac{v_1\epsilon}{w_b}$$

This distance will normally be held to a minimum so as to permit controlled flight of the attack vehicle for the longest possible time.

The applicability of this type of interception is difficult to assess inasmuch as the minimum destructive mass depends upon the nature of the target, while the maximum error $\epsilon$ depends upon the effectiveness of both the attack vehicle's control system and the ground equipment.

## NUMERICAL EXAMPLE FOR MASS DISPERSAL

In considering a numerical example it is convenient to rewrite equation (34) in the form

$$\frac{m}{m'} = \frac{m_{md}}{m'} \frac{2\pi\epsilon^2}{A_f}$$

where $m'$ is the target's mass. In view of the high relative speed, say $v_i = 50{,}000$ ft/sec, the ratio $m_{md}/m'$ will be taken as $10^{-5}$. Assume further that $A_f = 100$ ft$^2$ and that $\epsilon = 1{,}000$ feet. Then, for these values of the parameters,

$$\frac{m}{m'} = 10^{-5} \frac{6 \times 10^6}{10^2} = 0.6$$

That is, the mass required to be dispersed is of the order of the mass of the target.

Langley Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Langley Field, Va., July 23, 1958.

# APPENDIX

## RELATION OF APPARENT GRAVITATIONAL ACCELERATION TO TIDE-GENERATING ACCELERATION AND TO DISTURBING EFFECT OF A THIRD BODY

The apparent gravitational acceleration $\vec{\Gamma}$ experienced by the attack vehicle relative to a target moving in the earth's gravitational field is mathematically identical with the tide-generating acceleration field produced in the earth's oceans by the moon's gravitational field. Superimposed upon the tidal acceleration field caused by the moon is a similar but somewhat less intense field due to the sun. Each of these tide-generating accelerations has the approximate form (cf. eq. (10))

$$\vec{\Gamma} = -\frac{GMa}{\rho^3}\left[\frac{\vec{a}}{a} - 3\left(\frac{\vec{\rho}}{\rho} \cdot \frac{\vec{a}}{a}\right)\frac{\vec{\rho}}{\rho}\right] \tag{A1}$$

where $M$ now refers to the mass of the tide-producing body (moon or sun), $\vec{a}$ becomes the position vector of the ocean particle relative to the center of the earth, and $\vec{\rho}$ now refers to the position vector of the earth's center relative to the center of the tide-producing body.

The expression (A1) for $\vec{\Gamma}$ will be derived from the approximate potential (see, for example, ref. 4 or ref. 5):

$$U = \frac{3}{2}\frac{GM}{\rho^3}a^2\left(\frac{1}{3} - \cos^2\theta_{a\rho}\right) \tag{A2}$$

which is well-known in tidal theory. For a given value of $\rho$, equation (A2) defines a potential field about the point whose position vector is $\vec{\rho}$. Thus, the independent variables in equation (A2) are $a$ and $\theta_{a\rho}$, while $\rho$ occurs simply as a parameter. The radial variable $a$ and the colatitude $\theta_{a\rho}$ represent two of three possible spherical coordinates; the absence of the longitude angle $\varphi$ in equation (A2) attests to the rotational symmetry of the potential field about the line of $\vec{\rho}$. Hence, in spherical coordinates, the gradient of the potential $U$ is, by equation (A2),

$$\begin{aligned}\operatorname{grad} U &= \left(\frac{\partial U}{\partial a}, \frac{1}{a}\frac{\partial U}{\partial \theta_{a\rho}}, \frac{1}{a \sin\theta_{a\rho}}\frac{\partial U}{\partial \varphi}\right) \\ &= \frac{GMa}{\rho^3}\left(1 - 3\cos^2\theta_{a\rho}, 3\sin\theta_{a\rho}\cos\theta_{a\rho}, 0\right)\end{aligned} \tag{A3}$$

The first or radial component of grad U in equation (A3) is in the direction of $\vec{a}$ and can be expressed as

$$\frac{GMa}{\rho^3}\left(1 - 3\cos^2\theta_{a\rho}\right)\frac{\vec{a}}{a} \tag{A4}$$

The second component, which is in the direction of increasing $\theta_{a\rho}$ (see fig. 3), is perpendicular to $\vec{a}$, but can be expressed as the sum of a vector in the direction of $\vec{a}$ and a vector in the direction of $\vec{\rho}$; that is,

$$\frac{GMa}{\rho^3}\left[\left(3\cos^2\theta_{a\rho}\right)\frac{\vec{a}}{a} - \left(3\cos\theta_{a\rho}\right)\frac{\vec{\rho}}{\rho}\right] \tag{A5}$$

The third component of grad U in equation (A3) vanishes because of the rotational symmetry of the potential field. Adding expressions (A4) and (A5) gives

$$\begin{aligned}\text{grad } U &= \frac{GMa}{\rho^3}\left[\frac{\vec{a}}{a} - \left(3\cos\theta_{a\rho}\right)\frac{\vec{\rho}}{\rho}\right] \\ &= \frac{GMa}{\rho^3}\left[\frac{\vec{a}}{a} - 3\left(\frac{\vec{\rho}}{\rho}\cdot\frac{\vec{a}}{a}\right)\frac{\vec{\rho}}{\rho}\right]\end{aligned} \tag{A6}$$

Consequently, equations (A1) and (A6) show that

$$\vec{\Gamma} = -\text{grad } U$$

which was to be proved.

In celestial mechanics, the "disturbing effect of a third body" has a mathematical form which is identical with that of the tide-generating acceleration and the apparent gravitational acceleration of the present paper (see, for example, ref. 6, pp. 337-342). An example is the sun's disturbing effect on the moon's orbit about the earth. In this case $\vec{\Gamma}$ becomes the disturbing acceleration, M now denotes the mass of the sun, $\vec{a}$ becomes the position vector of the moon's center relative to the earth's center, while $\vec{\rho}$ becomes the position vector of the earth's center relative to the center of the sun.

Thus, equation (A1) represents a first approximation for an acceleration in each of the three cases discussed, that is, terminal motion in interception, tides, and lunar motion. In each application the ratio $a/\rho$ is sufficiently small for the first approximation to be useful.

For a given value of the distance a, equation (A1) shows that the magnitude of the acceleration $\vec{\Gamma}$ is mainly determined by the value of the quantity $GM/\rho^3$, which will be termed the "disturbance parameter." In a given situation, such as the motion of a satellite about the moon, the disturbing effect of the earth can be compared with that of the sun by comparing values of the disturbance parameter for the two cases. This comparison shows that the disturbing effect of the earth on a satellite of the moon is nearly two hundred times as great as the disturbing effect of the sun on the satellite's motion relative to the moon. (This is in marked contrast to the $GM/\rho^2$ values for the two cases; the sun's attraction at the moon's position is more than twice the earth's. However, each attracting body accelerates not only the moon's satellite but also the moon itself; as a result, the disturbance parameter, which is a measure of the net acceleration of the satellite, varies inversely as the cube, rather than the square, of the separating distance $\rho$.) Therefore, successively better approximations for the motion of a satellite relative to the moon would include (in this order): (1) the moon's gravitational field, (2) the disturbing effect of the earth, and (3) the disturbing effect of the sun.

Values of the disturbance parameter are given in table I to facilitate the comparison of disturbing effects within each of five cases. In the first case, with which the present paper is concerned, the three values of the disturbance parameter show that the disturbing effects due to moon and sun are both negligible compared with that of the earth. The second (or earth-satellite) case is similar to the third (tidal) case; the values of the disturbance parameter here reflect the well-known fact that the moon's effect on the tides is somewhat greater than the sun's. The fourth case has already been discussed, while the fifth is roughly analogous to the first case listed.

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TABLE I.- VALUES OF THE DISTURBANCE PARAMETER

| ① Disturbed body | ② In motion near | ③ Disturbing body (mass M) | ④ Approximate distance $\rho$ between ② and ③, miles | ⑤ Disturbance parameter, $GM/\rho^3$, $sec^{-2}$ |
|---|---|---|---|---|
| Attack vehicle | Satellite of earth | Earth | $>4 \times 10^3$ | $<1.5 \times 10^{-6}$ |
| | | Moon | $\approx 2.4 \times 10^5$ | $\approx 8.6 \times 10^{-14}$ |
| | | Sun | $\approx 9.3 \times 10^7$ | $\approx 3.9 \times 10^{-14}$ |
| Satellite of earth | Earth | Moon | $2.4 \times 10^5$ | $8.6 \times 10^{-14}$ |
| | | Sun | $9.3 \times 10^7$ | $3.9 \times 10^{-14}$ |
| Ocean particle | Earth | Moon | $2.4 \times 10^5$ | $8.6 \times 10^{-14}$ |
| | | Sun | $9.3 \times 10^7$ | $3.9 \times 10^{-14}$ |
| Satellite of moon | Moon | Earth | $2.4 \times 10^5$ | $7.0 \times 10^{-12}$ |
| | | Sun | $9.3 \times 10^7$ | $3.9 \times 10^{-14}$ |
| Attack vehicle | Satellite of moon | Moon | $>1.08 \times 10^3$ | $<9.3 \times 10^{-7}$ |
| | | Earth | $\approx 2.4 \times 10^5$ | $\approx 7.0 \times 10^{-12}$ |
| | | Sun | $\approx 9.3 \times 10^7$ | $\approx 3.9 \times 10^{-14}$ |

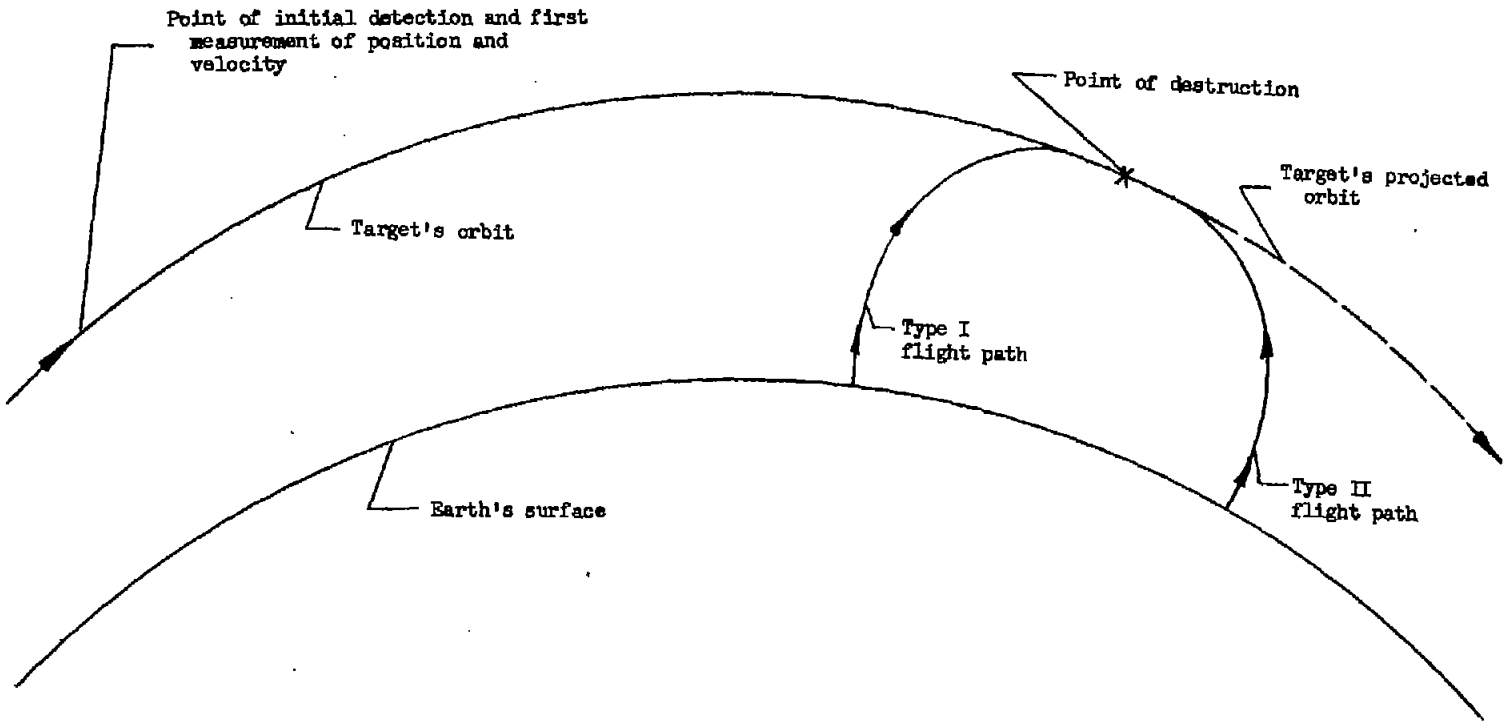


Figure 1.- Schematic diagram showing flight paths of two types of interceptor for the case of launching from the ground.

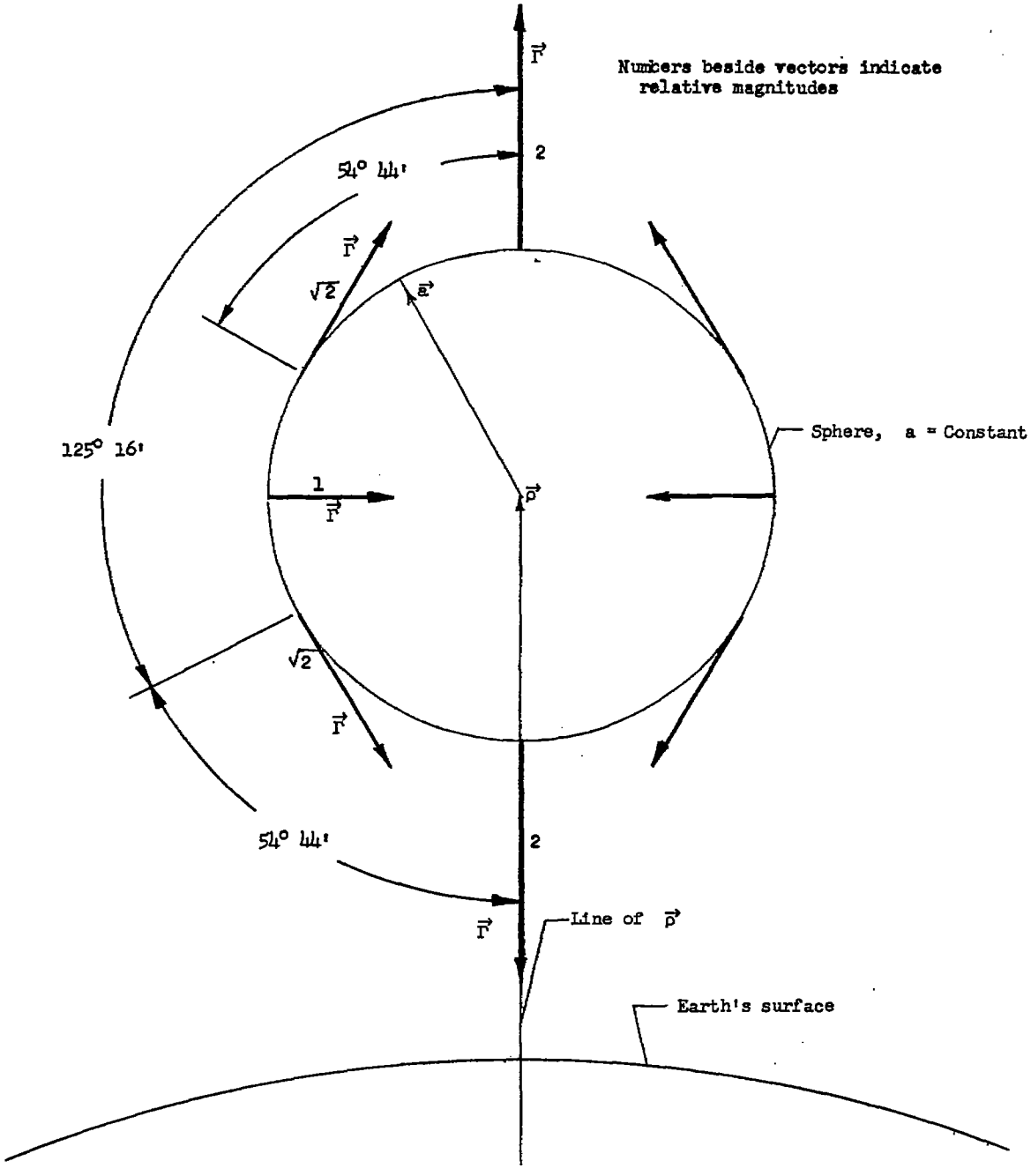


Figure 2.- Apparent gravitational acceleration $\vec{\Gamma}$ experienced by attack vehicle at small distance a from target. (Acceleration field is rotationally symmetric about line of $\vec{\rho}$ and varies in intensity as $a/\rho^3$.)

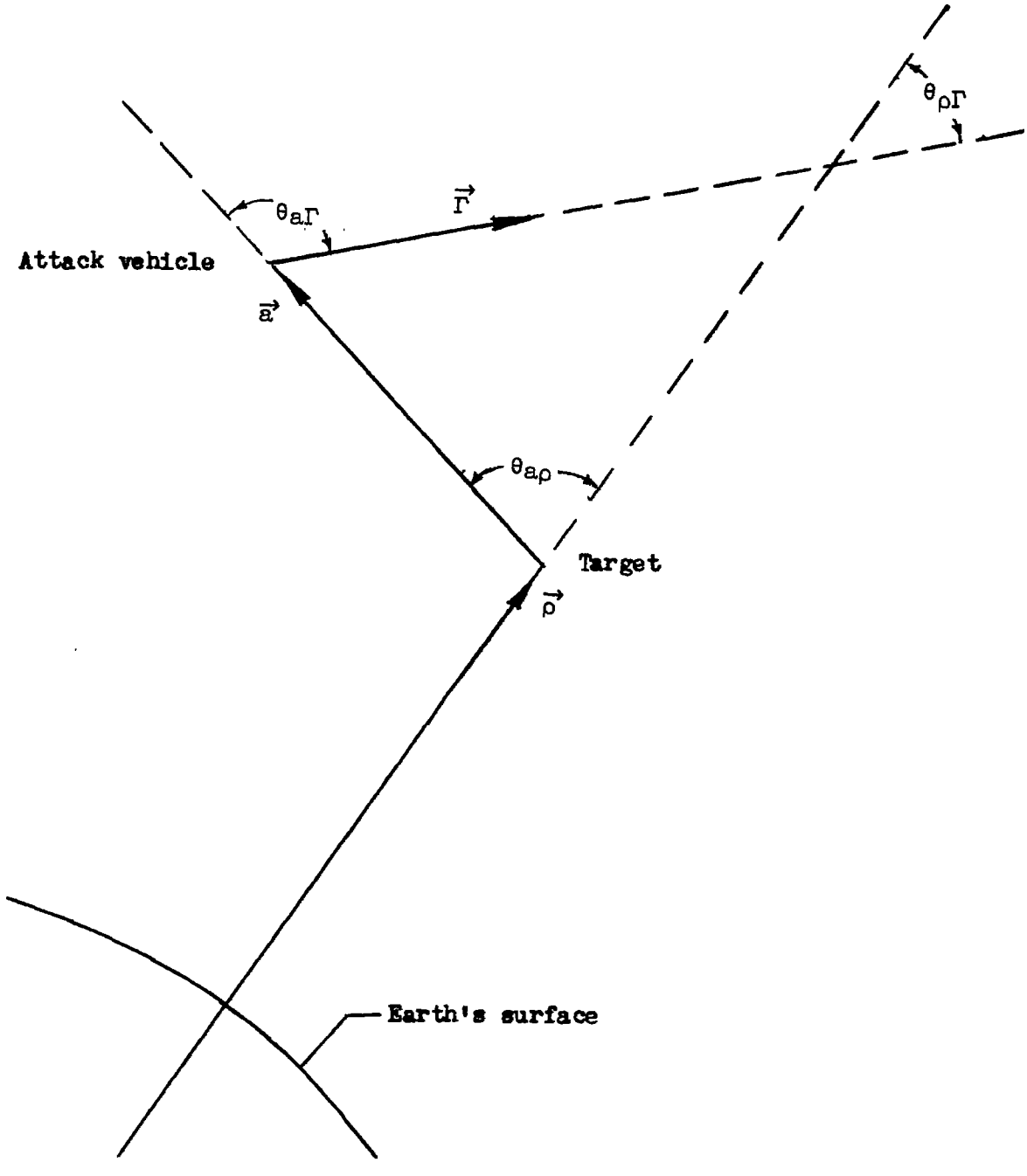


Figure 3.- Schematic relations of angles to positive directions of vectors. For situations arising herein $\theta_{a\rho} + \theta_{\rho\Gamma} = \theta_{a\Gamma}$ when $\theta_{a\rho}$ is acute, and $\theta_{a\rho} + \theta_{\rho\Gamma} = 360^{\circ} - \theta_{a\Gamma}$ when $\theta_{a\rho}$ is obtuse.

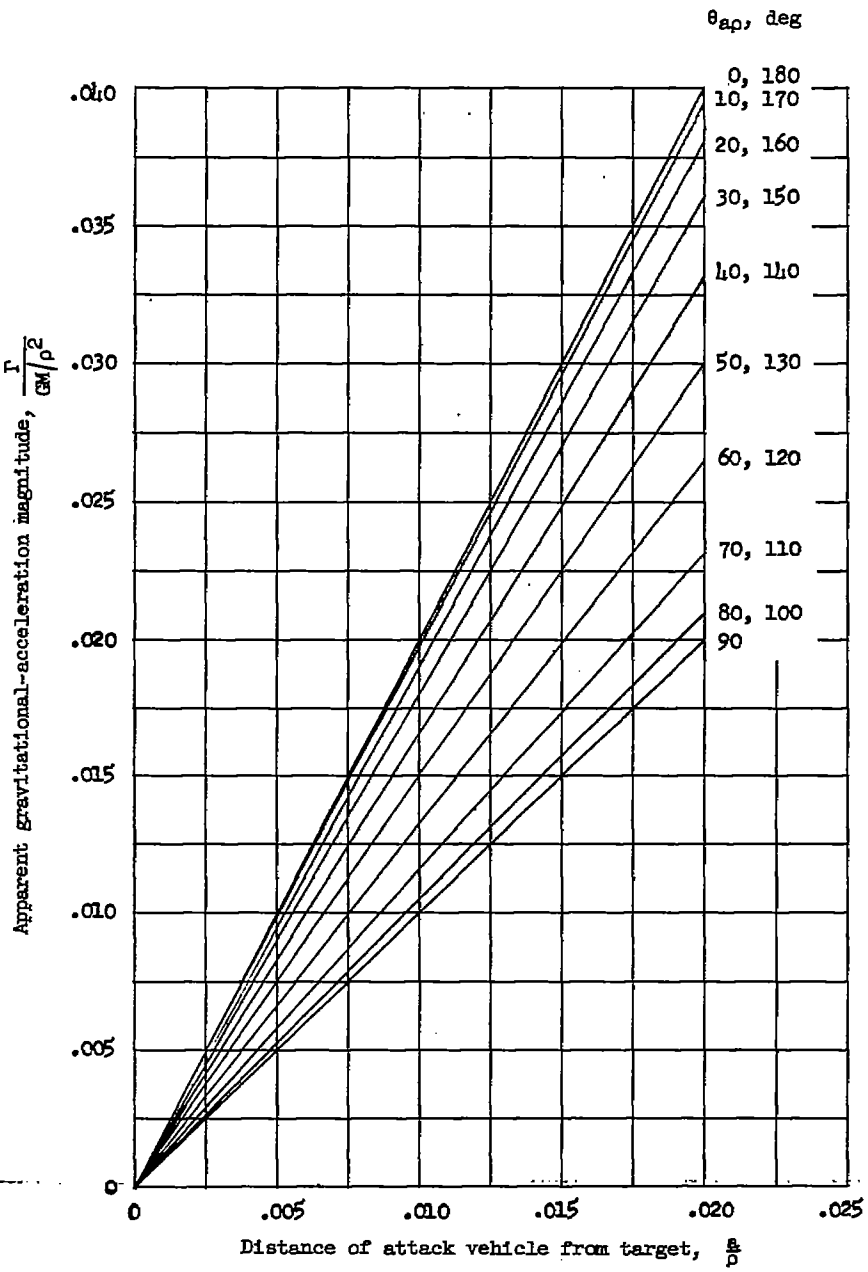


Figure 4.- Dimensionless relation of apparent gravitational-acceleration magnitude to distance from target for various values of $\theta_{a\rho}$. (See eq. (11).)

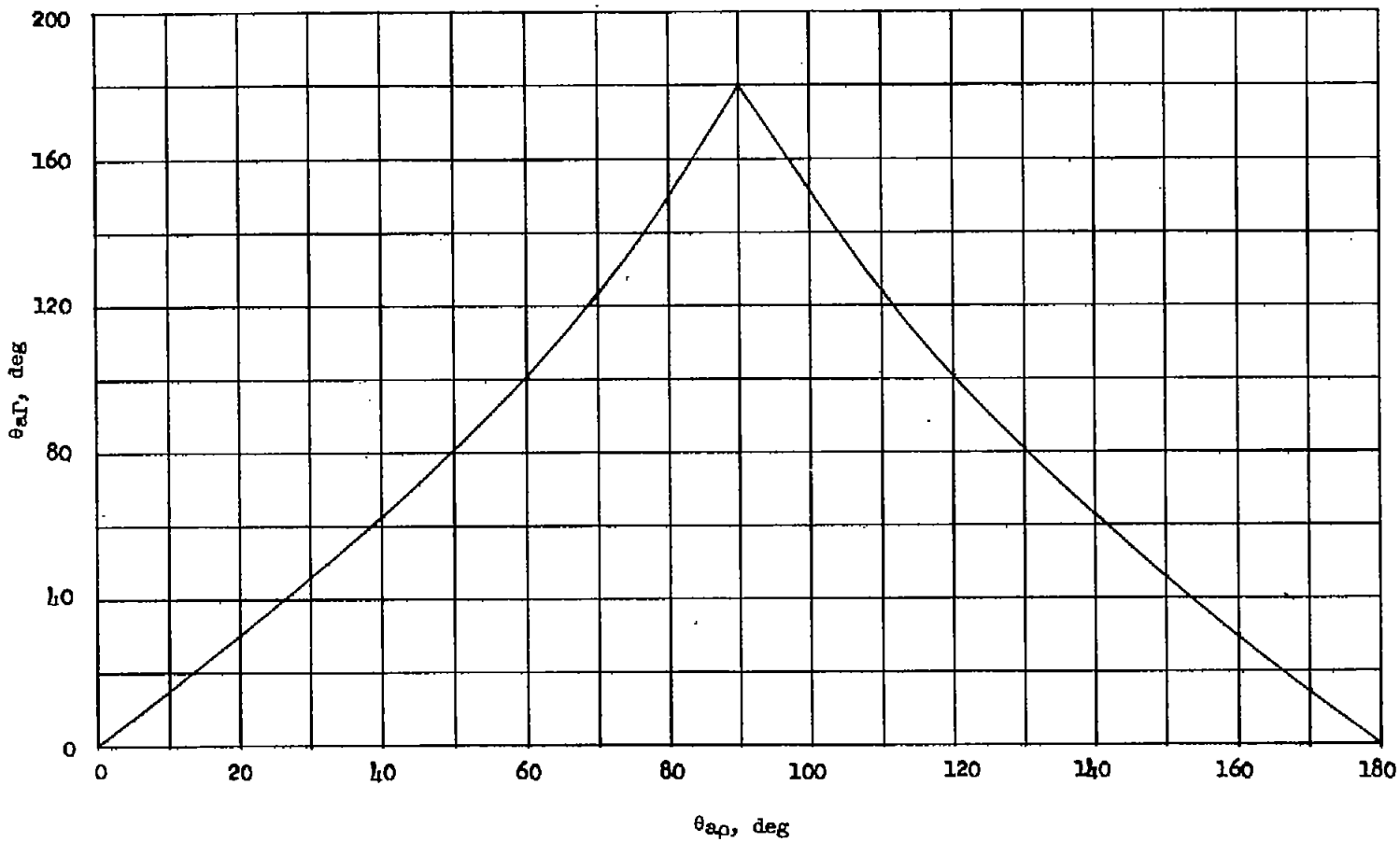


Figure 5.- Relation of angle $\theta_{a\Gamma}$ $\left(\text{which } \vec{\Gamma} \text{ makes with } \vec{a}\right)$ to angle $\theta_{a\rho}$ $\left(\text{which } \vec{a} \text{ makes with } \vec{\rho}\right)$. (See eq. (13).)

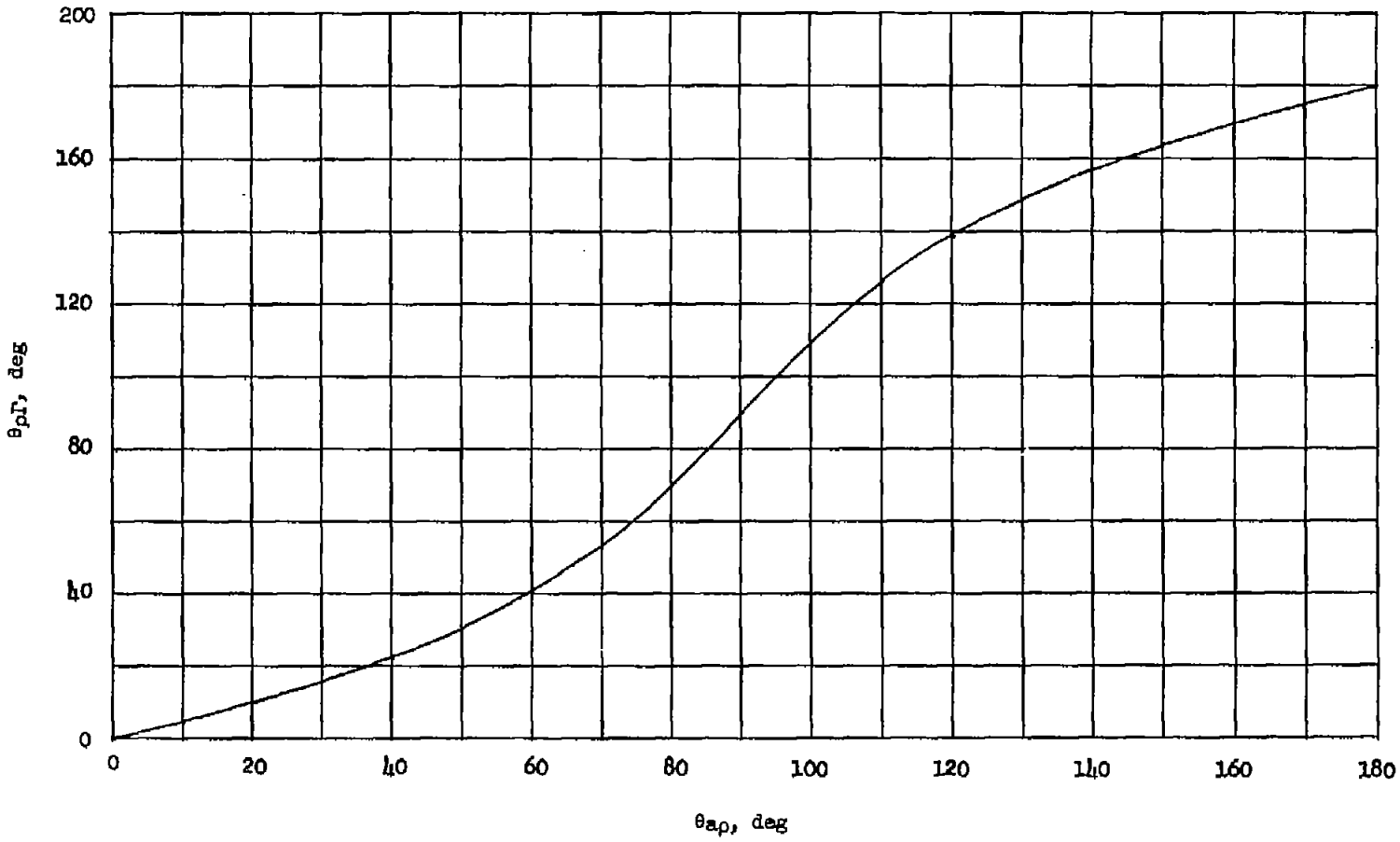


Figure 6.- Relation of angle $\theta_{\rho\Gamma}$ $\left(\text{which } \vec{\Gamma} \text{ makes with } \vec{\rho}\right)$ to angle $\theta_{a\rho}$ $\left(\text{which } \vec{a} \text{ makes with } \vec{\rho}\right)$. (See eq. (14).)

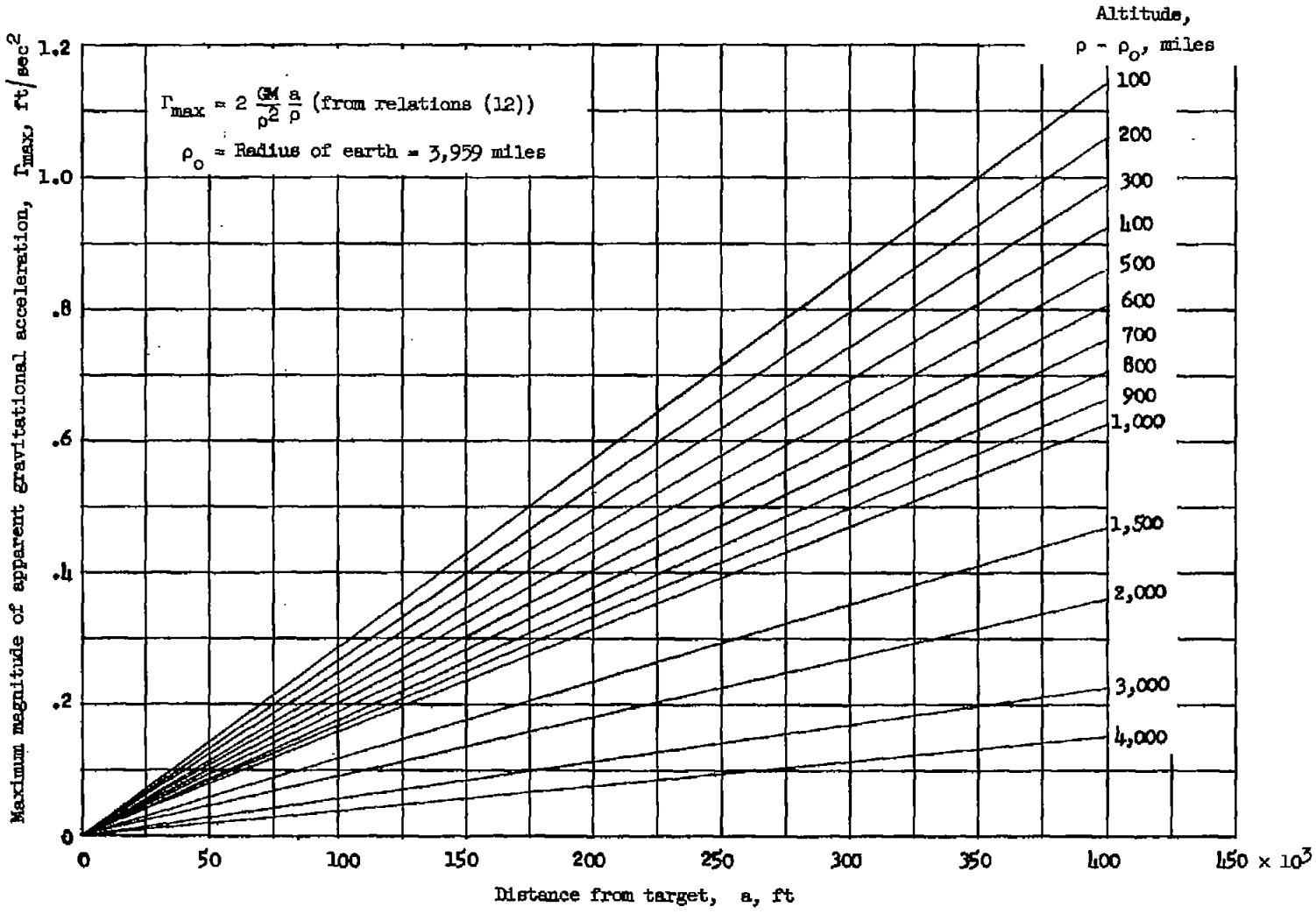


Figure 7.- Relation of upper bound of apparent gravitational-acceleration magnitude to distance from target for various altitudes.

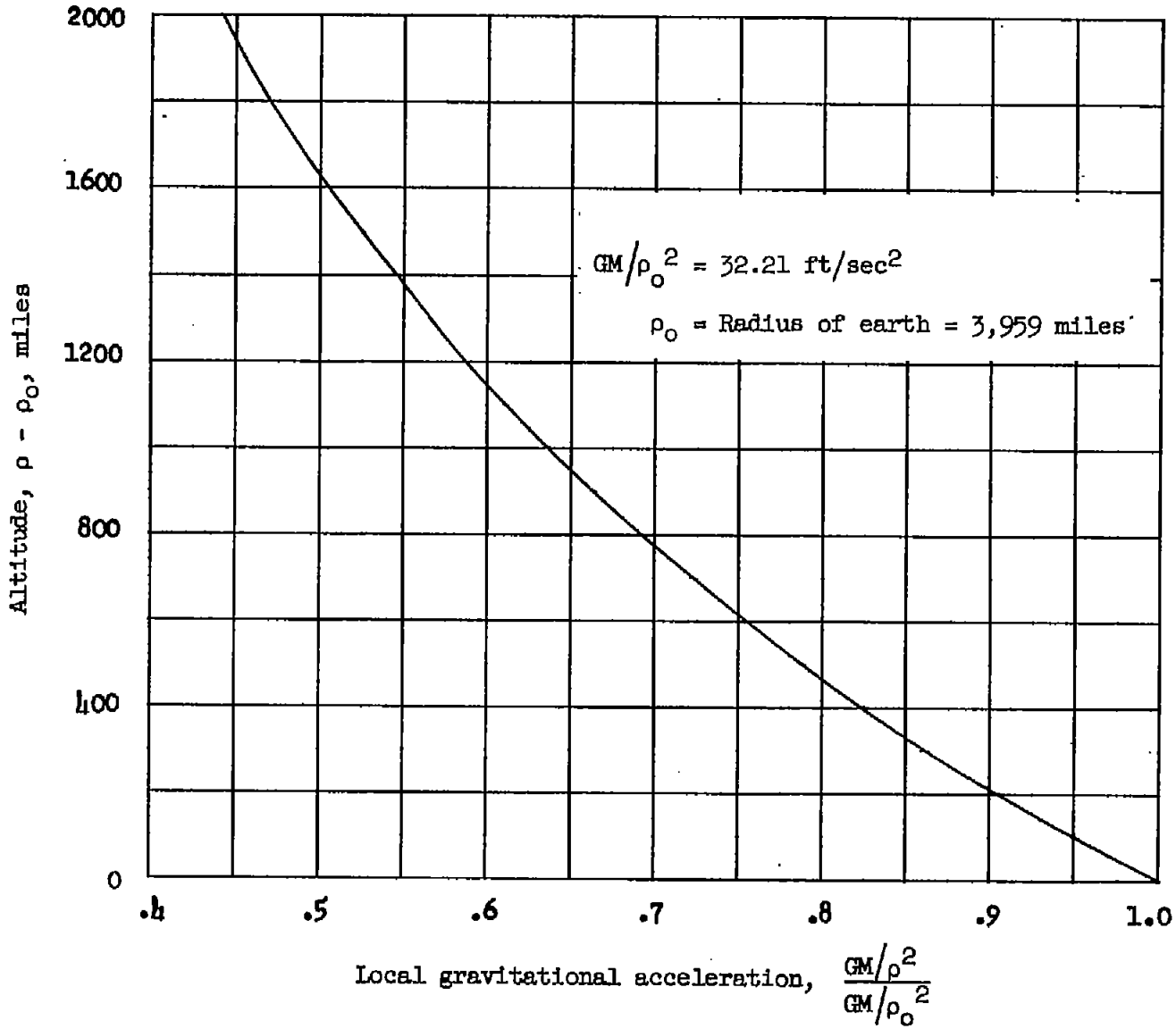


Figure 8.- Relation of local gravitational acceleration (dimensionless) to altitude.

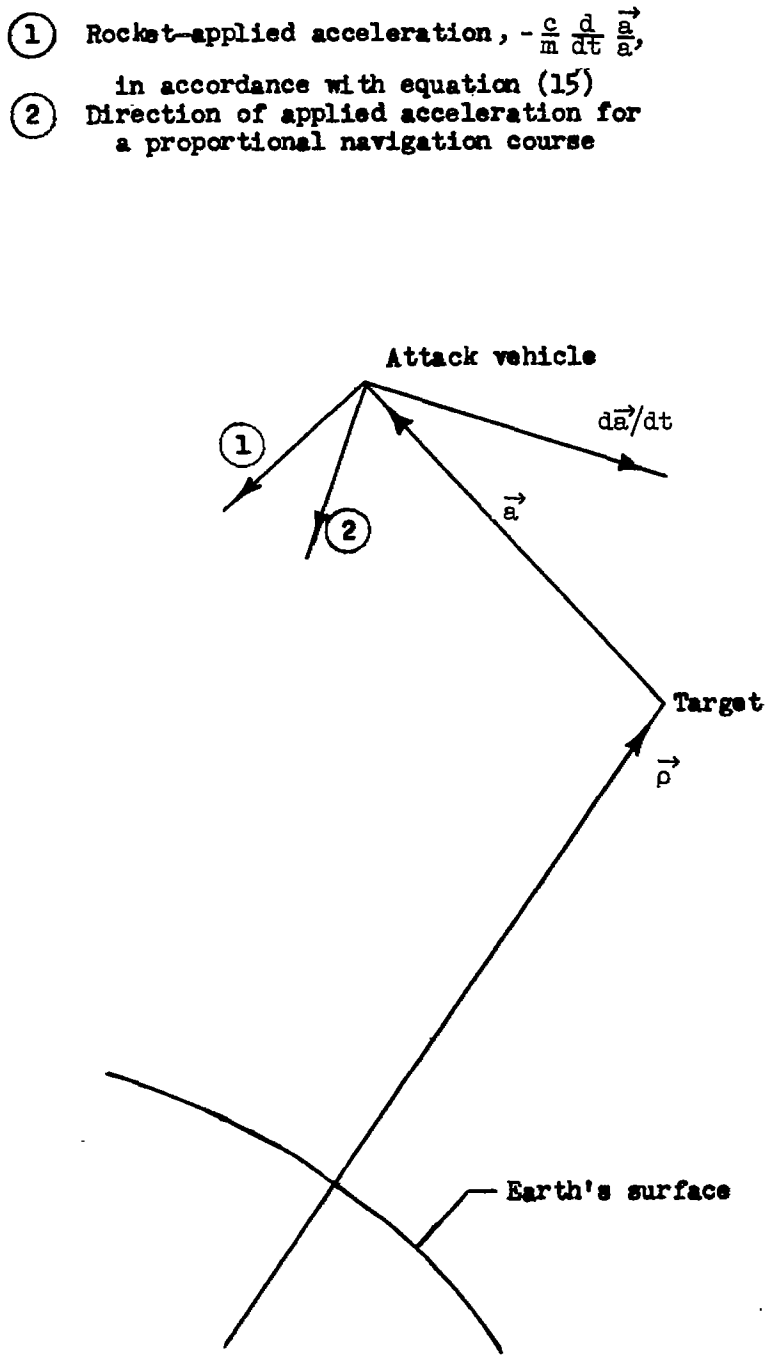


Figure 9.- Schematic comparison of direction of applied acceleration in present case (eq. (15)) with that in the case of a proportional navigation course.

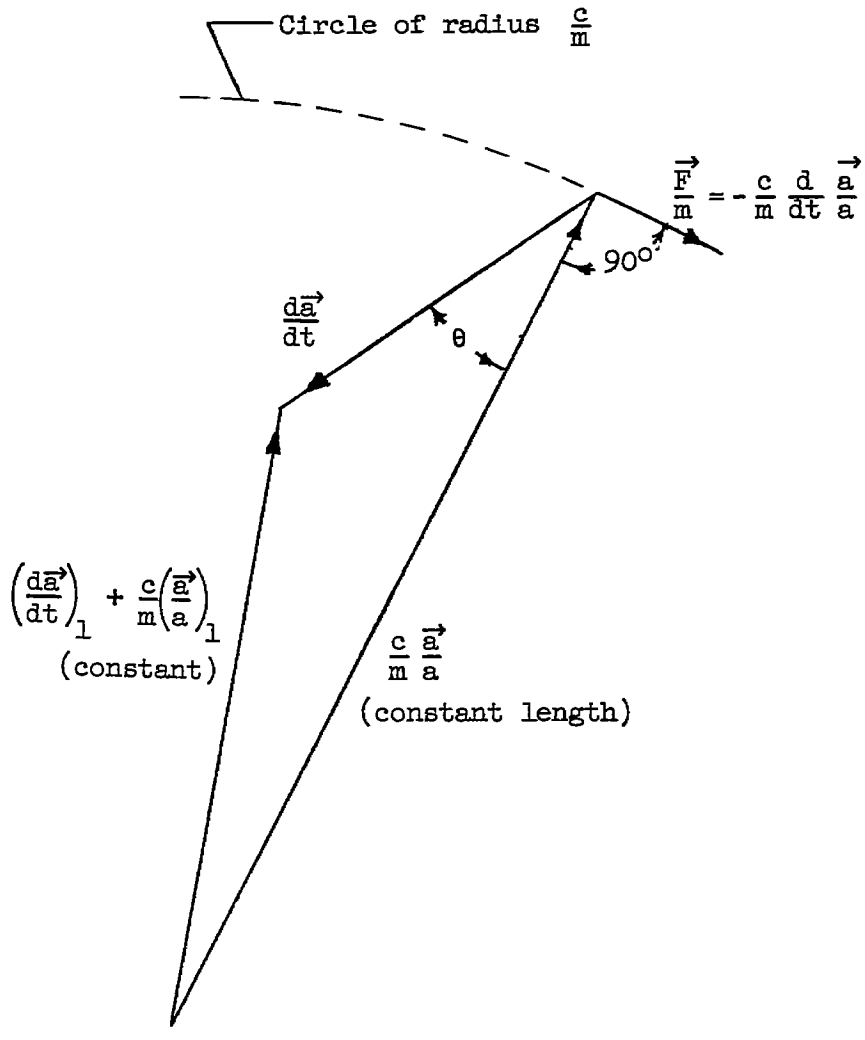


Figure 10.- Vector diagram corresponding to equation (20).